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On products in a real moment-angle manifold

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Abstract. In this paper we give a necessary and sufficient condition for a (real) moment-angle complex to be a topological manifold. The cup and cap products in a real moment-angle manifold are studied. Consequently, the cohomology ring (with coefficients integers) of a polyhedral product by pairs of disks and their bounding spheres is isomorphic to that of a differential graded algebra associated to K and the dimensions of the disks.

1. Introduction.

Let m be a positive integer, and let K be an *abstract simplicial complex* with vertex set $[m] = \{1, 2, \ldots, m\}$. Thus $\emptyset \in K$, and any subset of $\sigma \subset [m]$ is a simplex of K, if σ is. We use the notation $|K|$ for the geometric realization of K. Denote by $(\underline{X}, \underline{A})$ m pairs of topological spaces (X_i, A_i), $i = 1, 2, \ldots, m$, and let $\prod_{i=1}^m X_i$ be the m-fold Cartesian product of X_i. For $x = (x_i)_{i=1}^m \in \prod_{i=1}^m X_i$, define

$$\sigma_x = \{i \in [m] \mid x_i \in X_i \setminus A_i\},$$

then the corresponding *polyhedral product* (following [**BBCG10a**] and [**BP15**]) is given by

$$(\underline{X}, \underline{A})^K = \left\{ x \in \prod_{i=1}^m X_i \mid \sigma_x \in K \right\}. \tag{1}$$

If all pairs (X_i, A_i) are homeomorphic to a given one, (X, A), then $(\underline{X}, \underline{A})^K$ shall be denoted by $(X, A)^K$.

$(D^1, S^0)^K$ is referred to as a *real moment-angle complex* (cf. [**BP02**, Section 6.6]). If it is a topological manifold, then we call it a *real moment-angle manifold*. Following [**BP02**], [**GL13**] and [**BBCG12**], in this paper we focus on two problems:

(P-1) the characterization of a real moment-angle manifold, and

(P-2) cup and cap products in its (co)homology.

(P-1) was answered by M. Davis, under the assumption that K is a *flag complex*, i.e., any set of vertices of K that are pairwise connected by edges spans a simplex of

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K (cf. [**Dav08**, Theorem 10.6.1, p. 197]).[1] It follows that $(D^1, S^0)^K$ is a topological n-manifold if and only if $|K|$ is a *generalized homology* $(n-1)$-sphere (i.e., a homology $(n-1)$-manifold having the homology of an $(n-1)$-sphere), with $\pi_1(|K|) = 1$ when $n \neq 1, 2$.

In Section 2 we shall prove that Davis's characterization theorem still holds, without assuming the flagness of K (see Theorem 2.3). Together with the construction in [**BBCG10b**] given by Bahri, Bendersky, Cohen and Gitler (see Definition 2.6),[2] it follows that a *moment-angle complex* $(D^2, S^1)^K$ is a topological $(n+m)$-manifold if and only if $|K|$ is a generalized homology $(n-1)$-sphere.

Now we turn to (P-2). Unless otherwise stated, we always suppose that all coefficients taken in (co)homology groups are integers.

The additive structure of $H^*((D^1, S^0)^K)$ is well-known.[3] It turns out that, we have isomorphisms

$$H^p((D^1, S^0)^K) \cong \bigoplus_{\omega \subset [m]} H^{p-1}(K_\omega) \tag{2}$$

in all dimensions $p \geq 0$, where K_ω is the *full subcomplex* with respect to ω.

Let $J = (j_i)_{i=1}^m$ be an m-tuple of positive integers, and denote by $(\underline{D}, \underline{S})^K$ the polyhedral product with respect to pairs $(D^{j_i}, S^{j_i - 1})$ of j_i-disks and their bounding spheres, $i = 1, 2, \ldots, m$. With a general approach from homotopy theory, the information of the ring $H^*((D^1, S^0)^K)$ is given in [**BBCG12**], as well as its relation with other polyhedral products, including $(\underline{D}, \underline{S})^K$.

Our approach here follows that of Baskakov, Buchstaber and Panov [**BBP04**] and Panov [**Pan08**] on $H^*((D^2, S^1)^K)$: we will show that $H^*((D^1, S^0)^K)$ is also isomorphic to the cohomology a differential graded algebra R_K^*, which is *not* commutative in any sense (see (32), (33)). Technically, we use the result of Whitney [**Whi38**] on the cup and cap products in a Cartesian product of simplicial complexes, and prove that it applies to the situation here (see Theorem 3.1). Together with the method from [**BBCG12**], we will show that the ring $H^*((\underline{D}, \underline{S})^K)$ can be understood uniformly (see Theorem 5.1, Remark 5.2).

In the language of the intersections of submanifolds, rules for the cup products were understood by Gitler and López de Medrano [**GL13**]. We shall follow their approach to make a comparison of the two rings $H^*((D^1, S^0)^K)$ and $H^*((D^2, S^1)^K)$ in Example 5.4 (compare [**GPW01**]).

The paper is organized as follows. Section 2 is devoted to the characterization of a real moment-angle manifold; as a corollary, the characterization of a moment-angle manifold is given in Subsection 2.1. The cup and cap products in a real moment-angle complex, based on Whitney's formulae (13) and (14), is given in Section 3; while the proof the main theorem, Theorem 3.1, is postponed to Section 6. Most of the explicit

[1] The assumption implies that $(D^1, S^0)^K$ is *aspherical*, whose fundamental group is isomorphic to the commutator subgroup of the associated right-angled Coxeter group; see [**Dav08**, pp. 11–12].

[2] This construction was intensely used in [**LdM89**] and [**GL13**] in the language of simple polytopes.

[3] See for example, [**Dav83**], [**LdM89**] from reflection groups, [**BBCG10a**] from homotopy theory, also the Goresky–MacPherson Formula [**GM88**] together with the fact that $(D^1, S^0)^K$ is the deformation retract of the *coordinate subspace arrangement complement* $(\mathbb{R}, \mathbb{R} \setminus \{0\})^K$ (see [**BP02**, Theorem 8.9]).

calculations are taken in Section 4, based on a special cellular (co)chain complex given in Subsection 3.2.[4] In Section 5, we describe explicitly the cup products in the polyhedral product $(\underline{D}, \underline{S})^K$, via the multiplication in the corresponding differential graded algebra (see Theorem 5.1). Section 6 is devoted to the proof of Theorem 3.1.

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2. Davis's characterization theorem.

In this section, all manifolds are assumed to have no boundaries.

The dimension of a simplex $\sigma \in K$ is given by $\mathrm{card}(\sigma) - 1$. Thus $\dim(\sigma) = -1$ if and only if $\sigma = \emptyset$. Let K' be the *derived complex* of K, where a k-simplex is a chain $(\sigma_0, \sigma_1, \ldots, \sigma_k)$ of simplices in K, each of *non-negative* dimension, such that $\sigma_i \subset \sigma_{i+1}$ is a proper face, $i = 0, 1, \ldots, k-1$. Clearly $|K'|$ is the barycentric subdivision of $|K|$. Let K'_+ be the *augmentation* of K': the dimension of the starting simplex in each chain of K'_+ can be negative. Denote by $|K'_+|$ the cone over $|K'|$, with the collapsed end point corresponding to $(\emptyset)$. (While $\emptyset \in K$ has no geometric meanings in $|K|$.) In what follows, we treat $|K'|$ as a subcomplex of $|K'_+|$.

Let X_K be the intersection of $(D^1, S^0)^K$ with the first orthant of $\mathbb{R}^m$, namely the set $\{(x_i)_{i=1}^m \in \mathbb{R}^m \mid x_i \geq 0, i = 1, 2, \ldots, m\}$. By (1), X_K can be decomposed as the union of cubes

$$\bigcup_{\sigma \in K} C_\sigma; \quad C_\sigma = \prod_{i=1}^m Y_i,\ Y_i = \begin{cases} [0,1] & \text{if } i \in \sigma, \\ \{1\} & \text{otherwise.} \end{cases} \tag{3}$$

Lemma 2.1 (cf. [**BP02**, Chapter 4, pp. 53–55]). *Let $\varphi\colon K \to X_K$ be the map sending each simplex σ to the point $\{x_i\}_{i=1}^m$, where $x_i = 0$ if $i \in \sigma$ and $x_i = 1$ otherwise. Then the mapping $\varphi'\colon |K'_+| \to X_K$ which sends each k-simplex $|(\sigma_0, \sigma_1, \ldots, \sigma_k)|$ to the linear simplex spanned by $\varphi(\sigma_0), \varphi(\sigma_1), \ldots, \varphi(\sigma_k)$, yields a triangulation of X_K.*

Proof. Suppose that $\sigma \in K$ is a k-simplex. Let $K'_{\leq \sigma} \subset K'_+$ be the subcomplex with maximal chains starting from $\emptyset$ and ending with σ, i.e., each is of the form

$$\sigma' = (\sigma_{-1}, \sigma_0, \ldots, \sigma_k)$$

with $\dim(\sigma_i) = i$, such that $\sigma_k = \sigma$. For $i = -1, 0, \ldots, k$, suppose $\varphi(\sigma_i) = (x_{i,j})_{j=1}^m$. By definition, $x_{i.j}$ has value of 0 or 1, $x_{i,j} \geq x_{i+1,j}$ and two adjacent points $\varphi(\sigma_i)$ and $\varphi(\sigma_{i+1})$ have Euclidean distance 1. We see that $\varphi'(|\sigma'|)$ is spanned by lattice points on a path connecting the two end-points on the diagonal of a cube. Then the standard way of triangulating a product of simplices implies that (cf. Eilenberg and Steenrod

[4]The cellular (co)chain complex here is used by Choi and Park [**CP13**] to show that, any odd torsion can appear in the cohomology of a *real toric manifold* or a *small cover*.

[**ES52**, p. 68]), the restriction $\varphi'|_{|K'_{\leq\sigma}|}$ triangulates C_σ. In this way X_K is triangulated by φ'. $\square$

Let $\mathbb{R}^m$ be endowed with the $\mathbb{Z}_2^m$-action generated by $\{s_i\}_{i=1}^m$, with s_i the reflection changing the sign of the i-th coordinate of each point. Denote by X_i the subspace of X_K fixed by s_i, $i = 1, 2, \ldots, m$. It can be checked that the inverse image of X_i under φ' in the lemma above is the star of $|(i)|$ in $|K'|$. Suppose that $Y_i = (\varphi')^{-1}(X_i)$. Clearly $|K'| = \bigcup_{i=1}^m Y_i$, and for $x \in |K'_+|$, denote by I_x the set $\{i \in [m] \mid x \in Y_i\}$ (which can be empty). Let $\mathcal{U}(\mathbb{Z}_2^m, |K'_+|)$ be the *basic construction* with respect to $\mathbb{Z}_2^m$, $|K'_+|$ and $\{Y_i\}_{i=1}^m$ (cf. [**Dav08**, Chapter 5]), which is a $\mathbb{Z}_2^m$-space given by

$$\mathcal{U}(\mathbb{Z}_2^m, |K'_+|) = \left(\mathbb{Z}_2^m \times |K'_+|\right) / \sim, \tag{4}$$

where $(g, x) \sim (g', x')$ if and only if $x = x' \in |K'|$ and $g^{-1}g' \in \langle s_i \rangle_{i \in I_x}$ ($g = g'$ if $I_x = \emptyset$), and the action follows $g'[g, x] = [g'g, x]$. It can be checked directly that the map

$$\begin{aligned} u_{\varphi'} \colon \mathcal{U}(\mathbb{Z}_2^m, |K'_+|) &\longrightarrow (D^1, S^0)^K \\ [g, x] &\longmapsto g\varphi'(x), \end{aligned} \tag{5}$$

is a homeomorphism preserving the $\mathbb{Z}_2^m$-actions on both sides. Then it follows from Lemma 2.1 that $u_{\varphi'}$ triangulates $(D^1, S^0)^K$, such that $|K'|$ appears as the link of $\varphi'(|(\emptyset)|) = (1, 1, \ldots, 1)$.

In what follows, a *polyhedron* is a subset of Euclidean space, in which each point has a neighborhood being a compact and linear cone. It is well-known that a polyhedron can be triangulated.

Definition 2.2. A triangulated polyhedron X is a *homology n-manifold* if the link of each p-simplex, $0 \leq p \leq n$, has the homology of an $(n-1-p)$-sphere.[5] A homeomorphism f between two polyhedra is *piecewise linear* (abbreviated PL), if there exist suitable triangulations on both sides such that f is simplicial.

A polyhedral homology n-manifold X is called a *generalized homology n-sphere* (resp. a PL n-sphere) if it has the homology of an n-sphere (resp. is PL homeomorphic to the boundary of an $(n+1)$-simplex). X (triangulated) is a *piecewise linear n-manifold* if the link of each vertex is a PL $(n-1)$-sphere.

Let $(D^1, S^0)^K$ be equipped with the triangulation $u_{\varphi'}$ (see (5)). We see that a necessary condition for $(D^1, S^0)^K$ to be a homology n-manifold (resp. a PL n-manifold) is that $|K'|$ is a generalized homology $(n-1)$-sphere (resp. a PL $(n-1)$-sphere), where we can replace $|K'|$ by $|K|$ since they are PL homeomorphic to each other. In fact these conditions are also sufficient:

Theorem 2.3. *The real moment-angle complex $(D^1, S^0)^K$ is a homology n-manifold* (*resp. a* PL *n-manifold*)*, if and only if $|K|$ is a generalized homology $(n-1)$-sphere* (*resp. a* PL *$(n-1)$-sphere*); *suppose that $(D^1, S^0)^K$ is a homology n-manifold, then it is a topological manifold if and only if $|K|$ is simply connected when $n \geq 3$.*

[5]This is equivalent to the definition using local homology groups, see [**Mun84**, Exercise 64.1, p. 377].

REMARK 2.4. The argument of Panov and Ustinovsky [**PU12**, Theorem 2.2] can be used directly to prove that $(D^1, S^0)^K$ is homeomorphic to a smooth manifold, if K is induced from a *complete simplicial fan* (see also Tambour [**Tam12**]), including the well-known case that $|K|$ bounds a convex polytope.

The famous Edwards–Freedman Theorem (see [**Dav08**, Theorem 10.4.10, p. 194] and references therein) asserts that, a triangulated polyhedral homology n-manifold ($n \geq 3$) is a topological manifold, if and only if the link of each vertex is simply connected. Therefore, it suffices to check the link of each vertex.

We proceed with the proposition below, whose proof will be given after that of Theorem 2.3. Recall that for two disjoint polyhedra X and Y embedded in $\mathbb{R}^N$, their *exterior join exists*, if any two line segments, each joining two points in X and Y respectively, meet in at most one common endpoint, or coincide otherwise. If exists, then their *join* $X * Y$ is given by $\{tx + (1-t)y \mid x \in X,\ y \in Y,\ t \in [0,1]\}$.

PROPOSITION 2.5. *With the triangulation $u_{\varphi'}$, the link of a vertex in $(D^1, S^0)^K$ is either $|K'|$, or is* PL *homeomorphic to the join*

$$\underbrace{S^0 * S^0 * \cdots * S^0}_{k+1} * |\mathrm{Lk}(\sigma', K')|, \tag{6}$$

where $\sigma' \in K'$ is a k-simplex ($k \geq 0$), $\mathrm{Lk}(\sigma', K')$ *the link of σ' in K'.*

PROOF OF THEOREM 2.3. First if $n \leq 2$, then $|K|$ can always be realized as the boundary of a convex polytope in $\mathbb{R}^n$, thus the statement follows from Remark 2.4.

Suppose that $n \geq 3$. It is easy to see that the link of any k-simplex ($k \geq 0$) in a homology $(n-1)$-manifold (resp. a PL $(n-1)$-manifold) is a generalized homology $(n-k-2)$-sphere (resp. a PL $(n-k-2)$-sphere).[6] In particular, $|\mathrm{Lk}(\sigma', K')|$ is connected if $k = 0$, thus the space (6) is always simply connected. Together with the fact that the exterior join of a generalized homology sphere (resp. a PL sphere) with S^0 will be a sphere of the same type,[7] the statement follows from Proposition 2.5, and the Edwards–Freedman Theorem. □

PROOF OF PROPOSITION 2.5. Since $\mathbb{Z}_2^m$ acts on $\mathcal{U}(\mathbb{Z}_2^m, |K'_+|)$ simplicially, it suffices to consider the links of each vertex in the image of φ'. The link of $\varphi'(|\emptyset|)$ is $|K'|$. In what follows we consider other links.

Consider a vertex $(\sigma) \in K'_+$, where $\sigma \in K$ is a k-simplex, $k \geq 0$. Let $K'_{<\sigma}$ (resp. $K'_{\geq\sigma}$) be the subcomplex of K'_+ consisting of chains that end with a *proper face* of σ (resp. *begins* with σ). By definition, $\varphi'(|K'_{<\sigma}|) * \varphi'(|K'_{\geq\sigma}|)$ exists, which is $\varphi'(|K'_{<\sigma} * K'_{\geq\sigma}|)$, where we see that $K_{<\sigma} * K'_{\geq\sigma}$ is the subcomplex of chains containing σ, i.e., it is the star of (σ) in K'_+. It follows that $|K'_{\geq\sigma}| = \bigcap_{i\in\sigma} Y_i$, where $\varphi'(Y_i)$ is fixed by s_i (see (4)), thus the star of $|(\sigma)|$ in $\mathcal{U}(\mathbb{Z}_2^m, |K'_+|)$ is given by

[6] For instance, by induction on dimension k, with the observation that the link of a simplex σ in the manifold is that of a vertex in the link of a codimension-1 face of σ.

[7] the first case is easy; see [**RS72**, Proposition 2.23, pp. 23–24] for the PL case

$$\bigcup_{g\in\langle s_i\rangle_{i\in\sigma}} g|K'_{<\sigma} * K'_{\geq\sigma}|. \tag{7}$$

Since $\varphi(|K'_{\geq\sigma}|)$ is invariant under the subgroup $\langle s_i\rangle_{i\in\sigma}$, the image of (7) under φ' is

$$\bigcup_{g\in\langle s_i\rangle_{i\in\sigma}} g\varphi'(|K'_{<\sigma}|) * \varphi'(|K'_{\geq\sigma}|) = \left(\bigcup_{g\in\langle s_i\rangle_{i\in\sigma}} g\varphi'(|K'_{<\sigma}|)\right) * \varphi'(|K'_{\geq\sigma}|). \tag{8}$$

Let $\sigma' \in K'$ be any k-simplex (as a chain of length $k+1$) ending with σ. Then

$$\varphi'(|K'_{\geq\sigma}|) = \varphi'(|(\sigma)|) * \varphi'(|\mathrm{Lk}(\sigma', K')|) = \varphi'(|(\sigma)|) * |\mathrm{Lk}(\sigma', K')|. \tag{9}$$

At last, let $K'_{\leq\sigma}$ be the subcomplex of K'_+ consisting of chains ending with a face of σ. It is straightforward to see that $\widetilde{C}_\sigma = \bigcup_{g\in\langle s_i\rangle_{i\in\sigma}} g\varphi'(|K'_{\leq\sigma}|)$ is a $(k+1)$-cube centered at $\varphi'(|(\sigma)|)$, i.e., C_σ in (3) with $[0,1]$ replaced by $[-1,1]$. It follows that $\bigcup_{g\in\langle s_i\rangle_{i\in\sigma}} g\varphi'(|K'_{<\sigma}|)$ is the boundary of $\widetilde{C}_\sigma$, hence it is a k-sphere PL homeomorphic to the joins of $k+1$ copies of S^0. Together with (9) and (8), Proposition 2.5 follows. $\square$

2.1. An application of Theorem 2.3.

Suppose the vertex set of K is $[m]$. A subset $\tau \subset [m]$ not contained in K is called a *missing face*, if any proper subset of τ is a simplex of K. Clearly K is determined by its missing faces.

Let $J = (j_k)_{k=1}^m$ be an m-tuple of positive integers, with $d(J) = \sum_{k=1}^m j_k$. For $i = 1, 2, \ldots, m$, let B_i be the block of j_i integers (we set $j_0 = 0$)

$$\sum_{k=1}^{i-1} j_k + 1,\ \sum_{k=1}^{i-1} j_k + 2,\ \ldots,\ \sum_{k=1}^{i} j_k.$$

DEFINITION 2.6 (cf. [**BBCG10b**]). Suppose that $\tau = \{i_1, i_2, \ldots, i_l\}$ is a missing face of K. Denote by $\tau(J)$ the set $\{B_{i_1}, B_{i_2}, \ldots, B_{i_l}\}$. With respect to K and J, we define an abstract simplicial complex $K(J)$ in such a way that, as τ runs through all missing faces of K, $\tau(J)$ gives all missing faces of $K(J)$. The vertex set of $K(J)$ is $[d(J)]$.[8]

In what follows, let $(\underline{D}, \underline{S})$ be the pairs (D^{j_i}, S^{j_i-1}), i.e., the unit j_i-disk with its boundary, $i = 1, 2, \ldots, m$.

LEMMA 2.7 (cf. [**BBCG10b**]). *The polyhedral product $(\underline{D}, \underline{S})^K$ is homeomorphic to the real moment-angle complex $(D^1, S^0)^{K(J)}$ (see* (1) *for definition).*

PROOF. Let $f = (f_1, f_2, \ldots, f_m)$ be the product of canonical homeomorphisms $f_i\colon I^{j_i} \to D^{j_i}$, $I = [-1,1]$. We write $f\colon I^{d(J)} \to \prod_{i=1}^m D^{j_i}$, which is clearly a homeomorphism. We claim that the restriction of f to $(D^1, S^0)^{K(J)}$ induces a homeomorphism onto $(\underline{D}, \underline{S})^K$. By (1), $x = (x_i)_{i=1}^m \in (\underline{D}, \underline{S})^K$ if and only if $\sigma_x = \{i \in [m] \mid x_i \in D^{j_i} \setminus S^{j_i-1}\}$ does not contain any missing faces of K. This happens if and only if $\sigma_x(J) = \{B_i \mid$

[8] The definition here is slightly different from [**BBCG10b**, Definition 2.1]: we identify the vertices of $K(J)$ with $[d(J)]$ a priori and explicitly.

$f_i^{-1}(x_i) \in I^{j_i} \setminus \partial I^{j_i}\}$ does not contain any missing faces of $K(J)$. Therefore, the claim holds, from which the statement follows. $\square$

Following [**BBCG10b**], $K(J)$ can be understood in another way.

Recall that the *simplicial join* of two disjoint complexes K_1 and K_2 is the complex $K_1 * K_2 = \{\sigma_1 \cup \sigma_2 \mid \sigma_i \in K_i, i = 1, 2\}$. Suppose that $v_i = \{i\}$ is the i-th vertex of K, $i = 1, 2, \ldots, m$. Let $K(v_i)$ be the *simplicial wedge* of K on v_i, which is a simplicial complex given by

$$K(v_i) = \{i\} * K_{[m]\setminus\{i\}} \bigcup \{i+1\} * K_{[m]\setminus\{i\}} \bigcup \{i, i+1\} * \mathrm{Lk}(v_i, K), \tag{10}$$

where $K_{[m]\setminus\{i\}} = \{\sigma \in K \mid \sigma \subset [m] \setminus \{i\}\}$ is the *full subcomplex* containing the link $\mathrm{Lk}(v_i, K)$, such that for all $j > i$, the label of the original j-th vertex is shifted to $j+1$ (labels $\leq i$ are preserved). Thus the vertex set of $K(v_i)$ is $[m+1]$.

For $i = 1, 2, \ldots, m$, let $J_i = (j_k)_{k=1}^m$ be the tuple with $j_i = 2$, and $j_k = 1$ for all $k \neq i$. One can check that $K(J_i) = K(v_i)$. In this way, $K(J)$ can be obtained via consecutive simplicial wedge constructions. Moreover, it can be shown that $|K(v_i)|$ is PL homeomorphic to the suspension $S^0 * |K|$ (see for example, [**CP15**, Proposition 2.2]). As a conclusion, we have the lemma below, the proof of which is omitted (see the proof of Theorem 2.3).

LEMMA 2.8. *$|K(v_i)|$ is a generalized homology sphere* (*resp. a* PL *sphere*) *if and only if $|K|$ is* (*see Definition* 2.2).

PROPOSITION 2.9. *The polyhedral product $(D^1, S^0)^{K(J)}$ is a homology manifold* (*resp. a* PL *manifold*) *of dimension $n + d(J) - m$, if and only if $|K|$ is a generalized homology sphere* (*resp. a* PL *sphere*) *of dimension $n - 1$. Moreover, $(D^1, S^0)^{K(J)}$ is a topological manifold if additionally $d(J) > m$.*

PROOF. Since every simplicial wedge construction increases the dimension by one, the first statement follows directly from Theorem 2.3, and the lemma above. For the second one, from the Van Kampen theorem, we only need to consider the case when $|K|$ is the 0-sphere ($m = 2$), with $d(J) - m = 1$. In this case $|K(J)|$ bounds the 2-simplex, thus $(D^1, S^0)^{K(J)}$ is the 2-sphere. $\square$

In particular, suppose that all pairs are taken as (D^2, S^1). Together with Lemma 2.7, we have (cf. [**BP02**, Problem 6.14]):

COROLLARY 2.10. *The moment-angle complex $(D^2, S^1)^K$ is a topological $(n+m)$-manifold if and only if $|K|$ is a generalized homology $(n-1)$-sphere.*

3. Cup and cap products.

In this section we consider the cohomology of a real moment-angle complex $(D^1, S^0)^K$, with products involved.

3.1. Whitney's formulae.

We say that an abstract simplicial complex K is of *finite type*, if the number of simplices in each dimension is finite. Let $C_*(K)$ (resp. $C^*(K)$) denote the simplicial chain complex (resp. cochain complex) of K, where $C_*(K) = \bigoplus_{i=0}^{\infty} C_i(K)$ (resp. $C^*(K) = \bigoplus_{i=0}^{\infty} C^i(K)$). If K is of finite type, then each $C_i(K)$ is finitely generated, whose dual basis generates $C^i(K) = \mathrm{Hom}(C_i(K), \mathbb{Z})$.

Suppose that $\boldsymbol{p} = (p_i)_{i=1}^m$ and $\boldsymbol{q} = (q_i)_{i=1}^m$ are two vectors of integers. The notation $(\boldsymbol{p}, \boldsymbol{q})$ means the mod 2 integer from the shuffle

$$p_1, p_2, \ldots, p_m, q_1, q_2, \ldots, q_m \to p_1, q_1, p_2, q_2, \ldots, p_m, q_m,$$

namely by interchanging adjacent integers to make the left sequence into the right one, each interchange yields a summand, being the product of the two integers involved, and $(\boldsymbol{p}, \boldsymbol{q})$ is the sum of these summands. A straightforward calculation shows that

$$(\boldsymbol{p}, \boldsymbol{q}) = \sum_{i=1}^{m} q_i \sum_{j>i} p_j \mod 2. \tag{11}$$

Let $X = \prod_{i=1}^m |K_i|$ be a product of polyhedra, with each K_i of finite type. Denote by $C_*(X) = \bigoplus_{p=0}^{\infty} C_p(X)$ the tensor product $\bigotimes_{i=1}^m C_*(K_i)$, which is a differential graded $\mathbb{Z}$-module with the boundary operator ∂ subject to

$$\partial \left(\bigotimes_{i=1}^{m} c_{p_i} \right) = (-1)^{\sum_{j<i} p_j} c_{p_1} \otimes \cdots \otimes c_{p_{i-1}} \otimes \partial c_{p_i} \otimes c_{p_{i+1}} \otimes \cdots \otimes c_{p_m}, \tag{12}$$

where $c_{p_i} \in C_{p_i}(K)$. Let $(C^*(X), \mathrm{d})$ be the *graded dual* of $(C_*(X), \partial)$. By assumption, we have

$$C^p(X) = \bigoplus_{\sum_{i=1}^m p_i = p} \bigotimes_{i=1}^{m} C^{p_i}(K_i).$$

The notation $(C_*(X), C^*(X), \smile, \frown)$ means that $C_*(X)$ and $C^*(X)$ are endowed with products

$$\smile\colon C^*(X) \otimes C^*(X) \to C^*(X) \quad \text{and} \quad \frown\colon C^*(X) \otimes C_*(X) \to C_*(X)$$

respectively, such that

$$\left(\bigotimes_{i=1}^{m} c^{p_i} \right) \smile \left(\bigotimes_{i=1}^{m} c^{q_i} \right) = (-1)^{(\boldsymbol{p}, \boldsymbol{q})} \bigotimes_{i=1}^{m} c^{p_i} \smile c^{q_i} \tag{13}$$

with $c^{p_i} \in C^{p_i}(K_i)$, $c^{q_i} \in C^{q_i}(K_i)$, $\boldsymbol{p} = (p_i)_{i=1}^m$ and $\boldsymbol{q} = (q_i)_{i=1}^m$, and

$$\left(\bigotimes_{i=1}^{m} c^{p_i} \right) \frown \left(\bigotimes_{i=1}^{m} c_{r_i} \right) = (-1)^{(\boldsymbol{r} - \boldsymbol{p}, \boldsymbol{p})} \bigotimes_{i=1}^{m} c^{p_i} \frown c_{r_i} \tag{14}$$

with $c_{r_i} \in C_{r_i}(K_i)$ and $\boldsymbol{r} = (r_i)_{i=1}^m$, where each product $\smile$ and $\frown$ on the right-hand sides of (13) and (14) means the *simplicial cup and cap products*, respectively (see (23) and (24)).

We treat X as a CW complex, with each cell of the form $\prod_{i=1}^m |\sigma_i|$, $\sigma_i \in K_i$. Let A be a subcomplex of X. Denote by $C_*(A)$ the restriction of $C_*(X)$ to A; clearly $C_*(A)$ is closed under ∂. Let $\upsilon_c\colon C_*(A) \to C_*(X)$ be the chain inclusion, and let $(C^*(A), \mathrm{d})$ be the graded dual of $(C_*(A), \partial)$. Since the dual υ_c^* is surjective, the quadruple $(C_*(A), C^*(A), \smile, \frown)$ can be defined uniquely, such that diagrams

$$\begin{array}{ccc} C^*(X) \otimes C^*(X) & \xrightarrow{\ \smile\ } & C^*(X) \\ {\scriptstyle \upsilon_c^* \otimes \upsilon_c^*}\big\downarrow & & {\scriptstyle \upsilon_c^*}\big\downarrow \\ C^*(A) \otimes C^*(A) & \xrightarrow{\ \smile\ } & C^*(A) \end{array} \quad \text{and} \quad \begin{array}{ccc} C^*(X) \otimes C_*(X) & \xrightarrow{\ \frown\ } & C_*(X) \\ {\scriptstyle \upsilon_c^*}\big\downarrow \quad {\scriptstyle \upsilon_c}\big\uparrow & & {\scriptstyle \upsilon_c}\big\uparrow \\ C^*(A) \otimes C_*(A) & \xrightarrow{\ \frown\ } & C_*(A) \end{array}$$

commute.

THEOREM 3.1. *Let $(H_*(A), H^*(A), \smile, \frown)$ be the quadruple of singular (co)homology of A, endowed with cup and cap products. Then we have an isomorphism*

$$(H_*(C_*(A), \partial), H^*(C^*(A), \mathrm{d}), \smile, \frown) \cong (H_*(A), H^*(A), \smile, \frown) \tag{15}$$

on passage to (co)homology, preserving the products.

REMARK. The quadruple $(H_*(C_*(X), \partial), H^*(C^*(X), \mathrm{d}), \smile, \frown)$ with formulae (13) and (14) were given in Whitney [**Whi38**, pp. 424–426].

Theorem 3.1 will be illustrated in what follows; the proof will given in Section 6.

3.2. The (co)homology of a real moment-angle complex.

Henceforth, occasionally we will not distinguish a geometric cell (resp. geometric simplex) or the associated cellular chain (resp. simplicial chain).

Let $I = [-1, 1]$ be a simplicial complex with a single 1-simplex u connecting the two endpoints $\underline{t} = \{-1\}$ and $t = \{1\}$, respectively, and they are obviously oriented that $\partial u = t - \underline{t}$. Let I^m be the CW-complex, with each cell an m-fold product of simplices of the form above.

By (1), $(D^1, S^0)^K$ is a cellular subcomplex embedded in I^m, such that a cell $e = \prod_{i=1}^m e_i \subset I^m$ belongs to $(D^1, S^0)^K$ if and only if $\sigma_e = \{i \in [m] \mid e_i = u\}$ is a simplex of K.

In the remainder of this section, let (X, A) be the pair $(I^m, (D^1, S^0)^K)$.

Now we perform a change of basis for $C_*(A)$ and $C^*(A)$, respectively. For $(C_*(I), \partial)$ indicated above, denote by

$$\varepsilon = \partial u = t - \underline{t}, \tag{16}$$

then $C_*(I)$ is generated by u, ε and $\underline{t}$. After dualizing, for $(C^*(I), \mathrm{d})$, we use basis elements u^*, t^* and

$$\delta^* = t^* + \underline{t}^*, \tag{17}$$

where u^*, $\underline{t}^*$ and t^* are the dual simplices of u, $\underline{t}$ and t, respectively. Immediately we have $\mathrm{d}t^* = u^*$ and $\mathrm{d}u^* = 0 = \mathrm{d}\delta^*$.

Definition 3.2. Given a basis element $c = \bigotimes_{i=1}^m c_i \in C_*(X)$, $c_i \in C_*(I)$, we define $\sigma_c = \{i \in [m] \mid c_i = u\}$ and $\tau_c = \{i \in [m] \mid c_i = \varepsilon\}$. Then we denote by c the word

$$u_{\sigma_c}\varepsilon_{\tau_c}\underline{t}_{[m]\setminus(\sigma_c\cup\tau_c)},$$

which we shall abbreviate as $u_{\sigma_c}\varepsilon_{\tau_c}$, omitting the obvious part $\underline{t}_{[m]\setminus(\sigma_c\cup\tau_c)}$. Clearly each abbreviated word $u_\sigma\varepsilon_\tau$ with $\sigma\cap\tau=\emptyset$ corresponds uniquely to a chain of $C_*(X)$.

Analogously, we denote a dual basis element $c = \bigotimes c^i \in C^*(X)$, $c^i \in C^*(I)$, by the word

$$u^{\sigma_c}t^{\tau_c}\delta^{[m]\setminus(\sigma_c\cup\tau_c)},$$

with $\sigma_c = \{i \in [m] \mid c^i = u^*\}$ and $\tau_c = \{i \in [m] \mid c^i = t^*\}$, and write it as $u^{\sigma_c}t^{\tau_c}$.

By definition, we see that $C_*(A)$ and $C^*(A)$ are generated by $\{u_\sigma\varepsilon_\tau \mid \sigma\in K, \sigma\cap\tau=\emptyset\}$ and $\{u^\sigma t^\tau \mid \sigma\in K, \sigma\cap\tau=\emptyset\}$ respectively, with the *void word* corresponding to $\underline{t}_{[m]}$ or $\delta^{[m]}$.

For $i\in[m]$, we use the notation $(i,\sigma) = \mathrm{card}\{j\in\sigma \mid j<i\}$. Now take $u_\sigma\varepsilon_\tau \in C_*(A)$ and $u^\sigma t^\tau \in C^*(A)$ respectively, by (12) and its dual form, we have

$$\partial(u_\sigma\varepsilon_\tau) = \sum_{i\in\sigma}(-1)^{(i,\sigma)}u_{\sigma\setminus\{i\}}\varepsilon_{\tau\cup\{i\}}, \tag{18}$$

and

$$\mathrm{d}(u^\sigma t^\tau) = \sum_{\substack{(\sigma\cup\{i\})\in K\\ i\in\tau}}(-1)^{(i,\sigma)}u^{\sigma\cup\{i\}}t^{\tau\setminus\{i\}}. \tag{19}$$

Recall that for each $\omega\subset[m]$, the *full subcomplex* K_ω is given by $\{\sigma\in K \mid \sigma\subset\omega\}$. Consider the augmented chain complex $(\widetilde{C}_*(K_\omega),\partial)$ and its dual $(\widetilde{C}^*(K_\omega),\mathrm{d})$. Each simplex $\sigma\in K_\omega$ can be treated as a subset of $[m]$ equipped with the natural ordering (thus $|\sigma|$ is positively oriented if the indices contained in σ are written in the increasing order), then the simplicial boundary operator satisfies

$$\partial(\sigma) = \sum_{i\in\sigma}(-1)^{(i,\sigma)}\sigma\setminus\{i\}, \tag{20}$$

together with $\partial(\{i\}) = \emptyset_\omega$ for all $\{i\}\in K_\omega$, due to the augmentation. After dualizing, for a dual simplex $\sigma^*\in\widetilde{C}^*(K_\omega)$, we have $\mathrm{d}(\emptyset_\omega) = \sum_{i\in\omega}\{i\}$, and otherwise

$$\mathrm{d}(\sigma^*) = \sum_{\substack{(\sigma\cup\{i\})\in K\\ i\in\omega\setminus\sigma}}(-1)^{(i,\sigma)}\left(\sigma\cup\{i\}\right)^*. \tag{21}$$

A comparison of (18) and (20) (resp. (19) and (21)) yields the following:

Proposition 3.3. *The mapping*

$$\mu\colon \bigoplus_{\omega\subset[m]} \widetilde{C}_*(K_\omega) \longrightarrow C_*(A)$$

given by sending each $\sigma_\omega \in \widetilde{C}_*(K_\omega)$ to the word $u_\sigma\varepsilon_{\omega\setminus\sigma}$ yields a chain isomorphism that shifts the degrees up by one. Analogously, we have the degree-shifting cochain isomorphism

$$\eta\colon \bigoplus_{\omega\subset[m]} \widetilde{C}^*(K_\omega) \xrightarrow{\cong} C^*(A)$$

sending each $\sigma_\omega^* \in \widetilde{C}^*(K_\omega)$ to $u^\sigma t^{\omega\setminus\sigma}$.

Combined with Theorem 3.1, we have isomorphisms

$$H_p(C_*(A),\partial) \cong \bigoplus_{\omega\subset[m]} \widetilde{H}_{p-1}(K_\omega) \quad \text{and} \quad H^p(C^*(A),\mathrm{d}) \cong \bigoplus_{\omega\subset[m]} \widetilde{H}^{p-1}(K_\omega) \tag{22}$$

in each dimension $p \geq 0$.

Recall that, by definition, $\widetilde{H}_{-1}(K_\omega)$ is non-trivial only when $\omega = \emptyset$; $\widetilde{H}_{p-1}(K_\emptyset)$ vanishes if $p > 0$, and is infinite cyclic if $p = 0$. Therefore, we see that $\widetilde{H}_{-1}(K_\emptyset)$ corresponds to $H_0(A)$; the representative of $\widetilde{H}_{-1}(K_\emptyset)$ is the "empty set" of $K_\emptyset$, which is sent by μ to the void word associated to $\underline{t}_{[m]}$, i.e., the point in I^m with constant coordinates -1.

4. Applications of Whitney's formulae.

In order to apply Whitney's formulae (13) and (14) to the cup and cap products in a real moment-angle complex, by Theorem 3.1, it remains to understand the products in the simplicial complex $I = [-1, 1]$.

Recall that the *simplicial cup and cap products* in a simplicial complex $|K|$, i.e., $\smile\colon C^*(K)\otimes C^*(K) \to C^*(K)$ and $\frown\colon C^*(K)\otimes C_*(K) \to C_*(K)$, are given as follows. Choose a partial ordering on the vertex set of K which induces a total ordering on each simplex, and let $[v_{i_0}, v_{i_1}, \ldots, v_{i_p}] \in C_*(K)$ denote such a p-simplex, with $v_0 < v_1 < \cdots < v_p$ in the given ordering. Then we have

$$(c^p \smile c^q)([v_{i_0},\ldots,v_{i_{p+q}}]) = c^p([v_{i_0},\ldots,v_{i_p}])c^q([v_{i_p},\ldots,v_{i_{p+q}}]) \tag{23}$$

and

$$c^p \frown [i_0,\ldots,i_r] = c^p([i_{r-p},\ldots,i_r])[i_0,\ldots,i_{r-p}] \quad (c^p([i_{r-p},\ldots,i_r]) \in \mathbb{Z}), \tag{24}$$

for $c^p \in C^p(K)$ and $c^q \in C^q(K)$.

For instance, let $(C_*(I),\partial)$ and $(C^*(I),\mathrm{d})$ be the simplicial (co)chain complexes given in Subsection 3.2, endowed with bases $\{u,\varepsilon,\underline{t}\}$ and $\{u^*,\delta^*,t^*\}$, respectively (see (16), (17)). It can be easily checked that the following formulae hold:

$$\begin{aligned}
&t^* \smile t^* = t^*, \quad t^* \smile u^* = 0, \quad u^* \smile t^* = u^*, \quad u^* \smile u^* = 0,\\
&\delta^* \smile u^* = u^* \smile \delta^* = u^*, \quad \delta^* \smile t^* = t^* \smile \delta^* = t^*, \quad \delta^* \smile \delta^* = \delta^*;\\
&\delta^* \frown \underline{t} = \underline{t}, \quad u^* \frown u = \underline{t}, \quad u^* \frown \varepsilon = 0, \quad u^* \frown \underline{t} = 0,\\
&t^* \frown \varepsilon = \varepsilon + \underline{t}, \quad t^* \frown u = u, \quad t^* \frown \underline{t} = 0, \quad \delta^* \frown \varepsilon = \varepsilon, \quad \delta^* \frown u = u.
\end{aligned}$$

For $\sigma \subset [m]$, let $\boldsymbol{v}(\sigma) = (v_i)_{i=1}^m \in \mathbb{Z}^m$ be the vector with $v_i = 1$ if $i \in \sigma$, and $v_i = 0$ otherwise. By (13), (14) and the products presented above, choosing two cochains $u^\sigma t^\tau$ and $u^{\sigma'} t^{\tau'}$ from $C^*((D^1, S^0)^K)$ (see Definition 3.2), we have (see (11) for notations)

$$u^\sigma t^\tau \smile u^{\sigma'} t^{\tau'} = (-1)^{\left(\boldsymbol{v}(\sigma), \boldsymbol{v}(\sigma')\right)} u^{\sigma\cup\sigma'} t^{\tau\cup(\tau'\setminus\sigma)} \tag{25}$$

if $\sigma' \cap (\sigma \cup \tau) = \emptyset$ and $(\sigma \cup \sigma') \in K$, otherwise the product vanishes. Additionally, choosing $u_{\sigma''}\varepsilon_{\tau''} \in C_*((D^1, S^0)^K)$, then

$$u^\sigma t^\tau \frown u_{\sigma''}\varepsilon_{\tau''} = (-1)^{\left(\boldsymbol{v}(\sigma''\setminus\sigma), \boldsymbol{v}(\sigma)\right)} \sum_{\gamma\subset(\tau\setminus\sigma'')} u_{\sigma''\setminus\sigma}\varepsilon_{\tau''\setminus\gamma}, \tag{26}$$

provided that $\sigma \subset \sigma''$ and $\tau \subset (\sigma'' \cup \tau'')$; otherwise $u^\sigma t^\tau \frown u_{\sigma''}\varepsilon_{\tau''}$ vanishes. Suppose that $\mathrm{card}(\sigma) = p$ and $\mathrm{card}(\sigma'') = r$, it turns out that (see (44))

$$\partial(u^\sigma t^\tau \frown u_{\sigma''}\varepsilon_{\tau''}) = (-1)^{r-p}\mathrm{d}(u^\sigma t^\tau) \frown u_{\sigma''}\varepsilon_{\tau''} + u^\sigma t^\tau \frown \partial(u_{\sigma''}\varepsilon_{\tau''}). \tag{27}$$

It is convenient to consider $\left(C^*((D^1, S^0)^K), \smile\right)$ as a differential graded algebra with $2m$ generators, such that $\delta^{[m]}$ is the unique identity (see Theorem 5.1 for more details).

Example 4.1. Let K be the pentagon with vertex set $[5]$, whose maximal simplices are

$$\{1,2\},\ \{2,3\},\ \{3,4\},\ \{4,5\},\ \{5,1\}.$$

Using mod 5 integers, it can be checked that full subcomplexes of K with non-vanishing reduced (co)homology are: $K_{i,i+2}$ with $i = 1, 2, 3$, $K_{i,i+3}$ with $i = 1, 2$, $K_{i,i+2,i+3}$ with $i = 1, 2, \ldots, 5$, together with $K_\emptyset$ and $K_{[5]}$. Therefore by (22), $H^1((D^1, S^0)^K)$ is torsion-free with ten generators

$$\alpha_1 = [u^1t^3],\ \alpha_2 = [u^1t^4],\ \alpha_3 = [u^2t^4],\ \alpha_4 = [u^2t^5],\ \alpha_5 = [u^3t^5],$$

and

$$\beta_i = [u^it^{i+2}(1 - t^{i+3})] = [u^it^{i+2} - u^it^{i+2,i+3}] \quad i = 1, 2, \ldots, 5 \mod 5;$$

we see that $H^2((D^1, S^0)^K)$ is generated by $\gamma = [u^{j,j+1}t^{j+2,j+3,j+4}]$, with j an arbitrary mod 5 integer. By Theorem 2.3, $(D^1, S^0)^K$ is an orientable surface of genus 5. Using (25) we can give a presentation of the cup products. Since $H^1(K_\omega)$ is non-trivial only when $\omega = \{1, 2, 3, 4, 5\}$, a direct calculation shows that for all mod 5 integers i, j, $\alpha_i \smile \alpha_j = \beta_i \smile \beta_j = 0$, then

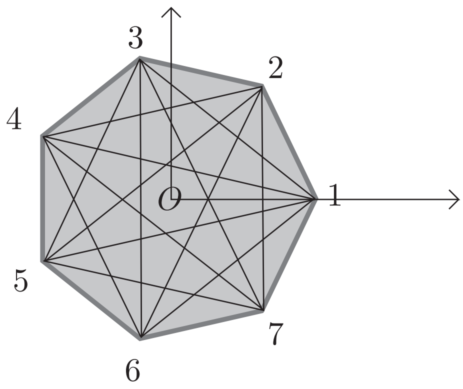

Figure 1. The Heptagon.

$$\gamma = -\alpha_1 \smile \beta_2 = \alpha_2 \smile \beta_5 = -\alpha_3 \smile \beta_3 = \alpha_4 \smile \beta_1 = -\alpha_5 \smile \beta_4$$

presents all non-trivial cup products.

Example 4.2 (cf. [**LdM89**], [**BLV13**], [**GL14**]). Let $\lambda = (\lambda_i)_{i=1}^7$ be the 7-tuple with $\lambda_i \in \mathbb{R}^2$, consisting of the real and imaginary parts of the solutions of the equation $z^7 = 1$. Consider the variety $Z(\Lambda) \subset \mathbb{R}^7$ given by the intersections of quadrics

$$\begin{cases} \sum_{i=1}^7 \lambda_i x_i^2 = \mathbf{0}, \\ \sum_{i=1}^7 x_i^2 = 1. \end{cases}$$

It can be checked directly that $Z(\Lambda)$ is a smooth 4-manifold.[9]

Let K be a simplicial complex with vertex set $[7]$, such that $\sigma \subset [7]$ belongs to K if and only if the origin of $\mathbb{R}^2$ is in the relative interior of the convex hull $\mathrm{conv}(\lambda_i)_{i\in[7]\setminus\sigma}$. It can be shown that $|K|$ bounds a convex polytope, and there is a piecewise differentiable homeomorphism $(D^1, S^0)^K \to Z(\Lambda)$.[10]

Observe that any subset of $[7]$ with cardinality 2 is a simplex of K, and any subset of $[7]$ not in K must contain three consecutive points of the form $\{i, i+1, i+2\}$, $i = 1, 2, \dots, 7$ mod 7 (see Figure 1). Therefore, $(D^1, S^0)^K$ is simply connected (see [**Dav08**, Chapter 1, p. 12]). Using (22), we can write down the orientation class of $(D^1, S^0)^K$ through $H_3(K)$: after listing all 14 3-faces of $|K|$ and assign to each of them a coefficient ± 1 to make a cycle, we see that (see Proposition 3.3)

$$\begin{aligned}\Gamma = [&u_{1,2,4,5}\varepsilon_{3,6,7} - u_{1,2,4,6}\varepsilon_{3,5,7} + u_{1,2,5,6}\varepsilon_{3,4,7} + u_{1,3,4,6}\varepsilon_{2,5,7} - u_{1,3,4,7}\varepsilon_{2,5,6} \\ &- u_{1,3,5,6}\varepsilon_{2,4,7} + u_{1,3,5,7}\varepsilon_{2,4,6} - u_{1,4,5,7}\varepsilon_{2,3,6} + u_{2,3,5,6}\varepsilon_{1,4,7} - u_{2,3,5,7}\varepsilon_{1,4,6} \\ &+ u_{2,3,6,7}\varepsilon_{1,4,5} + u_{2,4,5,7}\varepsilon_{1,3,6} - u_{2,4,6,7}\varepsilon_{1,3,5} + u_{3,4,6,7}\varepsilon_{1,2,5}].\end{aligned}$$

By Alexander duality, we have

[9] See for example, [**BM06**, Lemma 0.3, p. 58].

[10] See for example, [**BM06**, Lemma 0.12, p. 64] for the first statement, and [**Cai15**, Lemma 6.3] for the second.

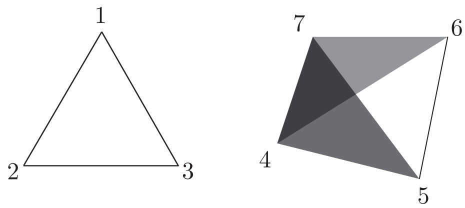

Figure 2. Full subcomplexes $K_{1,2,3}$ and $K_{4,5,6,7}$.

$$\widetilde{H}^{3-i-1}(K_\omega) \cong \widetilde{H}_i(K_{[7]\setminus\omega}). \tag{28}$$

It follows that $\widetilde{H}^*(K_\omega)$ is non-trivial only when $\mathrm{card}(\omega) = 0, 3, 4, 7$. In particular, $\widetilde{H}^1(K_\omega)$ is non-trivial if and only if ω consists of three or four consecutive mod 7 integers (see Figure 2).

As a conclusion, $H^2((D^1, S^0)^K)$ is torsion-free with 14 generators

$$\alpha_i = [u^{i,i+1}t^{i+2}] \quad \text{and} \quad \beta_i = [u^{i+1,i+2}t^{i,i+3}],$$

$i = 1, 2, \ldots, 7$ mod 7, according to $\mathrm{card}(\omega) = 3$ and $\mathrm{card}(\omega) = 4$, respectively. It can be checked that $\alpha_i \smile \alpha_j = 0$, and $\alpha_i \smile \beta_j$ is non-trivial if and only if $j = i + 3$, which generates $H^4((D^1, S^0)^K)$. For instance, $u^{1,2}t^3 \smile u^{5,6}t^{4,7} = u^{1,2,5,6}t^{3,4,7}$. Notice that $\beta_i \smile \beta_{i+4} = \pm u^{i+1,i+2,i+5,i+6}t^{i+3,i+4,i+7}$, which is non-trivial. If we change each β_i into

$$\beta_i' = [u^{i+1,i+2}t^i(1 - t^{i+3})] = [u^{i+1,i+2}t^i - u^{i+1,i+2}t^{i,i+3}],$$

then a straightforward calculation shows that each α_i has a unique pairing β'_{i+3} to make a non-trivial cup product, and $\beta_i' \smile \beta_j'$ vanish for all mod 7 integers i, j. As a conclusion, we see that $(D^1, S^0)^K$ and $\sharp_7 S^2 \times S^2$ have isomorphic cohomology rings, hence the two 4-manifolds are homeomorphic, by the classification theorem of Freedman [**Fre82**] (an explicit diffeomorphism between them is given by Gutiérrez and López de Medrano [**GL14**]).

By Poincaré duality, the intersection of submanifolds can be understood through cup products. For instance, by (26), we have

$$[u^{1,2}t^3 \frown \Gamma] = [\underbrace{u_{4,5}\varepsilon_{6,7} - u_{4,6}\varepsilon_{5,7} + u_{5,6}\varepsilon_{4,7}}_{K_{4,5,6,7}} + \underbrace{u_{4,5}\varepsilon_{3,6,7} - u_{4,6}\varepsilon_{3,5,7} + u_{5,6}\varepsilon_{3,4,7}}_{K_{3,4,5,6,7}}] \tag{29}$$

$$= [\underbrace{u_{4,5}\varepsilon_{6,7} - u_{4,6}\varepsilon_{5,7} + u_{5,6}\varepsilon_{4,7}}_{K_{4,5,6,7}}], \tag{30}$$

because $K_{3,4,5,6,7}$ is acyclic. Here the geometric meaning of the class on the right-hand side of (30) is as follows: consider the sphere

$$S = \{(x_i)_{i=1}^7 \in (D^1, S^0)^K \mid x_1 = x_2 = x_3 = -1, x_7 = 1\}$$

with suitable orientation, and let $s_7\colon \mathbb{R}^7 \to \mathbb{R}^7$ be the reflection changing the sign of the last coordinate, then $[u_{4,5}\varepsilon_{6,7} - u_{4,6}\varepsilon_{5,7} + u_{5,6}\varepsilon_{4,7}]$ corresponds to the class $[S] + s_7[S]$

(see Definition 3.2). Therefore, if we use the representative $[u^{1,2}t^3(1-t^7)]$ in (29) instead of $[u^{1,2}t^3]$, we will get the class $s_7[S]$, which is represented by a submanifold whose intersection with the sphere

$$S' = \{(x_i)_{i=1}^7 \in (D^1, S^0)^K \mid x_4 = x_5 = x_6 = x_7 = -1\}$$

is the point with constant coordinates -1. In the same way, by expanding $[u^{5,6}t^4(1-t^7) \frown \Gamma]$ we see that it coincides with suitable orientation class $[S']$. From the Poincaré duality we can read the plumbing of spheres $s_7(S)$ and S', which can also be checked directly.

5. Cohomology of certain polyhedral products.

Suppose $J = (j_i)_{i=1}^m$ is an m-tuple of positive integers. In this section we consider the relation between $H^*((D^1, S^0)^K)$ and $H^*((D^1, S^0)^{K(J)})$, where $K(J)$ is given in Definition 2.6.

Let $\big(\mathbb{Z}\langle \widetilde{v}^i, \widetilde{u}^i\rangle_{i=1}^m, \mathrm{d}\big)$ be the free algebra generated by $2m$ generators $\widetilde{v}^i$ with $\deg(\widetilde{v}^i) = j_i$ and $\widetilde{u}^i$ with $\deg(\widetilde{u}^i) = j_i - 1$, respectively, $i = 1, 2, \ldots, m$; for two homogeneous elements $x, y \in \mathbb{Z}\langle \widetilde{v}^i, \widetilde{u}^i\rangle_{i=1}^m$, the differential d satisfies

$$\mathrm{d}(xy) = (\mathrm{d}x)y + (-1)^{\deg(x)}x(\mathrm{d}y), \tag{31}$$

with $\mathrm{d}\widetilde{u}^i = \widetilde{v}^i$ and $\mathrm{d}\widetilde{v}^i = 0$. Let $(R^*(J), \mathrm{d})$ be the quotient of $\big(\mathbb{Z}\langle \widetilde{v}^i, \widetilde{u}^i\rangle_{i=1}^m, \mathrm{d}\big)$, subject to relations

$$x^i x^j = (-1)^{\deg(x^i)\deg(x^j)} x^j x^i, \quad (x^i)^2 = \begin{cases} x^i & \text{if } \deg(x^i) = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{32}$$

where x^i (resp. x^j) is either $\widetilde{u}^i$ or $\widetilde{v}^i$ (resp. $\widetilde{u}^j$ or $\widetilde{v}^j$), for *distinct* $i, j = 1, 2, \ldots, m$, together with

$$\widetilde{u}^i\widetilde{v}^i = 0, \quad \widetilde{v}^i\widetilde{u}^i = \begin{cases} \widetilde{v}^i & \text{if } \deg(\widetilde{u}^i) = 0, \\ 0 & \text{otherwise,} \end{cases} \qquad i = 1, 2, \ldots, m. \tag{33}$$

For a simplicial complex K with vertex set $[m]$, the corresponding *Stanley–Reisner ideal* $\mathcal{I}_K(J) \subset R^*(J)$ is generated by all square-free monomials of the form $\widetilde{v}^\tau = \prod_{i \in \sigma} \widetilde{v}^i$, where τ is *not* a simplex of K.

Let $(R^*_K(J), \mathrm{d})$ be the quotient $R^*(J)/\mathcal{I}_K(J)$. It is well-defined since $\mathcal{I}_K(J)$ is closed under d. Clearly by relations (32) and (33), each monomial in $R^*_K(J)$ can be uniquely written in the square-free form $\widetilde{v}^\sigma\widetilde{u}^\tau = \prod_{i=1}^m x^i$, where $x^i = \widetilde{v}^i$ if $i \in \sigma$, $x^i = \widetilde{u}^i$ if $i \in \tau$ and $x^i = 1$ otherwise, σ, τ disjoint subset of $[m]$, $\sigma \in K$.

First let $J = (\mathbf{1})$ be constant with integers 1, and denote $R^*_K(\mathbf{1})$ simply by R^*_K. A comparison of R^*_K with (25) shows that, if we identify $\widetilde{v}^i$ with u^i, $\widetilde{u}^i$ with t^i and the identity 1 with the void word, then we have the cochain isomorphism $(R^*_K, \cdot) \cong \big(C^*((D^1, S^0)^K), \smile\big)$. Together with Theorem 3.1, we see that

$$\Big(H^*(R^*_{K(J)}), \cdot\Big) \cong \Big(H^*((D^1, S^0)^{K(J)}), \smile\Big).$$

Notice that $R^*_{K(J)}$ has $2d(J)$ generators, $d(J) = \sum_{i=1}^m j_i$, which we shall still denote by u^i of degree 1 and t^i of degree 0, $i = 1, 2, \ldots, d(J)$. Actually, the algebra $H^*(R^*_{K(J)})$ can be refined as follows:

THEOREM 5.1. *We have isomorphisms*

$$H^*(R^*_K(J)) \cong H^*(R^*_{K(J)}) \cong H^*((D^1, S^0)^{K(J)}) \cong H^*((D^{j_i}, S^{j_i-1})_{i=1}^m)^K \tag{34}$$

of graded algebras. Moreover, by setting

$$(\sigma, \omega)_J = \sum_{k\in\omega}(j_k - 1)\mathrm{card}(\{i \in \sigma \mid i > k\}) + \sum_{k\in\omega}(j_k - 1) \sum_{r\in\omega, r>k}(j_r - 1) \tag{35}$$

for $\sigma \subset \omega \subset [m]$, the mapping

$$\begin{aligned} \eta_J\colon\ \textstyle\bigoplus_{\omega\subset[m]} \widetilde{C}^*(K_\omega) &\longrightarrow R^*_K(J) \\ \sigma^*_\omega &\longmapsto (-1)^{(\sigma,\omega)_J}\,\widetilde{v}^\sigma \widetilde{u}^{\omega\setminus\sigma} \end{aligned} \tag{36}$$

yields a cochain map ($\sigma^*_\omega \in \widetilde{C}^*(K_\omega)$ is a dual simplex) which induces an isomorphism

$$\bigoplus_{\omega\subset[m]} \widetilde{H}^*(K_\omega) \cong H^*(R^*_K(J))$$

of ungraded $\mathbb{Z}$-modules.

REMARK 5.2. The second isomorphism of (34) follows from Lemma 2.7. When j_i is even, $i = 1, 2, \ldots, m$, this can be deduced from [**BBCG10b**, Proposition 6.2]. In particular, when $J = (\mathbf{2})$ is constant with integers 2, $R^*_K(\mathbf{2})$ is the well-known algebra given in [**BBP04**] and [**Pan08**] for the cohomology of the moment-angle complex $(D^2, S^1)^K$.

In order to prove Theorem 5.1, we consider the simplicial wedge $K(v_i)$ on the i-th vertex (see (10)). Denote by $\varpi_i\colon R^*_K \to R^*_{K(v_i)}$ the additive homomorphism such that for each monomial $u^\sigma t^\tau \in R^p_K$,

$$\varpi_i(u^\sigma t^\tau) = \begin{cases} u^{\chi_i(\sigma)} t^{\chi_i(\tau)} \in R^p_{K(v_i)} & \text{if } i \notin \sigma \cup \tau, \\ u^{\chi_i(\sigma)} t^{\chi_i(\tau)} u^{i+1} \in R^{p+1}_{K(v_i)} & \text{otherwise,} \end{cases} \tag{37}$$

where χ_i is a label-shifting map such that each $\{i_k\}_{k=0}^l \subset [m]$ is sent to $\{i'_k\}_{k=0}^l \subset [m+1]$, in which $i'_k = i_k$ if $i_k \leq i$ and $i'_k = i_k + 1$ otherwise (thus the label $i+1$ is skipped in the image); $\chi_i(\emptyset) = \emptyset$. It can be checked easily that ϖ_i is well-defined and preserves the differential d on both sides.

In what follows, let $R^*_K|_\omega$ be the subalgebra generated by $u^\sigma t^{\omega\setminus\sigma}$, $\sigma \subset \omega$. Clearly as a $\mathbb{Z}$-module, with ω running through subsets of $[m]$, R^*_K is a direct sum with summands $R^*_K|_\omega$.

LEMMA 5.3. *The mapping $\varpi_i\colon R^*_K \to R^*_{K(v_i)}$ induces an additive isomorphism on passage to cohomology. More precisely, ϖ_i induces isomorphisms*

$$H^p(R^*_K|_\omega) \cong H^{p+1}(R^*_{K(v_i)}|_{\chi_i(\omega)\cup\{i+1\}}) \tag{38}$$

if $i \in \omega$, and isomorphisms

$$H^p(R^*_K|_\omega) \cong H^p(R^*_{K(v_i)}|_{\chi_i(\omega)}) \tag{39}$$

if $i \notin \omega$, for each $p \geq 0$.

Proof. First observe that if $i \notin \omega$, then the full subcomplex $K(v_i)_{\chi_i(\omega)}$ and K_ω are simplicially isomorphic. Then (39) follows from Proposition 3.3. It remains to prove (38). Suppose $i \in \omega$, and consider the short exact sequence

$$0 \longrightarrow R^*_K|_\omega \xrightarrow{\varpi_i} R^*_{K(v_i)}|_{\chi_i(\omega)\cup\{i+1\}} \longrightarrow Q^* \longrightarrow 0,$$

in which Q^* is the graded quotient. Let $f\colon \widetilde{C}^*(K(v_i)_{\chi_i(\omega)}) \to Q^*$ be the additive homomorphism sending each dual p-simplex $\sigma^*_{\chi_i(\omega)}$ to the element with representative $u^\sigma t^{\chi_i(\omega)\setminus\sigma} t^{i+1}$. We see that f is a cochain isomorphism shifting the degrees up by one, since

$$\mathrm{d}(u^\sigma t^\tau t^{i+1}) = \mathrm{d}(u^\sigma t^\tau) t^{i+1} + (-1)^{p+1} u^\sigma t^\tau \mathrm{d}t^{i+1},$$

where the second summand vanishes in Q^* (see (37)). It is easy to see that $K(v_i)_{\chi_i(\omega)}$ is the star of v_i in $K(v_i)$, which is acyclic, so is Q^*. Then (38) follows. □

Now we specify a sequence of simplicial wedge constructions for $K(J)$: it can be checked that

$$\begin{aligned} K(J) = K\, &\underbrace{(v_m)(v_{m+1})\cdots(v_{m+j_m-2})}_{j_m-1}(v_{m-1})(v_m)\cdots(v_{m-1+j_{m-1}-2})\cdots \\ &(v_{m-i+1+j_{m-i+1}-2})\,\underbrace{(v_{m-i})(v_{m-i+1})\cdots(v_{m-i+j_{m-i}-2})}_{j_{m-i}-1} \\ &(v_{m-i-1})\cdots(v_{2+j_2-2})\,\underbrace{(v_1)(v_2)\cdots(v_{1+j_1-2})}_{j_1-1}, \end{aligned} \tag{40}$$

where the block marked by $j_i - 1$ is deleted if $j_i = 1$.

Proof of Theorem 5.1. With respect to (40), we define m composite homomorphisms

$$\varpi_{j_k} = \varpi_{k+j_k-2}\varpi_{k+j_k-3}\cdots\varpi_{k+1}\varpi_k,$$

$k = 1, 2, \ldots, m$, and denote their composition by

$$\varpi_J = \varpi_{j_1}\varpi_{j_2}\cdots\varpi_{j_m}.$$

Suppose $\omega = \{i_1, i_2, \ldots, i_l\}$ is a subset of $[m]$, $i_1 < i_2 < \cdots < i_l$. Let $u^\sigma t^{\omega\setminus\sigma}$ ($\sigma \subset \omega$) be the monomial $\prod_{k=1}^l x^{i_k}$ with $x^{i_k} = u^{i_k}$ if $i_k \in \sigma$, and $x^{i_k} = t^{i_k}$ otherwise. By definition

(37) and a straightforward calculation, we have

$$\varpi_J(u^\sigma t^{\omega\setminus\sigma}) = \varpi_{j_1}\varpi_{j_2}\cdots\varpi_{j_{i_m}}(x^{i_1}\cdots x^{i_l}) = x^{\widetilde{i_1}}x^{\widetilde{i_2}}\cdots x^{\widetilde{i_l}}x^{\widetilde{B}_{i_l}}x^{\widetilde{B}_{i_{l-1}}}\cdots x^{\widetilde{B}_{i_1}}, \tag{41}$$

where $\widetilde{k} = k + \sum_{r<k}(j_r - 1)$ and

$$\widetilde{B}_k = \left\{k+1+\sum_{r<k}(j_r-1), k+2+\sum_{r<k}(j_r-1), \dots, k+j_k-1+\sum_{r<k}(j_r-1)\right\},$$

which is empty if $j_k = 1$. Let $(\sigma,\omega)_J$ be the mod 2 integer such that

$$x^{\widetilde{i_1}}x^{\widetilde{i_2}}\cdots x^{\widetilde{i_l}}x^{\widetilde{B}_{i_l}}x^{\widetilde{B}_{i_{l-1}}}\cdots x^{\widetilde{B}_{i_1}} = (-1)^{(\sigma,\omega)_J}x^{\widetilde{i_1}}x^{\widetilde{B}_{i_1}}x^{\widetilde{i_2}}x^{\widetilde{B}_{i_2}}\cdots x^{\widetilde{i_l}}x^{\widetilde{B}_{i_l}}.$$

Clearly by (32), $(\sigma,\omega)_J$ coincides with (35). As a conclusion, by setting $\widetilde{v}^k = u^{\widetilde{k}}u^{\widetilde{B}_k}$ and $\widetilde{u}^k = t^{\widetilde{k}}u^{\widetilde{B}_k}$, $k = 1,2,\dots,m$, we see that $R^*_K(J)$ is embedded in $R^*_{K(J)}$ as a subalgebra generated by the image of $\varpi_J\colon R^*_K \to R^*_{K(J)}$. Therefore, on passage to cohomology, the embedding above induces an isomorphism, by Lemma 5.3.

The second statement follows from the first one, together with Proposition 3.3. Here we check it directly. Note that by (35), for $i \in \omega\setminus\sigma$, we have

$$(\sigma\cup\{i\},\omega)_J - (\sigma,\omega)_J = \sum_{k\in\omega, k<i}(j_k - 1).$$

It turns out that

$$\begin{aligned}
\eta_J(\mathrm{d}\sigma^*_\omega) &= \sum_{\substack{i\in\omega\setminus\sigma\\ (\sigma\cup\{i\})\in K_\omega}} (-1)^{\mathrm{card}(\{k\in\sigma|k<i\})}\eta_J\left((\sigma\cup\{i\})^*_\omega\right)\\
&= \sum_{\substack{i\in\omega\setminus\sigma\\ (\sigma\cup\{i\})\in K_\omega}} (-1)^{\mathrm{card}(\{k\in\sigma|k<i\})+(\sigma\cup\{i\},\omega)_J}\widetilde{v}^{\sigma\cup\{i\}}\widetilde{u}^{\omega\setminus(\sigma\cup\{i\})}\\
&= \sum_{\substack{i\in\omega\setminus\sigma\\ (\sigma\cup\{i\})\in K_\omega}} (-1)^{\mathrm{card}(\{k\in\sigma|k<i\})+\sum_{k<i,k\in\omega}(j_k-1)+(\sigma,\omega)_J}\widetilde{v}^{\sigma\cup\{i\}}\widetilde{u}^{\omega\setminus(\sigma\cup\{i\})}\\
&= \sum_{\substack{i\in\omega\setminus\sigma\\ (\sigma\cup\{i\})\in K_\omega}} (-1)^{\sum_{k<i,k\in\sigma}j_k+\sum_{k<i,k\in\omega\setminus\sigma}(j_k-1)+(\sigma,\omega)_J}\widetilde{v}^{\sigma\cup\{i\}}\widetilde{u}^{\omega\setminus(\sigma\cup\{i\})}\\
&= \mathrm{d}\left((-1)^{(\sigma,\omega)_J}\widetilde{v}^\sigma\widetilde{u}^{\omega\setminus\sigma}\right) = \mathrm{d}\eta_J(\sigma^*_\omega),
\end{aligned}$$

where we have used (31) in the last two lines. □

In particular, when J is the constant tuple $(\mathbf{2})$, the sign $(\sigma,\omega)_{(\mathbf{2})}$ was understood in Bosio and Meersseman [**BM06**, Theorem 10.1], using the intersection of submanifolds (see also [**Pan08**, Theorem 5.1]).

EXAMPLE 5.4. Let K be the complex dual to the truncated cube in Figure 3,

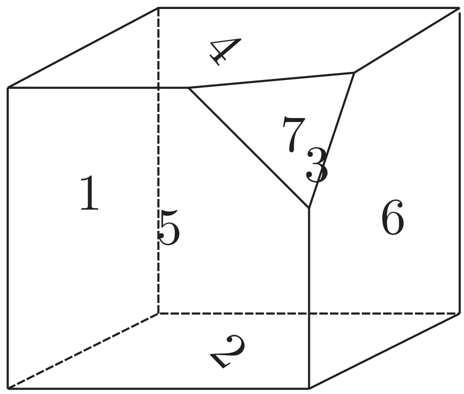

Figure 3. The truncated cube.

such that each vertex is labeled by the number in the center of the associated 2-face. In [**GL13**, Theorem 3.1], it was proved that $H^*(R_K^*)$ and $H^*(R_K^*(\mathbf{2}))$ are *not* isomorphic as ungraded rings, even with $\mathbb{Z}_2$ coefficients. Following their approach, here we illustrate the difference of the two rings, by Theorem 5.1. First by [**GL13**], we have the homeomorphisms

$$(D^1, S^0)^K \cong \left(S^1 \times S^1 \times S^1\right) \sharp \left(S^1 \times S^1 \times S^1\right) \sharp 7 \left(S^1 \times S^2\right),$$

and

$$(D^2, S^1)^K \cong \partial \left(M_{-1}^9 \times D^2\right) \sharp 3 \left(S^3 \times S^7\right) \sharp 3 \left(S^4 \times S^6\right) \sharp \left(S^5 \times S^5\right)$$

where M_{-1}^9 is the obtained from $S^3 \times S^3 \times S^3$ by subtracting an open ball.[11]

It is easy to check that $\widetilde{H}^*(K_\omega)$ is non-trivial only when:

(I) $\omega = I_{1,2,3} \cup \{7\}$ where $I_{1,2,3}$ can be any non-empty subsets of $\{1,2,3\}$ (7 cases); $\omega = \{1,2,3,4,5,6\} \setminus I_{1,2,3}$ (7 cases);

(II) $\omega = \{1,6,7\} \setminus I_7$, $\omega = \{2,4,7\} \setminus I_7$ and $\omega = \{3,5,7\} \setminus I_7$, where $I_7 = \emptyset$ or $I_7 = \{7\}$ (6 cases); $\omega = \{1,3,5,6,7\} \setminus I_7$, $\omega = \{2,3,4,5,7\} \setminus I_7$ and $\omega = \{1,2,4,6,7\} \setminus I_7$ (6 cases).

First we consider $H^*(R_K^*)$. With respect to (I), we have 7 pairs of representatives $u^7 t^{I_{1,2,3}}$ of degree 1 and $u^{4,5} t^{\{1,2,3,6\} \setminus I_{1,2,3}}$ of degree 2, respectively, with $I_{1,2,3}$ running through non-empty subsets of $\{1,2,3\}$, such that their product $-u^{4,5,7} t^{1,2,3,6}$ generates $H^3(R_K^*) \cong H^3((D^1, S^0)^K)$. For (II), we choose representatives $u^1 t^{6,7}$, $u^2 t^{4,7}$ and $u^3 t^{5,7}$ of degree 1, with their mutually two-fold products $u^{1,2} t^{4,6,7}$, $u^{1,3} t^{5,6,7}$ and $u^{2,3} t^{4,5,7}$, respectively, together with the three-fold product $u^{1,2,3} t^{4,5,6,7}$. Analogously, we choose representatives $u^1 t^6 (1-t^7)$, $u^2 t^4 (1-t^7)$ and $u^3 t^5 (1-t^7)$, together with their mutually two-fold and three-fold products. It can be checked that, in this way we give a presentation of $H^*(R_K^*)$.

[11] $(D^1, S^0)^K$ and $(D^2, S^1)^K$ can be smoothed canonically to make the two homeomorphisms above into diffeomorphisms; see [**GL13**].

Next we turn to $H^*(R_K^*(\mathbf{2}))$. For (I), we can simply use the previous argument, replacing u^i and t^i by $\widetilde{v}^i$ and $\widetilde{u}^i$, respectively. Notice that the degrees have changed accordingly. For (II), however, we have to modify the representatives for products, since the relations $(\widetilde{u}^i)^2 = \widetilde{u}^i$ do not hold any longer. Therefore, any non-trivial three-fold product must come from a partition of [7] by three parts: two of them have cardinality 2 and the remainder has cardinality 3. For instance, if we have chosen $[\widetilde{v}^1\widetilde{u}^6]$ and $[\widetilde{v}^2\widetilde{u}^{4,7}]$ associated to $\widetilde{H}^0(K_{1,6})$ and $\widetilde{H}^0(K_{2,4,7})$ by $\eta_{\mathbf{2}}$ (see (36)), respectively, then the remainder has to be $[\widetilde{v}^3\widetilde{u}^5]$ associated to $\widetilde{H}^0(K_{3,5})$.

6. Proof of Theorem 3.1.

We will use the same notations as in the statement of the theorem.

Let A^n be the n-th skeleton of A (we set $A^{-1} = \emptyset$), and let $(S_*(A), \partial_s)$ be the singular chain complex of A with differential ∂_s.

Proof of the additive isomorphism. The method here is well-known: it suffices to show that, there is a chain map $f_A\colon (C_*(A), \partial) \to (S_*(A), \partial_s)$ such that the image of $C_n(A)$ generates $H_n(A^n, A^{n-1})$ in each dimension $n \geq 0$. We construct f_A by induction on dimension n. This is clear when $n = 0$. Suppose $n > 0$ and f_A is well-defined for $k < n$, with the desired property. Choosing a cell e_n of dimension n, we see by hypothesis that the image of $\partial(e_n)$ under f_A is carried by the boundary $\mathrm{Bd}(e_n)$, which is topologically an $(n-1)$-sphere. Again by hypothesis, $f_A(\partial(e_n))$ generates the fundamental class of $\mathrm{Bd}(e_n)$. Therefore, by the long exact sequence of the pair $(e_n, \mathrm{Bd}(e_n))$, we can choose from $S_n(e_n)$ a representative of $H_n(e_n, \mathrm{Bd}(e_n))$, say c_n, such that $\partial_s c_n = f_A(\partial(e_n))$. Then we define $f_A(e_n) = c_n$, and in this way we extend f_A over all n-cells. Then a standard argument of cellular homology shows that, f_A and its dual induce the isomorphisms $H_*(C_*(A), \partial) \cong H_*(A)$ and $H^*(C^*(A), \mathrm{d}) \cong H^*(A)$, respectively. $\square$

On products, however, we have to work carefully at the (co)chain level. We shall proceed with singular (co)chains, and then compare it with simplicial ones.

Let Top be the category of topological spaces, and let Top^m be the category of m-fold Cartesian product spaces, with m-tuple of continuous maps as morphisms. Clearly Top^m is a subcategory of Top. Denote by C(Ab) the category of chain complexes with chain maps as morphisms.

Let $S_*\colon \mathrm{Top} \to \mathrm{C(Ab)}$ be the functor of singular chain complexes, assigning to a space X the singular chain complex $(S_*(X), \partial_s)$; let $S_*^m\colon \mathrm{Top}^m \to \mathrm{C(Ab)}$ be the functor assigning to a product $X = \prod_{i=1}^m X_i$ the tensor product $(\bigotimes_{i=1}^m S_*(X_i), \partial_{s'})$ ($\partial_{s'}$ follows (12), with suitable chains replaced).

Recall that for an object $X = \prod_{i=1}^m X_i$ from Top^m, the associated *Alexander–Whitney chain map* $T_X^m\colon\ (S_*(X), \partial_s) \to (S_*^m(X), \partial_{s'})$ is defined by sending each singular p-simplex $\sigma\colon \Delta^p \to X$ (the vertex set of Δ^p is $\{0, 1, \ldots, p\}$) to the sum

$$\sum_{0=k_0\leq k_1\leq\cdots\leq k_m=p} \bigotimes_{i=1}^m \pi_i\sigma|_{[k_{i-1},k_i]}, \tag{42}$$

where π_i is the projection onto the i-th component and $\sigma|_{[k_{i-1},k_i]}$ is the restriction of σ to the face spanned by $\{k_{i-1}, k_{i-1}+1, \ldots, k_i\}$. We see that $T^m\colon S_*|_{\mathrm{Top}^m} \to S_*^m$ is a natural transformation between the two functors. Moreover, by Eilenberg–Zilber Theorem [**EZ53**], T^m induces a natural chain-homotopy equivalence when the space is specified.

Let $(S^*(X), \delta_s)$ be the singular cochain complex of X. For $\mathbb{Z}$-modules M_i with their duals $M_i^* = \mathrm{Hom}(M_i, \mathbb{Z})$, $i = 1, 2, \ldots, m$, we denote by $\theta_m\colon \bigotimes_{i=1}^m M_i^* \to \mathrm{Hom}\left(\bigotimes_{i=1}^m M_i, \mathbb{Z}\right)$ the *evaluation map*, which is a homomorphism given by

$$\theta_m\left(\bigotimes_{i=1}^m f_i\right)\left(\bigotimes_{i=1}^m c_i\right) = \prod_{i=1}^m f_i(c_i),$$

where $f_i \in M_i^*$ and $c_i \in M_i$. Consider the sequence

$$S_*(X) \xrightarrow{d_*} S_*(X \times X) \xrightarrow{T^2_{X\times X}} S^2_*(X),$$

where d_* is induced by the diagonal map $d\colon X \to X \times X$. We refer to the composition $\tau_X = T^2_{X\times X} d_*$ as the *Alexander–Whitney diagonal approximation*. By definition, we see that $\tau\colon S_* \to S_*^2$ is a natural transformation between the two functors.

The cup product $\smile\colon S^*(X) \otimes S^*(X) \to S^*(X)$ is defined as the composition $\tau_X^* \theta_2$, where τ_X^* is the dual of τ_X.

To formulate the cap product, we need another homomorphism

$$\begin{aligned} h\colon M^* \otimes M \otimes M &\longrightarrow M \\ (m^*, m_1 \otimes m_2) &\longmapsto m^*(m_2)m_1, \end{aligned} \tag{43}$$

where M is a $\mathbb{Z}$-module with its dual M^*. It can be checked by a straightforward calculation that for $c^p \in S^p(X)$ and $c'_r \in S^2_r(X \times X)$, we have

$$\partial_s \circ h(c^p, c'_r) = (-1)^{r-p} h(\delta_s c^p, c'_r) + h(c^p, \partial_{s'} c'_r). \tag{44}$$

The cap product $\frown\colon S^*(X) \otimes S_*(X) \to S_*(X)$ is then given by

$$c^p \frown c_r = h(c^p, \tau_X(c_r)).$$

Now we consider functors from Top^m to $\mathrm{C(Ab)}$. For a space $X = \prod_{i=1}^m X_i$, we have the diagram

$$\begin{tikzcd} S_*(X) \arrow[r, "\tau_X"] \arrow[dd, "T^m_X"'] & S_*(X)\otimes S_*(X) \arrow[d, "T^m_X \otimes T^m_X"] \\ & S^m_*(X) \otimes S^m_*(X) \\ S^m_*(X) \arrow[r, "\bigotimes_{i=1}^m \tau_{X_i}"] & \bigotimes_{i=1}^m \left(S_*(X_i) \otimes S_*(X_i)\right), \arrow[u, "T_{Shuf}"'] \end{tikzcd} \tag{45}$$

in which the homomorphism T_{Shuf} is given by the shuffling:

$$T_{Shuf}\left(\bigotimes_{i=1}^{m}(c_{p_i}\otimes c_{q_i})\right)=(-1)^{(\boldsymbol{p},\boldsymbol{q})}\left(\bigotimes_{i=1}^{m}c_{p_i}\right)\otimes\left(\bigotimes_{i=1}^{m}c_{q_i}\right), \tag{46}$$

where $\boldsymbol{p}=(p_i)_{i=1}^m$, $\boldsymbol{q}=(q_i)_{i=1}^m$, $c_{p_i}\in S_{p_i}(X_i)$ and $c_{q_i}\in S_{q_i}(X_i)$, $i=1,2,\dots,m$; $(\boldsymbol{p},\boldsymbol{q})$ is given in (11). It can be checked directly that T_{Shuf} is a chain map. We see that chain maps

$$\psi_X=(T_X^m\otimes T_X^m)\circ\tau_X \quad \text{and} \quad \psi'_X=T_{Shuf}\circ\left(\bigotimes_{i=1}^{m}\tau_{X_i}\right)\circ T_X^m$$

can be treated as natural transformations $\mathrm{Top}^m\to\mathrm{C(Ab)}$ which are specified to X. Using Acyclic Model Theorem, with models as $2m$-product of simplices (see [**Spa66**, Section 6, Chapter 5, p. 252] for the details of the case $m=2$, and the proof for the general case is similar), it turns out that Diagram (45) commutes, up to natural chain homotopy. This means that there is a natural transformation $D\colon\ \mathrm{Top}^m\to\mathrm{C(Ab)}$, such that

$$\psi_X-\psi'_X=D_X\partial_s+\partial_{s'}D_X \tag{47}$$

when X is specified.

LEMMA 6.1. *Let $c^{\boldsymbol{p}}=\bigotimes_{i=1}^m c^{p_i}\in\bigotimes_{i=1}^m S^{p_i}(X_i), c^{\boldsymbol{q}}=\bigotimes_{i=1}^m c^{q_i}\in\bigotimes_{i=1}^m S^{q_i}(X_i)$ be two cochains of degrees $|\boldsymbol{p}|$ and $|\boldsymbol{q}|$, respectively. Then we have*

$$\begin{aligned}(T_X^m)^*\theta_m(c^{\boldsymbol{p}})\smile(T_X^m)^*\theta_m(c^{\boldsymbol{q}})-(-1)^{(\boldsymbol{p},\boldsymbol{q})}(T_X^m)^*\theta_m\left(\bigotimes_{i=1}^{m}c^{p_i}\smile c^{q_i}\right)\\ =\delta_sB_1\left(c^{\boldsymbol{p}},c^{\boldsymbol{q}}\right)+B_2\left(\delta_{s'}c^{\boldsymbol{p}},c^{\boldsymbol{q}}\right)+B_3\left(c^{\boldsymbol{p}},\delta_{s'}c^{\boldsymbol{q}}\right)\end{aligned} \tag{48}$$

where $(T_X^m)^*$ is the dual of T_X^m, $\delta_{s'}$ the coboundary operator of $\bigotimes_{i=1}^m S^*(X_i)$, and $B_i\colon\ S_*^m(X)\otimes S_*^m(X)\to S_*(X)$ are three bilinear forms associated to $|\boldsymbol{p}|$ and $|\boldsymbol{q}|$, $i=1,2,3$. Additionally, given $c_{\boldsymbol{r}}=\bigotimes_{i=1}^m c_{r_i}$, $c_{r_i}\in S_{r_i}(X_i)$, we have

$$\begin{aligned}(T_X^m)^*\theta_m(c^{\boldsymbol{p}})\frown(T_X^m)^{-1}(c_{\boldsymbol{r}})-(-1)^{(\boldsymbol{r}-\boldsymbol{p},\boldsymbol{p})}(T_X^m)^{-1}\left(\bigotimes_{i=1}^{m}c^{p_i}\frown c_{r_i}\right)\\ =\partial_sB'_1\left(c^{\boldsymbol{p}},c_{\boldsymbol{r}}\right)+B'_2\left(\delta_{s'}c^{\boldsymbol{p}},c_{\boldsymbol{r}}\right)+B'_3\left(c^{\boldsymbol{p}},\partial_{s'}c_{\boldsymbol{r}}\right)\end{aligned} \tag{49}$$

with three bilinear forms $B'_i\colon\ \bigotimes_{i=1}^m S^*(X_i)\otimes S_*^m(X)\to S_*^m(X)$ (with respect to $|\boldsymbol{p}|$, $|\boldsymbol{q}|$), $(T_X^m)^{-1}$ the chain-homotopy inverse of T_X^m.

PROOF. This follows by a diagram chasing on Diagram (45). Let $c^{\boldsymbol{p},\boldsymbol{q}}$ denote the cochain $c^{\boldsymbol{p}}\otimes c^{\boldsymbol{q}}$. By assumption, we have

$$\begin{aligned}\psi_X^*\left(\theta_{2m}(c^{\boldsymbol{p},\boldsymbol{q}})\right)&=\tau_X^*\theta_2\left((T_X^m)^*\theta_m(c^{\boldsymbol{p}})\otimes(T_X^m)^*\theta_m(c^{\boldsymbol{q}})\right)\\&=(T_X^m)^*\theta_m(c^{\boldsymbol{p}})\smile(T_X^m)^*\theta_m(c^{\boldsymbol{q}}),\end{aligned}$$

$$\begin{aligned}(\psi'_X)^*\left(\theta_{2m}(c^{\boldsymbol{p},\boldsymbol{q}})\right) &= (T_X^m)^*\theta_m\left((-1)^{(\boldsymbol{p},\boldsymbol{q})}\bigotimes_{i=1}^m \tau_{X_i}^*\theta_2(c^{p_i}\otimes c^{q_i})\right)\\ &= (-1)^{(\boldsymbol{p},\boldsymbol{q})}(T_X^m)^*\theta_m\left(\bigotimes_{i=1}^m c^{p_i}\smile c^{q_i}\right).\end{aligned}$$

Then (48) follows from a comparison of the two equations above, together with (47). It remains to prove (49). It can be checked directly that

$$\begin{aligned}(T_X^m)^*\theta_m(c^{\boldsymbol{p}})\frown(T_X^m)^{-1}(c_{\boldsymbol{r}}) &= h\left((T_X^m)^*\theta_m(c^{\boldsymbol{p}}),\tau_X(T_X^m)^{-1}(c_{\boldsymbol{r}})\right)\\ &= (T_X^m)^{-1}T_X^m h\left((T_X^m)^*\theta_m(c^{\boldsymbol{p}}),\tau_X(T_X^m)^{-1}(c_{\boldsymbol{r}})\right)\\ &\quad+\partial_s B'_{11}\left(c^{\boldsymbol{p}},c_{\boldsymbol{r}}\right)+B'_{12}\left(\delta_{s'}c^{\boldsymbol{p}},c_{\boldsymbol{r}}\right)+B'_{13}\left(c^{\boldsymbol{p}},\partial_{s'}c_{\boldsymbol{r}}\right),\end{aligned}$$

where bilinear forms B'_{1i} comes from the chain-homotopy between $(T_X^m)^{-1}T_X^m$ and the identity of $S_*(X)$, $i=1,2,3$, together with (44). On the other hand, again by (47), we see that

$$\begin{aligned}T_X^m h\left((T_X^m)^*\theta_m(c^{\boldsymbol{p}}),\tau_X(T_X^m)^{-1}(c_{\boldsymbol{r}})\right) &= h\left(\theta_m(c^{\boldsymbol{p}}),(T_X^m\otimes T_X^m)\,\tau_X(T_X^m)^{-1}(c_{\boldsymbol{r}})\right)\\ = h\left(\theta_m(c^{\boldsymbol{p}}),T_{Shuf}\left(\bigotimes_{i=1}^m\tau_{X_i}\right)(c_{\boldsymbol{r}})\right)&+\partial_s B'_{21}\left(c^{\boldsymbol{p}},c_{\boldsymbol{r}}\right)+B'_{22}\left(\delta_{s'}c^{\boldsymbol{p}},c_{\boldsymbol{r}}\right)\\ +B'_{23}\left(c^{\boldsymbol{p}},\partial_{s'}c_{\boldsymbol{r}}\right).&\end{aligned}$$

By expanding $\left(\bigotimes_{i=1}^m\tau_{X_i}\right)(c_{\boldsymbol{r}})$ with (42) and then matching the degrees, we have

$$h\left(\theta_m(c^{\boldsymbol{p}}),T_{Shuf}\left(\bigotimes_{i=1}^m\tau_{X_i}\right)(c_{\boldsymbol{r}})\right)=(-1)^{(\boldsymbol{r}-\boldsymbol{p},\boldsymbol{p})}\bigotimes_{i=1}^m c^{p_i}\frown c_{r_i}.$$

Clearly (49) follows from the equations above. □

Now let (X,A) be the pair in the statement of Theorem 3.1. Additionally, suppose that each K_i is given a partial ordering on the vertex set, which induces a total ordering on each simplex. To complete the proof, it remains to compare the singular (co)chain complex with the simplicial one.

Recall that there is a chain-homotopy equivalence $\iota_i\colon C_*(K_i)\to S_*(|K|)$ for each $i=1,2,\ldots,m$, such that ι_i sends each p-simplex $[v_0,v_1,\ldots,v_p]$ to the singular simplex linearly spanned by associated vertices, say $l(v_0,v_1,\ldots,v_p)$, with $v_0<v_1<\cdots<v_p$ in the given ordering; moreover, the chain-homotopy inverse ι_i^{-1} satisfies $\iota_i^{-1}(l(v_0,v_1,\ldots,v_p))=[v_0,v_1,\ldots,v_p]$ (see [**Mun84**, Theorem 34.3, pp. 194–195]). It can be checked that diagrams

$$\begin{array}{ccc} S^*(|K_i|)\otimes S^*(|K_i|) & \xrightarrow{\smile} & S^*(|K_i|)\\ \big\downarrow{\scriptstyle \iota_i^*\otimes\iota_i^*} & & \big\downarrow{\scriptstyle \iota_i^*}\\ C^*(K_i)\otimes C^*(K) & \xrightarrow{\smile} & C^*(K_i)\end{array}\quad\text{and}\quad\begin{array}{ccc} S^*(|K_i|)\otimes S_*(|K_i|) & \xrightarrow{\frown} & S_*(|K_i|)\\ \big\downarrow{\scriptstyle \iota_i^*\otimes\iota_i^{-1}} & & \big\downarrow{\scriptstyle \iota_i^{-1}}\\ C^*(K_i)\otimes C_*(K_i) & \xrightarrow{\frown} & C_*(K_i)\end{array}\qquad(50)$$

commute, where the simplicial cup and cap products in the bottom rows are given by (23) and (24), respectively.

It follows that the map $\iota\colon (C_*(X),\partial) \to (S^m_*(X),\partial_{s'})$, $\iota = \bigotimes_{i=1}^m \iota_i$, is a chain-homotopy equivalence with its inverse $\iota^{-1} = \bigotimes \iota_i^{-1}$.[12]

Proof of Theorem 3.1. Let $\upsilon_c\colon C_*(A) \to C_*(X)$ and $\upsilon_s\colon S_*(A) \to S_*(X)$ be the chain maps induced by the inclusion $A \to X$, respectively, and let υ_c^* and υ_s^* be their duals. We see that υ_c^* sends a dual basis $\bigotimes_{i=1}^m \sigma_{p_i}^*$ to the one of the same form, if $\prod_{i=1}^m |\sigma_{p_i}| \subset A$, and to zero otherwise (when using the dual basis of the form above, we have used the evaluation map θ_m implicitly). By definition, υ_c^* preserves the cup products given in (13).

Consider the composition $(T^m)^{-1}\iota\upsilon_c\colon C_*(A) \to S_*(X)$, by the naturality of $(T^m)^{-1}$, it gives rise to a chain map $C_*(A) \to S_*(A)$ (since the image of $S^m_*(\prod_{i=1}^m |\sigma_{p_i}|)$ under $(T^m)^{-1}$ lies in $S_*(\prod_{i=1}^m |\sigma_{p_i}|)$). Thus we define

$$f_A = (T^m)^{-1}\iota\upsilon_c;$$

it can be checked that f_A satisfies the desired property in the proof of the additive isomorphism $H_*(C_*(A),\partial) \cong H_*(A)$. The argument above shows that the chain-homotopy inverse of f_A, say f_A^{-1}, is then given by $\iota^{-1}T^m_X\upsilon_s\colon S_*(A) \to C_*(A)$, whose dual

$$(f_A^{-1})^* = \upsilon_s^*(T^m_X)^*(\iota^{-1})^*\colon C^*(A) \to S^*(A)$$

induces the isomorphism $H^*(C^*(A),\mathrm{d}) \cong H^*(A)$. Notice that $(f_A^{-1})^*$ can be defined over $C^*(X)$, which we shall denote by $(\widetilde{f}_A^{-1})^*$: we have

$$(f_A^{-1})^*\upsilon_c^* = (\widetilde{f}_A^{-1})^*. \tag{51}$$

Now we choose cochains $c^{\boldsymbol{p}} = \bigotimes_{i=1}^m c^{p_i} \in C^{|\boldsymbol{p}|}(X)$ and $c^{\boldsymbol{q}} = \bigotimes_{i=1}^m c^{q_i} \in C^{|\boldsymbol{q}|}(X)$, where $c^{p_i} \in C^{p_i}(|K_i|)$ and $c^{q_i} \in C^{q_i}(|K_i|)$. By (48) and Diagram (50), we have

$$\begin{aligned}
&(f_A^{-1})^*\upsilon_c^*(c^{\boldsymbol{p}}) \smile (f_A^{-1})^*\upsilon_c^*(c^{\boldsymbol{q}}) - (-1)^{(\boldsymbol{p},\boldsymbol{q})}(f_A^{-1})^*\upsilon_c^*\left(\bigotimes_{i=1}^m c^{p_i} \smile c^{q_i}\right)\\
&= (\widetilde{f}_A^{-1})^*(c^{\boldsymbol{p}}) \smile (\widetilde{f}_A^{-1})^*(c^{\boldsymbol{q}}) - (-1)^{(\boldsymbol{p},\boldsymbol{q})}(\widetilde{f}_A^{-1})^*\left(\bigotimes_{i=1}^m c^{p_i} \smile c^{q_i}\right)\\
&= \upsilon_s^*\delta_s B_1\left((\iota^{-1})^*(c^{\boldsymbol{p}}),(\iota^{-1})^*(c^{\boldsymbol{q}})\right) + \upsilon_s^* B_2\left(\delta_{s'}(\iota^{-1})^*(c^{\boldsymbol{p}}),(\iota^{-1})^*(c^{\boldsymbol{q}})\right)\\
&\quad + \upsilon_s^* B_3\left((\iota^{-1})^*(c^{\boldsymbol{p}}),\delta_{s'}(\iota^{-1})^*(c^{\boldsymbol{q}})\right);
\end{aligned}$$

due to the naturality, the summation in the last line above can be written as

[12] This is well-known: suppose D_i is the chain-homotopy between $\alpha_i = \iota_i^{-1}\iota_i$ and the identity id_i of $C_*(K_i)$, one can check that the homomorphism generated by

$$\widetilde{D}(c_{\boldsymbol{r}}) = \sum_{i=1}^m (-1)^{\sum_{k<i} r_k}\alpha_1(c_{r_1})\otimes\cdots\otimes\alpha_{i-1}(c_{r_{i-1}})\otimes D_i(c_{r_i})\otimes \mathrm{id}_{i+1}(c_{r_{i+1}})\cdots\otimes\mathrm{id}_m(c_{r_m}),$$

in which $c_{\boldsymbol{r}} = \bigotimes_{i=1}^m c_{r_i}$ $(c_{p_i} \in C_{r_i}(K_i))$, gives the desired homotopy between ι and the identity of $C_*(X)$. Similarly, we can define the chain-homotopy between $\iota\iota^{-1}$ and the identity of $S^m_*(X)$.

$$\delta_s \upsilon_s^* B_1 \left((\iota^{-1})^* \upsilon_c^*(c^{\boldsymbol{p}}), (\iota^{-1})^* \upsilon_c^*(c^{\boldsymbol{q}})\right) + \upsilon_s^* B_2 \left((\iota^{-1})^* \upsilon_c^*(\mathrm{d}c^{\boldsymbol{p}}), (\iota^{-1})^* \upsilon_c^*(c^{\boldsymbol{q}})\right)$$
$$+ \upsilon_s^* B_3 \left((\iota^{-1})^* \upsilon_c^*(c^{\boldsymbol{p}}), (\iota^{-1})^* \upsilon_c^*(\mathrm{d}c^{\boldsymbol{q}})\right). \tag{52}$$

Therefore, we have proved the part on cup products, since υ_c^* is surjective, and (52) will vanish on passage to cohomology (notice that $\upsilon_c^*(c^*) \in C^*(A)$ is closed means that $\upsilon_c^*(\mathrm{d}c^*) = 0$, while $\mathrm{d}c^*$ may not vanish in $C^*(X)$).

Now we prove the part on cap products. Suppose $c_{\boldsymbol{r}} = \bigotimes_{i=1}^m c_{r_i} \in C_{|\boldsymbol{r}|}(X)$, $c_{r_i} \in C_{r_i}(|K_i|)$, such that $c_{\boldsymbol{r}}$ is, under υ_c, the image of a chain of the same form in $C_*(A)$, which we shall denote as $\underline{c}_{\boldsymbol{r}}$. Then by definition (24), we see that under υ_c, $\bigotimes_{i=1}^m c^{p_i} \frown c_{r_i}$ has a unique preimage $\underline{\bigotimes_{i=1}^m c^{p_i} \frown c_{r_i}} \in C_*(A)$. Analogously to the previous case, by (49) and the naturality, we have

$$
\begin{aligned}
&(f_A^{-1})^* \upsilon_c^*(c^{\boldsymbol{p}}) \frown f_A(\underline{c}_{\boldsymbol{r}}) - (-1)^{(\boldsymbol{r}-\boldsymbol{p},\boldsymbol{p})} f_A\left(\underline{\bigotimes_{i=1}^m c^{p_i} \frown c_{r_i}}\right) \\
&= (\widetilde{f}_A^{-1})^*(c^{\boldsymbol{p}}) \frown (T_X^m)^{-1}\iota(c_{\boldsymbol{r}}) - (-1)^{(\boldsymbol{r}-\boldsymbol{p},\boldsymbol{p})} (T_X^m)^{-1}\iota\left(\bigotimes_{i=1}^m c^{p_i} \frown c_{r_i}\right) \\
&= \partial_s B_1'\left((\iota^{-1})^*(c^{\boldsymbol{p}}), \iota(c_{\boldsymbol{r}})\right) + B_2'\left(\delta_{s'}(\iota^{-1})^*(c^{\boldsymbol{p}}), \iota(c_{\boldsymbol{r}})\right) \\
&\quad + B_3'\left((\iota^{-1})^*(c^{\boldsymbol{p}}), \partial_{s'}\iota(c_{\boldsymbol{r}})\right) \\
&= \partial_s B_1'\left((\iota^{-1})^* \upsilon_c^*(c^{\boldsymbol{p}}), \iota(c_{\boldsymbol{r}})\right) + B_2'\left((\iota^{-1})^* \upsilon_c^*(\mathrm{d}c^{\boldsymbol{p}}), \iota(c_{\boldsymbol{r}})\right) \\
&\quad + B_3'\left((\iota^{-1})^* \upsilon_c^*(c^{\boldsymbol{p}}), \iota(\partial c_{\boldsymbol{r}})\right),
\end{aligned}
$$

where the summation in the last line shall vanish on passage to (co)homology. □

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