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A uniqueness of periodic maps on surfaces

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Abstract. Kulkarni showed that, if g is greater than 3, a periodic map on an oriented surface Σ_g of genus g with order not smaller than $4g$ is uniquely determined by its order, up to conjugation and power. In this paper, we show that, if g is greater than 30, the same phenomenon happens for periodic maps on the surfaces with orders more than $8g/3$, and, for any integer N, there is $g > N$ such that there are periodic maps of Σ_g of order $8g/3$ which are not conjugate up to power each other. Moreover, as a byproduct of our argument, we provide a short proof of Wiman's classical theorem: the maximal order of periodic maps of Σ_g is $4g+2$.

1. Introduction.

Let Σ_g be the oriented closed surface of genus $g \geq 2$. By the Nielsen–Thurston theory [**10**], orientation preserving homeomorphisms of Σ_g are classified into 3-types: (1) periodic, (2) reducible, (3) pseudo-Anosov. For each type, there exist several conjugacy invariants known. As for type (1), the *order* might be the most fundamental among such invariants. Kulkarni [**7**] showed that if the genus g is sufficiently large and the order is more than or equal to $4g$ then the order determines the conjugacy class of the periodic map up to power. The first author [**4**] showed the same type of result when the order is more than or equal to $3g$. In this paper, we investigate on the minimum M satisfying the following condition: *if the genus g is sufficiently large and $n > Mg$ (or $n \geq Mg$) then the order determines the conjugacy class of the periodic map up to power.*

Theorem 1.1. *Let $g > 30$, and $n > 8g/3$. If there is a periodic map of Σ_g of order n, then this map is unique up to conjugacy and power. On the other hand, let N be any positive integer. There is $g > N$ such that there are periodic maps of Σ_g of order $8g/3$ which are not conjugate up to power each other.*

We explain the outline of the proof of Theorem 1.1. By [**6**], the periodic map which satisfies the condition of Theorem 1.1 is irreducible, that is, the orbit surface of this periodic map is a 2-sphere with 3 branched points. Let n_1 be the minimum of the branching indices. In Section 3, it is shown that the order is included in one of certain disjoint ranges which is determined solely by the value of n_1 when the genus is sufficiently

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large. By using this result, we observe that n_1 should be at most 4 under the condition in Theorem 1.1. In Section 4, we discuss the uniqueness of periodic map by the order up to conjugacy and power.

It seems that our argument in Section 3, especially Theorem 3.2, is also useful for several known results on the distribution of periodic maps, in simplifying the subcases to be considered in their proofs. As an example, in Section 5, we provide a short and complete proof of Wiman's classical theorem: the maximal order of periodic maps of Σ_g is $4g+2$.

After we finished to write this paper, we were informed from Professor G. Gromadzki about their paper [**2**] and that Theorem 1.1 follows directly from their result in [**2**].

2. Preliminaries.

An orientation preserving homeomorphism f from the surface Σ_g to itself is said to be a *periodic map*, if there is a positive integer n such that $f^n = \mathrm{id}_{\Sigma_g}$. The *order* of f is the smallest positive integer which satisfies the above condition. Two periodic maps f and f' on Σ_g are *conjugate*, if there is an orientation preserving homeomorphism h from Σ_g to itself such that $f' = h \circ f \circ h^{-1}$. In this section, we will review the classification of conjugacy classes of periodic maps on surfaces by Nielsen [**8**]. We follow a description by Smith [**9**] and Yokoyama [**12**].

Let f be a periodic map on Σ_g, whose order is n. A point p on Σ_g is a *multiple point* of f, if there is a positive integer k less than n such that $f^k(p) = p$. Let M_f be the set of multiple points of f. The orbit space Σ_g/f of f is defined by identifying x in Σ_g with $f(x)$. Let $\pi_f : \Sigma_g \to \Sigma_g/f$ be the quotient map. Then π_f is an n-fold branched covering ramified at $\pi_f(M_f)$. The set $\pi_f(M_f)$ is denoted by B_f, and each element of B_f is called a *branch point* of f. We choose a point x in $\Sigma_g/f - B_f$, and a point $\tilde{x}$ in $\pi_f^{-1}(x)$. We define a homomorphism $\Omega_f : \pi_1(\Sigma_g/f - B_f) \to \mathbb{Z}_n$ as follows: Let l be a loop in $\Sigma_g/f - B_f$ with the base point x, and $[l]$ the element of $\pi_1(\Sigma_g/f - B_f)$ represented by l. Let $\tilde{l}$ be the lift of l on Σ_g which begins from $\tilde{x}$. There is a positive integer r less than or equal to n such that the terminal point of $\tilde{l}$ is $f^r(\tilde{x})$. We define $\Omega_f([l]) = r \bmod n$. Since $\mathbb{Z}_n$ is an abelian group, the homomorphism Ω_f induces a homomorphism ω_f from the abelianization of $\pi_1(\Sigma_g/f - B_f)$ to $\mathbb{Z}_n$. The abelianization of $\pi_1(\Sigma_g/f - B_f)$ is $H_1(\Sigma_g/f - B_f)$, therefore ω_f is a homomorphism from $H_1(\Sigma_g/f - B_f)$ to $\mathbb{Z}_n$. For each point of $B_f = \{Q_1, \ldots, Q_b\}$, let D_i be a disk in Σ_g/f, which contains Q_i in its interior and is sufficiently small so that no other points of B_f are in D_i. Let S_{Q_i} be the boundary of D_i with clockwise orientation.

Theorem 2.1 ([**8**, Section 11]). *Two periodic maps f and f' on Σ_g are conjugate to each other if and only if the following three conditions are satisfied.*

(1) *The order of f is equal to the order of f'.*
(2) *The number of elements in B_f is equal to that of $B_{f'}$.*
(3) *After renumbering the elements of $B_{f'}$, we have $\omega_f(S_{Q_i}) = \omega_{f'}(S_{Q_i})$ for each i.*

Let $\theta_i = \omega_f(S_{Q_i})$ for each i. By the above Theorem, the data $[g, n; \theta_1, \ldots, \theta_b]$ determines a periodic map up to conjugacy. The following proposition shows a sufficient and necessary condition for a data $[g, n; \theta_1, \ldots, \theta_b]$ to correspond to a periodic map.

PROPOSITION 2.2. *There is a periodic map with the data $[g, n; \theta_1, \ldots, \theta_b]$ if and only if the following conditions are satisfied.*

(1) $\theta_1 + \cdots + \theta_b \equiv 0 \mod n$.
(2) *Let $n_i = n/\gcd\{\theta_i, n\}$, then there exists a non-negative integer g' which satisfies*

$$2g - 2 = n\left(2g' - 2 + \sum\left(1 - \frac{1}{n_i}\right)\right),$$

where i runs through the branch points.
(3) *If $g' = 0$, then* $\gcd\{\theta_1, \ldots, \theta_b\} \equiv 1 \mod n$.

The necessity of three conditions in the above Proposition are shown as follows. (1) follows from the fact that ω_f is a homomorphism and $S_{Q_1} + \cdots + S_{Q_b}$ is null-homologous, (2) is the Riemann–Hurwitz formula, and (3) follows from the fact that ω_f is a surjection. The sufficiency of these conditions follows from the existence theorem of a branched covering space by Hurwitz [**5**]. The number n_i is called the *branching index* of Q_i.

In the following, we will use the expression $(n, \theta_1/n + \cdots + \theta_b/n)$ in place of $[g, n; \theta_1, \ldots, \theta_b]$. This data $(n, \theta_1/n + \cdots + \theta_b/n)$ is called the *total valency*, which is introduced by Ashikaga and Ishizaka [**1**]. In the above data, we call θ_i/n the *valency* of Q_i, and often rewrite this by an irreducible fraction. We remark that the denominator of the reduced θ_i/n is equal to the branching index n_i of Q_i, and the numerator of the reduced θ_i/n is well-defined modulo n_i. If k is an integer prime to n and $f = (n, m_1/n_1 + \cdots + m_b/n_b)$, then $f^k = (n, (k^* \cdot m_1)/n_1 + \cdots + (k^* \cdot m_b)/n_b)$ where k^* is an integer such that $k \cdot k^* \equiv 1 \bmod n$, and $k^* \cdot m_i$ is the remainder of $k^* m_i$ modulo n_i.

3. A discussion on branching indices.

Let f be an order n periodic map of Σ_g whose orbit space $\Sigma_g/f = \mathbb{S}^2(n_1, n_2, n_3)$. We assume that $n_1 \leq n_2 \leq n_3$. By the Riemann–Hurwitz formula, we see

$$2(g-1) = n\left(1 - \left(\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}\right)\right). \tag{1}$$

The branching indices n_1, n_2, n_3 satisfy the Harvey's lcm condition [**3**], that is,

$$\operatorname{lcm}\{n_1, n_2\} = \operatorname{lcm}\{n_2, n_3\} = \operatorname{lcm}\{n_3, n_1\} = n.$$

LEMMA 3.1. *Let $k_2 = n/n_2$, $k_3 = n/n_3$, then we see*:

(i) $n = (n_1/(n_1 - 1))(2g + (k_2 + k_3 - 2))$,
(ii) $k_2 \geq k_3$,
(iii) k_2, k_3 *are divisors of* n_1,
(iv) $\gcd\{k_2, k_3\} = 1$,
(v) $k_2 + k_3 \leq n_1 + 1$.

Proof. (i) is valid by (1).
(ii) is valid by $n_2 \leq n_3$.
(iii) Since $k_2 n_2 = \mathrm{lcm}\{n_1, n_2\}$, k_2 is a divisor of n_1. Similarly k_3 is a divisor of n_1.
(iv) Since $k_2 n_2 = k_3 n_3 = n = \mathrm{lcm}\{n_2, n_3\}$, k_2 and k_3 are prime each other.
(v) If $n_1 = 2$, then we see $k_2, k_3 \leq 2$ by (iii). Since $k_3 = 1$ by (iv), we see $k_2 + k_3 \leq n_1 + 1$. We assume $n_1 \geq 3$. If $k_2 = n_1$ then $k_3 = 1$ by (iii) (iv). Therefore $k_2 + k_3 = n_1 + 1$. If $k_2 \neq n_1$ then $k_2 \leq n_1/2$. Moreover $k_3 \leq k_2$ by (ii), hence, we see $k_2 + k_3 \leq 2k_2 \leq n_1 < n_1 + 1$. □

Theorem 3.2. *The inequality* $(2n_1/(n_1-1))g \leq n \leq (2n_1/(n_1-1))g + n_1$ *is valid.*

Proof. Since $n_2, n_3 \leq n$,

$$2(g-1) = n\left(1 - \left(\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}\right)\right) \leq n\left(1 - \left(\frac{1}{n_1} + \frac{2}{n}\right)\right) = n\left(\frac{n_1 - 1}{n_1}\right) - 2$$

by the equation (1), hence,

$$n \geq \frac{2n_1}{n_1 - 1} g.$$

On the other hand, by (i) (v) of the previous lemma, we see

$$n = \frac{n_1}{n_1 - 1}(2g + (k_2 + k_3 - 2)) \leq \frac{2n_1}{n_1 - 1} g + n_1.$$ □

Theorem 3.3. *For an integer* $N \geq 3$, *we assume* $g > (N-1)N(N+1)/2$. *Then*

$$n_1 = N \Longleftrightarrow \frac{2N}{N-1} g \leq n < \frac{2(N-1)}{N-2} g.$$

Remark 3.4. 1. Because $(N-1)N(N+1)/2$ is increasing for $N \geq 3$, we see that, for any N' such that $3 \leq N' \leq N$,

$$n_1 = N' \Longleftrightarrow \frac{2N'}{N'-1} g \leq n < \frac{2(N'-1)}{N'-2} g$$

under the assumption of the above theorem.

2. In the case where $g = (N-1)N(N+1)/2$, there is a periodic map of order $(2N/(N-1))g = N^2(N+1)$ such that $n_1 = N+1$ and whose valency data is

$$\left(N^2(N+1), \frac{N}{N+1} + \frac{N-1}{N^2} + \frac{1}{N^2(N+1)}\right).$$

Proof. We assume $n_1 = N$. By Theorem 3.2, $(2N/(N-1))g \leq n \leq (2N/(N-1))g + N$. By the assumption $g > (N-1)N(N+1)/2 > (N-2)(N-1)N/2$, we see $(2N/(N-1))g + N < (2(N-1)/(N-2))g$. Therefore $(2N/(N-1))g \leq n < (2(N-1)/(N-2))g$.

On the reverse order, we assume $(2N/(N-1))g \leq n < (2(N-1)/(N-2))g$. By Theorem 3.2, we see $n \geq (2n_1/(n_1-1))g$. If $n_1 \leq N-1$ then $2n_1/(n_1-1) \geq 2(N-1)/(N-2)$, hence $n \geq (2(N-1)/(N-2))g$, which contradicts $(2N/(N-1))g \leq n < (2(N-1)/(N-2))g$. Therefore $n_1 \geq N$.

Here, we show the following lemma.

Lemma 3.5. *If $n \geq (2N/(N-1))g$ and $N \geq 2$, then $n_1 \leq 3N-1$.*

Proof. By the equation (1)

$$\frac{2g-2}{n} = 1 - \left(\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}\right).$$

By the assumption that $n_1 \leq n_2 \leq n_3$ and $n \geq (2N/(N-1))g$, we see

$$\frac{(2g-2)(N-1)}{2Ng} \geq 1 - \frac{3}{n_1}.$$

By this inequality, we get an evaluation $n_1 \leq (3g/(g+N-1))N < 3N$. Therefore, we conclude $n_1 \leq 3N-1$. $\square$

Here we assume that $n_1 \geq N+1$. By Theorem 3.2 and the assumption $n \geq (2N/(N-1))g$, we see

$$\frac{2N}{N-1}g \leq \frac{2n_1}{n_1-1}g + n_1.$$

By the above inequality and the inequality $2N/(N-1) > 2n_1/(n_1-1)$ obtained from the assumption $n_1 \geq N+1$, we see

$$g \leq \frac{(N-1)n_1(n_1-1)}{2(n_1-N)}. \tag{2}$$

By the above lemma, $n_1 \leq 3N-1$. When $N \geq 3$, by the inequality $3N-1 \leq N^2$ and the assumption $n_1 \geq N+1$, we see $N+1 \leq n_1 \leq N^2$. From this inequality, $(n_1-(N+1))(n_1-N^2) \leq 0$, that is, $n_1^2 - (N^2+N+1)n_1 + N^2(N+1) \leq 0$, hence $n_1^2 - n_1 \leq N(N+1)(n_1-N)$. By dividing the last inequality by $n_1 - N > 0$, we obtain

$$\frac{n_1(n_1-1)}{n_1-N} \leq N(N+1).$$

By this inequality and the inequality (2), we see

$$g \leq \frac{(N-1)N(N+1)}{2},$$

which contradicts the assumption. Therefore $n_1 \leq N$.

We get a conclusion $n_1 = N$. □

4. The uniqueness by the order.

Theorem 1.1. *Let $g > 30$ and $n > 8g/3$. If there is a periodic map whose order is n, then this periodic map is unique up to conjugacy and power.*

Remark 4.1. In the sentence, *if the genus g is sufficiently large and $n > Mg$ (or $n \geq Mg$) then the order determines the conjugacy class of the periodic map up to power,* the condition of the order $n > 8g/3$ is best possible. In fact, when the genus $g = 3(2k+1)$, there are two periodic map of order $n = 8g/3$ whose total valencies are

$$\frac{1}{4} + \frac{1}{16k+8} + \frac{12k+5}{16k+8}, \qquad \frac{3}{4} + \frac{1}{16k+8} + \frac{4k+1}{16k+8}$$

and any power of the first one is not conjugate to the second one.

Proof. By [**6**], the periodic map f satisfying the condition in this theorem is irreducible, that is $\Sigma_g/f = \mathbb{S}^2(n_1, n_2, n_3)$. As in Section 3, we assume that $n_1 \leq n_2 \leq n_3$. Since we already discussed the case where $n \geq 3g$ in [**7**] and [**4**], we assume that $n < 3g$. Since $g > 30 = 4 \cdot (4^2 - 1)/2$, $(2 \cdot (4-1)/(4-2))g = 3g > n > (8/3)g = (2 \cdot 4/(4-1))g$, $n_1 = 4$ by Theorem 3.3. Let $k_2 = n/n_2$ and $k_3 = n/n_3$. By Lemma 3.1, there are 3 cases $(k_2, k_3) = (4,1)$, $(2,1)$, $(1,1)$.

(1) $(k_2, k_3) = (4,1)$: By (i) of Lemma 3.1, the order $n = (8/3)g + 4 > (8/3)g$. Since n is an integer, g should be a multiple of 3. Let $g = 3l$, then $n = 8l+4$, $n_2 = n/k_2 = 2l+1$, and $n_3 = n/k_3 = 8l+4$. We determine the numerators a, b, c of the valency data

$$\frac{a}{4} + \frac{b}{2l+1} + \frac{c}{8l+4}.$$

Since the branch point corresponding to $c/(8l+4)$ is the image of a fixed point of f by π_f, we fix $c = 1$ by taking a proper power of the periodic map f. Since $a/4$ is an irreducible fraction, $a = 1$ or 3. If $a = 3$, then $2b = l$. If $a = 1$, then $2b = 3l + 1$.

When l is even, we put $l = 2m$. If $a = 3$, then $b = m$. If $a = 1$, then $2b = 6m + 1$. Therefore, $a = 3$ and the total valency should be

$$\frac{3}{4} + \frac{m}{4m+1} + \frac{1}{16m+4}.$$

When l is odd, we put $l = 2m+1$. If $a = 3$, then $2b = 2m+1$. If $a = 1$, then $b = 3m+2$. Therefore, $a = 1$ and the total valency should be

$$\frac{1}{4} + \frac{3m+2}{4m+3} + \frac{1}{16m+12}.$$

(2) $(k_2, k_3) = (2,1)$: By (i) of Lemma 3.1, the order $n = 4(2g+1)/3 > (8/3)g$.

Since n is an integer, $2g+1$ should be a multiple of 3. Therefore $g \equiv 1 \mod 3$. Let $g = 3l+1$, then $n = 8l+4$, $n_2 = 4l+2$, and $n_3 = 8l+4$. We determine the numerators a, b, c of the valency data

$$\frac{a}{4} + \frac{b}{4l+2} + \frac{c}{8l+4}.$$

Since the branch point corresponding to $c/(8l+4)$ is the image of a fixed point of f by π_f, we fix $c = 1$ by taking a proper power of the periodic map f. Since $a/4$ is an irreducible fraction, $a = 1$ or 3. If $a = 1$, then $b = 3l+1$. If $a = 3$, then $b = l$. Since $b/(4l+2)$ is also an irreducible fraction, b should be an odd integer. If l is an even integer, $b = 3l+1$ and $a = 1$. Hence, the total valency should be

$$\frac{1}{4} + \frac{3l+1}{4l+2} + \frac{1}{8l+4}.$$

If l is an odd integer, $b = l$ and $a = 3$. Hence, the total valency should be

$$\frac{3}{4} + \frac{l}{4l+2} + \frac{1}{8l+4}.$$

(3) $(k_2, k_3) = (1, 1)$: By (i) of Lemma 3.1, the order $n = (8/3)g$, which contradicts the condition $n > (8/3)g$. □

By the proof of the above theorem, [**7**] and [**4**] we see:

Corollary 4.2. *Let $g > 30$ and $n > 8g/3$. If there is a periodic map f of Σ_g whose order is n, then f is conjugate to a power of one of periodic maps listed on Table 1.*

5. A proof of Wiman's Theorem [11].

We provide a short proof of Wiman's Theorem [**11**] using the argument in Section 3.

Theorem 5.1. *When $g \geq 2$, the order of any periodic map of Σ_g is at most $4g+2$.*

In the following, it is the subcase III)-iii) which is simplified by the argument mentioned above and seems to have been most involved. While the treatment of the other subcases is standard, we include them for completeness.

Proof. Let n be the order of a periodic map f of Σ_g, and $\Sigma_g/f = \Sigma_{g'}(n_1, \ldots, n_j)$, where $n_1 \leq n_2 \leq \cdots \leq n_j$. By the Riemann–Hurwitz formula,

$$\frac{2(g-1)}{n} = 2(g'-1) + j - \left(\frac{1}{n_1} + \cdots + \frac{1}{n_j}\right). \tag{3}$$

Table 1.

genus g	total valency
arbitrary	$\left(4g+2,\ \frac{1}{2}+\frac{g}{2g+1}+\frac{1}{4g+2}\right)$
arbitrary	$\left(4g,\ \frac{1}{2}+\frac{2g-1}{4g}+\frac{1}{4g}\right)$
$3k$	$\left(3g+3,\ \frac{2}{3}+\frac{k}{g+1}+\frac{1}{3g+3}\right)$
$3k+1$	$\left(3g+3,\ \frac{1}{3}+\frac{2k+1}{g+1}+\frac{1}{3g+3}\right)$
$3k$ or $3k+1$	$\left(3g,\ \frac{1}{3}+\frac{2g-1}{3g}+\frac{1}{3g}\right)$
$3k+2$	$\left(3g,\ \frac{2}{3}+\frac{g-1}{3g}+\frac{1}{3g}\right)$
$6m$	$\left(\frac{8}{3}g+4,\ \frac{3}{4}+\frac{m}{4m+1}+\frac{1}{16m+4}\right)$
$6m+3$	$\left(\frac{8}{3}g+4,\ \frac{1}{4}+\frac{3m+2}{4m+3}+\frac{1}{16m+12}\right)$
$6m+1$	$\left(\frac{4(2g+1)}{3},\ \frac{1}{4}+\frac{6m+1}{8m+2}+\frac{1}{16m+4}\right)$
$6m+4$	$\left(\frac{4(2g+1)}{3},\ \frac{3}{4}+\frac{2m+1}{8m+6}+\frac{1}{16m+12}\right)$

I) When $g' \geq 2$, the RHS of (3) $\geq 2(g'-1) \geq 2$. Therefore $2(g-1)/n \geq 2$, that is, $g-1 \geq n$, then we see $n \leq 4g+2$.

II) We discuss the case where $g' = 1$. If $j = 0$, then $g = 1$, which contradicts the assumption $g \geq 2$. Hence, $j \geq 1$. Since each $n_i \geq 2$, we see $1/n_1 + \cdots + 1/n_j \leq j/2$. Therefore, the RHS of (3) $\geq j - j/2 = j/2$, hence $2(g-1)/n \geq j/2$, and we see $n \leq 4(g-1)/j$. This shows that $n \leq 4g+2$ in this case.

III) We discuss the case where $g' = 0$. We first note that Proposition 2.2 implies $j \geq 3$.

i) $j \geq 5$: In this case, the RHS of (3) $\geq -2 + j/2 \geq 1/2$, hence $n \leq 4(g-1)$. Therefore, $n \leq 4g+2$.

ii) $j = 4$: We multiply n on both sides of (3) and change n/n_i into k_i for $i \neq 1$, then we see,

$$
\begin{aligned}
n &= \frac{n_1}{2n_1-1}(2g-2+k_2+k_3+k_4) \\
&= \frac{n_1}{2n_1-1}(2g-2) + \frac{1}{2n_1-1}(n_1k_2+n_1k_3+n_1k_4) \\
&\leq \frac{n_1}{2n_1-1}(2g-2) + \frac{3n}{2n_1-1}.
\end{aligned}
$$

In the last inequality, we use $n_1k_i \le n_ik_i = n$. In the case where $n_1 \ge 3$, since

$$n - \frac{3n}{2n_1-1} = n\left(1 - \frac{3}{2n_1-1}\right) = n\frac{2(n_1-2)}{2n_1-1},$$

we obtain

$$n \le \frac{n_1}{n_1-2}(g-1).$$

Because $n_1/(n_1-2) \le 3$, we have $n \le 3(g-1) < 4g+2$. In the case where $n_1 = 2$, we assume that $n \ge 4g+2 \ge 4\cdot 2+2 = 10$. By Harvey's lcm condition [**3**], we remark

$$\begin{aligned} n &= \mathrm{lcm}\{n_1,n_2,n_3\} = \mathrm{lcm}\{n_1,n_2,n_4\} \\ &= \mathrm{lcm}\{n_1,n_3,n_4\} = \mathrm{lcm}\{n_2,n_3,n_4\}. \end{aligned} \tag{4}$$

If all n_i are 2 or 3, then $n \le 6$ by (4) which contradicts the assumption $n \ge 10$. Therefore, there are some n_i's such that $n_i \ge 4$. By (4), there are at least 2 n_i's such that $n_i \ge 4$, hence $n_3, n_4 \ge 4$. We see $n_1k_3 = (n_1/n_3)n_3k_3 = (n_1/n_3)n \le n/2$, and in the same way, we see $n_1k_4 \le n/2$. Therefore,

$$\begin{aligned} n &= \frac{n_1}{2n_1-1}(2g-2) + \frac{1}{2n_1-1}(n_1k_2+n_1k_3+n_1k_4) \\ &\le \frac{n_1}{2n_1-1}(2g-2) + \frac{1}{2n_1-1}\left(n+\frac{n}{2}+\frac{n}{2}\right) = \frac{4g-4}{3} + \frac{2}{3}n. \end{aligned}$$

Hence we have $n \le 4g-4$ contradicting the assumption $n \ge 4g+2$. We conclude $n < 4g+2$.

iii) $j = 3$: At first, we show the following lemma:

Lemma 5.2. *If the orbit space of a periodic map f of Σ_g is a 2-sphere with 3 branch points, and let n_1 be the minimal branching index, then $n_1 \le 2g+1$.*

Proof. By (3), we see

$$2(g-1) = n\left(1 - \left(\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}\right)\right) = n - (k_1+k_2+k_3).$$

From the above equation, we have

$$n = 2g + (k_1+k_2+k_3) - 2 \ge 2g+1. \tag{5}$$

On the other hand, by the assumption $n_1 \le n_2 \le n_3$, we see $1/n_3 \le 1/n_2 \le 1/n_1$. It follows that $1 - (1/n_1+1/n_2+1/n_3) \ge 1 - 3/n_1$, and by the equation (3),

$$2(g-1)=n\left(1-\left(\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}\right)\right)\geq n\left(1-\frac{3}{n_1}\right).$$

If $(1-3/n_1)\leq 0$, then $n_1\leq 3$. Since $g\geq 2$, we see $2g+1\geq n_1$. If $(1-3/n_1)>0$, by the equation (5), we have

$$2(g-1)\geq n\left(1-\frac{3}{n_1}\right)\geq (2g+1)\left(1-\frac{3}{n_1}\right).$$

Therefore $2g+1\geq n_1$. □

By Theorem 3.2, we have

$$n\leq \frac{2n_1}{n_1-1}g+n_1.$$

By this inequality, we see

$$\begin{aligned}(4g+2)-n\geq (4g+2)-\left(\frac{2n_1}{n_1-1}g+n_1\right)&=\left(4-\frac{2n_1}{n_1-1}\right)g+(2-n_1)\\ =\frac{2n_1-4}{n_1-1}g+(2-n_1)&=(n_1-2)\left(\frac{2}{n_1-1}g-1\right).\end{aligned}$$

Since the branching index is at least 2, $n_1-2\geq 0$. By Lemma 5.2 we have $n_1\leq 2g+1$, that is, $(2/(n_1-1))g-1\geq 0$. By the above inequality, we conclude that $4g+2\geq n$. □

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