

L^p measure of growth and higher order Hardy–Sobolev–Morrey inequalities on \mathbb{R}^N

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Abstract. When the growth at infinity of a function u on \mathbb{R}^N is compared with the growth of $|x|^s$ for some $s \in \mathbb{R}$, this comparison is invariably made pointwise. This paper argues that the comparison can also be made in a suitably defined L^p sense for every $1 \leq p < \infty$ and that, in this perspective, inequalities of Hardy, Sobolev or Morrey type account for the fact that sub $|x|^{-N/p}$ growth of ∇u in the L^p sense implies sub $|x|^{1-N/p}$ growth of u in the L^q sense for well chosen values of q .

By investigating how sub $|x|^s$ growth of $\nabla^k u$ in the L^p sense implies sub $|x|^{s+j}$ growth of $\nabla^{k-j} u$ in the L^q sense for (almost) arbitrary $s \in \mathbb{R}$ and for q in a p -dependent range of values, a family of higher order Hardy/Sobolev/Morrey type inequalities is obtained, under optimal integrability assumptions.

These optimal inequalities take the form of estimates for $\nabla^{k-j}(u - \pi_u)$, $1 \leq j \leq k$, where π_u is a suitable polynomial of degree at most $k - 1$, which is unique if and only if $s < -k$. More generally, it can be chosen independent of (s, p) when s remains in the same connected component of $\mathbb{R} \setminus \{-k, \dots, -1\}$.

1. Introduction.

Unless specified otherwise, \mathbb{R}^N is the domain of all function spaces. If $s > -1$, $u \in \mathcal{D}'$ (distributions) and $\nabla u \in (L_{loc}^\infty)^N$ grows slower than $|x|^s$ at infinity for some $s > -1$, then u grows slower than $|x|^{s+1}$ at infinity. In this statement, growth is understood pointwise, outside a set of Lebesgue measure 0 and the precise result is that if $(1 + |x|)^{-s} \nabla u \in (L^\infty)^N$, then $(1 + |x|)^{-s-1} u \in L^\infty$. This property breaks down if $s \leq -1$ but, if $s < -1$ and $N > 1$, it is still true that $(1 + |x|)^{-s-1}(u - c_u) \in L^\infty$ for a unique constant c_u .

The pointwise criterion is only one of the ways to compare the growth of a function against the growth of the powers of $|x|$, but it is not necessarily the most useful one. For instance, it is notorious that pointwise growth has little relevance for functions of L_{loc}^p with $1 \leq p < \infty$, and, from the context, it is intuitively clear that an L^p evaluation of growth could only be more adequate.

Such an L^p measure of growth can be captured by various closely related but non-equivalent definitions. The option chosen in this paper is to say that $u \in L_{loc}^p$ grows slower than $|x|^s$ in the L^p sense if $(1 + |x|)^{-s-N/p} u \in L^p$. This is justified by the remarks that the function $u(x) := (1 + |x|)^t$ satisfies this condition if and only if $t < s$ and that the pointwise concept is recovered when $p = \infty$ although, in this case, $(1 + |x|)^{-s} u \in L^\infty$ still holds if u grows as fast as $|x|^s$ at infinity. Strictly slower growth requires the stronger

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$\lim_{R \rightarrow \infty} \operatorname{ess\,sup}_{|x| > R} |x|^{-s}|u| = 0$ or simply $\lim_{|x| \rightarrow \infty} |x|^{-s}u(x) = 0$ if u is continuous. In particular, $u \in L^p$ with $p < \infty$ ($p = \infty$) if and only if u grows slower than $|x|^{-N/p}$ (no faster than $|x|^0 = 1$) in the L^p sense. Of course, the choice of a scale based on $1 + |x|$ rather than $|x|$ is meant to avoid integrability issues near the origin, which have nothing to do with behavior at infinity.

It is a natural question whether the feature of the $p = \infty$ case highlighted in the first paragraph is preserved when $p < \infty$: If $s \neq -1$ and ∇u grows slower than $|x|^s$ in the L^p sense, is there a constant c_u such that $u - c_u$ grows slower than $|x|^{s+1}$ in the L^p sense, or in the L^q sense for some $q \neq p$? If $s > -1$, is it possible to choose $c_u = 0$? Although some widely explored issues, such as inequalities of Hardy, Sobolev or Morrey type, turn out to be intimately related to these questions, they have apparently not been tackled up front and the connection between familiar inequalities and growth transfer from gradient to function, while intuitively obvious, has nonetheless remained rather vague.

This paper investigates the more general growth transfer property in the $L^p - L^q$ sense, when ∇u is replaced with $\nabla^k u$ for some $k \in \mathbb{N}$ and $s \in \mathbb{R} \setminus \{-k, \dots, -1\}$. To deal with the excluded values, the discussion should incorporate a logarithmic scale and is omitted. Also, it will be necessary to assume $N > 1$ when $s < -1$, although this restriction can be lifted when \mathbb{R} is replaced with \mathbb{R}_\pm .

The space

$$L_s^q := \{u \in L_{loc}^q : (1 + |x|)^{-s-N/q}u \in L^q\}, \quad 1 \leq q \leq \infty, \tag{1.1}$$

that embodies sub $|x|^s$ growth in the L^q sense if $q < \infty$ (and up to $|x|^s$ growth if $q = \infty$) is equipped with the Banach space norm

$$\|u\|_{L_s^q} := \|(1 + |x|)^{-s-N/q}u\|_q, \tag{1.2}$$

where $\|\cdot\|_q$ is the L^q norm. If $q < \infty$, then $L_s^q = L^q(\mathbb{R}^N; (1 + |x|)^{-sq-N}dx)$, with identical norms.

A little more notation must be introduced to give a concise summary of the results. The number

$$\nu(k, N) := \binom{N + k - 1}{k}, \tag{1.3}$$

is the dimension of the space of real symmetric tensors of order $k \in \mathbb{N}$ and, for $d \in \mathbb{Z}$, \mathcal{P}_d denotes the space of polynomials of degree at most d , with the usual agreement that $\mathcal{P}_d = \{0\}$ if $d < 0$. Lastly, if $j \in \mathbb{N}$ and $1 \leq p \leq \infty$, we set $p^{*j} := Np/(N - jp)$ if $p < N/j$ and $p^{*j} := \infty$ otherwise and

$$I_{j,p} = \begin{cases} [p, p^{*j}] & \text{if } p \neq N/j \text{ or if } p = N = j = 1, \\ [p, \infty) & \text{if } p = N/j \text{ with } N > 1. \end{cases} \tag{1.4}$$

In particular, $I_{j,p} = [p, \infty)$ irrespective of j if $N = 1$.

The main result (Theorem 4.4) states that if $k \in \mathbb{N}$, $1 \leq p < \infty$ and either $s > -1$ or $N > 1$ and $s \notin \{-k, \dots, -1\}$ and if $\nabla^k u \in (L_s^p)^{\nu(k,N)}$, there is a polynomial $\pi_u \in \mathcal{P}_{k-1}$

such that $\nabla^{k-j}(u - \pi_u) \in (L^q_{s+j})^{\nu(k-j,N)}$ for every $1 \leq j \leq k$ and every $q \in I_{j,p}$ and there is a constant $C > 0$ independent of u such that

$$\|\nabla^{k-j}(u - \pi_u)\|_{L^q_{s+j}} \leq C \|\nabla^k u\|_{L^p_s}. \tag{1.5}$$

In particular, $\nabla^{k-j}(u - \pi_u)$ grows slower than $|x|^{s+j}$ in the L^q sense for every finite $q \in I_{j,p}$ and no faster than $|x|^{s+j}$ in the L^∞ sense when $p > N/j$ (so that $\infty \in I_{j,p}$). We also show that, in the latter case, $\nabla^{k-j}(u - \pi_u)$ still grows slower than $|x|^{s+j}$, that is, $\lim_{|x| \rightarrow \infty} |x|^{-(s+j)}(\nabla^{k-j}u(x) - \nabla^{k-j}\pi_u(x)) = 0$. When $j = k$ and $s = -N/p$ (i.e., $\nabla^k u \in (L^p)^{\nu(k,N)}$), this pointwise property was proved by Mizuta [14], by a different method.

If $s > -1$, then $\mathcal{P}_{j-1} \subset L^q_{s+j}$ irrespective of q and (1.5) implies $\nabla^{k-j}u \in L^q_{s+j}$ for every $q \in I_{j,p}$. Thus, π_u is irrelevant as regards the property $\nabla^{k-j}(u - \pi_u) \in (L^q_{s+k})^{\nu(k-j,N)}$, but it remains of course essential for the validity of (1.5).

The polynomial π_u may be chosen independent of (s, p) when s remains in any connected component of $\mathbb{R} \setminus \{-k, \dots, -1\}$ and its nature is different depending upon k and s and, to some extent, even p . When $s > -1$, there are many different ways to define a (generally different) polynomial π_u , each one being more or less reminiscent of a Taylor polynomial of u of order $k - 1$. That π_u may be chosen as a genuine Taylor polynomial is only true when $p > N$. Without this restriction (and, still, $s > -1$), the coefficients of π_u can be obtained by averaging the partial derivatives of u of order up to $k - 1$ on arbitrarily chosen balls independent of u . For details, see Theorem 2.3 when $k = 1$ and the comments following Theorem 4.4 in general.

In contrast, π_u is unique when $s < -k$ and its coefficients depend only upon the behavior at infinity of the partial derivatives of u of order up to $k - 1$. If $s \in (-k, -1)$, the part of π_u of higher degree is unique and depends upon the behavior of the higher order partial derivatives of u at infinity and its part of lower degree can be chosen as a Taylor polynomial of sorts, much like in the case when $s > -1$. Naturally, the meaning of higher and lower degree will be clarified.

All the spaces L^q_s are dilation-invariant, which allows for scaling arguments. In many cases, scaling produces inequalities (1.5) in which the weight $1 + |x|$ may be replaced with $|x|$. When $k = 1$ and $s = -N/p$, different choices of q produce the following sample of at least partially known inequalities:

(i) $\| |x|^{-1}(u - u(0)) \|_p \leq C \| |\nabla u| \|_p$ if $p > N$ (with $q = p$ and $\pi_u = u(0)$ in (1.5), plus scaling). This is Hardy’s inequality.

(ii) $\| u - c_u \|_{p^*} \leq C \| |\nabla u| \|_p$ for a unique constant c_u if $p < N$ (with $q = p^*$ and $\pi_u = c_u$ in (1.5)), a known generalization of Sobolev’s inequality ([13, Section 6.7.5]).

(iii) $\sup_{x \in \mathbb{R}^N} |x|^{-1+N/p} |u(x) - u(0)| \leq C \| |\nabla u| \|_p$ if $p > N$ (with $q = \infty$ and $\pi_u = u(0)$ in (1.5), plus scaling). This is Morrey’s inequality.

(iv) $\| (1 + |x|)^{-1}(u - c_u) \|_p \leq C \| |\nabla u| \|_p$ for a unique constant c_u if $p < N$ (with $q = p$ and $\pi_u = c_u$ in (1.5)), a variant (with $|x|$ replaced with $1 + |x|$) and generalization of the Hardy–Leray inequality¹ when $u \in \mathcal{C}_0^\infty$ (so that $c_u = 0$) ([10], [13, Section 2.8.1]). By scaling, the Hardy–Leray inequality $\| |x|^{-1}u \|_p \leq C \| |\nabla u| \|_p$ follows under the more

¹That is, Hardy’s inequality when $p < N$.

general assumption $c_u = 0$.

Other values of q, s or k produce inequalities of the same type. We shall refer to the texts by Maz'ya [13] and Opic and Kufner [16] for various related inequalities on \mathbb{R}^N when $N > 1$. The papers by Caffarelli, Kohn and Nirenberg [3], Catrina and Costa [4], Gatto, Gutiérrez and Wheeden [7], Lin [11] and the author [18], are in a similar spirit, but specifically devoted to inequalities involving pure power weights $|x|^s$. We do not mention work limited to Muckenhoupt weights since $(1 + |x|)^s$ need not belong to this class.

Aside from technical differences due to the choice of weights, the inequalities (1.5) depart from those in the above and other works in more basic aspects. In the literature, the focus has overwhelmingly been on inequalities of the form (1.5) when $\pi_u = 0$. Since this is not typical, such inequalities can only be true under restrictive assumptions. In fact, while (1.5) holds under the optimal integrability condition $\nabla^k u \in (L_s^p)^{\nu(k,N)}$, the others assume at least that u belongs to some weighted Sobolev space ([16], [18]) and, more often, $u \in C_0^\infty$ or $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ ([3], [4], [7], [11], [13], [16]), especially when $k > 1$ ([11], [13]). In that regard, it is instructive to observe that if $u \in C_0^\infty$, then $\pi_u = 0$ when π_u is determined by behavior at infinity (i.e., $s < -k$) and also $\pi_u = 0$ if $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ and π_u may be chosen as a Taylor polynomial at 0 (i.e., $s > -1$).

The proof of Theorem 4.4 is by induction on k . The case when $k = 1$ (Theorem 4.3) is more demanding and the proof has three steps. The first two consist in proving the theorem when $q = p$ or when u is radially symmetric and $q \in I_{1,p}$, and either $s > -1$ (Theorem 2.3) or $s < -1$ (Theorem 3.2). The main ingredients include a property of approximation by mollification in weighted spaces $L^p(\mathbb{R}^N; wdx)$ when $1 \leq p < \infty$ and $\log w$ is uniformly continuous (Lemma 2.1), two special cases of well-known one-dimensional Hardy-type inequalities (Lemmas 2.2 and 3.1) and the Poincaré–Wirtinger inequality on bounded open subsets of \mathbb{R}^N and on the sphere \mathbb{S}^{N-1} .

To prove Theorem 4.3 when $q > p$, we take advantage of the fact that the radially symmetric case has already been settled to reduce the problem when u has a vanishing radial symmetrization. Under this additional assumption, an elaboration on an argument first used by Caffarelli, Kohn and Nirenberg [3] (Lemma 4.2) completes the proof.

The existence of π_u depends only upon $\nabla^k u$ being in $(L_s^p)^{\nu(k,N)}$, but further assumptions about u may have an impact on π_u . In Section 5, we use this remark to sharpen and generalize known embedding theorems of weighted Sobolev spaces. The transfer of sub-exponential growth is briefly discussed in Section 6.

Throughout the paper, $C > 0$ denotes a constant whose value may change from place to place. The notation B_R refers to the open ball with center 0 and radius $R > 0$ in \mathbb{R}^N and $\tilde{B}_R := \mathbb{R}^N \setminus \bar{B}_R$. If $1 \leq p \leq \infty$, the Hölder conjugate of p is denoted by p' . We shall also make use of the norms $\|\cdot\|_{p,\Omega}$, $\|\cdot\|_{p,\mathbb{S}^{N-1}}$ and $\|\cdot\|_{1,p,\Omega}$ of $L^p(\Omega)$, $L^p(\mathbb{S}^{N-1})$ and (the classical Sobolev space) $W^{1,p}(\Omega)$, respectively.

2. Preliminary first order inequalities when $s > -1$.

We need a property of approximation by mollification in weighted Lebesgue spaces $L^p(\mathbb{R}^N; wdx)$ when $\log w$ is uniformly continuous. Just to put things in perspective, recall that $\log w \in BMO$ if w is a Muckenhoupt weight ([15]).

LEMMA 2.1. *Let $w > 0$ be a function such that $\log w$ is uniformly continuous on \mathbb{R}^N . The following properties hold:*

- (i) *For every $\varepsilon > 0$, there is $\delta > 0$ such that $w(x) \leq (1 + \varepsilon)w(y)$ whenever $|x - y| < \delta$.*
- (ii) *For every $\varepsilon > 0$, there is $\delta > 0$ such that $|w(x) - w(y)| \leq \varepsilon w(y)$ whenever $|x - y| < \delta$.*
- (iii) *If $\theta_n \in C_0^\infty$ is a sequence of mollifiers and if $u \in L^p(\mathbb{R}^N; wdx)$ for some $1 \leq p < \infty$, then $\theta_n * u \in L^p(\mathbb{R}^N; wdx)$ for n large enough and $\theta_n * u \rightarrow u$ in $L^p(\mathbb{R}^N; wdx)$.*

PROOF. (i) Choose $\delta > 0$ such that $|x - y| < \delta \Rightarrow |\log w(x) - \log w(y)| \leq \log(1 + \varepsilon)$.

(ii) If $|x - y| < \delta$ with $\delta > 0$ from (i), then $w(x) - w(y) \leq \varepsilon w(y)$ and $w(y) - w(x) \leq \varepsilon w(x) \leq \varepsilon(1 + \varepsilon)w(y)$. Thus, $|w(x) - w(y)| \leq \varepsilon(1 + \varepsilon)w(y)$ and it suffices to replace $\varepsilon(1 + \varepsilon)$ with ε .

(iii) With ε and δ from (i), let n be large enough that $\text{Supp } \theta_n \subset B_{\delta/2}$. For simplicity of notation, set $w_p := w^{1/p}$, so that $uw_p \in L^p$. Then,

$$\begin{aligned} |((\theta_n * u)w_p)(x)| &= \int_{B(x, \delta/2)} \theta_n(x - y)u(y)w_p(x)dy \\ &\leq (1 + \varepsilon)^{1/p} \int_{B(x, \delta/2)} \theta_n(x - y)|u(y)|w_p(y)dy \\ &= (1 + \varepsilon)^{1/p}(\theta_n * (|u|w_p))(x). \end{aligned}$$

This shows that $\theta_n * u \in L^p(\mathbb{R}^N; wdx)$. To prove that $\theta_n * u \rightarrow u$ in $L^p(\mathbb{R}^N; wdx)$, i.e., that $(\theta_n * u)w_p \rightarrow uw_p$ in L^p , write $(\theta_n * u)w_p - uw_p = [(\theta_n * u)w_p - \theta_n * (uw_p)] + [\theta_n * (uw_p) - uw_p]$. The latter bracket tends to 0 in L^p and it suffices to prove that the same thing is true for the former.

Since $\log w_p = (1/p)\log w$ is uniformly continuous on \mathbb{R}^N , part (ii) is applicable to w_p . Thus, given $\varepsilon > 0$, if $\delta > 0$ is small enough and if n is large enough that $\text{Supp } \theta_n \subset B_{\delta/2}$,

$$\begin{aligned} |((\theta_n * u)w_p)(x) - (\theta_n * (uw_p))(x)| &\leq \int_{B(x, \delta/2)} \theta_n(x - y)|u(y)||w_p(x) - w_p(y)|dy \\ &\leq \varepsilon \int_{B(x, \delta/2)} \theta_n(x - y)|u(y)|w_p(y)dy \\ &= \varepsilon(\theta_n * (|u|w_p))(x). \end{aligned}$$

As a result, $\|(\theta_n * u)w_p - \theta_n * (uw_p)\|_p \leq \varepsilon\|\theta_n * (|u|w_p)\|_p$. Since the right-hand side tends to $\varepsilon\|uw_p\|_p$ and $\varepsilon > 0$ is arbitrary, $\lim\|(\theta_n * u)w_p - \theta_n * (uw_p)\|_p = 0$ and the proof is complete. \square

REMARK 2.1. Obviously, Lemma 2.1 is valid when $w(x) = (1 + |x|)^a$ or $w(x) = e^{a|x|}$ and $a \in \mathbb{R}$.

We shall also need a special case of a known one-dimensional weighted Hardy inequality. Lemma 2.2 below follows from Bradley [2, Theorem 1] or Maz'ya [13, p. 40 ff]. Since the weights r^t and $(1 + r)^t$ are equivalent on $[\rho, \infty)$ with $\rho > 0$, it also follows directly from Opic and Kufner [16, Example 6.9, p. 70] when $q < \infty$.

LEMMA 2.2. *Suppose that $s > -1$ and that $1 \leq p < \infty$ and let $\rho > 0$ be given.*

(i) *If $p \leq q < \infty$, there is a constant $C > 0$ such that*

$$\begin{aligned} & \left(\int_{\rho}^{\infty} (1+r)^{-(s+1)q-N} r^{N-1} |f(r) - f(\rho)|^q dr \right)^{1/q} \\ & \leq C \left(\int_{\rho}^{\infty} (1+r)^{-sp-N} r^{N-1} |f'(r)|^p dr \right)^{1/p}, \end{aligned} \quad (2.1)$$

for every locally absolutely continuous function f on $[\rho, \infty)$.

(ii) *There is a constant $C > 0$ such that*

$$\sup_{r \geq R} (1+r)^{-(s+1)} |f(r) - f(R)| \leq C \left(\int_R^{\infty} (1+r)^{-sp-N} r^{N-1} |f'(r)|^p dr \right)^{1/p}, \quad (2.2)$$

for every $R \geq \rho$ and every locally absolutely continuous function f on $[\rho, \infty)$.

In (2.2), the result when $R > \rho$ follows from the case when $R = \rho$ with f replaced with $f_R = f(R)$ on $[\rho, R)$ and $f_R = f$ on $[R, \infty)$, so that $f_R(\rho) = f(R)$, $f'_R = 0$ on $[\rho, R)$ and $f'_R = f'$ on (R, ∞) . This does not affect C .

THEOREM 2.3. *Suppose that $s > -1$ and that $1 \leq p < \infty$. If $u \in \mathcal{D}'$ and $\nabla u \in (L_s^p)^N$, set*

$$c_u := |B_{\rho}|^{-1} \int_{B_{\rho}} u,$$

where $\rho > 0$ is chosen once and for all and independent of u . Then:

(i) *$u \in L_{s+1}^p$ and there is a constant $C = C(s, p) > 0$ independent of u such that*

$$\|u - c_u\|_{L_{s+1}^p} \leq C \|\nabla u\|_{L_s^p}. \quad (2.3)$$

(ii) *If $N = 1$ or if u is radially symmetric, $u \in L_{s+1}^q$ for every $q \in I_{1,p}$ (see (1.4)) and there is a constant $C = C(s, p, q) > 0$ independent of u such that*

$$\|u - c_u\|_{L_{s+1}^q} \leq C \|\nabla u\|_{L_s^p}. \quad (2.4)$$

Furthermore, if $p > N$ or $p = N = 1$ (so that $\infty \in I_{1,p}$), $\lim_{|x| \rightarrow \infty} |x|^{-(s+1)} u(x) = 0$.

PROOF. Suppose first that $u \in C^{\infty}$ and let $u = u(r, \sigma)$ with $r \geq 0$ and $\sigma \in \mathbb{S}^{N-1}$. If $p \leq q < \infty$, it follows from (2.1) that

$$\begin{aligned} & \int_{\rho}^{\infty} (1+r)^{-(s+1)q-N} r^{N-1} |u(r, \sigma) - u(\rho, \sigma)|^q dr \\ & \leq C \left(\int_{\rho}^{\infty} (1+r)^{-sp-N} r^{N-1} |\partial_r u(r, \sigma)|^p dr \right)^{q/p}, \end{aligned}$$

for every $\sigma \in \mathbb{S}^{N-1}$, where $\partial_r u$ is the radial derivative of u . Since $|u(r, \sigma)|^q \leq$

$2^{q-1}[|u(r, \sigma) - u(\rho, \sigma)|^q + |u(\rho, \sigma)|^q]$ and since $\int_\rho^\infty (1+r)^{-(s+1)q-N} r^{N-1} dr < \infty$ (recall $s > -1$), we infer that

$$\begin{aligned} & \int_\rho^\infty (1+r)^{-(s+1)q-N} r^{N-1} |u(r, \sigma)|^q dr \\ & \leq C \left[|u(\rho, \sigma)|^q + \left(\int_\rho^\infty (1+r)^{-sp-N} r^{N-1} |\partial_r u(r, \sigma)|^p dr \right)^{q/p} \right] \\ & \leq C \left[|u(\rho, \sigma)|^q + \left(\int_\rho^\infty (1+r)^{-sp-N} r^{N-1} |\nabla u(r, \sigma)|^p dr \right)^{q/p} \right]. \end{aligned} \quad (2.5)$$

(i) If $q = p$ above, integration on \mathbb{S}^{N-1} yields

$$\|u\|_{L^p_{s+1}(\tilde{B}_\rho)} \leq C(\|u(\rho, \cdot)\|_{p, \mathbb{S}^{N-1}} + \|\nabla u\|_{L^p_s(\tilde{B}_\rho)}), \quad (2.6)$$

when $u \in C^\infty$. Suppose now that $u \in \mathcal{D}'$ and that $\nabla u \in (L^p_s)^N$. Let θ_n denote a mollifying sequence and set $u_n := \theta_n * u$. By Remark 2.1 and Lemma 2.1 (iii), $\nabla u_n = \theta_n * \nabla u \rightarrow \nabla u$ in $(L^p_s)^N$. In particular, $\nabla u_n \rightarrow \nabla u$ in $(L^p_s(\tilde{B}_\rho))^N$. On the other hand, since $\nabla u \in (L^p_s)^N \subset (L^p_{loc})^N$, then $u \in W^{1,p}_{loc}$ ([13, p. 21]). Thus, $u_n \rightarrow u$ in $W^{1,p}_{loc}$ which, by the continuity of the trace (even when $p = 1$; see [1, p. 164]) implies $u_n(\rho, \cdot) \rightarrow u(\rho, \cdot)$ in $L^p(\mathbb{S}^{N-1})$. As a result,

$$\lim_{n,m \rightarrow \infty} \|u_n(\rho, \cdot) - u_m(\rho, \cdot)\|_{p, \mathbb{S}^{N-1}} + \|\nabla(u_n - u_m)\|_{L^p_s(\tilde{B}_\rho)} = 0$$

and so, by (2.6), u_n is a Cauchy sequence in $L^p_{s+1}(\tilde{B}_\rho)$. Call v its limit, so that $u_n \rightarrow v$ in $L^1_{loc}(\tilde{B}_\rho)$. Since also $u_n \rightarrow u$ in $W^{1,p}_{loc} \hookrightarrow L^1_{loc}$, it follows that $u = v \in L^p_{s+1}(\tilde{B}_\rho)$ and (2.6) holds. The remark that $L^p_{s+1}(B_\rho) = L^p(B_\rho)$ now yields $u \in L^p_{s+1}$.

It remains to prove (2.3). Upon replacing u with $u - c_u$, it is not restrictive to assume $\int_{B_\rho} u = 0$, so that $c_u = 0$. By the continuity of the trace, $\|u(\rho, \cdot)\|_{p, \mathbb{S}^{N-1}} \leq C\|u\|_{1,p,B_\rho}$ and, by the Poincaré–Wirtinger inequality, the seminorm $\|\nabla u\|_{p,B_\rho}$ is equivalent to the norm $\|u\|_{1,p,B_\rho}$ on the subspace of functions of $W^{1,p}(B_\rho)$ with zero mean. Thus, $\|u(\rho, \cdot)\|_{p, \mathbb{S}^{N-1}} \leq C\|\nabla u\|_{p,B_\rho}$ and $\|u\|_{p,B_\rho} \leq C\|\nabla u\|_{p,B_\rho}$. Since $(1+|x|)^t$ is bounded above and below on B_ρ for every t , it follows that

$$\|u\|_{L^p_{s+1}(B_\rho)} \leq C\|\nabla u\|_{L^p_s} \text{ and } \|u\|_{L^p_{s+1}(\tilde{B}_\rho)} \leq C\|\nabla u\|_{L^p_s},$$

where (2.6) was used to obtain the second inequality. This proves (2.3) when $\int_{B_\rho} u = 0$ and, hence, in general.

(ii) Suppose $q \in I_{1,p}$ and either $N = 1$ or u is radially symmetric. We only discuss the latter case since it will be clear that the former can be handled similarly. Since $\nabla u \in (L^p_s)^N$ implies that $u \in W^{1,p}_{loc}$ is a function, there is no need to introduce a distribution definition of radial symmetry.

If $q < \infty$, the proof of (2.4) proceeds as in (i), with only minor modifications. When $u \in C^\infty$, (2.5) is still valid but, since now u is radially symmetric, both u and $\partial_r u$ depend only upon r and the inequality becomes

$$\begin{aligned} & \int_{\rho}^{\infty} (1+r)^{-(s+1)q-N} r^{N-1} |u(r)|^q dr \\ & \leq C \left[|u(\rho)|^q + \left(\int_{\rho}^{\infty} (1+r)^{-sp-N} r^{N-1} |\partial_r u(r)|^p dr \right)^{q/p} \right]. \end{aligned}$$

Up to a factor independent of u , the integrals $\int_{\rho}^{\infty} (1+r)^{-(s+1)q-N} r^{N-1} |u(r)|^q dr$ and $\int_{\rho}^{\infty} (1+r)^{-sp-N} r^{N-1} |\partial_r u(r)|^p dr$ are $\|u\|_{L_{s+1}^q(\tilde{B}_{\rho})}^q$ and $\|\partial_r u\|_{L_s^p(\tilde{B}_{\rho})}^p$, respectively. Hence,

$$\|u\|_{L_{s+1}^q(\tilde{B}_{\rho})} \leq C(|u(\rho)| + \|\nabla u\|_{L_s^p(\tilde{B}_{\rho})}).$$

Up to another factor independent of u , the number $|u(\rho)|$ is the $L^p(\mathbb{S}^{N-1})$ norm of the constant function $u(\rho)$. Therefore, since $W^{1,p}(B_{\rho}) \hookrightarrow L^q(B_{\rho})$ for $q \in I_{1,p}$ and since the radial symmetry is preserved in approximations $u_n = \theta_n * u$ by simply choosing radially symmetric mollifiers, the proof can be completed exactly as before.

If $p > N$ or $p = N = 1$, then $\infty \in I_{1,p}$ and the proof when $q = \infty$ is similar: Just use (2.2) with $R = \rho$ instead of (2.1) to get $\|u\|_{L_{s+1}^{\infty}(\tilde{B}_{\rho})} \leq C(|u(\rho)| + \|\nabla u\|_{L_s^p(\tilde{B}_{\rho})})$. (The approximation by mollification is only used in L_s^p with $p < \infty$.)

To see that, in addition, $\lim_{|x| \rightarrow \infty} |x|^{-(s+1)} u(x) = 0$, observe first that, by radial symmetry, $u(x) = f_u(|x|)$ with f_u locally absolutely continuous on $(0, \infty)$ and $f_u'(|x|) = \partial_r u(x)$ (for more details, see the proof of Lemma 4.1 later), so that, by (2.2),

$$\begin{aligned} & \sup_{r \geq R} (1+r)^{-(s+1)} |f_u(r) - f_u(R)| \\ & \leq C \left(\int_R^{\infty} (1+r)^{-sp-N} r^{N-1} |f_u'(r)|^p dr \right)^{1/p} \leq C \|\nabla u\|_{L_s^p(\tilde{B}_R)}, \end{aligned}$$

where $C > 0$ is also independent of R . Thus, if $r \geq R$,

$$(1+r)^{-(s+1)} |f_u(r)| \leq (1+r)^{-(s+1)} |f_u(R)| + C \|\nabla u\|_{L_s^p(\tilde{B}_R)}.$$

Given $\varepsilon > 0$, choose R large enough that $C \|\nabla u\|_{L_s^p(\tilde{B}_R)} \leq \varepsilon$. Since $s > -1$, it follows that $\limsup_{r \rightarrow \infty} (1+r)^{-(s+1)} |f_u(r)| \leq \varepsilon$ and so $\lim_{r \rightarrow \infty} (1+r)^{-(s+1)} f_u(r) = 0$. Equivalently, $\lim_{r \rightarrow \infty} r^{-(s+1)} f_u(r) = 0$, whence $\lim_{|x| \rightarrow \infty} |x|^{-(s+1)} u(x) = 0$. \square

The choice $c_u = |B_{\rho}|^{-1} \int_{B_{\rho}} u$ in (2.3) and (2.4) is not the only possible one. By translation, we may choose $c_u = |B_{\rho}|^{-1} \int_{B(x_0, \rho)} u$ where $x_0 \in \mathbb{R}^N$ is arbitrary but independent of u . The constant C depends upon ρ and x_0 , but it does not necessarily blow up as $\rho \rightarrow 0$:

REMARK 2.2. If $p > N$ or if $N = 1$ (and in no other case), the obvious variants of the inequalities (2.1) and (2.2) continue to hold on $[0, \infty)$ ([**16**, Theorem 5.9, p. 63]) and minor modifications of the proof of Theorem 2.3 yield $\|u - u(0)\|_{L_{s+1}^p} \leq C \|\nabla u\|_{L_s^p}$ for every $u \in \mathcal{D}'$ such that $\nabla u \in (L_s^p)^N$ as well as $\|u - u(0)\|_{L_{s+1}^q} \leq C \|\nabla u\|_{L_s^p}$ if $N = 1$ or if u is radially symmetric and $q \in I_{1,p} = [p, \infty]$. Naturally, $u(0)$ may also be replaced with $u(x_0)$ where x_0 is independent of u .

3. Preliminary first order inequalities when $s < -1$.

We begin with a different version of Lemma 2.2, a special case of [16, Theorem 6.2, p. 65] that can also be found in [2, Theorem 2] or [13, p. 40 ff]. However, the proof is sketched to show how the restriction $q \in I_{1,p}$ (not needed in Lemma 2.2) arises.

LEMMA 3.1. *Suppose that $s < -1$ and that $1 \leq p < \infty$.*

(i) *For every finite $q \in I_{1,p}$ (see (1.4)), there is a constant $C > 0$ such that*

$$\begin{aligned} & \left(\int_0^\infty (1+r)^{-(s+1)q-N} r^{N-1} |f(r)|^q dr \right)^{1/q} \\ & \leq C \left(\int_0^\infty (1+r)^{-sp-N} r^{N-1} |f'(r)|^p dr \right)^{1/p}, \end{aligned} \tag{3.1}$$

for every locally absolutely continuous function f on $(0, \infty)$ such that $\lim_{r \rightarrow \infty} f(r) = 0$.

(ii) *If $p > N$ (so that $\infty \in I_{1,p}$), there is a constant $C > 0$ such that*

$$\sup_{r \geq R} (1+r)^{-(s+1)} |f(r)| \leq C \left(\int_R^\infty (1+r)^{-sp-N} r^{N-1} |f'(r)|^p dr \right)^{1/p}, \tag{3.2}$$

for every $R \geq 0$ and every locally absolutely continuous function f on $(0, \infty)$ such that $\lim_{r \rightarrow \infty} f(r) = 0$.

PROOF. From [16, Theorem 6.2, p. 65], (3.1) and (3.2) with $R = 0$ hold when $q \geq p$ if and only if $\sup_{\xi > 0} A(\xi)B(\xi) < \infty$, where $A(\xi) := \|(1+r)^{-(s+1)-N/q} r^{(N-1)/q}\|_{q,(0,\xi)}$ and $B(\xi) := \|(1+r)^{s+N/p} r^{(1-N)/p}\|_{p',(\xi,\infty)}$. Thus, everything boils down to showing that $A(\xi)B(\xi)$ is bounded when $\xi \rightarrow 0$ and when $\xi \rightarrow \infty$. Note that $B(\xi) < \infty$ since $s < -1$.

If $\xi > 0$ is small, a routine verification shows that $A(\xi) = O(\xi^{N/q})$ and that $B(\xi) = O(1)$ if $p > N$, $B(\xi) = O(|\log \xi|^{(N-1)/N})$ if $p = N$ and $B(\xi) = O(\xi^{1-N/p})$ if $p < N$. Thus, $A(\xi)B(\xi)$ is bounded near 0 if $p > N$ or $p = N = 1$, or if $p = N > 1$ and $q < \infty$, or if $p < N$ and $N/q + 1 - N/p \geq 0$, i.e., $q \leq p^*$. In other words, $A(\xi)B(\xi)$ is bounded near the origin if and only if $q \in I_{1,p}$. For large ξ , $A(\xi) = O(\xi^{-(s+1)})$ and $B(\xi) = O(\xi^{s+1})$, so that $A(\xi)B(\xi)$ is always bounded.

In (3.2), the result when $R > 0$ follows from the case when $R = 0$ with f replaced with $f_R = f(R)$ on $[0, R)$ and $f_R = f$ on $[R, \infty)$, so that $f'_R = 0$ on $[0, R)$ and $f'_R = f'$ on (R, ∞) . \square

THEOREM 3.2. *Suppose $N > 1$, $s < -1$ and $1 \leq p < \infty$. If $u \in \mathcal{D}'$ and $\nabla u \in (L^p_s)^N$, then:*

(i) *There is a unique constant $c_u \in \mathbb{R}$ such that $u - c_u \in L^p_{s+1}$ and there is a constant $C = C(s, p) > 0$ independent of u such that*

$$\|u - c_u\|_{L^p_{s+1}} \leq C \|\nabla u\|_{L^p_s}. \tag{3.3}$$

(ii) *If also u is radially symmetric, then for every $q \in I_{1,p}$, c_u in (i) is the unique constant such that $u - c_u \in L^q_{s+1}$ and there is a constant $C = C(s, p, q) > 0$ independent of u such*

that

$$\|u - c_u\|_{L^q_{s+1}} \leq C \|\nabla u\|_{L^p_s}. \quad (3.4)$$

Furthermore, if $p > N$ (so that $\infty \in I_{1,p}$), $\lim_{|x| \rightarrow \infty} |x|^{-(s+1)}(u(x) - c_u) = 0$.

PROOF. The uniqueness of c_u is obvious since L^q_{s+1} contains no nonzero constant when $s < -1$ and $1 \leq q \leq \infty$. We focus on the existence part. Some preliminary properties must be established to prove parts (i) and (ii) of the theorem.

It is well-known that if $u \in W^{1,1}_{loc}$, then u is locally absolutely continuous on almost every line parallel to the coordinate axes x_i and that, on such lines, the classical and weak derivatives $\partial_i u$ coincide. Together with the local equivalence of the measures dr and $r^{N-1}dr$ away from the origin, this implies that, when passing to spherical coordinates, $u(\cdot, \sigma)$ is locally absolutely continuous on $(0, \infty)$ (but not necessarily on $[0, \infty)$) for a.e. $\sigma \in \mathbb{S}^{N-1}$, with classical radial derivative $\partial_r u(r, \sigma) = \nabla u(r, \sigma) \cdot \sigma$. In particular, this holds if $\nabla u \in (L^p_s)^N$.

From now on, we assume $s < -1$ and $\nabla u \in (L^p_s)^N$, so that $(1 + |x|)^{-s-N/p} \partial_r u \in L^p$. By Fubini's theorem in spherical coordinates, $(1 + r)^{-s-N/p} r^{(N-1)/p} \partial_r u(\cdot, \sigma) \in L^p(0, \infty)$ for a.e. $\sigma \in \mathbb{S}^{N-1}$. Since $(1 + r)^{s+N/p} r^{(1-N)/p} \in L^{p'}(\varepsilon, \infty)$ for every $\varepsilon > 0$ when $s < -1$, it follows that $\partial_r u(\cdot, \sigma) \in L^1(\varepsilon, \infty)$. Consequently,

$$v(r, \sigma) := \int_{\infty}^r \partial_r u(t, \sigma) dt,$$

is a.e. defined and measurable on $(0, \infty) \times \mathbb{S}^{N-1}$. For a.e. $\sigma \in \mathbb{S}^{N-1}$, the function $v(\cdot, \sigma)$ is locally absolutely continuous and a.e. differentiable on $(0, \infty)$ with $\partial_r v(\cdot, \sigma) = \partial_r u(\cdot, \sigma)$ and $\lim_{r \rightarrow \infty} v(r, \sigma) = 0$. In particular,

$$c_u(\sigma) := u(\cdot, \sigma) - v(\cdot, \sigma), \quad (3.5)$$

is a function independent of $r > 0$ (difference of two locally absolutely continuous functions with the same a.e. derivative).

Next, $v(r, \cdot) \in L^p(\mathbb{S}^{N-1})$ for every $r > 0$ and $\lim_{r \rightarrow \infty} \|v(r, \cdot)\|_{p, \mathbb{S}^{N-1}} = 0$. To see this, use the estimate

$$|v(r, \sigma)| \leq \lambda(r) \left(\int_r^{\infty} (1+t)^{-sp-N} t^{N-1} |\partial_r u(t, \sigma)|^p \right)^{1/p},$$

where $\lambda(r) := \|(1+t)^{s+N/p} t^{(1-N)/p}\|_{p', (r, \infty)} \rightarrow 0$ when $r \rightarrow \infty$. By taking p^{th} powers and integrating on \mathbb{S}^{N-1} , we get $\|v(r, \cdot)\|_{p, \mathbb{S}^{N-1}} \leq \lambda(r) \|\nabla u\|_{L^p_s} \rightarrow 0$ when $r \rightarrow \infty$, as claimed. Thus, by (3.5),

$$\lim_{r \rightarrow \infty} \|u(r, \cdot) - c_u\|_{p, \mathbb{S}^{N-1}} = 0. \quad (3.6)$$

The next step is to show that c_u is actually constant. (When $1 < p < N$ and $\nabla u \in (L^p)^N$, i.e., $s = -N/p$, this goes back to Uspenskii [20]; see also Fefferman [5].) We shall use the Poincaré–Wirtinger inequality on the sphere \mathbb{S}^{N-1} : If $N > 1$,

$$\|w - \bar{w}\|_{p, \mathbb{S}^{N-1}} \leq C \|\nabla_{\mathbb{S}^{N-1}} w\|_{p, \mathbb{S}^{N-1}}, \quad (3.7)$$

for every $w \in W^{1,p}(\mathbb{S}^{N-1})$, where $\nabla_{\mathbb{S}^{N-1}}$ is the gradient of w for the natural Riemannian structure of the unit sphere, $C > 0$ is a constant independent of w and \bar{w} is the average of w on \mathbb{S}^{N-1} . In the literature, the Poincaré–Wirtinger inequality on compact manifolds is mostly quoted when $p = 2$ (Osserman [17]), but an elementary proof for arbitrary p follows, by contradiction, from the connectedness of \mathbb{S}^{N-1} and the compactness of the embedding $W^{1,p}(\mathbb{S}^{N-1}) \hookrightarrow L^p(\mathbb{S}^{N-1})$.

Assume $u \in C^\infty$ (in addition to $\nabla u \in (L^p_s)^N$). When $r > 0$ is fixed, $\nabla_{\mathbb{S}^{N-1}} u(r, \sigma)$ is the orthogonal projection of $\nabla u(r, \sigma)$ on the tangent space $\{\sigma\}^\perp$ of \mathbb{S}^{N-1} at σ , whence $|\nabla_{\mathbb{S}^{N-1}} u(r, \sigma)| \leq |\nabla u(r, \sigma)|$. Thus, by (3.7), $\|u(r, \cdot) - \bar{u}(r)\|_{p, \mathbb{S}^{N-1}} \leq C \|\nabla u(r, \cdot)\|_{p, \mathbb{S}^{N-1}}$ where $\bar{u}(r)$ is the average of $u(r, \cdot)$ on \mathbb{S}^{N-1} and so

$$\begin{aligned} & \int_0^\infty (1+r)^{-sp-N} r^{N-1} \|u(r, \cdot) - \bar{u}(r)\|_{p, \mathbb{S}^{N-1}}^p dr \\ & \leq C \int_0^\infty (1+r)^{-sp-N} r^{N-1} \|\nabla u(r, \cdot)\|_{p, \mathbb{S}^{N-1}}^p dr = C \|\nabla u\|_{L^p_s}^p. \end{aligned}$$

Since the left-hand side is finite, there is a sequence $r_n \rightarrow \infty$ such that $\lim (1+r_n)^{-sp-N} r_n^{N-1} \|u(r_n, \cdot) - \bar{u}(r_n)\|_{p, \mathbb{S}^{N-1}} = 0$, which in turn implies $\lim \|u(r_n, \cdot) - \bar{u}(r_n)\|_{p, \mathbb{S}^{N-1}} = 0$ because $\lim (1+r_n)^{-sp-N} r_n^{N-1} = \infty$ when $s < -1$. Together with $\lim \|u(r_n, \cdot) - c_u\|_{p, \mathbb{S}^{N-1}} = 0$ from (3.6), this yields $\lim \|\bar{u}(r_n) - c_u\|_{p, \mathbb{S}^{N-1}} = 0$ and, since $\bar{u}(r_n)$ is independent of σ , it follows that c_u is constant (under the additional assumption $u \in C^\infty$ at this point).

We are now in a position to prove (i) and (ii) of the theorem.

(i) Recall that $\lim_{r \rightarrow \infty} v(r, \sigma) = 0$ for a.e. $\sigma \in \mathbb{S}^{N-1}$. Since $v(r, \sigma) = u(r, \sigma) - c_u(\sigma)$, the choice $f(r) = u(r, \sigma) - c_u(\sigma)$ and $q = p$ in (3.1) yields

$$\begin{aligned} & \int_0^\infty (1+r)^{-(s+1)p-N} r^{N-1} |u(r, \sigma) - c_u(\sigma)|^p dr \\ & \leq C \int_0^\infty (1+r)^{-sp-N} r^{N-1} |\partial_r u(r, \sigma)|^p dr, \end{aligned}$$

whence, by integration on \mathbb{S}^{N-1} ,

$$\|u - c_u\|_{L^p_{s+1}} \leq C \|\nabla u\|_{L^p_s}. \quad (3.8)$$

Set $u_n := \theta_n * u$ where θ_n is a mollifying sequence. By Lemma 2.1 for ∇u and Remark 2.1, it follows from (3.8) that $u_n - c_{u_n}$ is a Cauchy sequence in L^p_{s+1} . Call \tilde{u} its limit. Then, $u_n - c_{u_n} \rightarrow \tilde{u}$ in L^1_{loc} and, since $u_n \rightarrow u$ in L^1_{loc} , we infer that $c_{u_n} \rightarrow u - \tilde{u}$ in L^1_{loc} . From the above, c_{u_n} is constant because $u_n \in C^\infty$. Therefore, $u - \tilde{u}$ is a constant \tilde{c} and $u - \tilde{c} = \tilde{u} \in L^p_{s+1}$. Thus, by (3.8), $c_u - \tilde{c} \in L^p_{s+1}$ and, since neither c_u nor \tilde{c} depends upon r and $\int_0^\infty (1+r)^{-(s+1)p-N} r^{N-1} dr = \infty$ when $s < -1$, this can only happen if $c_u = \tilde{c}$ a.e. on \mathbb{S}^{N-1} . This shows that c_u is constant and so (3.8) is the inequality (3.3).

(ii) If u is radially symmetric and $q \in I_{1,p}$ is finite, it follows from (3.1) with $f(r) = u(r) - c_u$ that

$$\begin{aligned} & \left(\int_0^\infty (1+r)^{-(s+1)q-N} r^{N-1} |u(r) - c_u|^q dr \right)^{1/q} \\ & \leq C \left(\int_0^\infty (1+r)^{-sp-N} r^{N-1} |\partial_r u(r)|^p dr \right)^{1/p}, \end{aligned}$$

which, up to a constant factor independent of u , is just the inequality (3.4).

If $p > N$, the same inequality when $q = \infty$ follows from (3.2) with $R = 0$ instead of (3.1). In addition, by (3.2) with $R > 0$, we also get $\|u - c_u\|_{L_{s+1}^\infty(\bar{B}_R)} \leq C \|\nabla u\|_{L_s^p(\bar{B}_R)}$ with $C > 0$ independent of R and so $\lim_{R \rightarrow \infty} \|u - c_u\|_{L_{s+1}^\infty(\bar{B}_R)} = \lim_{R \rightarrow \infty} \|\nabla u\|_{L_s^p(\bar{B}_R)} = 0$. By the continuity of u (recall $p > N$), this amounts to $\lim_{|x| \rightarrow \infty} |x|^{-(s+1)}(u(x) - c_u) = 0$. \square

By (3.6) and since $L^p(\mathbb{S}^{N-1}) \hookrightarrow L^1(\mathbb{S}^{N-1})$, it follows that

$$c_u = \lim_{r \rightarrow \infty} (N\omega_N)^{-1} \int_{\mathbb{S}^{N-1}} u(r, \sigma) d\sigma, \tag{3.9}$$

where $u(r, \cdot)$ is the trace of u on ∂B_r and ω_N is the measure of the unit ball of \mathbb{R}^N . In particular, c_u is independent of $s < -1$ and $1 \leq p < \infty$ such that $\nabla u \in (L_s^p)^N$.

REMARK 3.1. Although Theorem 3.2 is false when $N = 1$ (and indeed (3.7) breaks down), it is readily checked that it remains true on \mathbb{R}_\pm . Its failure on \mathbb{R} is only due to the fact that the restrictions of u to \mathbb{R}_- and \mathbb{R}_+ need not involve the same constant c_u .

4. The general inequalities.

In this section, Theorems 2.3 and 3.2 are complemented and subsumed in a single statement (Theorem 4.3). Next, the result is generalized when $\nabla^k u \in (L_s^p)^{\nu(N,k)}$ for some integer $k \in \mathbb{N}$ (Theorem 4.4).

Recall that ω_N is the measure of the unit ball of \mathbb{R}^N and suppose $u \in L_{loc}^p$ with $1 \leq p < \infty$. By Fubini's theorem in spherical coordinates,

$$f_u(t) := (N\omega_N)^{-1} \int_{\mathbb{S}^{N-1}} u(t\sigma) d\sigma,$$

is defined for a.e. $t > 0$ and $f_u \in L_{loc}^p([0, \infty), t^{N-1} dt) \subset L_{loc}^p(0, \infty)$. The radial symmetrization u_S of u is the radially symmetric function

$$u_S(x) := f_u(|x|) = (N\omega_N)^{-1} \int_{\mathbb{S}^{N-1}} u(|x|\sigma) d\sigma.$$

LEMMA 4.1. *If $u \in \mathcal{D}'$ and $\nabla u \in (L_s^p)^N$ with $s \in \mathbb{R}$ and $1 \leq p < \infty$, then $\nabla u_S \in (L_s^p)^N$ and $\|\nabla u_S\|_{L_s^p} \leq \|\nabla u\|_{L_s^p}$.*

PROOF. First, $u \in W_{loc}^{1,p}$ since $\nabla u \in (L_{loc}^p)^N$. We claim that $u_S \in W_{loc}^{1,p}$, which is obvious if $N = 1$. If $N > 1$, then $W^{1,p}(B_R) = W^{1,p}(B_R \setminus \{0\})$ (see for instance [8, p. 52]) and it suffices to show that $u_S \in W^{1,p}(B_R \setminus \{0\})$ for every $R > 0$. That $u_S \in$

$L^p(B_R \setminus \{0\}) = L^p(B_R)$ is clear from $u \in W_{loc}^{1,p}$. Since $\partial_r u = \nabla u \cdot |x|^{-1}x \in L_{loc}^p$, the formal calculation

$$\nabla u_S(x) = (N\omega_N)^{-1} \left(\int_{\mathbb{S}^{N-1}} \partial_r u(|x|\sigma) d\sigma \right) |x|^{-1}x, \quad (4.1)$$

yields $\nabla u_S \in (L^p(B_R \setminus \{0\}))^N = (L^p(B_R))^N$ and so, as claimed, $u_S \in W^{1,p}(B_R \setminus \{0\})$. This formula is justified below when ∇u_S is understood as a distribution on $\mathbb{R}^N \setminus \{0\}$, but since $W^{1,p}(B_R \setminus \{0\}) = W^{1,p}(B_R)$, it also gives ∇u_S as a distribution on \mathbb{R}^N .

If $\varphi \in C_0^\infty(0, \infty)$, set $\psi(x) := \varphi(|x|)$, so that $\psi \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ and that $\partial_r \psi(x) = \varphi'(|x|)$. Then, $\langle f'_u, \varphi \rangle = -(N\omega_N)^{-1} \langle u, |x|^{1-N} \partial_r \psi \rangle = (N\omega_N)^{-1} \langle |x|^{1-N} \partial_r u, \psi \rangle$ (use $\partial_r = |x|^{-1}x \cdot \nabla$ and $\nabla \cdot (|x|^{-N}x) = 0$). Since $\partial_r u \in L_{loc}^p$, this shows that $\langle f'_u, \varphi \rangle = \langle f_{\partial_r u}, \varphi \rangle$, that is, $f'_u = f_{\partial_r u} \in L_{loc}^p(0, \infty)$ and so $f_u \in W_{loc}^{1,p}(0, \infty)$. In particular, f_u is locally absolutely continuous on $(0, \infty)$. As a result, by Marcus and Mizel [12, Theorem 4.3], $\nabla u_S(x) = |x|^{-1} f'_u(|x|)x = |x|^{-1} f_{\partial_r u}(|x|)x$ as a distribution on $\mathbb{R}^N \setminus \{0\}$ and (4.1) is proved.

To see that $\nabla u_S \in (L_s^p)^N$, use (4.1) and Hölder's inequality to get $|\nabla u_S(x)| \leq (N\omega_N)^{-1} \int_{\mathbb{S}^{N-1}} |\nabla u(|x|\sigma)| d\sigma \leq (N\omega_N)^{-1/p} \left(\int_{\mathbb{S}^{N-1}} |\nabla u(|x|\sigma)|^p d\sigma \right)^{1/p}$. Hence, $(1 + |x|)^{-sp-N} |\nabla u_S(x)|^p \leq (N\omega_N)^{-1} \int_{\mathbb{S}^{N-1}} (1 + |x|)^{-sp-N} |\nabla u(|x|\sigma)|^p d\sigma$ and so, by integration in spherical coordinates, $\|\nabla u_S\|_{L_s^p} \leq \|\nabla u\|_{L_s^p}$. \square

If $u \in \mathcal{D}'$ and $\nabla u \in (L_s^p)^N$, Lemma 4.1 yields $\nabla u_S \in (L_s^p)^N$ and it then follows from Theorem 2.3 (Theorem 3.2) that $u_S \in L_{s+1}^q$ for every $q \in I_{1,p}$ if $s > -1$ ($u_S - c_{u_S} \in L_{s+1}^q$ for every $q \in I_{1,p}$ and a unique constant c_{u_S} if $s < -1$). Thus, to show that $u \in L_{s+1}^q$ or that $u - c_u \in L_{s+1}^q$ when $q > p$, it suffices to prove the same result for $u - u_S$. The difference between u and $u - u_S$ is that $(u - u_S)_S = 0$ and that, for functions with vanishing radial symmetrization, a result originating in the work of Caffarelli, Kohn and Nirenberg [3] and generalized in [18] is applicable. We only spell out the special case relevant to the issue of interest here and give a proof of it when $q = \infty$, not considered elsewhere.

LEMMA 4.2. *Suppose that $N > 1$. If $u \in L_{loc}^1$ and $u_S = 0$ and if $|x|^{a/p}u \in L^p$ and $|x|^{1+a/p}\nabla u \in (L^p)^N$ for some $a \in \mathbb{R}$ and $1 \leq p < \infty$, there is a constant $C = C(a, p, q) > 0$ independent of u such that*

$$\| |x|^{(a+N)/p-N/q} u \|_q \leq C \| |x|^{1+a/p} |\nabla u| \|_p, \quad (4.2)$$

for every $q \in I_{1,p}$ (see (1.4)). Furthermore, if $p > N$, then $\lim_{|x| \rightarrow \infty} |x|^{(a+N)/p} u(x) = 0$.

PROOF. If $q < \infty$, the result follows by letting $b = a + p$ and by substituting $q = p, r = q$ in part (ii) of [18, Corollary 6.1]. The proof when $q = \infty$ (hence $p > N$) is given below.

For $\tau > 0$, let $\Omega_\tau := \{x \in \mathbb{R}^N : \tau < |x| < 2\tau\}$. Since power weights are bounded above and below on Ω_1 , it follows that $u \in W^{1,p}(\Omega_1) \hookrightarrow L^\infty(\Omega_1)$. Thus, $\|u\|_{\infty, \Omega_1} \leq C \|u\|_{1,p, \Omega_1}$ with a constant $C > 0$ independent of u .

The assumption $u_S = 0$ entails $\int_{\Omega_\tau} u = 0$ for every $\tau > 0$ and so, since Ω_1 is

connected when $N > 1$, $\|u\|_{\infty, \Omega_1} \leq C \|\nabla u\|_{p, \Omega_1}$ by the Poincaré–Wirtinger inequality. Once again by the boundedness of power weights on Ω_1 , this yields $\| |x|^{(a+N)/p} u \|_{\infty, \Omega_1} \leq C \| |x|^{1+a/p} |\nabla u| \|_{p, \Omega_1}$. More generally, by scaling,

$$\| |x|^{(a+N)/p} u \|_{\infty, \Omega_\tau} \leq C \| |x|^{1+a/p} |\nabla u| \|_{p, \Omega_\tau}, \tag{4.3}$$

with the same $C > 0$ independent of τ . The right-hand side is majorized by $C \| |x|^{1+a/p} |\nabla u| \|_p$ and then (4.2) when $q = \infty$ follows from $\| |x|^{(a+N)/p} u \|_\infty = \sup_{\tau > 0} \| |x|^{(a+N)/p} u \|_{\infty, \Omega_\tau}$. The proof of this equality is a simple exercise.

More generally, $\|v\|_{\infty, \tilde{B}_R} = \sup_{\tau \geq R} \|v\|_{\infty, \Omega_\tau}$. Thus, when $v = |x|^{(a+N)/p} u$ with u as above, $\| |x|^{(a+N)/p} u \|_{\infty, \tilde{B}_R} \leq C \| |x|^{1+a/p} |\nabla u| \|_{p, \tilde{B}_R}$ by (4.3). Since $p < \infty$, $\lim_{R \rightarrow \infty} \| |x|^{(a+N)/p} u \|_{\infty, \tilde{B}_R} = \lim_{R \rightarrow \infty} \| |x|^{1+a/p} |\nabla u| \|_{p, \tilde{B}_R} = 0$. By the continuity of u away from 0 (recall $p > N$) this means $\lim_{|x| \rightarrow \infty} |x|^{(a+N)/p} u(x) = 0$. \square

THEOREM 4.3. *Suppose that $s \neq -1$ and that $1 \leq p < \infty$ and let $u \in \mathcal{D}'$ be such that $\nabla u \in (L^p_s)^N$.*

(i) *If $s > -1$, then $u \in L^q_{s+1}$ for every $q \in I_{1,p}$ (see (1.4)) and there are a constant c_u independent of s and p (the same as in Theorem 2.3) and a constant $C = C(s, p, q) > 0$ independent of u such that*

$$\|u - c_u\|_{L^q_{s+1}} \leq C \|\nabla u\|_{L^p_s}. \tag{4.4}$$

Furthermore, if $p > N$ or $p = N = 1$, then $\lim_{|x| \rightarrow \infty} |x|^{-(s+1)} u(x) = 0$.

(ii) *If $s < -1$ and $N > 1$, there is a unique constant $c_u \in \mathbb{R}$ independent of s and p (the same as in Theorem 3.2) such that $u - c_u \in L^q_{s+1}$ for every $q \in I_{1,p}$ and there is a constant $C = C(s, p, q) > 0$ independent of u such that (4.4) holds. Furthermore, if $p > N$, then $\lim_{|x| \rightarrow \infty} |x|^{-(s+1)} (u(x) - c_u) = 0$.*

PROOF. It is obvious that the constant $c_u = |B_\rho|^{-1} \int_{B_\rho} u$ of Theorem 2.3 is independent of $s > -1$ and p . For the constant c_u of Theorem 3.2, this independence of $s < -1$ and p was noticed in the comments following that theorem. Also, in (ii), the uniqueness of c_u follows from L^q_{s+1} containing no nonzero constant irrespective of q when $s < -1$.

If $q = p$, or if u is radially symmetric, or if $N = 1$ in (i), everything was proved in Theorems 2.3 and 3.2. Accordingly, we henceforth assume $N > 1$ and $q \in I_{1,p}$. The formula $c_u = |B_\rho|^{-1} \int_{B_\rho} u$ if $s > -1$ (Theorem 2.3) shows that $c_u = c_{u_S}$ and, by (3.9), the same thing is true if $s < -1$. Thus, $\|u - c_u\|_{L^q_{s+1}} \leq \|u - u_S\|_{L^q_{s+1}} + \|u_S - c_{u_S}\|_{L^q_{s+1}}$ and, since the theorem is true in the radially symmetric case, it follows from Lemma 4.1 that $\|u_S - c_{u_S}\|_{L^q_{s+1}} \leq C \|\nabla u\|_{L^p_s}$. Consequently, the proof of (4.4) is reduced to showing that $\|u - u_S\|_{L^q_{s+1}} \leq C \|\nabla u\|_{L^p_s}$. Since $\|\nabla(u - u_S)\|_{L^p_s} \leq 2 \|\nabla u\|_{L^p_s}$ by Lemma 4.1, it suffices to show that $\|u - u_S\|_{L^q_{s+1}} \leq C \|\nabla(u - u_S)\|_{L^p_s}$. From the remark that $(u - u_S)_S = 0$, this will follow from

$$\|u\|_{L^q_{s+1}} \leq C \|\nabla u\|_{L^p_s}, \tag{4.5}$$

when $\nabla u \in (L_s^p)^N$ and $u_S = 0$.

Likewise, since $\lim_{|x| \rightarrow \infty} |x|^{-(s+1)}(u_S(x) - c_{u_S}) = 0$ when $p > N$ is known (by radial symmetry and Theorems 2.3 and 3.2) and since $c_u = c_{u_S}$, the proof that $\lim_{|x| \rightarrow \infty} |x|^{-(s+1)}u(x) = 0$ or that $\lim_{|x| \rightarrow \infty} |x|^{-(s+1)}(u(x) - c_u) = 0$ when $p > N$ is reduced to showing that $\lim_{|x| \rightarrow \infty} |x|^{-(s+1)}(u(x) - u_S(x)) = 0$, i.e. that $\lim_{|x| \rightarrow \infty} |x|^{-(s+1)}u(x) = 0$ when $\nabla u \in (L_s^p)^N$ and $u_S = 0$.

From now on $\nabla u \in (L_s^p)^N$ and $u_S = 0$. We shall make repeated use, without further mention, of the elementary properties that for every $t \in \mathbb{R}$, the weights $(1 + |x|)^t$ are bounded above and below on every bounded subset of \mathbb{R}^N and that they are equivalent to $|x|^t$ when $|x|$ is bounded away from 0.

By Theorem 2.3, $u \in L_{s+1}^p$ if $s > -1$ and, by Theorem 3.2, $u \in L_{s+1}^p$ if $s < -1$ because $c_u = c_{u_S}$ and $u_S = 0$ show that $c_u = 0$. Thus, $u \in L_{s+1}^p$ when $s \neq -1$.

Let $\varphi \in C^\infty$ be radially symmetric, with $\varphi = 0$ on a neighborhood of 0 and $\varphi = 1$ outside B_1 . Since $\nabla u \in (L_s^p)^N$ and $u \in L_{s+1}^p$, it is readily checked that $\varphi u \in L_{s+1}^p$ and $\nabla(\varphi u) \in (L_s^p)^N$ and that $(\varphi u)_S = 0$. By Lemma 4.2 with $a = -(s+1)p - N$, we infer that $|x|^{-(s+1)-N/q}\varphi u \in L^q$ for every $q \in I_{1,p}$ and that

$$\| |x|^{-(s+1)-N/q}\varphi u \|_q \leq C \| |x|^{-s-N/p} |\nabla(\varphi u)| \|_p.$$

From the equivalence of weights away from 0 and since $\varphi = 1$ outside B_1 , this implies $\lim_{|x| \rightarrow \infty} |x|^{-(s+1)}u(x) = 0$ if $p > N$ (by Lemma 4.2) and, irrespective of p ,

$$\| u \|_{L_{s+1}^q(\tilde{B}_1)} \leq C \| |\nabla(\varphi u)| \|_{L_s^p} \leq C \| u \|_{p,\Omega} + C \| |\nabla u| \|_{L_s^p}, \tag{4.6}$$

where Ω is an annulus centered at the origin (not a ball, so that $|x|^{-sp-N}$ is bounded above on Ω) containing $\text{Supp } \nabla \varphi$.

Note that $u_S = 0$ implies $\int_\Omega u = 0$. Thus, since Ω is connected, $\| u \|_{p,\Omega} \leq C \| |\nabla u| \|_{p,\Omega} \leq C \| |\nabla u| \|_{L_s^p}$ by the Poincaré–Wirtinger inequality and so, by (4.6),

$$\| u \|_{L_{s+1}^q(\tilde{B}_1)} \leq C \| |\nabla u| \|_{L_s^p}. \tag{4.7}$$

On the other hand, $\| u \|_{L_{s+1}^q(B_1)} \leq C \| u \|_{q,B_1}$ and $\| u \|_{q,B_1} \leq C \| u \|_{1,p,B_1}$ since $q \in I_{1,p}$ and $u \in W_{loc}^{1,p}$. In addition, $\int_{B_1} u = 0$ and so $\| u \|_{L_{s+1}^q(B_1)} \leq C \| |\nabla u| \|_{p,B_1} \leq C \| |\nabla u| \|_{L_s^p}$ by the Poincaré–Wirtinger inequality on B_1 . Together with (4.7), this proves (4.5). \square

REMARK 4.1. If $s > -1$ and $p > N$, one may also choose $c_u = u(0)$ in (4.4). See Remark 2.2 and notice that since u is continuous, $u_S(0) = u(0)$ if u_S is extended by continuity at 0. Thus, the property $c_u = c_{u_S}$ is preserved (in particular, $c_u = 0$ if $u_S = 0$) and the above proof can be repeated verbatim. Once again, by translation, one may also choose $c_u = u(x_0)$ with $x_0 \in \mathbb{R}^N$ independent of u .

When $s > -1$ and $p > N$, (4.4) with $q \in I_{1,p} = [p, \infty]$ and $c_u = u(0)$ (Remark 4.1) reads $\| (1 + |x|)^{-(s+1)-N/q}(u - u(0)) \|_q \leq C \| (1 + |x|)^{-s-N/p} |\nabla u| \|_p$. This inequality for $u_\lambda(x) = u(\lambda x)$, $\lambda > 0$, yields $\| (\lambda + |x|)^{-(s+1)-N/q}(u - u(0)) \|_q \leq C \| (\lambda + |x|)^{-s-N/p} |\nabla u| \|_p$ (same C) and so $\| |x|^{-(s+1)-N/q}(u - u(0)) \|_q \leq C \| (|x|)^{-s-N/p} |\nabla u| \|_p$ by Fatou’s lemma and monotone convergence ($s > -N/p$) or dominated convergence ($s \leq -N/p$). If

$s < -N/p$, then $|x|^{-s-N/p} |\nabla u| \in L^p$ does not imply $\nabla u \in (L_s^p)^N$ unless $\nabla u \in (L_{loc}^p)^N$, which must then be assumed. Hardy's (Morrey's) inequality is recovered when $s = -N/p$ (> -1) and $q = p$ ($q = \infty$).

When $s > -1$ and $p \leq N$ (hence $s > -N/p$) and if 0 is in the Lebesgue set of u , the constant $c_{u,\lambda} = |B_\rho|^{-1} \int_{B_\rho} u_\lambda = |B_{\lambda\rho}|^{-1} \int_{B_{\lambda\rho}} u$ tends to some finite value $\bar{u}(0)$ independent of ρ as $\lambda \rightarrow 0$. Then, by scaling, $\||x|^{-(s+1)-N/q}(u - \bar{u}(0))\|_q \leq C\||x|^{-s-N/p} |\nabla u|\|_p$ follows as before. This extends the previous inequality when $p > N$ and $\bar{u}(0) = u(0)$, but $s = -N/p$ (the classical case) and $q = \infty \notin I_{1,p}$ are now ruled out.

When $s < -1$ and $N > 1$, (4.4) is $\|(1 + |x|)^{-(s+1)-N/q}(u - c_u)\|_q \leq C\|(1 + |x|)^{-s-N/p} |\nabla u|\|_p$ with c_u now given by (3.9). If $1 \leq p < N$, $q = p^*$ and $s = -N/p$ (< -1), this is Sobolev's inequality $\|u - c_u\|_{p^*} \leq C\|\nabla u\|_p$. In fact, since $c_{u,\lambda} = c_u$ by (3.9), scaling yields $\||x|^{-(s+1)-N/q}(u - c_u)\|_q \leq C\||x|^{-s-N/p} |\nabla u|\|_p$ for $q \in I_{1,p}$, a general Hardy-Sobolev inequality. Once again, $\nabla u \in (L_{loc}^p)^N$ must be assumed if $s < -N/p$. When $u \in C_0^\infty$ (hence $c_u = 0$ and $\nabla u \in (L_{loc}^p)^N$) and $1 \leq p < N$, another proof is given by Maz'ya (case $m = N, n = 0$ in [13, Corollary 2, p. 139]). If $q = p < N$ and $s = -N/p$ (< -1), the Hardy-Leray inequality $\||x|^{-1}u\|_p \leq C\|\nabla u\|_p$ is recovered when $c_u = 0$.

The next theorem generalizes Theorem 4.3. Before stating it, a cautionary remark is in order. If, in Theorem 4.3, $\nabla u \in (L_{s_1}^{p_1})^N \cap (L_{s_2}^{p_2})^N$ and $s_1 < -1 < s_2$, both parts (i) and (ii) of the theorem are applicable. This yields two constants $c_{u,i}$ independent of $s_i, i = 1, 2$, with $c_{u,1}$ unique, such that (4.4) holds with $s = s_i$ and $q \in I_{1,p_i}$. Although there are many ways to define $c_{u,2}$ as a function of u , there is no reason why $c_{u,2} = c_{u,1}$ would be an admissible choice whenever both constants exist. Indeed, in Theorem 4.3, the constant c_u is only independent of s in each connected component of $\mathbb{R} \setminus \{-1\}$ and its definition must be changed when s crosses -1 .

A similar issue arises if $s_1, s_2 > -1$ and $p_1 \leq N < p_2$. If so, it is possible to choose $c_{u,1} = c_{u,2} = |B_\rho|^{-1} \int_{B_\rho} u$ with $\rho > 0$. However, by Remark 2.2, $c_{u,2} = u(0)$ is another possible choice, but since $p_1 \leq N$, this does not mean that $c_{u,1} = u(0)$ is admissible.

THEOREM 4.4. *Suppose that $k \in \mathbb{N}$, that $s \notin \{-k, \dots, -1\}$ and that $N > 1$ if $s < -1$. Let $u \in \mathcal{D}'$ be such that $\nabla^k u \in (L_s^p)^{\nu(k,N)}$ with $1 \leq p < \infty$. Then, there is a polynomial $\pi_u \in \mathcal{P}_{k-1}$, independent of p and independent of s in each connected component of $\mathbb{R} \setminus \{-k, \dots, -1\}$, unique if $s < -k$, such that $\nabla^{k-j}(u - \pi_u) \in (L_{s+j}^q)^{\nu(k-j,N)}$ for every $1 \leq j \leq k$ and every $q \in I_{j,p}$ (see (1.4)) and there is a constant $C = C(s, j, p, q) > 0$ independent of u such that*

$$\|\nabla^{k-j}(u - \pi_u)\|_{L_{s+j}^q} \leq C\|\nabla^k u\|_{L_s^p}. \quad (4.8)$$

Furthermore, $\lim_{|x| \rightarrow \infty} |x|^{-(s+j)}(\nabla^{k-j}u(x) - \nabla^{k-j}\pi_u(x)) = 0$ if $p = N = j = 1$ or if $1 \leq j \leq k$ and $p > N/j$. (In particular, $\lim_{|x| \rightarrow \infty} |x|^{-(s+j)}\nabla^{k-j}u(x) = 0$ if also $s > -1$.)

PROOF. The uniqueness of π_u when $s < -k$ follows from the remark that L_{s+k}^q contains no nonzero polynomial for any q .

By Theorem 4.3, $\pi_u = c_u$ exists when $k = 1$. Suppose $k > 1$ and that π_u exists when k is replaced with $k - 1$. The hypothesis $\nabla^k u \in (L_s^p)^{\nu(k,N)}$ implies $\nabla(\partial^\alpha u) \in (L_s^p)^N$ for every multi-index α with $|\alpha| = k - 1$. Since $s \neq -1$, it follows from Theorem 4.3

that there is a constant $c_\alpha := c_{\partial^\alpha u}$, independent of s in each connected component of $\mathbb{R} \setminus \{-1\}$ (and independent of p , but we will return to this point later), such that $\partial^\alpha u - c_\alpha \in L_{s+1}^{p_1}$ for every $p_1 \in I_{1,p}$ and there is a constant $C_\alpha > 0$ independent of u such that $\|\partial^\alpha u - c_\alpha\|_{L_{s+1}^{p_1}} \leq C_\alpha \|\nabla(\partial^\alpha u)\|_{L_s^p}$. Upon replacing C_α with $\max_\alpha C_\alpha$, this yields

$$\|\partial^\alpha u - c_\alpha\|_{L_{s+1}^{p_1}} \leq C \|\nabla^k u\|_{L_s^p}, \tag{4.9}$$

with C independent of u and α . For future use, note also that, still by Theorem 4.3,

$$\lim_{|x| \rightarrow \infty} |x|^{-(s+1)} (\partial^\alpha u(x) - c_\alpha) = 0 \text{ if } p > N \text{ or } p = N = 1. \tag{4.10}$$

Set

$$\pi_{u,k-1}(x) := \sum_{|\alpha|=k-1} (\alpha!)^{-1} c_\alpha x^\alpha \tag{4.11}$$

and let $v := u - \pi_{u,k-1}$. Then, $\partial^\alpha v = \partial^\alpha u - c_\alpha$ for every α with $|\alpha| = k - 1$ and, by (4.9), $\nabla^{k-1} v \in (L_{s+1}^{p_1})^{\nu(k-1,N)}$ for every $p_1 \in I_{1,p}$, with

$$\|\nabla^{k-1} v\|_{L_{s+1}^{p_1}} \leq C \|\nabla^k u\|_{L_s^p}. \tag{4.12}$$

Since $s \notin \{-k, \dots, -1\}$ implies $s + 1 \notin \{-k + 1, \dots, -1\}$, it follows from the hypothesis of induction with s replaced with $s + 1$ that, as long as p_1 above is finite, there is a polynomial $\pi_v \in \mathcal{P}_{k-2}$ independent of p_1 and independent of $s + 1$ in each connected component of $\mathbb{R} \setminus \{-k + 1, \dots, -1\}$, such that $\nabla^{k-1-j}(v - \pi_v) \in (L_{s+1+j}^q)^{\nu(k-1,N)}$ for every $1 \leq j \leq k - 2$ and every $q \in I_{j,p_1}$ and that, for every such j and q , there is a constant $C = C(j, s, p_1, q) > 0$ independent of v such that $\|\nabla^{k-1-j}(v - \pi_v)\|_{L_{s+1+j}^q} \leq C \|\nabla^{k-1} v\|_{L_{s+1}^{p_1}}$.

Upon changing j into $j - 1$, this may be rewritten as

$$\|\nabla^{k-j}(v - \pi_v)\|_{L_{s+j}^q} \leq C \|\nabla^{k-1} v\|_{L_{s+1}^{p_1}}, \tag{4.13}$$

for $2 \leq j \leq k$ and $q \in \bigcup_{p_1 \in I_{1,p}, p_1 < \infty} I_{j-1,p_1}$. A routine verification shows that $\bigcup_{p_1 \in I_{1,p}, p_1 < \infty} I_{j-1,p_1} = I_{j,p}$. Thus, (4.13) holds for $2 \leq j \leq k$ and $q \in I_{j,p}$. By (4.12) and since $v - \pi_v = u - \pi_u$ with $\pi_u := \pi_v + \pi_{u,k-1}$, it follows that

$$\|\nabla^{k-j}(u - \pi_u)\|_{L_{s+j}^q} \leq C \|\nabla^k u\|_{L_s^p},$$

for $2 \leq j \leq k$ and $q \in I_{j,p}$. Since (4.9) is the same inequality when $j = 1$ (with q called $p_1 \in I_{1,p}$), the proof of (4.8) is complete.

As noted, $\pi_{u,k-1}$ is independent of s in each connected components of $\mathbb{R} \setminus \{-1\}$ and π_v is independent of $s + 1$ in each connected components of $\mathbb{R} \setminus \{-k + 1, \dots, -1\}$, that is, of s in each connected components of $\mathbb{R} \setminus \{-k, \dots, -2\}$. Thus, $\pi_u := \pi_v + \pi_{u,k-1}$ is independent of s in each connected component of $\mathbb{R} \setminus \{-k, \dots, -1\}$. That π_u is also independent of p will be obvious when we discuss how π_u can be calculated, after Remark 4.3.

We now prove the “furthermore” part. By (4.10), $\lim_{|x| \rightarrow \infty} |x|^{-(s+j)}(\nabla^{k-j}u(x) - \nabla^{k-j}\pi_u(x)) = 0$ if $p > N/j$ or $p = N = 1$ holds if $j = 1$. In general, the proof goes by induction on $j \in \{1, \dots, k\}$. Suppose $j > 1$ (hence $k > 1$) and $p > N/j$. Then, $p^* > N/(j - 1)$ (recall $p^* = \infty$ if $p \geq N$ and $p^* = Np/(N - p)$ if $p < N$) and so the interval $I_{1,p}$ contains some $p_1 \in (N/(j - 1), \infty)$. Therefore, with v as above, it follows from the hypothesis of induction with s replaced with $s + 1$ and j replaced with $j - 1$ (and since $(s + 1) + (j - 1) = s + j$) that $\lim_{|x| \rightarrow \infty} |x|^{-(s+j)}(v(x) - \pi_v(x)) = 0$. Since $v - \pi_v = u - \pi_u$ by definition of π_u , this is $\lim_{|x| \rightarrow \infty} |x|^{-(s+j)}(\nabla^{k-j}u(x) - \nabla^{k-j}\pi_u(x)) = 0$.

If $s > -1$, then $-(s + j) < 1 - j$. Hence, $\lim_{|x| \rightarrow \infty} |x|^{-(s+j)}\nabla^{k-j}\pi_u(x) = 0$ and so $\lim_{|x| \rightarrow \infty} |x|^{-(s+j)}\nabla^{k-j}u(x) = 0$. □

REMARK 4.2. By Remark 3.1, Theorem 4.3 and Theorem 4.4 remain true on \mathbb{R}_\pm , even if $s < -1$.

REMARK 4.3. A generalization of Theorem 4.4, with the same proof, is as follows: If $\ell \in \{1, \dots, k\}$ and $s \notin \{-\ell, \dots, -1\}$ and if $N > 1$ when $s < -1$, a polynomial $\pi_u \in \mathcal{P}_{k-1}$ still exists, which is unique modulo² $\mathcal{P}_{k-\ell-1}$ if $s < -\ell$, such that $\nabla^{k-j}(u - \pi_u) \in (L_{s+j}^q)^{\nu(k-j,N)}$ and (4.8) holds for every $1 \leq j \leq \ell$ and every $q \in I_{j,p}$. Furthermore, $\lim_{|x| \rightarrow \infty} |x|^{-(s+j)}(\nabla^{k-j}u(x) - \nabla^{k-j}\pi_u(x)) = 0$ if $p = N = j = 1$ or if $1 \leq j \leq \ell$ and $p > N/j$. The part of π_u of degree at least $k - \ell$ is obtained as in the proof of Theorem 4.4 and its part of degree at most $k - \ell - 1$ is irrelevant since it vanishes under the action of ∇^{k-j} when $j \leq \ell$. Theorem 4.4 is recovered when $\ell = k$.

Further comments about the polynomial π_u of Theorem 4.4 are in order. If $s < -k$, then $s < -1$ and the coefficients $(\alpha!)^{-1}c_\alpha$ of $\pi_{u,k-1}$ in (4.11) are given by the formula (3.9) with u replaced with $(\alpha!)^{-1}\partial^\alpha u$ and $|\alpha| = k - 1$. If $k > 1$, finding $\pi_u = \pi_{u,k-1} + \pi_v$ amounts to finding π_v where $v = u - \pi_{u,k-1}$. Since $\pi_v \in \mathcal{P}_{k-2}$ and $\nabla^{k-1}v \in (L_{s+1}^p)^{\nu(k-1,N)}$ and since $s+1 < -(k-1) < -1$, the coefficients of the homogeneous part $\pi_{v,k-2}$ of degree $k-2$ of π_v are given by (3.9) with u replaced with $(\alpha!)^{-1}\partial^\alpha v$ for $|\alpha| = k - 2$. The (unique) polynomial π_u is fully determined after k steps. Its definition is obviously independent of $1 \leq p < \infty$ such that $\nabla^k u \in (L_s^p)^{\nu(k,N)}$.

If $s > -1$, then $\mathcal{P}_{k-1} \subset L_{s+k}^q$ for every q . In particular, if $q \in I_{k,p}$, it follows from $u - \pi_u \in L_{s+k}^q$ that $u \in L_{s+k}^q$. There are now many different ways to define a suitable polynomial π_u . By Theorem 2.3, a possible choice for the coefficients $(\alpha!)^{-1}c_\alpha$ of $\pi_{u,k-1}$ in (4.11) is $(\alpha!)^{-1}c_\alpha = |B_\rho|^{-1} \int_{B_\rho} (\alpha!)^{-1}\partial^\alpha u$ where $|\alpha| = k - 1$ and $\rho > 0$ is independent of u , but there are other options. Indeed, one could as well define $(\alpha!)^{-1}c_\alpha = |B_{\rho_\alpha}|^{-1} \int_{B(x_\alpha, \rho_\alpha)} (\alpha!)^{-1}\partial^\alpha u$, where x_α and $\rho_\alpha > 0$ are independent of u but depend upon α ; see the comments after Theorem 2.3.

Once $\pi_{u,k-1}$ has been chosen, $v = u - \pi_{u,k-1}$ is known and the problem is reduced to finding π_v . This is the same problem with s replaced with $s + 1$ and k replaced with $k - 1$. Since $s > -1$ implies $s + 1 > -1$, the coefficients of the homogeneous part $\pi_{v,k-2}$ of π_v can be defined by $(\alpha!)^{-1}c_\alpha = |B_{\rho_\alpha}|^{-1} \int_{B(x_\alpha, \rho_\alpha)} (\alpha!)^{-1}\partial^\alpha v$ where $|\alpha| = k - 2$ and x_α and $\rho_\alpha > 0$ are once again arbitrarily chosen. A polynomial π_u is obtained in k steps. Clearly, different choices of x_α and ρ_α produce different polynomials π_u , but no

²This simply means that its part of degree at least $k - \ell$ is unique.

matter how these choices are made, they are always independent of $1 \leq p < \infty$ such that $\nabla^k u \in (L^p_s)^{\nu(k,N)}$.

Still when $s > -1$, but only when $p > N$, the Taylor polynomial of u at any point x_0 is another possible choice for π_u . This is most easily seen when $x_0 = 0$. First, by Remark 4.1, $(\alpha!)^{-1} \partial^\alpha u(0)$ is a possible choice for the coefficients $(\alpha!)^{-1} c_\alpha$ of $\pi_{u,k-1}$ when $|\alpha| = k - 1$. Next, $(\alpha!)^{-1} \partial^\alpha v(0)$ is a possible choice for the coefficients of $\pi_{v,k-2}$ when $|\alpha| = k - 2$, but since $v = u - \pi_{u,k-1}$ and $\partial^\alpha \pi_{u,k-1}(0) = 0$ when $|\alpha| = k - 2$, these coefficients are just $(\alpha!)^{-1} \partial^\alpha u(0)$. By repeating this argument, $\sum_{|\alpha| \leq k-1} (\alpha!)^{-1} \partial^\alpha u(0) x^\alpha$ is a possible choice for π_u and, by changing $u(x)$ into $u(x + x_0)$, it follows that $\sum_{|\alpha| \leq k-1} (\alpha!)^{-1} \partial^\alpha u(x_0) (x - x_0)^\alpha$ is an equally possible choice.

When $s \in (-k, -1)$, the procedure to find a suitable polynomial π_u combines the approaches of the previous two cases. Because $s < -1$, the homogeneous part $\pi_{u,k-1}$ of degree $k - 1$ of π_u is still unique, but the homogeneous part $\pi_{v,k-2}$ of degree $k - 2$ of π_v is unique only if $s + 1 < -1$. Otherwise, it must be determined as indicated above when $s > -1$ after replacing s with $s + 1$. More generally, if $k_s := E(s + k + 1)$ where E denotes integer part, so that $1 \leq k_s \leq k - 1$, the part of π_u of degree greater than or equal to k_s is unique and the part of degree less than or equal to $k_s - 1$ is determined as indicated above when $s > -1$ after replacing s with $s + k - k_s > -1$. Once again, the calculation of π_u does not depend on $1 \leq p < \infty$ such that $\nabla^k u \in (L^p_s)^{\nu(k,N)}$.

The next corollary singles out the more familiar case when $s = -N/p$, i.e., when $\nabla^k u \in (L^p)^{\nu(k,N)}$. Since s and p are now related, s lies in some connected component of $\mathbb{R} \setminus \{-k, \dots, -1\}$ if and only if p lies in the corresponding connected component of $\mathbb{R} \setminus \{N/k, N/(k - 1), \dots, N\}$.

COROLLARY 4.5. *Suppose that $k \in \mathbb{N}$ and that $1 \leq p < \infty$ with $p \neq N/j$ for $j = 1, \dots, k$. If $u \in \mathcal{D}'$ and $\nabla^k u \in (L^p)^{\nu(k,N)}$, there is a polynomial $\pi_u \in \mathcal{P}_{k-1}$ independent of p in each connected component of $\mathbb{R} \setminus \{N/k, N/(k - 1), \dots, N\}$, unique if $p < N/k$, such that $\nabla^{k-j}(u - \pi_u) \in (L^q_{j-N/p})^{\nu(k-j,N)}$ for every $1 \leq j \leq k$ and every $q \in I_{j,p}$ and there is a constant $C = C(s, j, p, q) > 0$ independent of u such that*

$$\|\nabla^{k-j}(u - \pi_u)\|_{L^q_{j-N/p}} \leq C \|\nabla^k u\|_p. \tag{4.14}$$

Furthermore, $\lim_{|x| \rightarrow \infty} |x|^{-j+N/p} (\nabla^{k-j} u(x) - \nabla^{k-j} \pi_u(x)) = 0$ if $1 \leq j \leq k$ and $p > N/j$. (In particular, $\lim_{|x| \rightarrow \infty} |x|^{-j+N/p} \nabla^{k-j} u(x) = 0$ for every $1 \leq j \leq k$ if $p > N$.)

When $j = k$, the “furthermore” part of Corollary 4.5 was proved by Mizuta [14]. If $p < N/k$, (4.14) for $q = p^{*j} = Np/(N - jp)$ and $j \in \{1, \dots, k\}$ becomes $\|\nabla^{k-j}(u - \pi_u)\|_{p^{*j}} \leq C \|\nabla^k u\|_p$ for $1 \leq j \leq k$, a Sobolev inequality of order j easily proved directly by induction. It can be found in [6], in the more general form given in Remark 4.3 (i.e., when $p < N/\ell$ for some $\ell \leq k$ and $1 \leq j \leq \ell$).

The scaling trick used when $k = 1$ yields numerous inequalities (4.8) with $1 + |x|$ replaced with $|x|$. We only give a small sample when $s = -N/p$ (so that (4.8) is (4.14)) and $j = k$. The earlier discussion about the calculation of π_u is crucial. If $k < N$ and $p < N/k$, then $s = -N/p < -k$ and, with $c_{\partial^\alpha u}$ given by (3.9) for $\partial^\alpha u$, it follows that $\pi_u = 0$ if $c_{\partial^\alpha u} = 0$ for $|\alpha| \leq k - 1$ (in particular, if u has compact support). By

scaling (4.14) with $j = k$, we get $\| |x|^{-k+N/p-N/q}u \|_q \leq C \| |\nabla^k u| \|_p$ for $q \in I_{k,p}$, a Hardy-Sobolev inequality of order k .

If ($p \geq 1$ and) $N/\ell < p < N/(\ell - 1)$ for some $1 \leq \ell \leq k$ and with $c_{\partial^\alpha u}$ given by (3.9), then $\deg \pi_u \leq k - \ell$ if $c_{\partial^\alpha u} = 0$ for $k - \ell + 1 \leq |\alpha| \leq k - 1$ (vacuous if $\ell = 1$ and trivially true if $\ell \geq 2$ and $\nabla^{k-\ell+1}u$ has compact support) and π_u may be chosen as the Taylor polynomial of u of order $k - \ell$ at 0. Then, scaling in (4.14) with $j = k$ yields $\| |x|^{-k+N/p-N/q}(u - \pi_u) \|_q \leq C \| |\nabla^k u| \|_p$ for $q \in I_{k,p} = [p, \infty]$. If $\ell = 1$ (i.e., $p > N$), π_u is the Taylor polynomial of u of order $k - 1$ at 0 and the inequality is a Hardy (Morrey) inequality of order k if $q = p$ ($q = \infty$). If $\ell > 1$, the required conditions are non-standard in inequalities of this sort.

5. An application to embeddings of weighted Sobolev spaces.

For $k \in \mathbb{N}, s \in \mathbb{R}$ and $1 \leq p < \infty, 1 \leq q \leq \infty$, consider the space

$$W_s^{k,q,p} := \{u \in L_{s+k}^q : \nabla^k u \in (L_s^p)^{\nu(k,N)}\},$$

equipped with the Banach space norm $\|u\|_{L_{s+k}^q} + \| |\nabla^k u| \|_{L_s^p}$. When $k = 1, s < -1$ and $q \in I_{1,p}$ is finite, it follows from [16, Example 21.10 (i), p.309] that $W_s^{1,p,p} \hookrightarrow W_s^{1,q,p}$ and that the norm of $W_s^{1,p,p}$ is equivalent to the norm $\| |\nabla u| \|_{L_s^p}$. In Theorem 5.2 below, we show that if $s \neq -1$ -not just $s < -1$ - and $q \in I_{1,p}$ (possibly ∞), then in fact $W_s^{1,p,p} = W_s^{1,q,p}$ with equivalent norms³ and that the reverse embedding $W_s^{1,q,p} \hookrightarrow W_s^{1,p,p}$ holds for every $1 \leq q \leq \infty$. We also show that the spaces $W_s^{k,q,p}, k \in \mathbb{N}$, have similar (and other) properties if $s \notin \{-k, \dots, -1\}$.

LEMMA 5.1. *Suppose $k \in \mathbb{N}$ and $s \in \mathbb{R}, s \notin \{-k, \dots, -1\}$. For a polynomial $\pi \in \mathcal{P}_{k-1}$, the following conditions are equivalent:*

- (i) $\pi \in L_{s+k}^q$ for every $1 \leq q \leq \infty$.
- (ii) $\pi \in L_{s+k}^{q_1} + L_{s+k}^{q_2}$ for some $1 \leq q_1, q_2 \leq \infty$.

PROOF. (i) \Rightarrow (ii) is obvious.

(ii) \Rightarrow (i). With no loss of generality, assume $\pi \neq 0$ and let $d := \deg \pi \leq k - 1$. Since $\pi \in L_{s+k}^q$ for every $1 \leq q \leq \infty$ if $d < s + k$, it suffices to prove that, indeed, $d < s + k$.

We shall use the remark that $|\pi(x)|$ grows (pointwise) as fast as $|x|^d$ on some sector (open cone with vertex at the origin) Σ . Specifically, there are $\delta > 0$ and $R > 0$ such that $|\pi(x)| \geq \delta(1 + |x|)^d$ for $x \in \Sigma_R := \{x \in \Sigma : |x| \geq R\}$.

Write $\pi = f + g$ with $f \in L_{s+k}^{q_1}$ and $g \in L_{s+k}^{q_2}$. Since $|\pi| \leq |f| + |g|$, it follows that if $x \in \Sigma_R$, then either $|f(x)| \geq (\delta/2)(1 + |x|)^d$ or $|g(x)| \geq (\delta/2)(1 + |x|)^d$. Set $E_f := \{x \in \mathbb{R}^N : |f(x)| \geq (\delta/2)(1 + |x|)^d\}$. If $q_1 < \infty$, then $\int_{E_f} (1 + |x|)^{-(s+k-d)q_1-N} dx \leq (2/\delta)^{q_1} \|f\|_{L_{s+k}^{q_1}}^{q_1} < \infty$, i.e., E_f has finite $w_1(x)dx$ measure, where $w_1(x) := (1 + |x|)^{-(s+k-d)q_1-N}$. Likewise, if $E_g := \{x \in \mathbb{R}^N : |g(x)| \geq (\delta/2)(1 + |x|)^d\}$ and $q_2 < \infty$, then E_g has finite $w_2(x)dx$ measure where $w_2(x) := (1 + |x|)^{-(s+k-d)q_2-N}$. Thus, both E_f and E_g have finite $w_0(x)dx$ measure, where $w_0(x) := (1 + |x|)^{-(s+k-d)q_0-N}$ and $q_0 = q_1$ or $q_0 = q_2$, depending upon which of the two exponents $-(s + k - d)q_i - N, i = 1, 2$, is smaller.

³However, the norms are equivalent to $\| |\nabla u| \|_{L_s^p}$ only when $s < -1$.

As a result, $\Sigma_R \subset E_f \cup E_g$ has finite $w_0(x)dx$ measure $\int_{\Sigma_R} (1 + |x|)^{-(s+k-d)q_0-N} dx$ and a calculation in spherical coordinates shows at once that this happens if and only if $d < s + k$.

Suppose now that $q_1 = \infty$, so that $(1 + |x|)^d(1 + |x|)^{-(s+k)}$ is bounded on E_f . This can only happen if E_f is bounded or if $d \leq s + k$. In the latter case, $d < s + k$ since $0 \leq d \leq k - 1$ and $s \notin \{-k, \dots, -1\}$. Assume then that E_f is bounded. If $q_2 < \infty$, the result that E_g has finite $w_2(x)dx$ measure continues to hold and then the same thing is true of E_f (bounded). Hence, Σ_R has finite $w_2(x)dx$ measure and the same argument as before yields $d < s + k$. By symmetry, this remains true if $q_2 = \infty$ and $q_1 < \infty$.

Lastly, suppose that $q_1 = q_2 = \infty$. Since $\Sigma_R \subset E_f \cup E_g$ is unbounded, one at least among E_f and E_g is unbounded and so, as was seen above, $d < s + k$. \square

REMARK 5.1. Lemma 5.1 is also true, with the same proof, on \mathbb{R}_\pm .

THEOREM 5.2. *Suppose that $k \in \mathbb{N}$, that $1 \leq p < \infty$ and that $s \in \mathbb{R}, s \notin \{-k, \dots, -1\}$ (it is not assumed that $N > 1$ if $s < -1$). Define $k_s := k$ if $s > -1$, $k_s := E(s + k + 1)$ (integer part) if $s \in (-k, -1)$ and $k_s := 0$ if $s < -k$. Then,*

- (i) $W_s^{k,q,p} \hookrightarrow W_s^{k,p,p}$ for every $1 \leq q \leq \infty$.
- (ii) If $u \in W_s^{k,p,p}$, then $\nabla^{k-j}u \in (L_{s+j}^q)^{\nu(k-j,N)}$ for every $1 \leq j \leq k$ and every $q \in I_{j,p}$ (see (1.4)) and there is a constant $C > 0$ independent of u such that

$$\|\nabla^{k-j}u\|_{L_{s+j}^q} \leq C \|\nabla^k u\|_{L_s^p} \text{ if } 1 \leq j \leq k - k_s \tag{5.1}$$

and that

$$\|\nabla^{k-j}u\|_{L_{s+j}^q} \leq C(\|u\|_{L_{s+k}^p} + \|\nabla^k u\|_{L_s^p}) \text{ if } 1 \leq j \leq k. \tag{5.2}$$

- (iii) $W_s^{k,q,p} = W_s^{k,p,p}$ for every $q \in I_{k,p}$, with equivalent norms as q is varied. Furthermore, if $s < -k$, the norm of $W_s^{k,p,p}$ is equivalent to $\|\nabla^k u\|_{L_s^p}$.

PROOF. (i) In a first step, we also assume that $N > 1$ if $s < -1$. If $u \in W_s^{k,q,p}$ with $1 \leq q \leq \infty$, then $u \in L_{s+k}^q$ and $\nabla^k u \in (L_s^p)^{\nu(k,N)}$. By Theorem 4.4, there is a polynomial $\pi_u \in \mathcal{P}_{k-1}$ such that $u - \pi_u \in L_{s+k}^p$ and, since $u \in L_{s+k}^q$, it follows that $\pi_u \in L_{s+k}^p + L_{s+k}^q$. Thus, $\pi_u \in L_{s+k}^p$ by (ii) \Rightarrow (i) in Lemma 5.1, so that $u \in L_{s+k}^p$. This shows that $W_s^{k,q,p} \subset W_s^{k,p,p}$.

If $s < -1$ and $N = 1$, the above still shows, by Remarks 4.2 and 5.1, that $W_s^{k,q,p}(\mathbb{R}_\pm) \subset W_s^{k,p,p}(\mathbb{R}_\pm)$ for every $1 \leq q \leq \infty$. Thus, if $u \in W_s^{k,q,p}(\mathbb{R})$ with $1 \leq q \leq \infty$, then $u \in W_{loc}^{k,p}(\mathbb{R})$ and $u \in W_s^{k,p,p}(\mathbb{R}_-) \cap W_s^{k,p,p}(\mathbb{R}_+)$, whence $u \in W_s^{k,p,p}(\mathbb{R})$. This shows that $W_s^{k,q,p} \subset W_s^{k,p,p}$ for every $1 \leq q \leq \infty$ still holds when $N = 1$.

That the above embeddings are continuous follows from the closed graph theorem, for if $u_n \rightarrow u$ in $W_s^{k,q,p}$ and $u_n \rightarrow v$ in $W_s^{k,p,p}$, then u_n tends to both u and v in \mathcal{D}' , so that $u = v$.

- (ii) If $s < -k$, so that $k_s = 0$, the polynomial π_u of Theorem 4.4 (when $N > 1$) is unique, whence $\pi_u = 0$ if $u \in W_s^{k,p,p} \subset L_{s+k}^p$ and then (5.1) follows from (4.8). If $N = 1$, use the same arguments on \mathbb{R}_\pm (Remark 4.2). Evidently, (5.2) follows from (5.1).

Suppose now that $s \in (-k, -1)$, so that s is not an integer and $k_s = E(s + k + 1) <$

$s+k+1$. Assume $N > 1$. From the discussion after Theorem 4.4, the part of π_u of degree greater than or equal to k_s is unique. If $u \in W_s^{k,p,p} \subset L_{s+k}^p$, this part is 0, for then $\pi_u \in L_{s+k}^p$ by (4.8), so that $\pi_u \in \mathcal{P}_{k_s-1}$ and (5.1) follows from (4.8) since $\nabla^{k-j}\pi_u = 0$ when $1 \leq j \leq k - k_s$.

To prove (5.2) (obvious from (5.1) when $1 \leq j \leq k - k_s$), we first show that $\nabla^{k-j}u \in (L_{s+j}^q)^{\nu(k-j,N)}$ for $1 \leq j \leq k$. By Theorem 4.4, $\nabla^{k-j}u - \nabla^{k-j}\pi_u \in (L_{s+j}^q)^{\nu(k-j,N)}$ and, since $\pi_u \in \mathcal{P}_{k_s-1}$, it follows that $\nabla^{k-j}\pi_u \in (\mathcal{P}_{k_s-1-k+j})^{\nu(k-j,N)}$. But $\mathcal{P}_{k_s-1-k+j} \subset L_{s+j}^q$ since $s+j > k_s - 1 - k + j$ (recall $k_s < s+k+1$) and so $\nabla^{k-j}u \in (L_{s+j}^q)^{\nu(k-j,N)}$, as claimed.

This shows that $W_s^{k,p,p} \subset W_{s+j}^{k-j,p,q}$. By the closed graph theorem, the embedding is continuous and (5.2) follows. As before, if $N = 1$, repeat the same arguments on \mathbb{R}_\pm .

Suppose now that $s > -1$, so that $k_s = k$ and (5.1) is trivial. The proof of (5.2) proceeds as above, based on the remark that $\nabla^{k-j}\pi_u \in (\mathcal{P}_{j-1})^{\nu(k-j,N)}$ and that $\mathcal{P}_{j-1} \subset L_{s+j}^q$ since $s+j > j-1$.

(iii) By (i), $W_s^{k,q,p} \hookrightarrow W_s^{k,p,p}$ and, by (ii) with $j = k$, $W_s^{k,p,p} \subset W_s^{k,q,p}$ if $q \in I_{k,p}$. The continuity of the embedding (hence the equivalence of norms) follows from (5.2) with $j = k$. If also $s < -k$, then $k_s = 0$ and, by (5.1) with $j = k$ and $q = p \in I_{k,p}$, the norm of $W_s^{k,p,p}$ is equivalent with $\|\nabla^k u\|_{L_s^p}$. \square

6. Exponential growth.

If $s \in \mathbb{R}$ and $1 \leq p \leq \infty$, define

$$L_{\text{exp},s}^p := \{u \in L_{\text{loc}}^p : e^{-s|x|}u \in L^p\},$$

equipped with the Banach space norm $\|u\|_{L_{\text{exp},s}^p} := \|e^{-s|x|}u\|_p$. If $p < \infty$, $L_{\text{exp},s}^p = L^p(\mathbb{R}^N; e^{-sp|x|}dx)$ and $L_{\text{exp},0}^p = L^p = L_{-N/p}^p$.

It is readily checked that $u(x) = e^{t|x|}$ is in $L_{\text{exp},s}^p$ for every $t < s$ and that $u(x) = (1+|x|)^t$ is in $L_{\text{exp},s}^p$ for every $t \in \mathbb{R}$ if $s > 0$. Thus, when $p < \infty$, it is appropriate to say that the functions of $L_{\text{exp},s}^p$ grow slower than $e^{s|x|}$ at infinity in the L^p sense. The functions of $L_{\text{exp},s}^\infty$ do not grow faster than $e^{s|x|}$, and grow slower if $\lim_{R \rightarrow \infty} \text{ess sup}_{|x| > R} e^{-s|x|}|u| = 0$, or simply $\lim_{|x| \rightarrow \infty} e^{-s|x|}u(x) = 0$ if u is continuous.

By Remark 2.1, Lemma 2.1 (approximation by mollification) holds in all the spaces $L_{\text{exp},s}^p$ with $p < \infty$. Also, the one-dimensional Hardy inequalities of Lemmas 2.2 and 3.1 have the following counterpart ([16, Theorems 5.9 and 6.2]):

LEMMA 6.1. *Suppose that $1 \leq p < \infty$ and that $s \neq 0$.*

(i) *If $s > 0$ and $\rho > 0$, there is a constant $C > 0$ such that*

$$\left(\int_\rho^\infty e^{-spr} r^{N-1} |f(r) - f(\rho)|^p dr \right)^{1/p} \leq C \left(\int_\rho^\infty e^{-spr} r^{N-1} |f'(r)|^p dr \right)^{1/p}, \quad (6.1)$$

for every locally absolutely continuous function f on $[\rho, \infty)$.

(ii) *If $s < 0$, there is a constant $C > 0$ such that*

$$\left(\int_0^\infty e^{-spr} r^{N-1} |f(r)|^p dr \right)^{1/p} \leq C \left(\int_0^\infty e^{-spr} r^{N-1} |f'(r)|^p dr \right)^{1/p}, \quad (6.2)$$

for every absolutely continuous function f on $(0, \infty)$ such that $\lim_{r \rightarrow \infty} f(r) = 0$.

Both (6.1) and (6.2) remain true, but will not be needed, when p is replaced with $q \in [p, \infty]$ in the left-hand side, with the usual modification when $q = \infty$.

THEOREM 6.2. *Suppose that $k \in \mathbb{N}$, that $s \neq 0$ and that $N > 1$ if $s < 0$. Let $u \in \mathcal{D}'$ be such that $\nabla^k u \in (L^p_{\text{exp},s})^{\nu(k,N)}$ with $1 \leq p < \infty$. Then, there is a polynomial $\pi_u \in \mathcal{P}_{k-1}$ independent of s and p , unique if $s < 0$, such that $\nabla^{k-j}(u - \pi_u) \in (L^q_{\text{exp},s})^{\nu(k-j,N)}$ for every $1 \leq j \leq k$ and every $q \in I_{j,p}$ and there is a constant $C = C(s, j, p, q) > 0$ independent of u such that*

$$\| |\nabla^{k-j}(u - \pi_u)| \|_{L^q_{\text{exp},s}} \leq C \| |\nabla^k u| \|_{L^p_{\text{exp},s}}.$$

Furthermore, $\lim_{|x| \rightarrow \infty} e^{-s|x|}(\nabla^{k-j}u(x) - \nabla^{k-j}\pi_u(x)) = 0$ if $p = N = j = 1$ or if $1 \leq j \leq k$ and $p > N/j$. (In particular, $\lim_{|x| \rightarrow \infty} e^{-s|x|}\nabla^{k-j}u(x) = 0$ if also $s > 0$.)

PROOF. Suppose first that $k = 1$, so that π_u is a constant c_u . The case $q = p$ can be handled along the lines of the proof of Theorem 2.3 (when $s > 0$) or Theorem 3.2 (when $s < 0$), upon merely using Lemma 6.1 instead of Lemma 2.2 or Lemma 3.1. We skip the details.

It follows from $\nabla u \in (L^p_{\text{exp},s})^N$ and $u - c_u \in L^p_{\text{exp},s}$ that $e^{-s|x|}(u - c_u) \in W^{1,p}$ (use $\nabla e^{-s|x|} = -se^{-s|x|}|x|^{-1}x$). Thus, by the Sobolev embedding theorem, $e^{-s|x|}(u - c_u) \in L^q$, i.e., $u - c_u \in L^q_{\text{exp},s}$, for every $q \in I_{1,p}$. If $p > N$, it is well-known that the functions of $W^{1,p}$ tend to 0 at infinity, so that $\lim_{|x| \rightarrow \infty} e^{-s|x|}(u(x) - c_u) = 0$. If $s > 0$, then $\lim_{|x| \rightarrow \infty} e^{-s|x|}c_u = 0$ and so $\lim_{|x| \rightarrow \infty} e^{-s|x|}u(x) = 0$. This proves the theorem when $k = 1$. The general case follows by induction; see the proof of Theorem 4.4. \square

When $s = 0$, Theorem 4.4 with $s = -N/p$ (that is, Corollary 4.5) must be substituted for Theorem 6.2, at least when $p \neq N/j, j = 1, \dots, k$.

If $s < 0$ and $k = 1$, the polynomial π_u is a constant c_u given by (3.9). If $s > 0$ and $k = 1$, a formula for c_u is $|B_\rho|^{-1} \int_{B(x_0, \rho)} u$ where $x_0 \in \mathbb{R}^N$ and $\rho > 0$ are chosen independent of u . When $k \in \mathbb{N}$, these formulas can be used to find the coefficients of π_u ; see the comments after Theorem 4.4.

There is also an analog of Theorem 5.2, with an entirely similar proof. Define

$$W_{\text{exp},s}^{k,q,p} := \{u \in L^q_{\text{exp},s} : \nabla^k u \in (L^p_{\text{exp},s})^{\nu(k,N)}\},$$

with Banach space norm $\|u\|_{L^q_{\text{exp},s}} + \| |\nabla^k u| \|_{L^p_{\text{exp},s}}$.

THEOREM 6.3. *Suppose that $k \in \mathbb{N}$, that $1 \leq p < \infty$ and that $s \in \mathbb{R}, s \neq 0$ (it is not assumed that $N > 1$ if $s < 0$). Then,*

- (i) $W_{\text{exp},s}^{k,q,p} \hookrightarrow W_{\text{exp},s}^{k,p,p}$ for every $1 \leq q \leq \infty$.
- (ii) If $u \in W_{\text{exp},s}^{k,p,p}$, then $\nabla^{k-j}u \in (L^q_{\text{exp},s})^{\nu(k-j,N)}$ for every $1 \leq j \leq k$ and every $q \in I_{j,p}$ and there is a constant $C > 0$ independent of u such that

$$\|\nabla^{k-j}u\|_{L_{\text{exp},s}^q} \leq C\|\nabla^k u\|_{L_{\text{exp},s}^p},$$

when $s < 0$ and that

$$\|\nabla^{k-j}u\|_{L_{\text{exp},s}^q} \leq C(\|u\|_{L_{\text{exp},s}^p} + \|\nabla^k u\|_{L_{\text{exp},s}^p}).$$

(iii) $W_{\text{exp},s}^{k,q,p} = W_{\text{exp},s}^{k,p,p}$ for every $q \in I_{k,p}$, with equivalent norms as q is varied. Furthermore, if $s < 0$, the norm of $W_{\text{exp},s}^{k,p,p}$ is equivalent to $\|\nabla^k u\|_{L_{\text{exp},s}^p}$.

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