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More on 2-chains with 1-shell boundaries in rosy theories

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Abstract. In [**4**], B. Kim, and the authors classified 2-chains with 1-shell boundaries into either RN (renamable)-type or NR (non renamable)-type 2-chains up to renamability of support of subsummands of a 2-chain and introduced the notion of chain-walk, which was motivated from graph theory : a directed walk in a directed graph is a sequence of edges with compatible condition on initial and terminal vertices between sequential edges. We consider a directed graph whose vertices are 1-simplices whose supports contain 0 and edges are plus/minus of 2-simplices whose supports contain 0. A chain-walk is a 2-chain induced from a directed walk in this graph. We reduced any 2-chains with 1-shell boundaries into chain-walks having the same boundaries.

In this paper, we reduce any 2-chains of 1-shell boundaries into chain-walks of the same boundary with support of size 3. Using this reduction, we give a combinatorial criterion determining whether a minimal 2-chain is of RN- or NR-type. For a minimal RN-type 2-chains, we show that it is equivalent to a 2-chain of Lascar type (coming from model theory) if and only if it is equivalent to a planar type 2-chain.

1. Introduction.

In [**2**], [**3**], J. Goodrick, B. Kim, and A. Kolesnikov defined homology groups of a strong type $p \in S(A)$ in any rosy theory T and they addressed that those groups are related with amalgamation property. More precisely, they proved that the $(n-1)$-th homology group of a strong type p consists of $(n-1)$-shells of p with support $n+1 = \{0, \ldots, n\}$ whenever T has n-CA over $A = \mathrm{acl}(A)$ $(n \geq 2)$. In particular the first homology group consists of 1-shells of p. Therefore as is known if T is simple then due to 3-amalgamation the first homology group is trivial. Moreover in [**4**], B. Kim, and the authors proved that in any rosy theory T, the first homology group of a Lascar type p is also trivial. We classified 2-chains with a 1-shell boundary into two types : NR (non-renamable)- and RN (renamable)-types, and we reduce 2-chains with 1-shell boundaries into chain-walks having the same boundaries. Using this classification, we showed that the minimal lengths of 2-chains with 1-shell boundaries are not bounded in rosy theories.

In this paper, we give geometric and combinatorial criteria determining the types of 2-chains. Using the notion of *matrix expression*, we give a combinatorial criterion for determining whether a given minimal 2-chain having a 1-shell boundary is of RN-type. We deduce that when the length of a given 2-chain is 3 modulo 4, the given chain must be of RN-type. We also show that a Lascar 2-chain (a model theoretic notion crucially

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used in the proof of Fact 1.9) is equivalent to a planar 2-chain (a geometric notion). We are working in model theoretic setting but our classification results hold in more general categorical setting, that is, an amenable class of functors in [**3**].

In the rest of this section, we review some notions and facts from [**2**], [**3**] and [**4**]. We first recall the definitions of simplices and the corresponding homology groups introduced in [**2**], [**3**]. Throughout we work with a large saturated model $\mathcal{M} = \mathcal{M}^{\mathrm{eq}}$ whose theory T is rosy with the thorn-independence relation $\downarrow$ on the small sets of $\mathcal{M}$.

From now on, we fix a small algebraically closed set $A = \mathrm{acl}(A)$ and $p(x) \in S(A)$ (with possibly infinite x). Let $\mathcal{C}_A$ denote the category, where

1. objects are small subsets of $\mathcal{M}$ containing A, and
2. morphisms are elementary maps which fix A pointwise.

For some finite $s \subseteq \omega$, the power set of s, $\mathcal{P}(s)$ forms the category as an ordered set :

1. $\mathrm{Ob}(\mathcal{P}(s)) = \mathcal{P}(s)$, and
2. for $u, v \in \mathcal{P}(s)$, $\mathrm{Mor}(u, v) = \{\iota_{u,v}\}$, where $\iota_{u,v}$ is the single inclusion map for $u \subseteq v$, or $= \emptyset$ otherwise.

For a functor $f : \mathcal{P}(s) \to \mathcal{C}_A$ and $u \subseteq v \in \mathcal{P}(s)$, we write $f_v^u := f(\iota_{u,v}) \in \mathrm{Mor}(f(u), f(v))$ and $f_v^u(u) := f_v^u(f(u)) \subseteq f(v)$.

DEFINITION 1.1. A functor $f : \mathcal{P}(s) \to \mathcal{C}_A$ for some finite $s \subseteq \omega$ is said to be a *closed independent* (*regular*) *n-simplex* in p if

1. $|s| = n + 1$.
2. $f(\emptyset) \supseteq A$; and for $i \in s$, $f(\{i\})$ is of the form $\mathrm{acl}(Ca)$ where $a(\models p)$ is independent with $C = f_{\{i\}}^{\emptyset}(\emptyset)$ over A.
3. For all non-empty $u \in \mathcal{P}(s)$, we have
$$f(u) = \mathrm{acl}\left(A \cup \bigcup_{i \in u} f_u^{\{i\}}(\{i\})\right);$$
and $\{f_u^{\{i\}}(\{i\}) |\ i \in u\}$ is independent over $f_u^{\emptyset}(\emptyset)$.

We shall call a closed independent n-simplex simply by an *n-simplex*. The set s is called the *support of* f, denoted by $\mathrm{supp}(f)$.

Let $S_n(p)$ denote the collection of all n-simplices in p and $C_n(p)$ the free abelian group generated by $S_n(p)$; its elements are called *n-chains* in p.

A non-zero n-chain c is uniquely written (up to permutation of terms) as $c = \sum_{1 \le i \le k} n_i f_i$, where n_i is a non-zero integer and $f_1, \ldots, f_k$ are distinct n-simplices. (This form is called the *standard form* of the chain c.) We call $|c| := |n_1| + \cdots + |n_k|$ the *length* of the chain c, and define the *support* of c as the union of $\mathrm{supp}(f_i)$'s.

We use $a, b, c, \ldots, f, g, h, \ldots, \alpha, \beta, \ldots$ to denote simplices and chains. Now we define the boundary operators and using the boundary operators we will define homology groups.

Definition 1.2. Let $n \geq 1$ and $0 \leq i \leq n$. The i-th *boundary operator* $\partial_n^i : C_n(p) \to C_{n-1}(p)$ is defined so that if f is an n-simplex with domain $\mathcal{P}(s)$ with $s = \{s_0 < \cdots < s_n\}$, then

$$\partial_n^i(f) = f \restriction \mathcal{P}(s \setminus \{s_i\})$$

and extended linearly to all n-chains in $C_n(p)$.

The *boundary map* $\partial_n : C_n(p) \to C_{n-1}(p)$ is defined by the rule

$$\partial_n(c) = \sum_{0 \leq i \leq n} (-1)^i \partial_n^i(c).$$

We write ∂^i and ∂ for ∂_n^i and ∂_n, respectively, if n is clear from context.

Definition 1.3. The kernel of ∂_n is denoted $Z_n(p)$, and its elements are called (n-)*cycles*. The image of ∂_{n+1} in $C_n(p)$ is denoted by $B_n(p)$ and its elements are called (n-)*boundaries*.

Since $\partial_n \circ \partial_{n+1} = 0$, $B_n(p) \subseteq Z_n(p)$ and we can define simplicial homology groups in p.

Definition 1.4. The n-th (*simplicial*) *homology group* in p is

$$H_n(p) := Z_n(p)/B_n(p).$$

Definition 1.5. For $n \geq 1$, an n-chain c is called an *n-shell* if it is in the form

$$c = \pm \sum_{0 \leq i \leq n+1} (-1)^i f_i,$$

where $f_0, \cdots, f_{n+1}$ are n-simplices such that whenever $0 \leq i < j \leq n+1$, we have $\partial^i f_j = \partial^{j-1} f_i$. Specially, a 1-shell c is of the form

$$c = f_0 - f_1 + f_2.$$

Remark 1.6. The boundary of a 2-simplex is a 1-shell, and the boundary of any 1-shell is 0.

Definition 1.7. Let $n \geq 0$.

1. p has $(n+2)$-*amalgamation* if any n-shell in p is the boundary of some $(n+1)$-simplex in p.

2. p has $(n+2)$-*complete amalgamation* (or simply $(n+2)$-CA) if p has k-amalgamation for every $2 \leq k \leq n+2$.

By extension axiom of thorn-independence, whenever $f : \mathcal{P}(s) \to \mathcal{C}_A$, $g : \mathcal{P}(t) \to \mathcal{C}_A \in S(p)$ and $f \restriction \mathcal{P}(s \cap t) = g \restriction \mathcal{P}(s \cap t)$, then f and g can be extended to a simplex $h : \mathcal{P}(s \cup t) \to \mathcal{C}_A$ in p. This property is called *strong* 2-*amalgamation*.

The following fact shows why the notion of shells is important.

Fact 1.8 ([**2**], [**3**]). *If p has $(n+1)$-CA for some $n \geq 1$, then*

$$H_n(p) = \{[c] : c \text{ is an } n\text{-shell over } A \text{ with } \operatorname{supp}(c) = \{0, \dots, n+1\}\ \}.$$

We have that $H_1(p)$ is trivial if and only if any 1-shell in p is the boundary of some 2-chain in p. Therefore, if T is simple, due to 3-amalgamation $H_1(p)$ is trivial. The following shows that the same result holds in any rosy theory.

Fact 1.9 ([**4**]). *Suppose that p is any Lascar strong type. Then $H_1(p) = 0$.*

There are two fundamental operations used in the classification of 2-chains in [**4**] : *crossing* and *renaming-of-support* operations.

Remark/Definition 1.10. Given any bijection $\sigma \colon \omega \to \omega$ (not necessarily order-preserving), we may define an automorphism $\sigma_n^* \colon C_n(p) \to C_n(p)$ for each n as follows: for any n-chain $c = \sum_i n_i f_i \in C_n(p)$, where each f_i is an n-simplex with $s_i := \operatorname{supp}(f_i) = \{s_{i,0} < \cdots < s_{i,n}\}$, we let $\sigma_i := \sigma \restriction s_i$ and $t_i := \sigma_i(s_i) = \{t_{i,0} < \cdots < t_{i,n}\}$. We define

$$\sigma^*(c) := \sum_i n_i |\sigma_i| f_i \circ \sigma_i^{-1}$$

with $|\sigma_i| := \operatorname{sign}(\sigma_i') \ (= \pm 1)$ where $\sigma_i' \in \operatorname{Sym}(n+1)$ such that for $j \leq n$, $\sigma_i(s_{i,j}) = t_{i,\sigma_i'(j)}$. For example

$$\sigma^*(f_i) = |\sigma_i| f_i \circ \sigma_i^{-1}.$$

Moreover, σ^* commutes with the boundary map, i.e., $\partial \circ \sigma^* = \sigma^* \circ \partial$.

Definition 1.11. Let $v \in C_2(p)$ be a 2-chain and let $w := \epsilon_1 \alpha_1 + \epsilon_2 \alpha_2$ be a subsummand of v, where α_i's are 2-simplices with for $i = 1, 2$, $\epsilon_i = \pm 1$, $\operatorname{supp}(\alpha_i) = \{\ell_1, \ell_2, k_i\}$ (k_i, ℓ_i being all distinct numbers) such that α_1 and α_2 agree on the intersection of their domains, namely $\mathcal{P}(\{\ell_1, \ell_2\})$. Further assume that, if we let $\gamma := \alpha_i \restriction \mathcal{P}(\{\ell_1, \ell_2\})$, then γ does not appear in $\partial(w)$, i.e., the two γ terms in $\partial(w)$ have opposite signs and cancel each other.

Now by strong 2-amalgamation, there exists a 3-simplex μ extending both α_i. For $i = 1, 2$, let $\beta_i := \mu \restriction \mathcal{P}(\{k_1, k_2, \ell_i\})$ and

$$w' := \begin{cases} \epsilon_2\ \beta_1 + \epsilon_1\ \beta_2 & \text{if } \epsilon_1\epsilon_2 = -1, \text{and exactly one of } k_2, \ell_1 \text{ is in between } k_1, \text{and } \ell_2 \\ \epsilon_1\ \beta_1 + \epsilon_2\ \beta_2 & \text{otherwise.} \end{cases}$$

Then the operation of replacing the subsummand w in v by w' is called the *crossing operation* (or simply *CR-operation*).

Definition 1.12. Let c be an n-chain in $C_n(p)$ and let d be a subsummand of c. Let $j \in \mathrm{supp}(d)$ such that $j \notin \mathrm{supp}(\partial_n(d))$. (In this situation, we say that d has a *vanishing support*, namely j, in its boundary.) Choose any $k \notin \mathrm{supp}(c)$ and any bijection $\sigma\colon \omega \to \omega$ which sends $j \mapsto k$ but which fixes the rest of the elements in $\mathrm{supp}(c)$. Then the operation of replacing the subsummand d in c by $\sigma_n^*(d)$ is called the *renaming-of-support operation* (or simply *RS-operation*). (See Remark/Definition 1.10 to recall the definition of σ_n^*.)

Remark/Definition 1.13. A 2-chain c is called *proper* if its length $|c|$ does not change after any finitely many applications of CR/RS-operations to its subsummands. This allows us to define an equivalence relation $\sim$ among proper 2-chains by: $c \sim c' \Leftrightarrow$ c can be obtained from c' by finitely many applications of the CR/RS-operations to its subsummands. Note that $c \sim c'$ implies $\partial(c) = \partial(c')$ and $|c| = |c'|$. A proper 2-chain α is said to be *minimal* if for any proper 2-chain α' with $\alpha' \sim \alpha$ there does not exist a subsummand β of α' such that $\partial(\beta) = 0$.

We classify 2-chains into two types.

Definition 1.14. Let α be a 2-chain having a 1-shell boundary. We call α *renamable type* (or *RN-type*) if a subsummand of α has a vanishing support. If α is not an RN-type 2-chain (so $|\mathrm{supp}(\alpha)| = 3$) we call α *non-renamable* (*NR-*) type.

Notation. Let f be any simplex. For any subset $\{j_0, \dots, j_k\} \subseteq \mathrm{supp}(f)$, we shall abbreviate $f \restriction \mathcal{P}(\{j_0, \dots, j_k\})$ as $f^{j_0,\cdots,j_k}$. Also, given a chain $c = \sum_{i\in I} n_i f_i$ (in its standard form), and any subset $\{j_0, \dots, j_k\} \subseteq \mathrm{supp}(c)$, we shall write $c^{j_0,\dots,j_k}$ to denote the subchain $\sum_{i\in J} n_i f_i$, where $J := \{i \in I \mid \mathrm{supp}(f_i) = \{j_0, \dots, j_k\}\}$.

Remark 1.15. If α is any 2-chain with a 1-shell boundary, then its length is always an odd positive number.

Now we introduce the notion of a chain-walk. At first, we recall the notion of *directed walk* from graph theory.

Definition 1.16. A *directed (multi) graph*, abbreviated to *digraph*, G is a pair (V, E) of disjoint sets (of *vertices* and *edges*) together with two maps $\mathrm{init} : E \to V$ and $\mathrm{ter} : E \to V$ assigning to every edge $e \in E$ an *initial vertex* $\mathrm{init}(e)$ and a *terminal vertex* $\mathrm{ter}(e)$. The edge e is said to be (*directed*) *from* $\mathrm{init}(e)$ *to* $\mathrm{ter}(e)$ (or is written as $e\colon \mathrm{init}(e) \to \mathrm{ter}(e)$). Indeed, we loosely write e as a triple $(e, \mathrm{init}(e), \mathrm{ter}(e))$. For any $e \in E$, let $-e$ denote the triple $(e, \mathrm{ter}(e), \mathrm{init}(e))$. We may write $G = (V, \{\pm e \mid e \in E\})$.

A *directed walk* in G from a vertex v_0 to a vertex v_{n+1} is a sequence of the form $\langle \epsilon_0 e_0, \dots, \epsilon_n e_n \rangle$ where $\epsilon_i = \pm 1$ and $e_i \in E$ for each i, such that $v_0 = \mathrm{init}(\epsilon_0 e_0)$, $v_{n+1} = \mathrm{ter}(\epsilon_n e_n)$ and $\mathrm{ter}(\epsilon_i e_i) = \mathrm{init}(\epsilon_{i+1} e_{i+1})$ for all $i = 0, \cdots, n-1$.

Remark/Definition 1.17. Let $(x_i)_{i=0}^n$ be a finite sequence of natural numbers with $n \geq 0$. For $k \leq n$, the k-th sign of this sequence is the number $(-1)^{I_k}$, where I_k is the number of places $x_j < x_{j+1}$ in the sequence of $(x_i)_{i=0}^{k+1}$ with $x_{n+1} = x_0$, denoted by $\mathrm{sign}_k(x_i)_{i=0}^n$. The n-th sign is called the sign of this sequence, denoted by $\mathrm{sign}(x_i)_{i=0}^n$.

Fix $n_0 \geq 0$. Consider a digraph $G_{n_0} = (V_{n_0}, E_{n_0})$, where V_{n_0} is *the set of* 1*-simplices with support containing* n_0. Let $s_j \in V_{n_0}(j = 1, 2)$ be with $\mathrm{supp}(s_j) = \{n_0, n_j\}$. An edge from s_0 to s_1 is the 2-chain ϵa, where a is a 2-simplex with $\mathrm{supp}(a) = \{n_0, n_1, n_2\}$ such that $a^{n_0,n_j} = s_j$, and $\epsilon = \mathrm{sign}(n_i)_{i=0}^2$. In this case, if $n_1 = n_2$, there are no edges between s_1 and s_2. For $s, t \in V_{n_0}$, a 2-chain α is called a *chain-walk* from s to t if it is of the form $\sum_i \epsilon_i a_i$, where $(\epsilon_i a_i)$ is a directed walk in G_{n_0} from s to t. So, if $\mathrm{supp}(s) = \{n_0, n_1\}$ and $\mathrm{supp}(t) = \{n_0, n_2\}$, then $(\partial\alpha)^{n_0,n_1} = -\mathrm{sign}_0(n_0, n_1)s$ and $(\partial\alpha)^{n_0,n_2} = \mathrm{sign}_0(n_0, n_2)t$ (*). To emphasize (*), we call α a chain-walk from $-\mathrm{sign}_0(n_0, n_1)s$ to $\mathrm{sign}_0(n_0, n_2)t$ instead of a chain-walk from s to t.

Let β be a 2-chain which is a chain-walk. Then there is a sum $\sum_{i=0}^m \epsilon_i b_i$ with respect to the order of indices, which ensures the property of a chain-walk; for $0 \leq i < m$, $(\partial\epsilon_i b_i)^{n,n_{i+1}} + (\partial\epsilon_{i+1} b_{i+1})^{n,n_{i+1}} = 0$, where each $\mathrm{supp}(b_i) = \{n, n_i, n_{i+1}\}$. We call such sum a *chain-walk representation of* β, simply a *representation.* By a *section* of the chain-walk β, we shall mean a subchain of β of the form

$$\beta' := \sum_{i=j}^{m'} \epsilon_i b_i \quad \text{for some } 0 \leq j < m' \leq m.$$

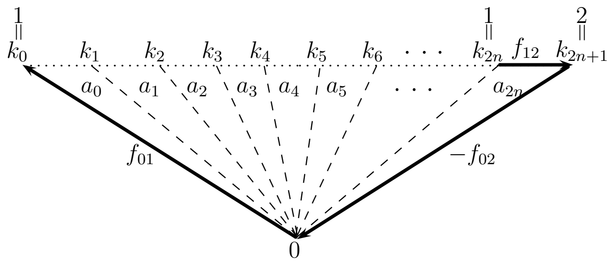

Figure 1. An example of a chain-walk 2-chain.

For the rest of this paper, we fix a 1-shell boundary $f_{12} - f_{02} + f_{01}$ with $\mathrm{supp}(f_{jk}) = \{j < k\}$. We reduce 2-chains having 1-shell boundaries into chain-walks with the same boundaries.

Fact 1.18 ([**4**]). *Let α be a minimal 2-chain with the boundary $f_{12} - f_{02} + f_{01}$.*

1. *Assume α is of NR-type. Then $|\alpha| = 1$ or $|\alpha| \geq 5$. If $|\alpha| \geq 5$ then any chain-walk in α from f_{01} to $-f_{02}$ is of the form $\sum_{i=0}^{2n}(-1)^i a_i$ which is as a chain equal to α such that $f_{12} = a_{2j}^{1,2}$ for some $1 \leq j \leq n-1$.*
2. *α is of RN-type if and only if α is equivalent to a 2-chain*

$$\alpha' = a_0 + \sum_{i=1}^{2n-1} \epsilon_i a_i + a_{2n}$$

$(n \geq 1)$ which is a chain-walk from f_{01} to $-f_{02}$ such that $\mathrm{supp}(a_{2n}) = \{0, 1, 2\}$ *and* $\partial^0 a_{2n} = f_{12}$, $\partial^1(a_{2n}) = -f_{02}$.

2. Reduction to 2-chains with support $\{0, 1, 2\}$.

Next we show that any 2-chain having a 1-shell boundary can be reduced to a 2-chain with support $\{0, 1, 2\}$. From this, any 2-chain having the 1-shell boundary $f_{12} - f_{02} + f_{01}$ is reduced to a 2-chain of the form $\sum_{i=0}^{2n}(-1)^i a_i$ which is a chain-walk from f_{01} to $-f_{02}$, and its support is $\{0, 1, 2\}$. This reduction is crucially used to develop a combinatorial criterion for RN-type 2-chains in the next section.

First we digress to recall some graph theoretical notions related with directed walk, and we consider a directed walk in a digraph, related with a given chain-walk, *which is different from one in Remark/Definition* 1.17.

Definition 2.1. Let $G = (V, E)$ be a digraph and consider a directed walk in G from a vertex v_0 to a vertex v_{n+1} of the form $\langle \epsilon_0 e_0, \ldots, \epsilon_n e_n\rangle$ where $\epsilon_i = \pm 1$ and $e_i \in E$ for each i. A directed walk from v_0 to v_{n+1} is *closed* if $v_0 = v_{n+1}$; is *reduced to* $\epsilon_{i_0} e_{i_0}$ if the commutative sum $\sum_{i \le n} \epsilon_i e_i = \epsilon_{i_0} e_{i_0} (0 \le i_0 \le n)$; and is a *balanced walk* (*from* v_0 *to* v_0) if $\sum_{i\le n} \epsilon_i e_i = 0$. A balanced walk is closed.

Remark 2.2. Let $\beta = \sum_{i\in I} \epsilon_i b_i$ be the chain-walk in Remark/Definition 1.17 from f_{01} to $-f_{02}$, and let $\beta' = \sum_{i\in J} \epsilon_i b_i$ be a section of β. Then β' canonically induces a directed walk in (V, E) where $V = \mathrm{supp}(\beta') \setminus \{0\}$ and $E = \{\pm b_i^{k_i,k_{i+1}} \mid i \in J\}$ with

$$(\partial \epsilon_i b_i)^{k_i,k_{i+1}} : k_i \to k_{i+1}.$$

(We may also loosely write $\epsilon_i b_i : k_i \to k_{i+1}$.) Such a directed walk is called the *induced walk from* β'.

Hence the directed walk induced from the chain-walk $\sum_{i\le 2n}(-1)^i a_i$ in Fact 1.18(1), is reduced to the edge f_{12}. The induced walk from the section $a_0 + \sum_{i=1}^{2n-1} \epsilon_i a_i$ of α' in Fact 1.18(2), is balanced.

We aim now to further reduce a given minimal 2-chain with a 1-shell boundary to an equivalent one having a support of size 3.

Theorem 2.3. *Let α be a minimal* 2*-chain having the* 1*-shell boundary* $f_{12} - f_{02} + f_{01}$*. Then α is of RN-type if and only if α is equivalent to a* 2*-chain*

$$\alpha' = \sum_{i=0}^{2n}(-1)^i a_i$$

($n \ge 1$) *which is a chain-walk from* f_{01} *to* $-f_{02}$ *such that* $\mathrm{supp}(\alpha') = \{0, 1, 2\}$ *and* $\partial^0 a_{2n} = f_{12}$, $\partial^1(a_{2n}) = -f_{02}$.

Proof. Suppose that $\alpha = \sum_{i<2n} \epsilon_i b_i + a_{2n}$ ($\epsilon_0 = 1$) is a chain-walk from f_{01} to $-f_{02}$ such that $\mathrm{supp}(a_{2n}) = \{0, 1, 2\}$ and $\partial^0 a_{2n} = f_{12}$, $\partial^1(a_{2n}) = -f_{02}$. Due to Fact 1.18, it suffices to show that there is $\alpha' = \sum_{i<2n} \epsilon'_i a_i + a_{2n} \sim \alpha$ as in the theorem (then it automatically follows $\epsilon'_i = (-1)^i$). We show this using induction on $|s(= s_\alpha)|$ where $s := \mathrm{supp}(\beta) \setminus \{0\}$ with $\beta := \alpha - a_{2n}$. Note that $1 \in s$ and $0 \notin s$. As was pointed out in Remark 2.2, the induced walk from β is a balanced walk in $V = s$ from 1 to 1 (*). If

$|s| = 2$ then $s = \{1 < k\}$ and k vanishes in $\partial(\beta)$. Hence by the RS-operation we rename k to 2 and we are done. Now let $|s| = m + 1$ with the induction hypothesis for $m \geq 2$ (**).

Fix $j(\neq 1) \in s$. We are now going to inductively remove all the edges induced from β, connecting 1 and j. For $s_0 \neq s_1 \in s$, let

$$I^{\alpha}_{s_0 s_1} = I_{s_0 s_1} := \{i < 2n \mid \epsilon_i b_i \colon s_0 \to s_1 \quad \text{or} \quad \epsilon_i b_i \colon s_1 \to s_0\} \quad \text{and}$$
$$I^{\alpha}_{s_0} = I_{s_0} := \{i < 2n \mid \text{for some } k \in s, \quad \epsilon_i b_i \colon s_0 \to k \quad \text{or} \quad \epsilon_i b_i \colon k \to s_0\}.$$

Let $I := I_{1j}$. We use induction on $|I|$. Due to (*), $|I|$ is even. Assume $|I| = 0$. Then $1 \notin I_j$ and in particular $1 \notin \operatorname{supp}(\beta_j)$ where $\beta_j := \sum_{i \in I_j} \epsilon_i b_i$. Now due to (*), it follows j vanishes in $\partial(\beta_j)$. Then by applying the RS-operation to β_j, we rename j to 1, and obtain $\alpha'' \sim \alpha$ with $|s_{\alpha''}| = m$. Then due to (**), there is a desired $\alpha' \sim \alpha''$.

Now let $|I| = m' + 2$ with the induction hypothesis that if $|I| = m'$ then we can find a desired $\alpha' \sim \alpha$ (***). Since $|s| \geq 3$, there are $j'(\neq 1, j) \in s$ and $\ell \in I$ such that either of $\epsilon_{\ell+1} b_{\ell+1} \colon j \to j'$, $\epsilon_{\ell+1} b_{\ell+1} \colon 1 \to j'$, $\epsilon_{\ell-1} b_{\ell-1} \colon j' \to j$, or $\epsilon_{\ell-1} b_{\ell-1} \colon j' \to 1$ holds. We will only show for the case $\epsilon_{\ell+1} b_{\ell+1} \colon j \to j'$ (the other cases can be checked by similar arguments). Now due to (*), we have $\epsilon_\ell b_\ell \colon 1 \to j$; and for some $t \in I$,

$$\epsilon_t b_t \colon j \to 1, \text{ and } (\partial \epsilon_\ell b_\ell)^{1,j} + (\partial \epsilon_t b_t)^{1,j} = 0.$$

We now can apply the CR-operation to $\epsilon_\ell b_\ell + \epsilon_{\ell+1} b_{\ell+1}$ and obtain $\epsilon'_\ell b'_\ell + \epsilon'_{\ell+1} b'_{\ell+1}$ preserving the boundary such that $\operatorname{supp}(b'_\ell) = \{0, 1, j'\}$ and $\operatorname{supp}(b'_{\ell+1}) = \{1, j, j'\}$. Hence we replace $\epsilon_\ell b_\ell + \epsilon_{\ell+1} b_{\ell+1}$ in β by $\epsilon'_\ell b'_\ell$, and obtain β'. Then $\beta' + a_{2n}$ is still a chain-walk, while the term $\epsilon'_{\ell+1} b'_{\ell+1}$ is left. We use it as follows. Since $(\partial \epsilon_\ell b_\ell)^{1,j} = (\partial \epsilon'_{\ell+1} b'_{\ell+1})^{1,j}$, we can apply the CR-operation to $\epsilon_t b_t + \epsilon'_{\ell+1} b'_{\ell+1}$, and obtain $\epsilon'_t b'_t + \epsilon''_{\ell+1} b''_{\ell+1}$ with $\operatorname{supp}(b'_t) = \{0, j, j'\}$ and $\operatorname{supp}(b''_{\ell+1}) = \{0, 1, j'\}$. Then we replace $\epsilon_t b_t$ in β' by $\epsilon'_t b'_t + \epsilon''_{\ell+1} b''_{\ell+1}$, and obtain β''. Thus $\alpha'' := \beta'' + a_{2n}$ is a chain-walk equivalent to α. Note now $|I^{\alpha''}_{1j}| = m'$. Hence due to the induction hypothesis (***), there is a desired $\alpha' \sim \alpha''$. □

Corollary 2.4. *Suppose that α is a minimal 2-chain with the 1-shell boundary $f_{12} - f_{02} + f_{01}$. Then there is an equivalent minimal 2-chain $\alpha' = \sum_{i=0}^{2n} (-1)^i a_i$ which is a chain-walk from f_{01} to $-f_{02}$, and* $\operatorname{supp}(\alpha') = \{0, 1, 2\}$.

Remark 2.5. In this remark we bring to our attention an issue of counting number of possible directed walks induced from chain-walks of the form α' in Corollary 2.4. We use the same notation. Let α' be an RN-type 2-chain (so $n \geq 1$) such that $\partial^0 a_{2n} = f_{12}$, $\partial^1(a_{2n}) = -f_{02}$. Let $\beta := \alpha' - a_{2n}$. Then β induces a balanced walk w (from 1 to 1) of length $2n$ on $V = \{1, 2\}$ with $E^+_\beta = E^+ := \{a_i^{1,2} | \ i < 2n\}$, the set of positive edges. So $|E^+| \leq n$, and $a_i^{1,2} : 1 \to 2$ for $a_i^{1,2} \in E^+$. One may ask how many such balanced walks of length $2n$ exist with a fixed set E^+. Let $|E^+| = m \leq n$, and rewrite $E^+ = \{b_1, \dots, b_m\}$. Recall that the walk w induced form β with $E^+ = E^+_\beta$ is a sequence of the form $w^+ := \langle (-1)^i e_i | i < 2n \rangle$. So $w^+ := \langle e_0, e_2, \dots, e_{2n-2} \rangle$ is an enumeration of E^+ using all the members. Also $w^- := \langle -e_1, -e_3, \dots, -e_{2n-1} \rangle$ is an enumeration of

$E^- := \{-b_1, \ldots, -b_m\}$. Now the number of occurrence of a given b_i in w^+ and $-b_i$ in w^- are the same, say $(0 <) r_i \leq n$. Hence given an assignment $r : E^+ \to \omega$ with $r_i = r(b_i)$ and $r_1 + \cdots + r_m = n$, there are at most

$$\left(\binom{n}{r_1}\binom{n-r_1}{r_2}\cdots\binom{n-(r_1+\cdots+r_{m-1})}{r_m}\right)^2 = \left(\frac{n!}{r_1!\cdots r_m!}\right)^2$$

many possible such walks. Thus if we only fix E^+ (so n, m fixed while r and $r_i > 0$ vary) then for $R := \{r : E^+ \to \{1, \ldots, n\} \mid\ r_1 + \cdots + r_m = n\}$ there are at most

$$\sum_{r \in R} \left(\frac{n!}{r_1!\cdots r_m!}\right)^2$$

many possible induced walks.

A similar computation can be made when α' is an NR-type 2-chain.

3. Combinatorial criterion for RN-type 2-chains.

In this section, by introducing the notion of *matrix expression*, we shall give a criterion for determining whether a given minimal 2-chain having a 1-shell boundary is of RN-type. We also give sufficient conditions for 2-chains to be of RN-type. According to Corollary 2.4, we are mainly interested in minimal 2-chains having 1-shell boundaries with support $\{0, 1, 2\}$, which are chain-walks. Let α be a chain-walk of length $2n+1$ having the 1-shell boundary $f_{12} - f_{02} + f_{01}$ with $\mathrm{supp}(\alpha) = \{0, 1, 2\}$. For $\{0, 1, 2\} = \{i, j, k\}$, f_i' denotes f_{jk} $(j < k)$. Fix $I = \{0, 1, 2\}$ and $J = \{0, \ldots, n\}$. Also, we write $\epsilon a \in \alpha$ $(\epsilon = \pm 1)$ to denote that a 2-simplex term ϵa is in α.

Now we assign a $3 \times (n+1)$ matrix to the 2-chain as follows:

Definition 3.1. Let $\sum_{j=0}^{2n} (-1)^j a_j$ be a representation of the given α which itself is a chain-walk from f_2' to $-f_1'$. By a *matrix expression* of (the representation of) α, we mean a function $M : I \times J \to J$ such that

1. for each $i \in I$, $M \restriction \{i\} \times\ J : (\{i\} \times)\ J \to J$ is a permutation of J;
2. for each $i \in I$, $M(i, 0)$ is an index such that $f_i' = \partial^i a_{2M(i,0)}$;
3. for each $i \in I,\ j \in J \setminus \{0\}$, $M(i, j)$ is an index such that $\partial^i a_{2j-1} = \partial^i a_{2M(i,j)}$.

Interpret $M(i, j)$ as an entry m_{ij} of a matrix in the $(i+1)$-th row and the $(j+1)$-th column, then $M = (m_{ij})_{I,J}$ is a $3 \times (n+1)$ matrix. Obviously, given a chain-walk representation of α, there is at least one (possibly more than one) matrix expression.

We may and will use $M(i, j)$ to represent both the image of (i, j) under the function M, and the $(i+1, j+1)$-entry of the matrix M.

Example 3.2. Let $\alpha = \sum_{j=0}^{8} (-1)^j a_j$ be a chain-walk from f_{01} to $-f_{02}$ such that $\partial^0 a_2 = f_{12}$, $\partial^1 a_8 = f_{02}$, $\partial^2 a_0 = f_{01}$, and $\partial^0 a_0 = \partial^0 a_4 = \partial^0 a_5 = \partial^0 a_7$, $\partial^0 a_1 = \partial^0 a_8$, $\partial^0 a_3 = \partial^0 a_6$. Then both

$$M_1 = \begin{pmatrix} 1\,4\,3\,0\,2 \\ 4\,0\,1\,2\,3 \\ 0\,1\,2\,3\,4 \end{pmatrix}, \; M_2 = \begin{pmatrix} 1\,4\,3\,2\,0 \\ 4\,0\,1\,2\,3 \\ 0\,1\,2\,3\,4 \end{pmatrix}$$

are matrix expressions of α and $(1,4)$, $(1,5)$-entries are swapped between M_1 and M_2.

Notice that matrix expressions are determined according to the *choices* of pairs of terms which cancel out each other. Therefore the second (similarly the third) row of a matrix expression need not always be of the form $(n\ 0\ 1 \cdots n-1)$; even the $(2,1)$-entry can be $n'(<n)$, if $f_{02} = \partial^1 a_{2n'} (= \partial^1 a_{2n})$.

Now we are ready to state a criterion for determining the type of α.

Theorem 3.3. *Let α be a minimal 2-chain of length $2n+1$ having the 1-shell boundary $f_{12} - f_{02} + f_{01}$ with* $\operatorname{supp}(\alpha) = \{0,1,2\}$*, which is a chain-walk. Then α is of RN-type if and only if there is a matrix expression M for a representation $\sum_{j=0}^{2n}(-1)^j a_j$ such that for some $0 \le i_0 < i_1 \le 2$, and non-empty $J_0 \subseteq \{1,\ldots,n\}$, $M(\{i_0\}\times J_0) = M(\{i_1\}\times J_0)$ as image set under the function M.*

Proof. $(\Rightarrow)$ Assume that $\alpha = \sum_{j=0}^{2n}(-1)^j a_j$ is of RN-type. Then there is a subchain α_1 of $\sum_{j=0}^{2n}(-1)^j a_j$ having a vanishing support $i_* \in \{0,1,2\}$. Hence $\partial^i \alpha_1 = 0$ for $i \in I \setminus \{i_*\}$ and $|\alpha_1| = 2m$. Therefore when we choose a matrix expression M satisfying Definition 3.1(3), we can let $a_{2M(i,j)} \in \alpha_1$ for each $i \in I\setminus\{i_*\}$ and $j \in J_0 := \{j \in J \mid -a_{2j-1} \in \alpha_1\}$.

Then by the choice, $M(\{i_0\}\times J_0) = M(\{i_1\}\times J_0)$, where $\{i_0, i_1, i_*\} = I$, as desired.

$(\Leftarrow)$ Let M be a given matrix expression of α so that for $J_0 \subseteq J \setminus \{0\}$ and $0 \le i_0 < i_1 \le 2$, we have $M(\{i_0\}\times J_0) = M(\{i_1\}\times J_0)$, say J_1. Let $\alpha_1 := \sum_{j\in J_1} a_{2j} + \sum_{j\in J_0} -a_{2j-1}$, a subchain of α. Then for i_0, i_1, we have $\partial^{i_0}\alpha_1 = \partial^{i_1}\alpha_1 = 0$, so α_1 has a vanishing support i_*, where $\{i_0, i_1, i_*\} = I$. Therefore α is of RN-type. □

Remark 3.4. The matrix expression M_1 from Example 3.2 does not satisfy the right-hand side in the above theorem. However M_2 is also a matrix expression of $\alpha = \sum_{j=0}^{8}(-1)^j a_j$ as well and the fact that $M_2(0,3) = M_2(1,3)$ ensures that the 2-chain α is of RN-type.

Let $M : I \times J \to J$ be a matrix expression. Then M induces a triple $(\sigma_{01}, \sigma_{12}, \sigma_{02})$ of permutations of J such that σ_{ik} is a map sending the $(i+1)$-th row to the $(k+1)$-th row, i.e., $\sigma_{ik}(m_{ij}) = m_{kj}$ for $j \in J$, and $0 \le i < k \le 2$. Notice that $\sigma_{02} = \sigma_{12} \circ \sigma_{01}$.

As is well-known that every element of the symmetric group $S_{|J|}$ has the unique cycle decomposition (up to obvious permutations), where each $j \in J$ appears exactly once in the decomposition (so it may contain a 1-cycle). Therefore we have the following fact : M is a matrix expression of an RN-type 2-chain α described in Theorem 3.3 if and only if there is a permutation σ_{ik} from the triple of M whose cycle decomposition has (more than) two disjoint cycles. On the other hand, any σ_{ik} from any matrix expressions of NR-type 2-chains cannot be disjointly decomposed.

Now we recall some basic facts about permutations which will be used in the proof of upcoming theorems.

FACT 3.5. *Let $A = \{a_0, \ldots, a_m\}$ and $B = \{b_0, \ldots, b_k\}$ be disjoint sets. Then for $0 < i \leq m$,*

1. $(a_0\ a_i) \circ (a_0\ a_1 \cdots a_i \cdots a_m) = (a_0\ a_1 \cdots a_{i-1}) \circ (a_i a_{i+1} \cdots a_m)$;
2. $(a_0\ a_1 \cdots\ a_i \cdots a_m) \circ (a_0\ a_i) = (a_0\ a_{i+1}\ a_{i+2} \cdots a_m) \circ (a_1\ a_2 \cdots a_i)$;
3. $(a_0\ b_0) \circ (a_0\ a_1 \cdots a_m) \circ (b_0\ b_1 \cdots b_k) = (a_0\ a_1 \cdots a_m\ b_0\ b_1 \cdots b_k)$;

NOTATION. For a permutation τ, let $\#_c(\tau)$ denote the number of disjoint cycles in the cycle decomposition of τ.

THEOREM 3.6. *Let α be a minimal 2-chain having the 1-shell boundary $f_{12} - f_{02} + f_{01}$. If the length of α is 3 modulo 4, then α is always of RN-type.*

PROOF. Suppose the length of α is $2n+1$ for odd n. Moreover, by Corollary 2.4, we may assume that α is a chain-walk from f_{01} to $-f_{02}$. Let $(\sigma_{01}, \sigma_{12}, \sigma_{02})$ be a triple induced from some matrix expression of α. If the cycle decomposition of σ_{01} or σ_{12} has disjoint cycles, then we are done. Otherwise, we can assume that $\sigma_{01} = (j_0\ j_1 \cdots j_n)$ and $\sigma_{12} = (0\ k_1 \cdots k_n)$, where $\{j_0, \ldots, j_n\} = \{0, k_1 \ldots, k_n\} = J$. For $i \geq 1$, let $\tau_i = (0\ k_i) \circ (0\ k_{i-1}) \circ \cdots \circ (0\ k_1) \circ \sigma_{01}$. Then $\tau_n = \sigma_{12} \circ \sigma_{01} = \sigma_{02}$, and it suffices to show that $\#_c(\tau_n) \neq 1$. Now we claim that $\#_c(\tau_{i+1})$ is either $\#_c(\tau_i) + 1$ or $\#_c(\tau_i) - 1$. Notice that $\#_c(\tau_1) = 2$ due to Fact 3.5(1). There now are two cases: If both 0 and k_{i+1} are contained in the same cycle μ of the decomposition of τ_i, then by Fact 3.5(1), $(0\ k_{i+1}) \circ \mu$ splits into two cycles which have 0 and k_{i+1}, respectively. Therefore, $\#_c(\tau_{i+1}) = \#_c(\tau_i) + 1$. If 0 and k_{i+1} are contained in different cycles μ_0 and μ_1, respectively, then by Fact 3.5(3), $(0\ k_{i+1}) \circ \mu_0 \circ \mu_1$ becomes a single cycle of size $|\mu_0| + |\mu_1|$, so we have $\#_c(\tau_{i+1}) = \#_c(\tau_i) - 1$. Inductively, we get

$$\#_c(\tau_i) \equiv \begin{cases} 0 \pmod 2 & \text{if } i \equiv 1 \pmod 2 \\ 1 \pmod 2 & \text{if } i \equiv 0 \pmod 2. \end{cases}$$

So for odd n, $\#_c(\tau_n)$ must be even. Therefore the cycle decomposition of σ_{02} cannot be a single cycle, and α must be of RN-type. □

REMARK 3.7. Let us summarize the previous theorem as follows : Let β be a minimal 2-chain with a 1-shell boundary. Then,

- if its length is 1 modulo 4, it *may be* of NR-type; or
- if its length is 3 modulo 4, it *must be* of RN-type.

For the first case, indeed we can find an NR-type 2-chain with a 1-shell boundary : Let α be a 2-chain with $\operatorname{supp}(\alpha) = \{0, 1, 2\}$, which is a chain-walk with a representation $\sum_{j=0}^{4k} (-1)^j a_j$ such that $\partial^0 a_{4k} = f_{12}$, $\partial^1 a_{4k} = f_{02}$, and $\partial^2 a_0 = f_{01}$; for each $0 \leq j_0 \neq j_1 < 2k$, and $i \in \{0, 1, 2\}$, $\partial^i a_{2j_0+1} \neq \partial^i a_{2j_1+1}$, $f_{01} \neq \partial^2 a_{2j_0+1}$, $f_{02} \neq \partial^1 a_{2j_0+1}$; for

$0 \le j < 2k$, $\partial^0 a_j = \partial^0 a_{2k+1+j} \ne f_{12}$; and no other relations between the boundaries of each 2-simplex terms. In this case, we obtain the unique matrix expression

$$M = \begin{pmatrix} k & k+1 \cdots & 2k & 0 & 1 & \cdots & k-1 \\ 2k & 0 & \cdots k-1 & k & k+1 \cdots & 2k-1 \\ 0 & 1 & \cdots & k & k+1 \, k+2 \cdots & 2k \end{pmatrix}.$$

In Example 3.2, α has two 2-simplex terms which have the same sign and the same image under the 0-th boundary operator ∂^0, for example $\partial^0 a_0 = \partial^0 a_4$, and then α is of RN-type. The following theorem says this does not happen by accident.

Theorem 3.8. *Let α be a minimal 2-chain of length $2n+1$ having the 1-shell boundary $f_{12} - f_{02} + f_{01}$ with $\mathrm{supp}(\alpha) = \{0, 1, 2\}$, which is a chain-walk with a representation $\sum_{j=0}^{2n} (-1)^j a_j$. Suppose one of the following holds :*

1. *$\partial^\ell a_{2j_0 - 1} = \partial^\ell a_{2j_1 - 1}$ for some $0 < j_0 < j_1 \le n$ and $0 \le \ell \le 2$;*
2. *$\partial^\ell a_{2j_0} = \partial^\ell a_{2j_1}$ for some $0 \le j_0 < j_1 \le n$ and $0 \le \ell \le 2$.*

Then α is of RN-type.

Proof. Assume (1) holds. Let M be a matrix expression of α. Consider the triple $(\sigma_{01}, \sigma_{12}, \sigma_{02})$ with respect to M. If one of the permutations can be decomposed into (more than) two disjoint cycles, then we are done. Therefore we can assume that all the three permutations are not properly decomposed. Now let $p := M(\ell, j_0) \ne q := M(\ell, j_1)$. Due to (1), we can swap the entries p and q from M to obtain a new matrix M'. Thus $M'(\ell, j_0) = M(\ell, j_1) = q$ and $M'(\ell, j_1) = M(\ell, j_0) = p$. Notice that $M'(i, j) = M(i, j)$ except for $(i, j) = (\ell, j_0), (\ell, j_1)$. Now we have three cases:

Case 1: $\ell = 2$: Since $\ell = 2$, only the third row is changed. So, $(\tau_{01}, \tau_{12}, \tau_{02}) := (\sigma_{01}, (p\ q) \circ \sigma_{12}, (p\ q) \circ \sigma_{02})$ is a triple of permutations induced from M', and τ_{12} (or τ_{02}) has two disjoint cycles due to Fact 3.5(1).

Case 2: $\ell = 1$: Similarly to Case 1, $((p\ q) \circ \sigma_{01},\ \sigma_{12} \circ (p\ q),\ \sigma_{02})$ is a triple induced from M' and again due to Fact 3.5(1), $(p\ q) \circ \sigma_{01}$ is decomposed into two disjoint cycles.

Case 3: $\ell = 0$: Similarly, $(\sigma_{01} \circ (p\ q), \sigma_{12}, \sigma_{02} \circ (p\ q))$ is a new triple and $\sigma_{01} \circ (p\ q)$ splits into two disjoint cycles by Fact 3.5(2).

In conclusion, for any case, new matrix expression M' witnesses α being of RN-type.

For the second condition, the theorem holds by the same argument. □

4. Lascar 2-chains.

In this section, we look closely at RN-type 2-chains. We present the notions of *planar type*, *Lascar type*, and *tower type* 2-chains, which are all of RN-type unless the length of any given 2-chain is 1. We shall show these three properties of RN-type 2-chains are all equivalent : Given a minimal 2-chain, if it is equivalent to a 2-chain satisfying one of the three properties, then it is also equivalent to 2-chains which satisfy the others. This is an interesting result as the notion of planar type comes from geometry while that of Lascar type comes from model theory (in particular, found in the proof of Fact 1.9).

REMARK/NOTATION 4.1. Let $\triangle \subset \mathcal{P}(\omega)$. We say $\triangle$ is *an abstract simplicial complex* if for $u \in \triangle$ and $v \subseteq u$, v is again in $\triangle$. The vertex set of $\triangle$ is the set $\bigcup \triangle$. For a fixed finite set $X = \{x_0 < x_1 < \cdots < x_n\} \subset \omega$, we say the power set of X, $\mathcal{P}(X)$ is called *an abstract n-simplex*. Let S_n denote the set of abstract n-simplices and let C_n the free abelian group generated by S_n; its elements are called *abstract n-chains*. Next, we define abstract n-th boundary maps $\partial_n : C_n \to C_{n-1}$ as follows : if $\mathcal{P}(X)$ is an n-simplex , the boundary of $\mathcal{P}(X)$ is defined as $\partial_n(\mathcal{P}(X)) = \sum_{i=0}^{n}(-1)^i\mathcal{P}(X \setminus \{x_i\})$ and we extend linearly to all n-chains in C_n.

Let $\mathcal{T}$ be a closed triangular plane region. Let $\triangle(\mathcal{T})$ be a triangulation, which is a triangular subdivision, of $\mathcal{T}$ in the plane with only three exterior vertices assigned $\{0, 1, 2\}$, and consider an abstract simplicial complex $\triangle_a(\mathcal{T})$ whose geometric realization is homeomorphic to $\triangle(\mathcal{T})$. Then $\triangle_a(\mathcal{T}) = \bigcup_i \mathcal{P}(s_i)$ for finite subsets $s_i \subset \omega$ and $|s_i| = 3$, and this induces an abstract 2-chain $\alpha(\triangle_a(\mathcal{T})) = \sum_i \epsilon_i\mathcal{P}(s_i)$ with its boundary $\mathcal{P}(\{1, 2\}) - \mathcal{P}(\{0, 2\}) + \mathcal{P}(\{0, 1\})$, and $\epsilon_i = \pm 1$ uniquely determined. For a 2-chain $\alpha = \sum_i \epsilon_i a_i$ in p with a 1-shell boundary, where $a_i : \mathcal{P}(s_i) \to \mathcal{C}_A$ is 2-simplex in p, we shall write α as $\alpha : \triangle_a(\mathcal{T}) \to \mathcal{C}_A$, and we say α has the domain of a triangulation of $\mathcal{T}$.

Of course we can consider a 2-chain having the domain of any triangular subdivision. But if such a 2-chain has a 1-shell boundary, then its triangulation must have only three exterior vertices.

And by a *simplicial map* between simplicial complexes L and K, we mean a map such that whenever the vertices of L span a simplex of L, their image span a simplex of K. We say two triangulations are *isomorphic* if there is a bijective simplicial map between two triangulations.

REMARK 4.2. Let $\mathcal{T}$ be a closed triangular plane region. If two triangulations of $\mathcal{T}$ are isomorphic, then they have the same abstract simplicial complex.

DEFINITION 4.3. Let α be a minimal 2-chain having the 1-shell boundary $f_{12} - f_{02} + f_{01}$.

1. We call α *planar type* (or simply *planar*) if $\alpha : \triangle_a(\mathcal{T}) \to \mathcal{C}_A$ for some closed triangular plane region $\mathcal{T}$, where $\triangle_a(\mathcal{T})$ is a planar triangulation of $\mathcal{T}$.

2. We call α *Lascar type* (or *Lascar*) if $|\alpha| = 1$ or α is an RN-type 2-chain of length $2n+1$, which is a chain-walk with a representation $\sum_{i=0}^{2n}(-1)^i a_i$ (see Fact 1.18(2)) such that for each $j < n$, $\partial^0(a_{2j}) - \partial^0(a_{2j+1}) = 0$. (Hence $1 \in \mathrm{supp}(a_i)$ for all $i \leq 2n$.)

3. We call α *tower type* if $\alpha = \sum_{i=-n}^{n} \epsilon_i a_i$ $(0 \leq n)$ with $\epsilon_0 = 1$, $\epsilon_i = \epsilon_{-i}$ $(i > 0)$, such that

   (a) $\partial^2 a_0 = f_{01}$, $(\partial\epsilon_n a_n)^{0,2} = -f_{02}$ and $(\partial\epsilon_{-n}a_{-n})^{1,2} = f_{12}$;

   (b) $\mathrm{supp}(a_0) = \{0, 1, k_0\}$; for $1 \leq i \leq n$, $\mathrm{supp}(a_i) = \{0, k_{i-1}, k_i\}$, $\mathrm{supp}(a_{-i}) = \{1, k_{i-1}, k_i\}$; $k_n = 2$;

(c) for $0 \leq i \leq n-1$, $(\partial \epsilon_i a_i)^{0,k_i} + (\partial \epsilon_{i+1} a_{i+1})^{0,k_i} = 0$ and $(\partial \epsilon_{-i} a_{-i})^{1,k_i} + (\partial \epsilon_{-(i+1)} a_{-(i+1)})^{1,k_i} = 0$; and

(d) $\epsilon_i(\partial^0 a_i) + \epsilon_{-i}(\partial^0 a_{-i}) = 0$ for $1 \leq i \leq n$.

The Lascar type 2-chains are so named, because such type chains are crucially used in the proof of Fact 1.9, which are to do with the Lascar distance. Note that each of the three type 2-chains is of RN-type if its length is ≥ 3.

We show all the three type 2-chains are equivalent. In above definition, if $|\alpha| = 3$ then α is equivalent to a Lascar 2-chain by Fact 1.18.

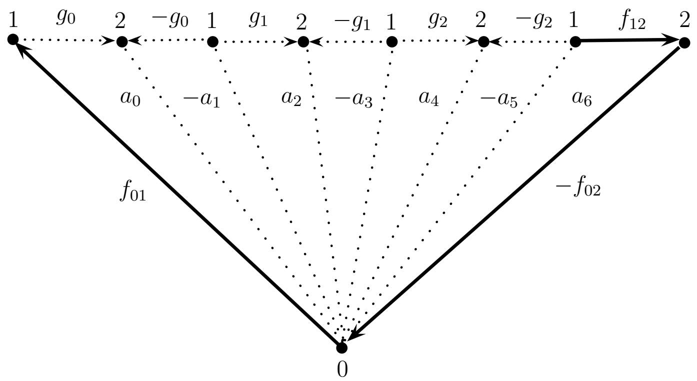

Figure 2. An example of a Lascar type 2-chain. The 0-th boundaries of adjacent 2-simplices are cancelled out in a pair after taking boundary map.

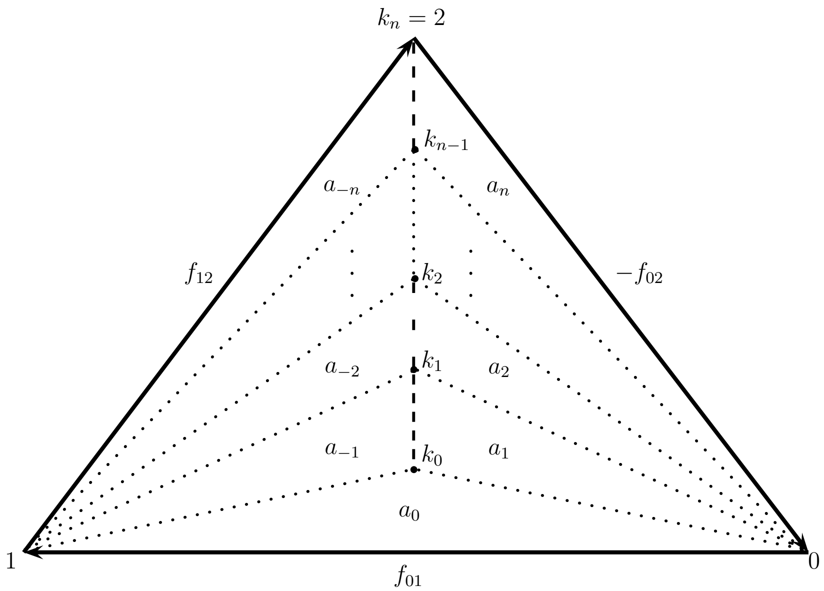

Figure 3. A tower type 2-chain (so planar).

THEOREM 4.4. *Let α be a minimal 2-chain with a 1-shell boundary $f_{12} - f_{02} + f_{01}$. The following are equivalent.*

1. *α is equivalent to a Lascar type 2-chain.*

2. *α is equivalent to a tower type 2-chain.*

3. *α is equivalent to a planar 2-chain.*

PROOF. When $|\alpha| = 1$, nothing to prove, so we assume $|\alpha| \geq 3$.

(1) $\Rightarrow$ (2) Assume $\alpha = \sum_{i=0}^{2n}(-1)^i a_i$ is a Lascar 2-chain as in Definition 4.3, with $\mathrm{supp}(a_i) = \{0, k_i, k_{i+1}\}$. Thus $k_0 = k_2 = \cdots = k_{2n} = 1$, and due to the RS-operation, we can assume that $2 < k_1 < k_3 < \cdots < k_{2n-1}$. Since $1, k_1, k_3$ are distinct, we can apply the CR-operation to $-a_1 + a_2$ and obtain $-b_{-1} + b_1$ with $\mathrm{supp}(b_{-1}) = \{1, k_1, k_3\}$ and $\mathrm{supp}(b_1) = \{0, k_1, k_3\}$. Similarly we apply the CR-operation to $-a_3 + a_4$ and obtain $-b_{-2} + b_2$ with $\mathrm{supp}(b_{-2}) = \{1, k_3, k_5\}$ and $\mathrm{supp}(b_2) = \{0, k_3, k_5\}$. Iterate this process and lastly we get $b_{-n} - b_n$ with $\mathrm{supp}(b_{-n}) = \{1, 2, k_{2n-1}\}$, $\mathrm{supp}(b_n) = \{0, 2, k_{2n-1}\}$ by applying the CR-operation to $-a_{2n-1} + a_{2n}$. Now put $b_0 := a_0$. Then it follows $b_0 + \sum_{i=1}^{n-1}(-b_{-i} + b_i) + (b_{-n} - b_n)$ is a tower type 2-chain.

(2) $\Rightarrow$ (1) This can be shown by reversely taking the process described in the proof of (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3) Clear from Figure 3.

(3) $\Rightarrow$ (2) Let the 2-chain $\alpha : \triangle_a(\mathcal{T}) \to \mathcal{C}$ be planar, where $\triangle_a(\mathcal{T})$ is an abstract simplicial complex of a closed triangular plane region $\mathcal{T}$ with exterior vertices $0, 1, 2$; and $\triangle(\mathcal{T})$ is a planar triangulation of $\mathcal{T}$. Now we prove using induction on $|\alpha|$. If $|\alpha| = 3$ then due to the comment before this theorem, we are done. So let us prove (2) when $|\alpha| = 2n + 1$ (*), with the induction hypothesis when $3 \leq |\alpha| \leq 2n - 1$ (**). We take a chain-walk β in α from $-f_{02}$ to f_{12}. (Since α is planar, such β is unique.) We prove (*) using induction on $|\beta|$.

We have $|\beta| \geq 2$ because α is planar and $|\alpha| \geq 3$. If $|\beta| = 2$, $\gamma(= \alpha - \beta)$ is again a planar 2-chain with 1-shell boundary and by (**) γ is equivalent to a tower type 2-chain γ'. So α is equivalent to $\gamma' + \beta$, which is a tower type 2-chain and we are done. Let us prove (*) when $|\beta| = m + 1$ with the induction hypothesis for $m \geq 2$ (†). Let $\beta = \sum_{i=0}^{m} \epsilon_i b_i$ with its representation.

Let $\{2, k_0, k_1, \ldots, k_{m+1}\}$ be the support of β such that $k_0 = 0$, $k_{m+1} = 1$ and $\mathrm{supp}(b_i) = \{2, k_i, k_{i+1}\}$ for $0 \leq i \leq m$ (moreover 2 and k_i's are all distinct). Then $\beta^{k_0, k_1, \ldots, k_{m+1}}$ is corresponding to a piecewise-linear graph Γ connecting two vertices 0 and 1 in $\mathcal{T}$. And we regard this graph as a graph on an interval corresponding to $\alpha^{0,1}$.

Case 1: Γ is locally concave upward : Let $(\epsilon_j b_j + \epsilon_{j+1} b_{j+1})^{k_j, k_{j+1}, k_{j+2}}$ be corresponding to a concave upward piece in Γ. Obtain $\epsilon'_j b'_j + \epsilon'_{j+1} b'_{j+1}$ by applying the CR-operation to $\epsilon_j b_j + \epsilon_{j+1} b_{j+1}$, so that $\beta' = \beta - (\epsilon_j b_j + \epsilon_{j+1} b_{j+1}) + (\epsilon'_j b'_j + \epsilon'_{j+1} b'_{j+1})$ forms a chain-walk in a planar $\alpha' := \alpha - (\epsilon_j b_j + \epsilon_{j+1} b_{j+1}) + (\epsilon'_j b'_j + \epsilon'_{j+1} b'_{j+1}) \sim \alpha$. Now $|\beta'| = m$. Hence due to (†), there is a tower type $\alpha'' \sim \alpha' \sim \alpha_0$.

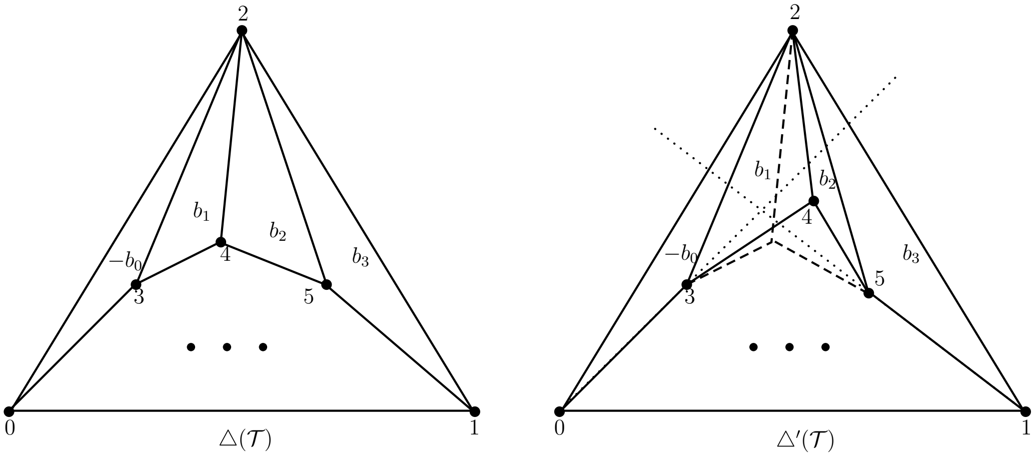

Figure 4. Two isomorphic triangulations. $(b_2 + b_3)^{1,4,5}$ corresponds to a concave upward piece in $\triangle'(\mathcal{T})$ but not in $\triangle(\mathcal{T})$.

Case 2: Γ is concave downward : There is a triangulation $\triangle'(\mathcal{T})$ isomorphic to $\triangle(\mathcal{T})$, where the corresponding graph Γ' is locally concave upward. By Remark 4.2 we may replace $\triangle(\mathcal{T})$ by $\triangle'(\mathcal{T})$ and apply the same process in the proof of Case 1 to Γ'.

$\square$

QUESTION 4.5. In an amenable category, is there an RN-type 2-chain not equivalent to a Lascar 2-chain?

Due to Theorem 4.4, it easily follows that any RN-type 2-chain of length 3 or 5 is equivalent to a Lascar type. We guess that there is an RN-type 2-chain of length 7 which is not equivalent to a Lascar 2-chain.

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