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Surface diffeomorphisms with connected but not path-connected minimal sets containing arcs

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Abstract. In 1955, Gottschalk and Hedlund introduced in their book that Jones constructed a minimal homeomorphism whose minimal set is connectd but not path-connected and contains infinitely many arcs. However the homeomorphism is defined only on this set. In 1991, Walker first constructed a homeomorphism of $S^1 \times \mathbf{R}$ with such a minimal set. In this paper, we will show that Walker's homeomorphism cannot be a diffeomorphism (Theorem 2). Furthermore, we will construct a C^∞ diffeomorphism of $S^1 \times \mathbf{R}$ with a compact connected but not path-connected minimal set containing arcs (Theorem 1) by using the approximation by conjugation method.

1. Introduction.

In order to examine the dynamical properties of homeomorphisms, compact invariant sets are keys to consider the asymptotic behavior of orbits. A minimal set is a compact invariant set which is minimal with respect to the inclusion. The minimal sets play important roles as cores of compact invariant sets.

In low dimensional dynamical systems, only few topological types of minimal sets have been found (Problem 1.6 in [**4**]). In this paper, we consider whether the Warsaw circle with infinitely many singular arcs (Figure 1) can be a minimal set of a surface diffeomorphism.

The Warsaw circle is the set obtained from the closure of the graph of

$$y = \sin \frac{1}{x} \quad (-1/\pi \le x \le 1/\pi, x \ne 0)$$

by identifying the ends. We call $\{(0, y)\,;\, |y| \le 1\}$ a *singular arc*. The Warsaw circle is famous as an example of a connected but not path-connected set. A Warsaw circle with infinity many singular arcs is obtained by inserting infinity many such singular arcs along the circle, denoted by X (the precise definition will be given in Section 2).

In 1955, Gottschalk and Hedlund introduced in their book ([**6**]) that Jones constructed a minimal homeomorphism of X (that is, the whole set X is a minimal set). Although this set X was embedded in $S^1 \times \mathbf{R}$, the homeomorphism is defined only on the set X. In 1991, Walker ([**10**]) constructed a homeomorphism of $S^1 \times \mathbf{R}$ whose minimal set

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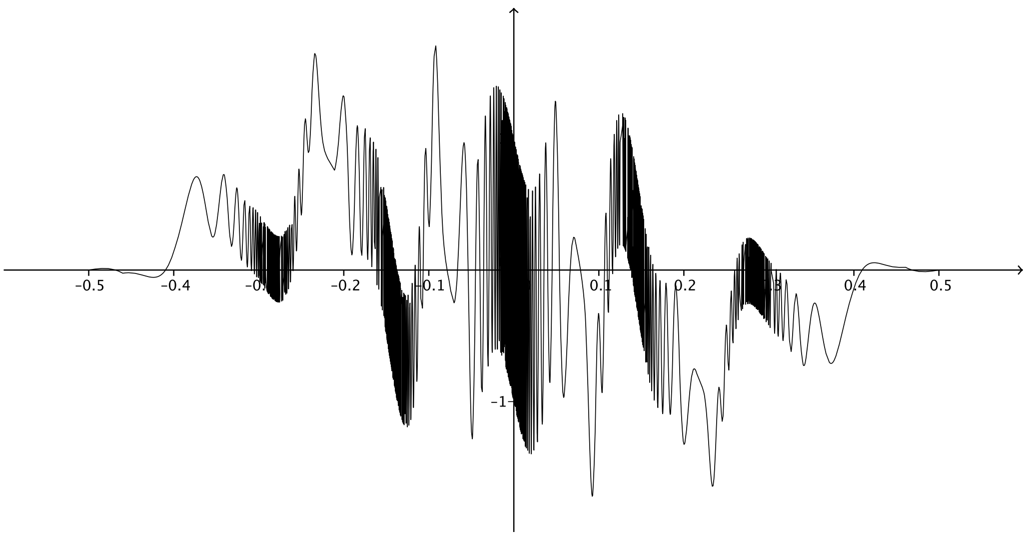

Figure 1. The Warsaw circle with infinitely many singular arcs.

is homeomorphic to X. However, his homeomorphism cannot be differentiable because the singular arcs keep the vertical directions invariant and the minimality destroys the differential structure (Theorem 2) in Section 2.

In the latter part of this paper, we will construct a C^∞ diffeomorphism of $S^1 \times \mathbf{R}$ with a compact connected but not path-connected minimal set containing arcs (Theorem 1). This minimal set is an inverse limit of circles. In general, inverse limits of circles are suitable for constructing minimal diffeomorphisms (for example, a pseudo-circle in [**7**]). Here we observe that the minimal set of Theorem 1 is similar to a Warsaw circle with infinitely many singular arcs but not always homeomorphic to it. The construction is not adequate to prove that they are homeomorphic or not.

Theorem 1. *There is a C^∞ diffeomorphism f of $S^1 \times \mathbf{R}$ with a compact connected but not path-connected minimal set containing arcs.*

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2. Difficulty for the construction of diffeomorphisms.

First recall the homeomorphism of Gottschalk and Hedlund, which was introduced in Section 14 of [**6**] as an example communicated by Jones.

We parametrize the circle by $S^1 = \mathbf{R}/\mathbf{Z}$. Let $\chi_0 : S^1 - \{0\} \to \mathbf{R}$ denote the function defined by

$$\begin{cases} \chi_0(x) = \sin \dfrac{1}{x} & \text{if } -\dfrac{1}{\pi} \le x \le \dfrac{1}{\pi}, x \neq 0, \\ \chi_0(x) = 0 & \text{if } \dfrac{1}{\pi} \le |x| \le \dfrac{1}{2}. \end{cases}$$

Then the closure of the graph of χ_0 is called the *Warsaw circle*, denoted by X_0.

Let ω be an irrational number. Let $\Lambda = \{n\omega \bmod \mathbf{Z}; n \in \mathbf{Z}\}$. We choose a sequence $\{c_n\}_{n\in\mathbf{Z}}$ of positive numbers satisfying $\sum_{n\in\mathbf{Z}} c_n < \infty$. We define a function $\chi_\omega : S^1 - \Lambda \to \mathbf{R}$ by

$$\chi_\omega(x) = \sum_{n\in\mathbf{Z}} c_n \chi_0(x - n\omega).$$

Let $\mathrm{graph}\,\chi_\omega$ denote the graph of χ_ω. The closure of $\mathrm{graph}\,\chi_\omega$ is called the *Warsaw circle with infinitely many singular arcs*, denoted by X. Let S_m ($m \in \mathbf{Z}$) denote the arc

$$\left\{ (m\omega, y) \in S^1 \times \mathbf{R}\,;\, -c_m \le y - \sum_{n\in\mathbf{Z}, n\neq m} c_n \chi_0(x - n\omega) \le c_m \right\},$$

which is called a *singular arc*. For $x \notin \Lambda$, χ_ω is continuous at x ([**6**]). Thus X consists of $\mathrm{graph}\,\chi_\omega$ and singular arcs S_m ($m \in \mathbf{Z}$).

The rotation by ω on S^1 induces a homeomorphism on $\mathrm{graph}\,\chi_\omega$. By [**6**], this homeomorphism is uniformly continuous on $\mathrm{graph}\,\chi_\omega$, and thus it can be extended on the closure of $\mathrm{graph}\,\chi_\omega$. This is the minimal homeomorphism of X introduced in [**6**].

We assume that f is a homeomorphism of $S^1 \times \mathbf{R}$ such that X is a minimal set of f. Then f maps each singular arc onto a singular arc. Let p_i ($i = 1, 2$) denote the projection to the i-th factor of $S^1 \times \mathbf{R}$. Then we can define an induced homeomorphism $\rho_f : S^1 \to S^1$ by $\rho_f(x) = p_1 f(x, y)$ for any $(x, y) \in X$.

Theorem 2. *Let ω be an irrational number. Let $\{c_n\}_{n\in\mathbf{Z}}$ be a sequence of positive numbers satisfying $\sum_{n\in\mathbf{Z}} c_n < \infty$. Let X denote the closure of the graph of χ_ω. If c_n satisfies that $\limsup_{n\to\infty} (c_{n+1})/c_n \le 1$ and $\limsup_{n\to-\infty} c_n/(c_{n+1}) \le 1$, then there is no C^1-diffeomorphism f of $S^1 \times \mathbf{R}$ such that the induced homeomorphism ρ_f of S^1 is a rotation and X is a minimal set of f.*

For the homeomorphism f constructed by Walker, ρ_f is a rotation and $c_n = 1/2^{|n|}$. Thus this cannot be of class C^1 by Theorem 2.

In the rest of this section, we will prove Theorem 2. We assume that there is a C^1 diffeomorphism f of $S^1 \times \mathbf{R}$ such that the induced homeomorphism ρ_f of S^1 is a rotation and $X = \overline{\mathrm{graph}\,\chi_\omega}$ is a minimal set for an irrational number ω. In the following, we will deduce the contradiction.

Let $\Omega_+ = \{(x, y)\,;\, y > y_0 \text{ for any } (x, y_0) \in X\}$. Since $S^1 \times \mathbf{R} - X$ consists of two connected open sets, Ω_+ is invariant under f or f^2.

Proposition 1. *X is a minimal set of f^2.*

Proof. Suppose that there is a compact subset C of X invariant under f^2. Then $C \cup f(C)$ is invariant under f, and thus $C \cup f(C) = X$. Since $C \cap f(C)$ is also invariant under f, either $X = C$ or $C \cap f(C) = \emptyset$ holds. Now X is connected. Thus $C \cap f(C)$ is not empty, and thus $X = C$. Therefore X is a minimal set of f^2. □

Thus we have only to prove Theorem 2 when $f(\Omega_+) = \Omega_+$.

Proof of Theorem 2. Let $S_n = p_1^{-1}(n\omega) \cap X$. Since $f(S_0)$ is a singular arc, there is $n_0 \in \mathbf{Z}$ such that $f(S_0) = S_{n_0}$. Thus $\rho_f(0) = n_0\omega$. We choose a universal covering $\widetilde{\rho}_f$ of ρ_f so that $\widetilde{\rho}_f(0) = n_0\omega$. Then $\widetilde{\rho}_f(x) = x + n_0\omega$ for any $x \in \mathbf{R}$ because ρ_f is a rotation. As a consequence, $f(S_i) = S_{i+n_0}$ for any $i \in \mathbf{Z}$.

We assume that $n_0 > 0$. We can prove the other case similarly.

Let (x, y) be a point of X such that $p_1^{-1}(x) \cap X$ consists of one point, i.e. $x \notin \Lambda$. We take an arbitrary $\varepsilon > 0$ and an arbitrary neighborhood W of (x, y) in $S^1 \times \mathbf{R}$. Since $p_1^{-1}(x) \cap X$ consists of one point, there is a neighborhood U of x in S^1 such that $p_1^{-1}(U) \cap X$ is contained in W. Since $\limsup_{n\to\infty} (c_{n+1})/c_n \leq 1$, there is $I > 0$ such that $(c_{i+1})/c_i < \sqrt[n_0]{1+\varepsilon}$ for any $i \geq I$. We choose an integer i_0 greater than or equal to I such that $i_0\omega \in U$. Then S_{i_0} is contained in W. By the mean value theorem, there is z_{i_0} of S_{i_0} such that $(\partial (p_2 \circ f)/\partial y)(z_{i_0}) = (c_{i_0+n_0})/c_{i_0}$. Now

$$\frac{c_{i_0+n_0}}{c_{i_0}} = \frac{c_{i_0+1}}{c_{i_0}} \frac{c_{i_0+2}}{c_{i_0+1}} \cdots \frac{c_{i_0+n_0}}{c_{i_0+n_0-1}} < 1 + \varepsilon.$$

Thus we conclude that, for any ε and neighborhood W of (x, y), there is a point in W such that $\partial (p_2 \circ f)/\partial y < 1 + \varepsilon$. Since ε and W can be chosen so small, we obtain $(\partial (p_2 \circ f)/\partial y)(x, y) \leq 1$.

The set $\{(x, y)\,;\, p_1^{-1}(x) \cap X$ consists of one point$\}$ is dense in X. Thus $\partial (p_2 \circ f)/\partial y$ is less than or equal to 1 on the whole X. Now $f(S_i) = S_{i+n_0}$ for any $i \in \mathbf{Z}$. Thus we obtain $\cdots \geq c_{-2n_0} \geq c_{-n_0} \geq c_0 \geq \cdots$. However this contradicts the assumption $\sum_{n\in\mathbf{Z}} c_n < \infty$. □

3. C^∞ construction.

3.1. Inverse limit of circles for the construction.

We will construct an inverse limit of circles which will be a minimal set of the diffeomorphism of Theorem 1.

Let $q_1 = 2$. We assume that large positive integers q_n $(n = 1, 2, \cdots)$ were already given inductively. Let L_n denote the positive numbers defined by $L_1 = 3$ and

$$L_n = q_n \left(\frac{2}{L_1 L_2 \cdots L_{n-1}} - \frac{1}{q_n} \right)$$

for $n > 1$. Although we need several conditions on q_n for our construction, we only assume here that $q_n = k_n q_{n-1} L_1 L_2 \cdots L_{n-1}$ for some positive integer k_n. Then $L_n = 2k_n q_{n-1} - 1$ is an integer, and q_n is a multiple of q_{n-1}. Let X_n be an annulus whose coordinate is given by $\{(x, y)\,;\, x \in \mathbf{R}/\mathbf{Z}, |y| \leq 1\}$ for $n = 1, 2, \cdots$ and let p_i denote the i-th projection of X_n $(i = 1, 2)$. Let R_θ denote the θ-rotation $R_\theta(x, y) = (x + \theta, y)$ in X_n. We define a simple closed curve $C_n : \mathbf{R}/L_n\mathbf{Z} \to X_n$ by

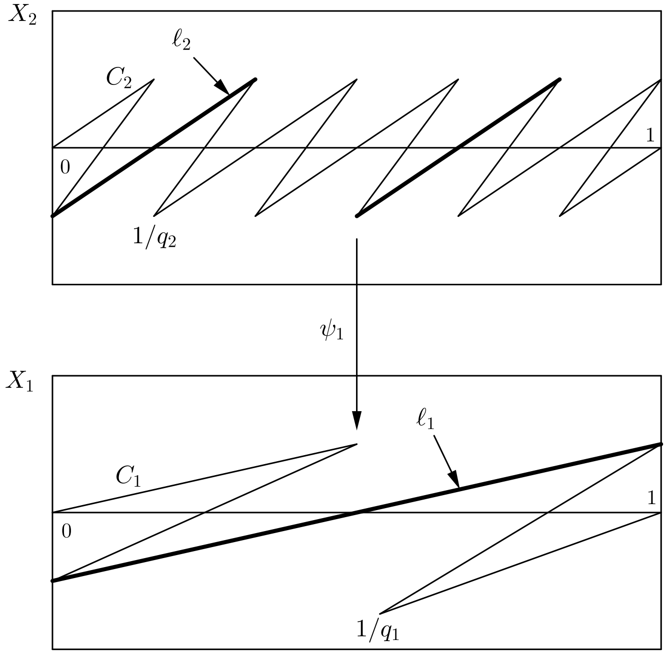

Figure 2. Circles C_n.

$$C_n(t) = \begin{cases} \left(t, L_1 \cdots L_{n-1} t - \dfrac{1}{2}\right) & \text{if } 0 \le t \le \dfrac{1}{L_1 \cdots L_{n-1}} \\ \left(-t + \dfrac{2}{L_1 \cdots L_{n-1}}, \dfrac{L_1 \cdots L_{n-1} q_n}{q_n - L_1 \cdots L_n}\left(t - \dfrac{L_n}{q_n}\right) - \dfrac{1}{2}\right) & \text{if } \dfrac{1}{L_1 \cdots L_{n-1}} \le t \le \dfrac{L_n}{q_n} \end{cases}$$

and $C_n\left(t + \dfrac{L_n}{q_n}\right) = R_{1/q_n} C_n(t)$(see Figure 2).

Then $C_n(0) = (0, -1/2)$, $C_n(1/(L_1 \cdots L_{n-1})) = (1/(L_1 \cdots L_{n-1}), 1/2)$, $C_n(L_n/q_n) = (1/q_n, -1/2)$ and C_n connects these points by line segments. Let $\ell_n = \{C_n(t)\,;\, 0 \le t \le 1/(L_1 L_2 \cdots L_{n-1})\}$. The slope of ℓ_n is $L_1 \cdots L_{n-1}$, which tends to ∞ very fast as $n \to \infty$. Furthermore, the curve C_n is invariant under R_{1/q_n} and is contained in $\mathbf{R}/\mathbf{Z} \times [-1/2, 1/2]$.

Let $\psi_n : X_{n+1} \to X_n$ denote the map defined by $\psi_n(x, y) = C_n(L_n x)$. Notice that $\psi_n(\ell_{n+1}) = \ell_n$ and ψ_n commutes with R_{1/q_n}. The latter implies that ψ_n commutes with R_{θ_n} if θ_n is a multiple of $1/q_n$.

We define a continuous map $\Psi_n : S^1 \to S^1$ by $\Psi_n(t) = p_1 C_{n+1}(L_{n+1} t)$. Then $\psi_n(C_{n+1}(L_{n+1} t)) = C_n(L_n \Psi_n(t))$ because, for $(x, y) = C_{n+1}(L_{n+1} t)$, $\psi_n(x, y) = C_n(L_n x) = C_n(L_n \Psi_n(t))$. Thus the following diagram commutes.

$$\begin{array}{ccc} S^1 & \xrightarrow{C_{n+1}(L_{n+1} t)} & X_{n+1} \\ \Big\downarrow{\scriptstyle \Psi_n} & \circlearrowright & \Big\downarrow{\scriptstyle \psi_n} \\ S^1 & \xrightarrow{C_n(L_n t)} & X_n \end{array}$$

We will use the inverse limit (S^1, Ψ_n) as a core for the construction of a C^∞ diffeomorphism in Theorem 1 (see [**1**]).

3.2. Overview of the construction.

We give an angle $\theta_n \in \mathbf{R}/\mathbf{Z}$ by $\theta_n = \sum_{i=1}^n 1/q_i$ for $n = 1, 2, \ldots$. Since θ_n is a multiple of $1/q_n$, ψ_n commutes with R_{θ_n}. We choose a C^∞ embedding $\varphi_n : X_{n+1} \to X_n$ sufficiently near ψ_n satisfying

(a) $R_{\theta_n} \circ \varphi_n = \varphi_n \circ R_{\theta_n}$,

(b) $\varphi_n(\ell_{n+1}) = \ell_n$,

(c) $\varphi_n(X_{n+1}) \subset \left\{(x, y) \in X_n \,;\, |y| < \dfrac{3}{4}\right\}$.

Let $\Phi_n = \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_n$. Then $\Phi_1(X_2) \supset \Phi_2(X_3) \supset \cdots$, and thus $\bigcap_n \Phi_n(X_{n+1})$ is a nonempty compact and connected set. Let X denote $\bigcap_n \Phi_n(X_{n+1})$. Then X will be the minimal set of the diffeomorphism in Theorem 1.

The set X contains the arc ℓ_1 because $\Phi_{n-1}(\ell_n) = \Phi_{n-2}(\ell_{n-1}) = \cdots = \varphi_1(\ell_2) = \ell_1$.

Moreover, X is not path-connected, which will be proved in Lemma 2. The idea of the proof is as follows: Let $z_1 = (1, 1/2) \in X_1$ and $z_2 = (1/2, -1/2) \in X_1$. Then z_1 and z_2 are points of X. Suppose that there is a path γ from z_1 to z_2 contained in X satisfying that $\gamma : [0, 1] \to X_1$ is homotopic to $t \mapsto (1 - (1/2)t, 1/2 - t)$ with the boundary fixed. Let N denote the number of the connected components of $\gamma \cap p_2^{-1}(-1/4, 1/4)$ such that one of the boundary points is contained in $p_2^{-1}(-1/4)$ and the other boundary points is contained in $p_2^{-1}(1/4)$. For an integer n satisfying $2n - 2 > N$, we consider the arc $\gamma_{n+1} = \Phi_n^{-1}(\gamma)$. By the condition of Φ_n given later in the precise construction, $\Phi_n(\gamma_{n+1})$ is a zigzag curve in X_1 passing through $n - 1$ points near z_1 and n points near $(0, -1/2)$ alternatingly. Then there is at least $2n - 2$ connected components of $\gamma \cap p_2^{-1}(-1/4, 1/4)$ as above. This contradicts the assumption $2n - 2 > N$.

We will give a diffeomorphism $f_n : X_1 \to X_1$ satisfying

(d) $f_{n+1} = f_n$ outside $\Phi_n(X_{n+1})$ and

(e) $f_{n+1}(x, y) = \Phi_n R_{\theta_{n+1}} \Phi_n^{-1}(x, y)$ if $(x, y) \in \Phi_n(X_{n+1})$ and $\Phi_n^{-1}(x, y) \in \left\{(x, y) \,;\, |y| \leq \dfrac{3}{4}\right\}$.

If we choose f_{n+1} sufficiently near f_n, then we can show that f_n converges to a C^∞ diffeomorphism f of X_1 as $n \to \infty$. The proof is based on the comparison of f_n and f_{n+1} in the middle part. Thanks to the condition $R_{\theta_n} \circ \varphi_n = \varphi_n \circ R_{\theta_n}$, the equation $f_{n-1} = \Phi_{n-1} R_{\theta_n} \Phi_{n-1}^{-1}$ can be written as $\Phi_n R_{\theta_n} \Phi_n^{-1}$, while $f_n = \Phi_n R_{\theta_{n+1}} \Phi_n^{-1}$. The crucial point is that we can choose the number q_{n+1} after the construction of Φ_n. Letting $|\theta_{n+1} - \theta_n|$ small enough compared with Φ_n, we get the desired convergence.

For $\theta_n = j_n/q_n$, the integers j_n and q_n are assumed to be relatively prime (see Section 3.3 (7)). Thus R_{θ_n} permutes the sets $\{(x, y) \in X_n \,;\, i/q_n \leq x \leq (i+1)/q_n\}$ $(i = 0, 1, 2, \ldots, q_n - 1)$ transitively. By using this property, we will show that X is a minimal set in Lemma 1.

In the following, we will give the precise construction of f and will show in detail that X is a connected but not path-connected minimal set of f containing the arc ℓ_1.

3.3. Precise construction.

Let X_n $(n = 1, 2, \cdots)$ be an annulus whose coordinate is given by $\{(x, y)\,;\, x \in \mathbf{R}/\mathbf{Z}, |y| \leq 1\}$. Let d denote the metric of X_n induced from the Euclidean metric, and let $\operatorname{diam} F$ denote the diameter of a set F. We define the rotation $R_\theta : X_n \to X_n$ by $R_\theta(x, y) = (x + \theta, y)$.

Let $q_1 = 2$ and $\theta_1 = 1/q_1$. We define $f_1 : X_1 \to X_1$ by $f_1(x, y) = R_{\theta_1}(x, y)$ for $x \in \mathbf{R}/\mathbf{Z}$ and $|y| \leq 1$. Let $L_1 = 3$. We define a simple closed curve $C_1 : \mathbf{R}/L_1\mathbf{Z} \to X_1$ by

$$C_1(t) = \begin{cases} \left(t, t - \dfrac{1}{2}\right) & \text{if } 0 \leq t \leq 1 \\ \left(-t + 2, \dfrac{q_1}{q_1 - L_1}\left(t - \dfrac{L_1}{q_1}\right) - \dfrac{1}{2}\right) & \text{if } 1 \leq t \leq \dfrac{L_1}{q_1} \end{cases}$$

and $C_1(t+(L_1/q_1)) = R_{1/q_1}C_1(t)$ for any $t \in \mathbf{R}/L_1\mathbf{Z}$ (see Figure 2). Then L_1 is the length of $p_1 \circ C_1$ and C_1 is invariant under R_{1/q_1}. Let ℓ_1 denote the segment $\{(t, t - 1/2)\,;\, 0 \leq t \leq 1\}$.

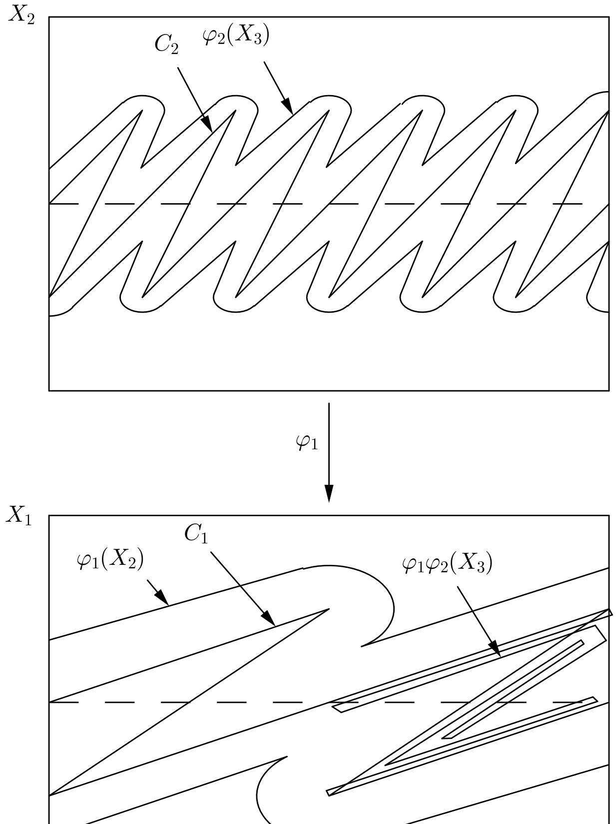

Figure 3. $\varphi_1 : X_2 \to X_1$.

We define φ_n and f_n inductively as follows: We assume that φ_i, ψ_i $(i = 1, 2, \ldots, n-2)$ if $n > 2$ and ℓ_i, q_i, θ_i f_i, L_i, C_i, $(i = 1, 2, \ldots, n-1)$ for $n \geq 2$ satisfying the following conditions have already been given:

There is $k_{n-1} \in \mathbf{Z}_+ = \{n \in \mathbf{Z}\,;\, n > 0\}$ such that $q_{n-1} = k_{n-1}q_{n-2}L_1 \cdots L_{n-2}$ $(\in \mathbf{Z}_+)$.

$$L_{n-1} = q_{n-1}\left(\frac{2}{L_1L_2\cdots L_{n-2}} - \frac{1}{q_{n-1}}\right) \in \mathbf{Z}.$$

$$C_{n-1} : \mathbf{R}/L_{n-1}\mathbf{Z} \to X_{n-1}$$

$$\begin{cases} C_{n-1}(t) = \begin{cases} \left(t, L_1\cdots L_{n-2}t - \dfrac{1}{2}\right) & \text{if } 0 \leq t \leq \dfrac{1}{L_1\cdots L_{n-2}} \\ \left(-t + \dfrac{2}{L_1\cdots L_{n-2}}, \dfrac{L_1\cdots L_{n-2}q_{n-1}}{q_{n-1} - L_1\cdots L_{n-1}}\left(t - \dfrac{L_{n-1}}{q_{n-1}}\right) - \dfrac{1}{2}\right) & \\ \qquad\qquad \text{if } \dfrac{1}{L_1\cdots L_{n-2}} \leq t \leq \dfrac{L_{n-1}}{q_{n-1}} & \end{cases} \\ C_{n-1}\left(t + \dfrac{L_{n-1}}{q_{n-1}}\right) = R_{1/(q_{n-1})}C_{n-1}(t) \quad \text{for any } t. \end{cases}$$

$$\ell_i = \left\{\left(t, L_1\cdots L_{i-1}t - \frac{1}{2}\right)\,;\, 0 \leq t \leq \frac{1}{L_1\cdots L_{i-1}}\right\} (i = 2, 3, \ldots, n-1).$$

$$\varphi_{n-2}|\ell_{n-1} = \psi_{n-2}|\ell_{n-1} \text{ (in particular, } \varphi_{n-2}(\ell_{n-1}) = \ell_{n-2}).$$

$$\theta_{n-1} = \sum_{i=1}^{n-1}\frac{1}{q_i}.$$

Since the integer q_{i+1} is a multiple of q_i $(i = 1, 2, \ldots, n-2)$. $q_{n-1}\theta_{n-1}$ is an integer.

We define $\psi_{n-1} : X_n \to X_{n-1}$ by $\psi_{n-1}(x, y) = C_{n-1}(L_{n-1}x)$. Then ψ_{n-1} maps X_n onto the curve C_{n-1}. Furthermore,

$$\begin{aligned} \psi_{n-1}R_{1/q_{n-1}}(x, y) &= C_{n-1}\left(L_{n-1}x + \frac{L_{n-1}}{q_{n-1}}\right) \\ &= R_{1/q_{n-1}}C_{n-1}(L_{n-1}x) \\ &= R_{1/q_{n-1}}\psi_{n-1}(x, y). \end{aligned}$$

Let ℓ_n denote the segment $\{(t, L_1L_2\cdots L_{n-1}t - 1/2)\,;\, 0 \leq t \leq 1/(L_1\cdots L_{n-1})\}$. Then $\psi_{n-1}(\ell_n) = \ell_{n-1}$. We choose a C^∞-embedding $\varphi_{n-1} : X_n \to X_{n-1}$ along the curve C_{n-1} satisfying

(1) $d(\varphi_1\cdots\varphi_{n-2}\varphi_{n-1}(x, y), \varphi_1\cdots\varphi_{n-2}\psi_{n-1}(x, y)) < 1/2^{n+2}$ for any $(x, y) \in X_n$, In particular, $\operatorname{diam}\{\varphi_1\cdots\varphi_{n-1}(x, y)\,;\, |y| \leq 1\} < 1/2^{n+1}$ because $\{\psi_{n-1}(x, y)\,;\, |y| \leq 1\}$ consists of one point.

(2) $\varphi_{n-1} \circ R_{1/q_{n-1}} = R_{1/q_{n-1}} \circ \varphi_{n-1}$. Since $q_{n-1}\theta_{n-1}$ is an integer, we have $\varphi_{n-1} \circ R_{\theta_{n-1}} = R_{\theta_{n-1}} \circ \varphi_{n-1}$.

(3) $\varphi_{n-1}|\ell_n = \psi_{n-1}|\ell_n$. In particular, $\varphi_{n-1}(\ell_n) = \ell_{n-1}$, $\varphi_{n-1}(0,-1/2) = (0,-1/2)$ and $\varphi_{n-1}(1/(L_1 \cdots L_{n-1}), 1/2) = (1/(L_1 \cdots L_{n-2}), 1/2)$, where $\varphi_1(1/L_1, 1/2)$ is assumed to be $(1, 1/2)$.

(4) $\varphi_{n-1}(X_n) \subset \{(x,y)\,;\, |y| < 3/4\}$.

Let $\Phi_{n-1} = \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_{n-1}$ for $n > 1$ ($\Phi_0 = \mathrm{id}$). We choose a large integer q_n satisfying that

(5) There is $k_n \in \mathbf{Z}_+$ such that $q_n = k_n q_{n-1} L_1 \cdots L_{n-1}$. In particular, $q_n > L_1 \cdots L_{n-1}$. Further we assume that $q_n > 2n$ for $n > 1$.

(6) If $z_1, z_2 \in X_n$ and $d(z_1, z_2) \le n/q_n$, then $d(\Phi_{n-1}(z_1), \Phi_{n-1}(z_2)) < 1/2^n$.

(7) For $\theta_n = \sum_{i=1}^n 1/q_i$, $q_n\theta_n$ and q_n are relatively prime. For example, if $q_n = k{q_{n-1}}^2$ and $\theta_n = (1/q_n) + (j/(q_{n-1}))$ for some integers k and j, then $\theta_n = (1 + kjq_{n-1})/k{q_{n-1}}^2$. Thus $q_n\theta_n = 1 + kjq_{n-1}$ and $q_n = kq_{n-1}^2$ are relatively prime.

Here we remark that φ_{n-1} has already been given independent of the choice of q_n.

We choose a smooth increasing function $\eta_n : [3/4, 1] \to \mathbf{R}$ so that $\eta_n(3/4) = \theta_n$, $\eta_n(1) = \theta_{n-1}$ and η_n is constant on neighborhoods of 3/4 and 1. We define a C^∞ diffeomorphism $f_n : X_1 \to X_1$ by

$$f_n(x,y) = \begin{cases} f_{n-1}(x,y) & \text{outside } \Phi_{n-1}(X_n) \\ \Phi_{n-1} R_{\eta_n(|t|)} \Phi_{n-1}^{-1}(x,y) & \text{if } (x,y) \in \Phi_{n-1}(X_n) \text{ and } \dfrac{3}{4} \le |p_2 \Phi_{n-1}^{-1}(x,y)| \le 1 \\ \Phi_{n-1} R_{\theta_n} \Phi_{n-1}^{-1}(x,y) & \text{if } (x,y) \in \Phi_{n-1}(X_n) \text{ and } |p_2 \Phi_{n-1}^{-1}(x,y)| \le \dfrac{3}{4}, \end{cases}$$

where f_n is well-defined by (2). We further assume that q_n is so large that f_n is assumed to be $1/2^n$-closed to f_{n-1} in the C^n-topology.

Let L_n denote the integer defined by $L_n = q_n\,(2/(L_1 \cdots L_{n-1}) - 1/q_n)$. Then

$$\begin{aligned} \frac{1}{L_1 \cdots L_{n-1}} &< \left(\frac{1}{L_1 \cdots L_{n-1}} - \frac{1}{q_n} \right) + \frac{1}{L_1 \cdots L_{n-1}} \text{ by (5)} \\ &= \frac{2}{L_1 \cdots L_{n-1}} - \frac{1}{q_n} \\ &= \frac{L_n}{q_n}. \end{aligned}$$

Thus $1/(L_1 \cdots L_n) < 1/q_n$. As a consequence, we have

(8) $0 < \dfrac{1}{L_1 \cdots L_n} < \dfrac{1}{q_n} < \dfrac{1}{L_1 \cdots L_n} + \dfrac{1}{q_n} < \dfrac{2}{q_n} < \cdots < 1.$

We define a simple closed curve $C_n : \mathbf{R}/L_n\mathbf{Z} \to X_n$ by

$$C_n(t) = \begin{cases} \left(t, L_1 \cdots L_{n-1} t - \dfrac{1}{2}\right) & \text{if } 0 \leq t \leq \dfrac{1}{L_1 \cdots L_{n-1}} \\ \left(-t + \dfrac{2}{L_1 \cdots L_{n-1}}, \dfrac{L_1 \cdots L_{n-1} q_n}{q_n - L_1 \cdots L_n}\left(t - \dfrac{L_n}{q_n}\right) - \dfrac{1}{2}\right) & \text{if } \dfrac{1}{L_1 \cdots L_{n-1}} \leq t \leq \dfrac{L_n}{q_n} \end{cases}$$

and $C_n(t+L_n/q_n) = R_{1/q_n} C_n(t)$ for any t. Then C_n is invariant under R_{θ_n}. We construct φ_n and f_n $(n = 1, 2, \cdots)$ inductively in this way.

By the same argument as in [**7**] and [**5**], we can choose q_n so large that f_n converges to a C^∞ diffeomorphism f as $n \to \infty$, and $d(f^k(x, y), f_n^k(x, y)) < 1/2^n$ for any $(x, y) \in X_1$ and $0 \leq k \leq q_n$.

Remark 1. We can extend f to a C^∞ diffeomorphism of any surface.

3.4. Properties of the minimal set.

Let $X = \bigcap_{n=2}^\infty \Phi_{n-1}(X_n)$. Then X is not empty because

$$\cdots \subset \Phi_n(X_{n+1}) \subset \Phi_{n-1}(X_n) \subset \cdots.$$

Furthermore, X contains the arc ℓ_1 because $\Phi_{n-1}(\ell_n) = \ell_1$. On the other hand, if $(x, y) \notin \Phi_k(X_{k+1})$ for some $k \in \mathbf{Z}_+$, then $f_n(x, y) = f_k(x, y)$ for any $n > k$. Since $\Phi_{n-1}(X_n)$ is connected, the set X is connected. Thus, in order to prove Theorem 1, we have only to show that X is a minimal set (Lemma 1) and X is not path-connected (Lemma 2).

Proposition 2. *For the subsets $D_i^n = \{(x, y) \in X_n\,;\, i/q_n \leq x \leq (i+1)/q_n\}$ $(i = 0, 1, \ldots, q_n - 1)$, the diameter of $\Phi_{n-1}(D_i^n)$ is less than $1/2^{n-1}$.*

Proof. Let $z_1, z_2 \in D_i^n$. Let $z_1' = (p_1(z_1), 0)$ and $z_2' = (p_1(z_2), 0)$. Since $d(z_1', z_2') \leq 1/q_n$, we have $d(\Phi_{n-1}(z_1'), \Phi_{n-1}(z_2')) < 1/2^n$ by (6). Since $\{\psi_{n-1}(x, y)\,;\, |y| \leq 1\}$ consists of one point, $\psi_{n-1}(z_i) = \psi_{n-1}(z_i')$ for $i = 1, 2$. Thus

$$\begin{aligned} d(\Phi_{n-1}(z_i), \Phi_{n-1}(z_i')) &\leq d(\Phi_{n-1}(z_i), \Phi_{n-2}\psi_{n-1}(z_i)) + d(\Phi_{n-2}\psi_{n-1}(z_i'), \Phi_{n-1}(z_i')) \\ &< \frac{1}{2^{n+2}} + \frac{1}{2^{n+2}} \text{ by (1)} \\ &= \frac{1}{2^{n+1}} \end{aligned}$$

for $i = 1, 2$. Therefore

$$\begin{aligned} &d(\Phi_{n-1}(z_1), \Phi_{n-1}(z_2)) \\ &\quad \leq d(\Phi_{n-1}(z_1), \Phi_{n-1}(z_1')) + d(\Phi_{n-1}(z_1'), \Phi_{n-1}(z_2')) + d(\Phi_{n-1}(z_2'), \Phi_{n-1}(z_2)) \\ &\quad < \frac{1}{2^{n-1}}. \end{aligned}$$

□

Proposition 3. *For any z of X,*

$$f_n^j(z) = \Phi_{n-1} R_{\theta_n}^j \Phi_{n-1}^{-1}(z)$$

for $n, j \in \mathbf{Z}_+$.

PROOF. Let z be a point of X. Then $z \in \Phi_n(X_{n+1})$ for any $n > 0$. Since $z \in \Phi_{n-1}(\varphi_n(X_{n+1}))$, we have $|p_2\Phi_{n-1}^{-1}(z)| < 3/4$ by (4). Therefore, $f_n(z) = \Phi_{n-1}R_{\theta_n}\Phi_{n-1}^{-1}(z)$ by definition. Suppose that $f_n^k(z) = \Phi_{n-1}R_{\theta_n}^k\Phi_{n-1}^{-1}(z)$ for some $k \in \mathbf{Z}_+$. Then $f_n^k(z) = \Phi_{n-1}(R_{\theta_n}^k\Phi_{n-1}^{-1}(z)) \in \Phi_{n-1}(X_n)$. Furthermore, $|p_2\Phi_{n-1}^{-1}f_n^k(z)| = |p_2R_{\theta_n}^k\Phi_{n-1}^{-1}(z)| < 3/4$ as above. Therefore, $f_n^{k+1}(z) = \Phi_{n-1}R_{\theta_n}\Phi_{n-1}^{-1}f_n^k(z)$ by the definition of f_n. Thus $f_n^{k+1}(z) = \Phi_{n-1}R_{\theta_n}^{k+1}\Phi_{n-1}^{-1}(z)$. By induction, $f_n^j(z) = \Phi_{n-1}R_{\theta_n}^j\Phi_{n-1}^{-1}(z)$ for any $j \in \mathbf{Z}_+$. □

LEMMA 1. *X is a minimal set of f.*

PROOF. First prove that X is invariant under f. Let $z \in X$. We fix $n \in \mathbf{Z}$ ($n \geq 2$). Let k be an integer greater than or equal to n. Then $f_k(z) = \Phi_{k-1}R_{\theta_k}\Phi_{k-1}^{-1}(z)$ by Proposition 3. Let $w = (\varphi_n \circ \cdots \circ \varphi_{k-1})R_{\theta_k}\Phi_{k-1}^{-1}(z) \in X_n$. Then $f_k(z) = \Phi_{n-1}(\varphi_n \circ \cdots \circ \varphi_{k-1})R_{\theta_k}\Phi_{k-1}^{-1}(z) = \Phi_{n-1}(w)$ is an element of $\Phi_{n-1}(X_n)$. Therefore $f(z) = \lim_{k\to\infty} f_k(z) \in \Phi_{n-1}(X_n)$. As a result, $f(z) \in \bigcap_{n=2}^{\infty}\Phi_{n-1}(X_n) = X$. Since $f^{-1}(z) = \lim_{k\to\infty} f_k^{-1}(z)$, we can also show that $f^{-1}(z) \in X$. Thus $f(X) = X$.

Next we will show that the orbit of any point z of X is dense in X. Let u be a point of X. For an arbitrary positive integer n, let $z_n = \Phi_{n-1}^{-1}(z) \in X_n$ and $u_n = \Phi_{n-1}^{-1}(u) \in X_n$. Then there is i ($0 \leq i < q_n$) such that $u_n \in D_i^n = \{(x,y) \in X_n\,;\, i/q_n \leq x \leq (i+1)/q_n\}$. For $j_n = q_n\theta_n$, the integers j_n and q_n are relatively prime by (7). Thus there is $k \in \mathbf{Z}$ ($0 \leq k < q_n$) such that $R_{\theta_n}^k(z_n) \in D_i^n$. Since $\operatorname{diam}\Phi_{n-1}(D_i^n) < 1/2^{n-1}$ by Proposition 2, we have $d(\Phi_{n-1}R_{\theta_n}^k(z_n), \Phi_{n-1}(u_n)) < 1/2^{n-1}$. On the other hand, by Proposition 3, $d(f_n^k(z), u) = d(\Phi_{n-1}R_{\theta_n}^k\Phi_{n-1}^{-1}(z), u) = d(\Phi_{n-1}R_{\theta_n}^k(z_n), \Phi_{n-1}(u_n)) < 1/2^{n-1}$ as above. Since $d(f^k(z), f_n^k(z)) < 1/2^n$ for $0 \leq k \leq q_n$ by construction, we conclude that $d(f^k(z), u) < 3/2^n$. Thus the orbit of z is dense in X □

We fix $n \geq 1$. Let $v_i = (i/q_n, -1/2) \in X_n$ and $w_i = ((1/(L_1\cdots L_{n-1})) + (i/q_n), 1/2) \in X_n$ for $i = 1, 2, \ldots, n$. Let $v_i' = (i/q_n, -1/2) \in X_{n+1}$ and $w_i' = ((1/(L_1\cdots L_n)) + (i/q_n), 1/2) \in X_{n+1}$ for $i = 1, 2, \ldots, n$. Then $p_1(v_1') < p_1(w_1') < p_1(v_2') < p_1(w_2') < \cdots$ by (8).

PROPOSITION 4. $\psi_n(v_i') = v_i$ *and* $\psi_n(w_i') = w_i$.

PROOF.

$$\psi_n(v_i') = \psi_n\left(\frac{i}{q_n}, -\frac{1}{2}\right) = C_n\left(i\frac{L_n}{q_n}\right) = (R_{1/q_n})^i C_n(0) = \left(\frac{i}{q_n}, -\frac{1}{2}\right) = v_i.$$

$$\begin{aligned}\psi_n(w_i') &= \psi_n\left(\frac{1}{L_1\cdots L_n} + \frac{i}{q_n}, \frac{1}{2}\right) = C_n\left(\frac{1}{L_1\cdots L_{n-1}} + \frac{iL_n}{q_n}\right)\\ &= (R_{1/q_n})^i C_n\left(\frac{1}{L_1\cdots L_{n-1}}\right) = \left(\frac{1}{L_1\cdots L_{n-1}} + \frac{i}{q_n}, \frac{1}{2}\right) = w_i.\end{aligned}$$

□

LEMMA 2. *X is not path-connected.*

PROOF. Let $z_1 = (1, 1/2) \in X_1$ and $z_2 = (1/2, -1/2) \in X_1$. The point z_1 is an end point of ℓ_1. Thus $z_1 \in X$. Furthermore, $\Phi_n(1/(L_1 \cdots L_n), 1/2) = z_1$ for any $n \in \mathbf{Z}_+$ because $\Phi_n(\ell_{n+1}) = \ell_1$ by (3). On the other hand, for any $j \geq 1$, $\varphi_j(z_2) = \varphi_j(1/q_1, -1/2) = R_{1/q_1}\varphi_j(0, -1/2) = R_{1/q_1}(0, -1/2) = (1/q_1, -1/2) = z_2$ by (2). Thus $\Phi_n(1/2, -1/2) = z_2$ for any $n \in \mathbf{Z}_+$. Therefore $z_2 \in X$.

Assume that there is a path γ connecting z_1 and z_2 contained in X. We further assume that $\gamma : [0, 1] \to X$ is homotopic to $t \mapsto (1 - (1/2)t, 1/2 - t)$ in X_1 with the boundary fixed (we can prove the other cases similarly).

Let N denote the number of connected components of $\gamma \cap p_2^{-1}(-1/4, 1/4)$ such that one of the boundary points is contained in $p_2^{-1}(-1/4)$ and the other boundary point is contained in $p_2^{-1}(1/4)$.

We choose an integer n satisfying $2n - 2 > N$ and $n \geq 3$. Let $\gamma_{n+1} = \Phi_n^{-1}(\gamma)$. Then γ_{n+1} connects $(1/(L_1 \cdots L_n), 1/2)$ with $(1/q_1, -1/2)$ in X_{n+1} as above. By (8) and (5), we obtain

$$\frac{1}{L_1 \cdots L_n} < \frac{1}{q_n} < \frac{1}{q_n} + \frac{1}{L_1 \cdots L_n} < \frac{2}{q_n} < \cdots < \frac{n-1}{q_n} + \frac{1}{L_1 \cdots L_n} < \frac{n}{q_n} < \frac{1}{q_1}.$$

We choose points $a'_i \in X_{n+1}$ in $p_1^{-1}(i/q_n) \cap \gamma_{n+1}$ for $i = 1, 2, \ldots, n$ and points $b'_j \in X_{n+1}$ in $p_1^{-1}(j/q_n + 1/(L_1 \cdots L_n)) \cap \gamma_{n+1}$ for $j = 1, 2, \ldots, n-1$ so that there are s_i and t_j of $[0, 1]$ satisfying $a'_i = \gamma_{n+1}(s_i)$, $b'_j = \gamma_{n+1}(t_j)$ and

$$0 < s_1 < t_1 < s_2 < t_2 < \cdots < t_{n-1} < s_n < 1.$$

Now $p_1(v'_i) = i/q_n$ and $p_1(a'_i) = i/q_n$. Since the diameter of $\{\Phi_n(x, y)\,;\, |y| \leq 1\} < 1/2^{n+2}$ by (1), we have $d(\Phi_n(v'_i), \Phi_n(a'_i)) < 1/2^{n+2}$ for $i = 1, 2, \ldots, n$. Furthermore, $d(\Phi_n(v'_i), \Phi_{n-1}(v_i)) = d(\Phi_{n-1}\varphi_n(v'_i), \Phi_{n-1}\psi_n(v'_i)) < 1/2^{n+3}$ again by (1) and Proposition 4. Moreover, $d(\Phi_{n-1}(v_i), (0, -1/2)) < 1/2^n$ by (6). As a result,

$$p_2\Phi_n(a'_i) < -\frac{1}{2} + \frac{1}{2^{n+2}} + \frac{1}{2^{n+3}} + \frac{1}{2^n} < -\frac{1}{4}$$

when $n \geq 3$.

On the other hand, for $j = 1, 2, \ldots, n-1$, $d(\Phi_n(w'_j), \Phi_n(b'_j)) < 1/2^{n+2}$ by (1), and $d(\Phi_n(w'_j), \Phi_{n-1}(w_j)) = d(\Phi_{n-1}\varphi_n(w'_j), \Phi_{n-1}\psi_n(w'_j)) < 1/2^{n+3}$ by (1) and Proposition 4. Since $d(\Phi_{n-1}(w_j), (1, 1/2)) = d(\Phi_{n-1}(w_j), \Phi_{n-1}(1/(L_1 \cdots L_{n-1}), 1/2)) < 1/2^n$ by (6), we have

$$p_2\Phi_n(b'_j) > \frac{1}{2} - \frac{1}{2^{n+2}} - \frac{1}{2^{n+3}} - \frac{1}{2^n} > \frac{1}{4}$$

when $n \geq 3$.

The points $\Phi_n(a'_i)$ and $\Phi_n(b'_j)$ of γ satisfy $p_2\Phi_n(a'_i) < -1/4$ and $p_2\Phi_n(b'_j) > 1/4$. Therefore, there are at least $2n - 2$ connected components of $\gamma \cap p_2^{-1}(-1/4, 1/4)$ such that one of the boundaries is contained in $p_2^{-1}(-1/4)$ and the other boundary point is contained in $p_2^{-1}(1/4)$. However, this contradicts the assumption, $2n - 2 > N$. Therefore, there is no path γ connecting z_1 and z_2. □

Remark 2. A locally connected complete metric space is path-connected (see

[**8**, Section 50]). Thus the minimal set of Theorem 1 is not locally connected.

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