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Identity families of multiple harmonic sums and multiple zeta star values

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Abstract. In this paper we present many new families of identities for multiple harmonic sums using binomial coefficients. Some of these generalize a few recent results of Hessami Pilehrood, Hessami Pilehrood and Tauraso [Trans. Amer. Math. Soc. **366** (2014), pp. 3131–3159]. As applications we prove several conjectures involving multiple zeta star values (MZSV): the Two-one formula conjectured by Ohno and Zudilin, and a few conjectures of Imatomi et al. involving **2**-3-**2**-1 type MZSV, where the boldfaced **2** means some finite string of 2's.

1. Introduction.

For over two hundred years, Euler's pioneering work on double zeta values [**5**] was largely neglected, until in the early 1990s when Zagier showed the importance of the more general multiple zeta values in his famous paper [**25**]. Since then these values have come up in many areas of current research in mathematics and physics, such as knot theory, motivic theory, mirror symmetry and Feynman integrals, to name just a few. One of the central problems is to determine various $\mathbb{Q}$-linear relations among these values, many of which have been discovered numerically first and then proved rigorously later. One such family that still defies a proof until now is the celebrated Two-one formula discovered by Ohno and Zudilin [**19**].

The main goal of this paper is to give a comprehensive study of multiple zeta star values of a few special types using the corresponding identities established first for multiple harmonic sums. As one of the applications, we give a concise proof of the Two-one formula.

We now recall some definitions. In order to unify MHS, MZV and their alternating versions we first define a sort of double cover of the set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ where $\mathbb{N}$ is the set of positive integers.

Definition 1.1. Let $\mathbb{D}_0 := \mathbb{N}_0 \cup \overline{\mathbb{N}}_0$ and $\mathbb{D} := \mathbb{N} \cup \overline{\mathbb{N}}$ be the sets of *signed nonnegative* and *signed positive numbers*, respectively, where

$$\overline{\mathbb{N}}_0 = \{\bar{k} : k \in \mathbb{N}_0\} \quad \text{and} \quad \overline{\mathbb{N}} = \{\bar{k} : k \in \mathbb{N}\}.$$

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In some sense, $\bar{k}$ is k dressed by a negative sign, but $\bar{k}$ is not a negative number. Define for all $k \in \mathbb{N}_0$ the absolute value function $|\cdot|$ on $\mathbb{D}_0$ by $|k| = |\bar{k}| = k$ and the sign function by $\operatorname{sgn}(k) = 1$ and $\operatorname{sgn}(\bar{k}) = -1$. We make $\mathbb{D}_0$ a semi-group by defining a commutative and associative binary operation $\oplus$ (called *O-plus*) as follows: for all $a, b \in \mathbb{D}_0$

$$a \oplus b = \begin{cases} \overline{|a| + |b|}, & \text{if only one of } a \text{ or } b \text{ is in } \mathbb{N}_0; \\ |a| + |b|, & \text{if } a, b \in \mathbb{N}_0 \text{ or if } a, b \in \overline{\mathbb{N}}_0. \end{cases} \tag{1}$$

For any $\ell \in \mathbb{N}$ and $\boldsymbol{s} = (s_1, s_2, \dots, s_\ell) \in \mathbb{D}^\ell$ we define the (*alternating*) *multiple harmonic sum* (MHS for short)

$$H_n(s_1, s_2, \dots, s_\ell) = \sum_{n \ge k_1 > k_2 > \cdots > k_\ell \ge 1} \prod_{i=1}^{\ell} \frac{\operatorname{sgn}(s_i)^{k_i}}{k_i^{|s_i|}}, \tag{2}$$

$$H_n^\star(s_1, s_2, \dots, s_\ell) = \sum_{n \ge k_1 \ge k_2 \ge \cdots \ge k_\ell \ge 1} \prod_{i=1}^{\ell} \frac{\operatorname{sgn}(s_i)^{k_i}}{k_i^{|s_i|}}. \tag{3}$$

This star-version has been denoted by S_n in the literature but it seems to be more appropriate to use $H^\star$ in this paper due to its close connection with multiple zeta star values to be defined momentarily. Conventionally, we call $\ell(\boldsymbol{s}) := \ell$ the depth and $|\boldsymbol{s}| := \sum_{i=1}^{\ell} |s_i|$ the weight. For convenience we set $H_n(\boldsymbol{s}) = 0$ if $n < l(\boldsymbol{s})$, $H_n(\emptyset) = H_n^\star(\emptyset) = 1$ for all $n \ge 0$, and $\{s_1, s_2, \dots, s_\ell\}^r$ the set formed by repeating the composition $(s_1, s_2, \dots, s_\ell)$ exactly r times.

When $\boldsymbol{s} = (s_1, s_2, \dots, s_\ell) \in \mathbb{D}^\ell$ with $(s_1, \operatorname{sgn}(s_1)) \neq (1, 1)$ we set, respectively, the (*alternating*) *Euler sum* and the (*alternating*) *star Euler sum* by

$$\zeta(\boldsymbol{s}) = \lim_{n \to \infty} H_n(\boldsymbol{s}), \qquad \zeta^\star(\boldsymbol{s}) = \lim_{n \to \infty} H_n^\star(\boldsymbol{s}). \tag{4}$$

When $\boldsymbol{s} \in \mathbb{N}^\ell$ they are called the *multiple zeta value* (MZV) and the *multiple zeta star value* (MZSV), respectively.

We now state the Two-one formula conjectured by Ohno and Zudilin [**19**].

THEOREM 1.2. *Let $r \in \mathbb{N}$ and $\boldsymbol{s} = (\{2\}^{a_1}, 1, \dots, \{2\}^{a_r}, 1)$ where $a_1 \in \mathbb{N}$ and $a_j \in \mathbb{N}_0$ for all $j \ge 2$. Then we have*

$$\zeta^\star(\boldsymbol{s}) = \sum_{\boldsymbol{p}} 2^{\ell(\boldsymbol{p})} \zeta(\boldsymbol{p}),$$

where $\boldsymbol{p}$ runs through all indices of the form $(2a_1 + 1) \circ \cdots \circ (2a_r + 1)$ with "$\circ$" being either the symbol "," or the symbol "+".

Until recently, not many nontrivial families of identities relating MZVs or MZSVs with truly alternating Euler sums have been proved. One such result is proved by Zlobin [**32**]

$$\zeta^\star(\{2\}^n) = -2\zeta(\overline{2n}) \quad \text{for all } n \geq 1. \tag{5}$$

Another is proved in [**30**]: $\zeta(\{3\}^n) = 8^n\zeta(\{\overline{2}, 1\}^n)$. Recently, two more appear as (27) and (28) of [**7**] one of which yields a new proof of an identity of [**26**]. Notice that [**7**, (22)] implies (5) easily (see Lemma 4.5). To provide more such families in this paper we need some book-keeping first. A boldface of a single digit number means the number is repeated a few times. We underline a string pattern to mean the whole pattern is repeated. Thus the Two-one formula should be written as $\underline{\mathbf{2}\text{-}1}$ formula and in each repetition the **2**'s may have different lengths.

Besides the $\underline{\mathbf{2}\text{-}1}$ formula in Theorem 1.2 we show many analogous formulas in this paper. For example, the following is the $\underline{\mathbf{2}\text{-}1}$-**2** *formula*.

THEOREM 1.3. *Let $r \in \mathbb{N}$ and $\boldsymbol{s} = (\{2\}^{a_1}, 1, \dots, \{2\}^{a_r}, 1, \{2\}^{a_{r+1}})$ where $a_1, a_{r+1} \in \mathbb{N}$ and $a_j \in \mathbb{N}_0$ for all $2 \leq j \leq r$. Then we have*

$$\zeta^\star(\boldsymbol{s}) = -\sum_{\boldsymbol{p}} 2^{\ell(\boldsymbol{p})}\zeta(\boldsymbol{p}),$$

where $\boldsymbol{p}$ runs through all indices of the form $(2a_1 + 1) \circ \cdots \circ (2a_r + 1) \circ \overline{2a_{r+1}}$ with "$\circ$" being either the symbol "," or the symbol O-plus "$\oplus$" defined by (1).

When $r = 2$, we have checked numerically the following identities for all $0 \leq a, b, c \leq 2$ and $ac \neq 0$ with the help of EZ-face [**3**]:

$$\begin{aligned}\zeta^\star(\{2\}^a, 1, \{2\}^b, 1, \{2\}^c) = &-2\zeta(\overline{2(a+b+c)+2}) - 4\zeta(2a+2+2b, \overline{2c}) \\ &- 4\zeta(2a+1, \overline{2b+1+2c}) - 8\zeta(2a+1, 2b+1, \overline{2c}).\end{aligned}$$

The main idea is to find out what happens when a new component is attached to the front of a composition of positive integers. To state our main theorems we need some additional notations first. For $\boldsymbol{s} = (s_1, \dots, s_\ell) \in \mathbb{D}_0^\ell$ we define the mollified companion of $H_n(\boldsymbol{s})$ by

$$\begin{aligned}\mathcal{H}_n(\boldsymbol{s}) &:= 2^\ell \sum_{n \geq k_1 > \cdots > k_\ell \geq 1} \frac{\binom{n}{k_1}}{\binom{n+k_1}{k_1}} \prod_{j=1}^{\ell} \frac{\operatorname{sgn}(s_j)^{k_j}}{k_j^{|s_j|}} \\ &= 2^\ell \sum_{k=1}^{n} \frac{\operatorname{sgn}(s_1)^k}{k^{|s_1|}} \frac{\binom{n}{k}}{\binom{n+k}{k}} H_{k-1}(s_2, \dots, s_\ell).\end{aligned} \tag{6}$$

We further define $\Pi(\boldsymbol{s})$ to be the set of all indices of the form $(s_1 \circ \cdots \circ s_\ell)$ where "$\circ$" being either the symbol "," or the symbol O-plus"$\oplus$" defined by (1). We also define a sequence of compositions of nonzero integers $\boldsymbol{s}^i$ $(i = \ell, \dots, 2, 1)$ (backward) inductively if $\boldsymbol{s} \in \mathbb{N}^\ell$:

$$\boldsymbol{s}^{(\ell)} = \begin{cases} (1), & \text{if } s_\ell = 1; \\ (\overline{2}, \{1\}^{s_m-2}), & \text{if } s_\ell \geq 2, \end{cases}$$

$$\boldsymbol{s}^{(i)} = \begin{cases} (1)\cdot \boldsymbol{s}^{(i+1)}, & \text{if } s_i = 1;\\ (2)\oplus \boldsymbol{s}^{(i+1)}, & \text{if } s_i = 2;\\ (\bar{2}, \{1\}^{s_m-3}, \bar{1})\oplus \boldsymbol{s}^{(i+1)}, & \text{if } s_i \geq 3, \end{cases}$$

for all $i < \ell$. Here, for any two compositions $\boldsymbol{a} = (a_1, \dots, a_r)$, $\boldsymbol{b} = (b_1, \dots, b_t)$

$$(1)\cdot \boldsymbol{a} = (1, a_1, \dots, a_r), \quad \boldsymbol{a}\oplus\boldsymbol{b} = (a_1, \dots, a_{r-1}, a_r \oplus b_1, b_2, \dots, b_t).$$

THEOREM 1.4. *For all* $\boldsymbol{s} = (s_1, \dots, s_\ell) \in \mathbb{N}^\ell$ *we have*

$$H_n^\star(\boldsymbol{s}) = \varepsilon(\boldsymbol{s}) \sum_{\boldsymbol{p}\in\Pi(\boldsymbol{s}^{(1)})} \mathcal{H}_n(\boldsymbol{p}), \quad \textit{where } \varepsilon(\boldsymbol{s}) = \begin{cases} 1, & \textit{if } s_\ell = 1;\\ -1, & \textit{if } s_\ell \geq 2. \end{cases}$$

Applying this theorem repeatedly we can derive many different types of identities. For ease of reference we list them as follows:

1. **2**-1: Corollary 4.1 for MHS, Theorem 1.2 for MZSV;
2. **2**-1-**2** (nontrivial substring **2** at the end): Corollary 4.3 for MHS, Theorem 1.3 for MZSV;
3. **2**-c-**2** ($c \geq 3$ and **2** at the end may be trivial): appeared in a joint work with my student Erin Linebarger [**16**];
4. **2**-c-**2**-1 and **2**-1-**2**-c-**2**-1: Corollary 4.7 for MHS, Theorem 4.8 for MZSV;
5. **2**-c-**2**-1-**2** and **2**-1-**2**-c-**2**-1-**2** (nontrivial **2** at the end): Corollary 4.9 for MHS, Theorem 4.10 for MZSV;
6. **2**-1-**2**-c-**2** and **2**-c-**2**-1-**2**-c-**2** (**2** at the end may be trivial): Corollary 4.11 for MHS, Theorem 4.12 for MZSV;
7. **1**-c-**1** ($c \geq 1$ and **1** at the end may be trivial): Theorem 7.1 for MHS.

One of the main results contained in [**7**, Theorem 2.3] is the following theorem.

THEOREM 1.5 ([**7**, Theorem 2.3]). *Let* $a \in \mathbb{N}_0$ *and* $b \in \mathbb{N}$. *Then for any* $n \in \mathbb{N}$

$$H_n^\star(\{2\}^a, 1) = 2\sum_{k=1}^n \frac{\binom{n}{k}}{k^{2a+1}\binom{n+k}{k}}, \tag{A}$$

$$H_n^\star(\{2\}^a, 1, \{2\}^b) = -2\sum_{k=1}^n \frac{(-1)^k\binom{n}{k}}{k^{2a+1+2b}\binom{n+k}{k}} - 4\sum_{k=1}^n \frac{H_{k-1}(\overline{2b})\binom{n}{k}}{k^{2a+1}\binom{n+k}{k}}. \tag{B}$$

We want to caution the reader that the convention of index ordering in [**7**] is opposite to ours in the definitions (2) and (3) of MHS. This is the reason why a and b in Theorem 1.5(B) is switched from the original statement in [**7**, Theorem 2.3].

Corollary 4.1 and Corollary 4.3 generalize Theorem 1.5(A) and (B) respectively by allowing the arguments to contain an arbitrary number of 2-strings, which lead to the **2**-1 and **2**-1-**2** formulas for MHS. The proofs are straight-forward, however, the difficult part is the discovery of the corollaries and the theorems (using a lot of Maple experiments).

By taking limits in these two theorems so that MHS become MZSV we can prove the <u>**2**-1</u> formula and the <u>**2**-1</u>-**2** formula for MZSV in Theorems 1.2 and 1.3, respectively.

In Section 5 and Section 6 we provide new and concise proofs of a few conjectures first formulated by Imatomi et al. in [**13**] concerning MZSV of types <u>**2**-3-**2**-1</u> and <u>**2**-3-**2**-1</u>-**2**.

In the last section we propose a few possible future research directions, one of which will be carried out in a sequel to this work in which we will study congruence properties of MHS as further applications of the results we have obtained in this paper.

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2. A combinatorial lemma.

In this short section we prove the following combinatorial identities which are similar in spirit to [**7**, Lemma 2.2].

LEMMA 2.1. *Let $k, n \in \mathbb{N}$, $a \in \mathbb{N}_0$, $c \in \mathbb{N}$, $A_{n,k}^{(m)} = (-1)^k \binom{mn}{n-k} c_n^{(m)}$ where $c_n^{(m)}$ is an arbitrary sequence independent of k, and $B_{n,k}^{(m)} = \binom{mn}{n-k} c_n^{(m)}$. Then for any composition $\boldsymbol{v}$ the following statements hold.*

(i) *We have*

$$\frac{1}{n^c} \sum_{k=1}^{n} \frac{H_{k-1}(\boldsymbol{v}) A_{n,k}^{(m)}}{k^a} = \sum_{k=1}^{n} \frac{H_{k-1}(\boldsymbol{v}) A_{n,k}^{(m)}}{k^{a+c}} + \sum_{\substack{j+|\boldsymbol{x}|=a+c \\ j \geq 0, x_r > a}} m^{l(\boldsymbol{x})} \sum_{k=1}^{n} \frac{H_{k-1}(\boldsymbol{x}, \boldsymbol{v}) A_{n,k}^{(m)}}{k^j}, \quad (7)$$

where x_r denotes the last component of $\boldsymbol{x} \in \mathbb{N}^r$.

(ii) *We have*

$$n \sum_{k=1}^{n} \frac{H_{k-1}(\boldsymbol{v}) B_{n,k}^{(2)}}{k^a} = \sum_{k=1}^{n} \frac{H_{k-1}(\boldsymbol{v}) B_{n,k}^{(2)}}{k^{a-1}} + 2 \sum_{k=1}^{n} H_{k-1}(a, \boldsymbol{v}) k B_{n,k}^{(2)}. \quad (8)$$

(iii) *We have*

$$\frac{1}{n^c} \sum_{k=1}^{n} \frac{H_{k-1}(\boldsymbol{v}) B_{n,k}^{(m)}}{k^a} = \sum_{k=1}^{n} \frac{H_{k-1}(\boldsymbol{v}) B_{n,k}^{(m)}}{k^{a+c}} + \sum_{\substack{j+|\boldsymbol{x}|=a+c \\ j \geq 0, x_r < -a}} m^{l(\boldsymbol{x})} \sum_{k=1}^{n} \frac{H_{k-1}(\boldsymbol{x}, \boldsymbol{v}) A_{n,k}^{(m)}}{k^j}, \quad (9)$$

where x_r denotes the last component of $\boldsymbol{x} \in \mathbb{N}^{r-1} \times \mathbb{D}$.

(iv) *We have*

$$n\sum_{k=1}^{n}\frac{H_{k-1}(\boldsymbol{v})A^{(2)}_{n,k}}{k^a}=\sum_{k=1}^{n}\frac{H_{k-1}(\boldsymbol{v})A^{(2)}_{n,k}}{k^{a-1}}+2\sum_{k=1}^{n}H_{k-1}(\bar{a},\boldsymbol{v})kB^{(2)}_{n,k}. \tag{10}$$

PROOF. We need to mention again that the ordering is reversed in this paper so s_1 in [**7**, Lemma 2.2] should be the last component of $\boldsymbol{x}$, namely, x_r in our setup. Now, equation (7) follows from [**7**, Lemma 2.2] directly. We may also use this proof for (9) by taking the sign of x_r into consideration.

Now by the identity proved in [**7**, Lemma 2.1]

$$2\sum_{k=l+1}^{n}\frac{k\binom{n}{k}}{\binom{n+k}{k}}=\frac{n\binom{n-1}{l}}{\binom{n+l}{l}}=\frac{(n-l)\binom{n}{l}}{\binom{n+l}{l}} \tag{11}$$

we see that

$$\begin{aligned}2\sum_{k=1}^{n}H_{k-1}(a,\boldsymbol{v})kB^{(2)}_{n,k}&=\sum_{l=1}^{n}\frac{H_{l-1}(\boldsymbol{v})}{l^a}\sum_{k=l+1}^{n}2kB^{(2)}_{n,k}\\&=\sum_{l=1}^{n}\frac{H_{l-1}(\boldsymbol{v})}{l^a}(n-l)B^{(2)}_{n,l}\\&=n\sum_{l=1}^{n}\frac{H_{l-1}(\boldsymbol{v})B^{(2)}_{n,l}}{l^a}-\sum_{l=1}^{n}\frac{H_{l-1}(\boldsymbol{v})B^{(2)}_{n,l}}{l^{a-1}}\end{aligned}$$

which is (8). Similar argument yields (10). We leave the details of the rest of the proof to the interested reader. □

REMARK 2.2. In this paper we will always choose $c^{(1)}_n=1$ so that $A^{(1)}_{n,k}=(-1)^k\binom{n}{k}$ and $B^{(1)}_{n,k}=\binom{n}{k}$, and $c^{(2)}_n=(n!)^2/(2n)!$ so that $A^{(2)}_{n,k}=(-1)^k\binom{n}{k}/\binom{n+k}{k}$ and $B^{(2)}_{n,k}=\binom{n}{k}/\binom{n+k}{k}$.

Taking $m=2$ in Lemma 2.1 we immediately get the following results.

COROLLARY 2.3. *Let $a\in\mathbb{D}_0$, $n,c\in\mathbb{N}$. Then for any composition $\boldsymbol{v}$, we have*

$$\frac{1}{n^c}\mathcal{H}_n(a,\boldsymbol{v})=\sum_{\boldsymbol{p}\in\{\bar{0}\circ 1^{\circ(c-1)}\circ(a\oplus\bar{1})\}}\mathcal{H}_n(\boldsymbol{p},\boldsymbol{v}).$$

COROLLARY 2.4. *Let $c\in\mathbb{N}_0$ and $\Pi(c)=\{\bar{0}\circ 1^{\circ c}\}$. Then*

$$\frac{1}{n^c}=-\sum_{\boldsymbol{p}\in\Pi(c)}\mathcal{H}_n(\boldsymbol{p}).$$

PROOF. By [**7**, (2.12)] we have

$$2\sum_{k=l+1}^{n} A_{n,k}^{(2)} = \left(\frac{l}{n}-1\right)A_{n,l}^{(2)}.$$

The case $c=0$ follows from this by setting $l=0$. For $c\geq 1$, using the $c=0$ case we get

$$\frac{1}{n^c} = -\frac{1}{n^c}\mathcal{H}_n(\bar{0}) = -\sum_{\boldsymbol{p}\in\Pi(c)} \mathcal{H}_n(\boldsymbol{p})$$

by taking $a=\bar{0}$ and $\boldsymbol{v}=\emptyset$ in Corollary 2.3. Hence the corollary is proved. □

3. Proof of Theorem 1.4.

We now prove the key result in Theorem 1.4. We break it into a series of lemmas.

LEMMA 3.1. *Let $\boldsymbol{s}=(\{2\}^a,c)$ with $a,c\in\mathbb{N}_0$ and $c\geq 3$. Then*

$$H_n^\star(\boldsymbol{s}) = -\sum_{\boldsymbol{p}\in\Pi(\boldsymbol{s})} \mathcal{H}_n(\boldsymbol{p}), \tag{12}$$

where the set $\Pi(\boldsymbol{s})=\{(\overline{2a+2})\circ 1^{\circ(c-2)}\}$.

PROOF. We proceed by induction on n. When $n=1$ we have $H_1^\star(\{2\}^a,c)=1$. On the other hand,

$$\sum_{\boldsymbol{p}\in\Pi(\boldsymbol{s})} \mathcal{H}_1(\boldsymbol{p}) = \mathcal{H}_1(\overline{2a+c}) = -1,$$

and therefore the formula is true. Suppose the statement is true for $n-1$. Then by definition

$$H_n^\star(\boldsymbol{s}) = \sum_{i=0}^{a} \frac{1}{n^{2(a-i)}} H_{n-1}^\star(\{2\}^i,c) + \frac{1}{n^{2a+c}}.$$

Applying inductive hypothesis, we obtain

$$H_n^\star(\boldsymbol{s}) = -\sum_{i=0}^{a} \frac{1}{n^{2(a-i)}} \sum_{\boldsymbol{p}\in\Pi(\{2\}^i,c)} \mathcal{H}_{n-1}(\boldsymbol{p}) + \frac{1}{n^{2a+c}}. \tag{13}$$

Set $\Pi(\boldsymbol{u}_{-1})=\{0\circ 1^{\circ(c-2)}\}$. Then the inner sum in (13) becomes

$$\sum_{\boldsymbol{p}\in\Pi(\{2\}^i,c)} \mathcal{H}_{n-1}(\boldsymbol{p}) = \sum_{\boldsymbol{p}=(p_1,p_2,\dots,p_r)\in\Pi(\boldsymbol{u}_{-1})} \mathcal{H}_{n-1}(\overline{2i+2+p_1},p_2,\dots,p_r)$$

$$= \sum_{\boldsymbol{p}=(p_1,p_2,\dots,p_r)\in\Pi(\boldsymbol{u}_{-1})} \sum_{n>k_1>\cdots>k_r\ge 1} \frac{A^{(2)}_{n-1,k_1}}{k_1^{2i+2+p_1}} \prod_{j=2}^{r} \frac{2}{k_j^{p_j}},$$

where $A^{(2)}_{n,k} = (-1)^k\binom{n}{k}/\binom{n+k}{k}$ (see Remark 2.2). Plugging this into (13) and summing over i by the formula

$$A^{(2)}_{n-1,k} \sum_{i=0}^{a} \frac{n^{2i}}{k^{2i}} = A^{(2)}_{n,k}\left(\frac{n^{2a}}{k^{2a}} - \frac{k^2}{n^2}\right), \tag{14}$$

we obtain

$$H^\star_n(\boldsymbol{s}) = -\sum_{\boldsymbol{p}=(p_1,p_2,\dots,p_r)\in\Pi(\boldsymbol{u}_{-1})} \sum_{n\ge k_1>\cdots>k_r\ge 1} \frac{A^{(2)}_{n,k_1}}{k_1^{2a+2+p_1}} \prod_{j=2}^{r} \frac{2}{k_j^{p_j}}$$
$$+ \frac{1}{n^{2a+2}} \sum_{\boldsymbol{p}=(p_1,p_2,\dots,p_r)\in\Pi(\boldsymbol{u}_{-1})} \sum_{n\ge k_1>\cdots>k_r\ge 1} \frac{A^{(2)}_{n,k_1}}{k_1^{p_1}} \prod_{j=2}^{r} \frac{2}{k_j^{p_j}} + \frac{1}{n^{2a+c}},$$

which implies

$$H^\star_n(\boldsymbol{s}) = -\sum_{\boldsymbol{p}\in\Pi(\boldsymbol{s})} \mathcal{H}_n(\boldsymbol{p}) + \frac{1}{n^{2a+2}} \sum_{\boldsymbol{p}\in\Pi(\hat{\boldsymbol{u}}_{-1})} \mathcal{H}_n(\boldsymbol{p}) + \frac{1}{n^{2a+c}},$$

where $\Pi(\hat{\boldsymbol{u}}_{-1}) = \{\overline{0}\circ 1^{\circ(c-2)}\}$. Hence the theorem follows from Corollary 2.4 immediately (by replacing c by $c-2$ there). $\square$

LEMMA 3.2. *Suppose* $\boldsymbol{s}$ *is a composition of positive integers and there exists* $\boldsymbol{\lambda} = (\lambda_1,\dots,\lambda_m)$ *such that there is an expansion of the form*

$$H^\star_n(\boldsymbol{s}) = \varepsilon(\boldsymbol{s}) \sum_{\boldsymbol{p}\in\{\lambda_1\circ\cdots\circ\lambda_m\}} \mathcal{H}_n(\boldsymbol{p}).$$

Then for any integers $a\in\mathbb{N}_0$ *we have*

$$H^\star_n(\{2\}^a,\boldsymbol{s}) = \varepsilon(\boldsymbol{s}) \sum_{\boldsymbol{p}\in\Pi_{\boldsymbol{2}^a}} \mathcal{H}_n(\boldsymbol{p}), \tag{15}$$

where $\Pi_{\boldsymbol{2}^a} = \{(2a\oplus\lambda_1)\circ\lambda_2\circ\cdots\circ\lambda_m\}$.

PROOF. Set $\Pi(\boldsymbol{s}) = \{\lambda_1\circ\cdots\circ\lambda_m\}$. The proof of the identities is by induction on $n+a$ or $n+b$. When $n=1$ the theorem is clear. Assume formulas (15) and (17) are true for all $a+n\le N$ where $N\ge 2$. Suppose now we have $n\ge 2$ and $n+a=N+1$. We start proving the first identity. By definition, we have

$$H^\star_n(\{2\}^a,\boldsymbol{s}) = \sum_{i=1}^{a} \frac{1}{n^{2a-2i}} H^\star_{n-1}(\{2\}^i,\boldsymbol{s}) + \frac{1}{n^{2a}} H^\star_n(\boldsymbol{s}).$$

Applying induction assumption, we obtain

$$\varepsilon(\boldsymbol{s})H_n^\star(\{2\}^a,\boldsymbol{s}) = \sum_{i=1}^{a} \frac{1}{n^{2a-2i}} \sum_{\boldsymbol{p}\in\Pi_{\boldsymbol{2}^i}} \mathcal{H}_{n-1}(\boldsymbol{p}) + \frac{1}{n^{2a}} \sum_{\boldsymbol{p}\in\Pi(\boldsymbol{s})} \mathcal{H}_n(\boldsymbol{p}). \tag{16}$$

Expanding the inner sum

$$\begin{aligned}\sum_{\boldsymbol{p}\in\Pi_{\boldsymbol{2}^i}} \mathcal{H}_{n-1}(\boldsymbol{p}) &= \sum_{\boldsymbol{p}=(p_1,p_2,\dots,p_r)\in\Pi(\boldsymbol{s})} \mathcal{H}_{n-1}(2i\oplus p_1,p_2,\dots,p_r)\\ &= \sum_{\boldsymbol{p}=(p_1,p_2,\dots,p_r)\in\Pi(\boldsymbol{s})} \sum_{n>k_1>\cdots>k_r\ge 1} \frac{\binom{n-1}{k_1}}{\binom{n-1+k_1}{k_1}} \frac{1}{k_1^{2i}} \prod_{j=1}^{r} \frac{2\operatorname{sgn}(p_j)^{k_j}}{k_j^{|p_j|}}\end{aligned}$$

and summing over i in (16), we obtain

$$\begin{aligned}\varepsilon(\boldsymbol{s})H_n^\star(\{2\}^a,\boldsymbol{s}) = &\sum_{\boldsymbol{p}=(p_1,p_2,\dots,p_r)\in\Pi(\boldsymbol{s})} \sum_{n\ge k_1>\cdots>k_r\ge 1} \frac{\binom{n}{k_1}}{\binom{n+k_1}{k_1}} \left(\frac{1}{k_1^{2a}} - \frac{1}{n^{2a}}\right) \prod_{j=1}^{r} \frac{2\operatorname{sgn}(p_j)^{k_j}}{k_j^{|p_j|}}\\ &+ \frac{1}{n^{2a}} \sum_{\boldsymbol{p}\in\Pi(\boldsymbol{s})} \mathcal{H}_n(\boldsymbol{p}),\end{aligned}$$

which implies (15) by definition and straightforward cancelation. □

Lemma 3.3. *Suppose $\boldsymbol{s}$ is a composition of positive integers and there exists $\boldsymbol{\lambda} = (\lambda_1,\dots,\lambda_m)$ such that there is an expansion of the form*

$$H_n^\star(\boldsymbol{s}) = \varepsilon(\boldsymbol{s}) \sum_{\boldsymbol{p}\in\{\lambda_1\circ\cdots\circ\lambda_m\}} \mathcal{H}_n(\boldsymbol{p}).$$

Then for any integers $a\in\mathbb{N}_0$ we have

$$H_n^\star(\{2\}^a,1,\boldsymbol{s}) = \varepsilon(\boldsymbol{s}) \sum_{\boldsymbol{p}\in\Pi_{\boldsymbol{2}^a1}} \mathcal{H}_n(\boldsymbol{p}), \tag{17}$$

where $\Pi_{\boldsymbol{2}^a1} = \{(2a+1)\circ\lambda_1\circ\cdots\circ\lambda_m\}$,

Proof. By definition

$$H_n^\star(\{2\}^a,1,\boldsymbol{s}) = \sum_{i=0}^{a} \frac{1}{n^{2a-2i}}\, H_{n-1}^\star(\{2\}^i,1,\boldsymbol{s}) + \frac{1}{n^{2a+1}}\, H_n^\star(\boldsymbol{s}).$$

Applying induction assumption, we obtain

$$\varepsilon(\boldsymbol{s})H_n^\star(\{2\}^a,1,\boldsymbol{s}) = \sum_{i=0}^{a} \frac{1}{n^{2a-2i}} \sum_{\boldsymbol{p}\in\Pi_{\boldsymbol{2}^i1}} \mathcal{H}_{n-1}(\boldsymbol{p}) + \frac{1}{n^{2a+1}} \sum_{\boldsymbol{p}\in\Pi(\boldsymbol{s})} \mathcal{H}_n(\boldsymbol{p}). \tag{18}$$

Setting $\Pi(\boldsymbol{s})=\{\lambda_1\circ\cdots\circ\lambda_m\}$ we have

$$\begin{aligned}
&\sum_{\boldsymbol{p}\in\Pi_{\boldsymbol{2}^i1}}\mathcal{H}_{n-1}(\boldsymbol{p})\\
&=\sum_{\boldsymbol{p}\in\Pi(\boldsymbol{s})}\mathcal{H}_{n-1}(2i+1,\boldsymbol{p})+\sum_{\boldsymbol{p}=(p_1,p_2,\dots,p_r)\in\Pi(\boldsymbol{s})}\mathcal{H}_{n-1}((2i+1)\oplus p_1,p_2,\dots,p_r)\\
&=\sum_{\boldsymbol{p}=(p_1,p_2,\dots,p_r)\in\Pi(\boldsymbol{s})}\Bigg(\sum_{n>k_0>k_1>\cdots>k_r\ge1}\frac{\binom{n-1}{k_0}}{\binom{n-1+k_0}{k_0}}\frac{2}{k_0^{2i+1}}\prod_{j=1}^{r}\frac{2\operatorname{sgn}(p_j)^{k_j}}{k_j^{|p_j|}}\\
&\qquad+\sum_{n>k_1>\cdots>k_r\ge1}\frac{\binom{n-1}{k_1}}{\binom{n-1+k_1}{k_1}}\frac{2\operatorname{sgn}(p_1)^{k_1}}{k_1^{2i+1+|p_1|}}\prod_{j=2}^{r}\frac{2\operatorname{sgn}(p_j)^{k_j}}{k_j^{|p_j|}}\Bigg).
\end{aligned}$$

Substituting the above expression into (18) and summing over i by (14) we obtain

$$\begin{aligned}
&\varepsilon(\boldsymbol{s})H_n^\star(\{2\}^a,1,\boldsymbol{s})-\frac{1}{n^{2a+1}}\sum_{\boldsymbol{p}\in\Pi(\boldsymbol{s})}\mathcal{H}_n(\boldsymbol{p})\\
&=\sum_{\boldsymbol{p}=(p_1,p_2,\dots,p_r)\in\Pi(\boldsymbol{s})}\Bigg(\sum_{n\ge k_0>k_1>\cdots>k_r\ge1}\frac{\binom{n}{k_0}}{\binom{n+k_0}{k_0}}\frac{2}{k_0^{2a+1}}\prod_{j=1}^{r}\frac{2\operatorname{sgn}(p_j)^{k_j}}{k_j^{|p_j|}}\\
&\qquad-\frac{1}{n^{2a+2}}\sum_{n\ge k_0>k_1>\cdots>k_r\ge1}\frac{\binom{n}{k_0}}{\binom{n+k_0}{k_0}}2k_0\prod_{j=1}^{r}\frac{2\operatorname{sgn}(p_j)^{k_j}}{k_j^{|p_j|}}\\
&\qquad+\sum_{n\ge k_1>\cdots>k_r\ge1}\frac{\binom{n}{k_1}}{\binom{n+k_1}{k_1}}\frac{2\operatorname{sgn}(p_1)^{k_1}}{k_1^{2a+1+|p_1|}}\prod_{j=2}^{r}\frac{2\operatorname{sgn}(p_j)^{k_j}}{k_j^{|p_j|}}\\
&\qquad-\frac{1}{n^{2a+2}}\sum_{n\ge k_1>\cdots>k_r\ge1}\frac{\binom{n}{k_1}}{\binom{n+k_1}{k_1}}\frac{2\operatorname{sgn}(p_1)^{k_1}}{k_1^{|p_1|-1}}\prod_{j=2}^{r}\frac{2\operatorname{sgn}(p_j)^{k_j}}{k_j^{|p_j|}}\Bigg).
\end{aligned}$$

Noticing that the first and third sums on the right-hand side of the above add up to

$$\sum_{\boldsymbol{p}\in\Pi_{\boldsymbol{2}^a1}}\mathcal{H}_n(\boldsymbol{p}),$$

we have

$$\begin{aligned}
&\varepsilon(\boldsymbol{s})H_n^\star(\{2\}^a,1,\boldsymbol{s})-\frac{1}{n^{2a+1}}\sum_{\boldsymbol{p}\in\Pi(\boldsymbol{s})}\mathcal{H}_n(\boldsymbol{p})-\sum_{\boldsymbol{p}\in\Pi_{\boldsymbol{2}^a1}}\mathcal{H}_n(\boldsymbol{p})\\
&=-\frac{1}{n^{2a+2}}\sum_{\boldsymbol{p}\in\Pi(\boldsymbol{s})}\sum_{n\ge k_1>\cdots>k_r\ge1}\Bigg(\prod_{j=1}^{r}\frac{2\operatorname{sgn}(p_j)^{k_j}}{k_j^{|p_j|}}\sum_{k_0=k_1+1}^{n}2k_0\frac{\binom{n}{k_0}}{\binom{n+k_0}{k_0}}\\
&\qquad+\frac{\binom{n}{k_1}}{\binom{n+k_1}{k_1}}\frac{2\operatorname{sgn}(p_1)^{k_1}}{k_1^{|p_1|-1}}\prod_{j=2}^{r}\frac{2\operatorname{sgn}(p_j)^{k_j}}{k_j^{|p_j|}}\Bigg).
\end{aligned}$$

Observe that [**7**, (2.2)] can be rewritten as

$$\sum_{k=l+1}^{n} 2k\frac{\binom{n}{k}}{\binom{n+k}{k}} = (n-l)\frac{\binom{n}{l}}{\binom{n+l}{l}}.$$

Using this to simplify the sum over k_0 in the above we obtain

$$\varepsilon(\boldsymbol{s})H_n^\star(\{2\}^a, 1, \boldsymbol{s}) = \sum_{\boldsymbol{p}\in\Pi_{\boldsymbol{2}^a 1}} \mathcal{H}_n(\boldsymbol{p}).$$

This proves identity (17) by induction. $\square$

LEMMA 3.4. *Suppose* $\boldsymbol{s}$ *is a composition of positive integers and there exists* $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$ *such that there is an expansion of the form*

$$H_n^\star(\boldsymbol{s}) = \varepsilon(\boldsymbol{s}) \sum_{\boldsymbol{p}\in\{\boldsymbol{\lambda}^\circ\}} \mathcal{H}_n(\boldsymbol{p}).$$

Then for any integers $b \in \mathbb{N}_0$ *and* $c \geq 3$ *we have*

$$H_n^\star(\{2\}^b, c, \boldsymbol{s}) = \varepsilon(\boldsymbol{s}) \sum_{\boldsymbol{p}\in\Pi_{\boldsymbol{2}^b c}} \mathcal{H}_n(\boldsymbol{p}), \tag{19}$$

where $\Pi_{\boldsymbol{2}^b c} = \{(\overline{2b+2}) \circ 1^{\circ(c-3)} \circ (\lambda_1 \oplus \bar{1}) \circ \boldsymbol{\lambda}^\circ_{\hat{1}}\}$.

PROOF. We proceed by induction on $n+b$. Assume formula (19) is true for all $b+n \leq N$. Now suppose $b+n = N+1$. By definition we have

$$H_n^\star(\{2\}^b, c, \boldsymbol{s}) = \sum_{i=0}^{b} \frac{1}{n^{2b-2i}} H_{n-1}^\star(\{2\}^i, c, \boldsymbol{s}) + \frac{1}{n^{2b+c}} H_n^\star(\boldsymbol{s}).$$

By the induction assumption, we see that

$$\varepsilon(\boldsymbol{s})H_n^\star(\{2\}^b, c, \boldsymbol{s}) = \sum_{i=0}^{b} \frac{1}{n^{2b-2i}} \sum_{\boldsymbol{p}\in\Pi_{\boldsymbol{2}^i c}} \mathcal{H}_{n-1}(\boldsymbol{p}) + \frac{1}{n^{2b+c}} \sum_{\boldsymbol{p}\in\Pi(\boldsymbol{s})} \mathcal{H}_n(\boldsymbol{p}). \tag{20}$$

Setting $\Pi_1 = \{0 \circ 1^{\circ(c-3)} \circ (\lambda_1 \oplus \bar{1}) \circ \lambda_2 \circ \cdots \circ \lambda_m\}$ we have

$$\begin{aligned}
\sum_{\boldsymbol{p}\in\Pi_{\boldsymbol{2}^i c}} \mathcal{H}_{n-1}(\boldsymbol{p}) &= \sum_{\boldsymbol{p}=(p_1,\dots,p_r)\in\Pi_1} \mathcal{H}_{n-1}(\overline{2i+2} \oplus p_1, p_2, \dots, p_r) \\
&= \sum_{\boldsymbol{p}=(p_1,\dots,p_r)\in\Pi_1} \sum_{n>k_1>\cdots>k_r\geq 1} \frac{\binom{n-1}{k_1}}{\binom{n-1+k_1}{k_1}} \frac{1}{(-\operatorname{sgn}(p_1))^{k_1} k_1^{2i+2+|p_1|}} \prod_{j=2}^{r} \frac{2\operatorname{sgn}(p_j)^{k_j}}{k_j^{|p_j|}}.
\end{aligned}$$

Plugging this into (20) and summing over i by (14), we obtain

$$\varepsilon(\boldsymbol{s})H_n^\star(\{2\}^b,c,\boldsymbol{s}) = \sum_{\boldsymbol{p}=(p_1,\dots,p_r)\in\Pi_1} \mathcal{H}_n((\overline{2b+2})\oplus p_1,p_2,\dots,p_r)) + \frac{1}{n^{2b+c}}\sum_{\boldsymbol{p}\in\Pi(\boldsymbol{s})}\mathcal{H}_n(\boldsymbol{p}) - \frac{1}{n^{2b+2}}\sum_{\boldsymbol{p}\in\Pi_1}\mathcal{H}_n(\overline{p}_1,p_2,\dots,p_r),$$

which implies

$$\varepsilon(\boldsymbol{s})H_n^\star(\{2\}^b,c,\boldsymbol{s}) = \sum_{\boldsymbol{p}\in\Pi_{\boldsymbol{2}^b c}}\mathcal{H}_n(\boldsymbol{p}) - \frac{1}{n^{2b+2}}\sum_{\boldsymbol{p}\in\Pi_2}\mathcal{H}_n(\boldsymbol{p}) + \frac{1}{n^{2b+c}}\sum_{\boldsymbol{p}\in\Pi(\boldsymbol{s})}\mathcal{H}_n(\boldsymbol{p}), \tag{21}$$

where $\Pi_2 = \{\overline{0}\circ 1^{\circ(c-3)}\circ(\lambda_1\oplus\overline{1})\circ\lambda_2\circ\cdots\circ\lambda_m\}$. Expanding the second sum from (21), we have

$$\sum_{\boldsymbol{p}\in\Pi_2}\mathcal{H}_n(\boldsymbol{p}) = \sum_{\boldsymbol{p}\in\Pi(\boldsymbol{s})}\ \sum_{\boldsymbol{w}=\overline{0}\circ 1^{\circ(c-3)}\circ(p_1\oplus\overline{1})}\mathcal{H}_n(\boldsymbol{w},p_2,\dots,p_r),$$

where $\Pi(\boldsymbol{s}) = \{\lambda_1\circ\cdots\circ\lambda_m\}$. Applying Lemma 7 to the inner sum with $a=p_1$, c replaced by $c-2$, and $\boldsymbol{x}=(p_2,\dots,p_r)$, we obtain

$$\sum_{\boldsymbol{p}\in\Pi_2}\mathcal{H}_n(\boldsymbol{p}) = \sum_{\boldsymbol{p}\in\Pi(\boldsymbol{s})}\frac{1}{n^{c-2}}\mathcal{H}_n(p_1,\boldsymbol{x}) = \frac{1}{n^{c-2}}\sum_{\boldsymbol{p}\in\Pi(\boldsymbol{s})}\mathcal{H}_n(\boldsymbol{p}). \tag{22}$$

Now by (21) and (22) we see that (19) is true when $n+b=N+1$. We have completed the proof of the lemma. □

By combining the four lemmas we have proved in this section we can finally derive Theorem 1.4 by a straight-forward case by case analysis. We leave the details to the interested reader.

4. A few corollaries.

The following result generalizes Theorem 1.5(A).

COROLLARY 4.1. *Let $r\in\mathbb{N}$ and $\boldsymbol{s}=(\{2\}^{a_1},1,\dots,\{2\}^{a_r},1)$ where $a_1\in\mathbb{N}$ and $a_j\in\mathbb{N}_0$ for all $j\ge 2$. Then we have*

$$H_n^\star(\boldsymbol{s}) = \sum_{\boldsymbol{p}\in\Pi(2a_1+1,\dots,2a_r+1)}\mathcal{H}_n(\boldsymbol{p}).$$

PROOF. Apply Theorem 1.4 repeatedly. □

REMARK 4.2. (a). When $r=1$ Corollary 4.1 becomes Theorem 1.5(A).
(b). When $r=2$ we get: for all $n\in\mathbb{N}$ and $a,b\in\mathbb{N}_0$

$$H_n^\star(\{2\}^a,1,\{2\}^b,1)=2\sum_{k=1}^n\frac{\binom{n}{k}}{k^{2(a+b)+2}\binom{n+k}{k}}+4\sum_{k=1}^n\frac{H_{k-1}(2b+1)\binom{n}{k}}{k^{2a+1}\binom{n+k}{k}}.$$

When $r=3$ we have: for all $n\in\mathbb{N}$ and $a,b,c\in\mathbb{N}_0$

$$\begin{aligned}&H_n^\star(\{2\}^a,1,\{2\}^b,1,\{2\}^c,1)\\&\quad=2\sum_{k=1}^n\frac{\binom{n}{k}}{k^{2(a+b+c)+3}\binom{n+k}{k}}+4\sum_{k=1}^n\frac{H_{k-1}(2c+1)\binom{n}{k}}{k^{2a+2b+2}\binom{n+k}{k}}\\&\qquad+4\sum_{k=1}^n\frac{H_{k-1}(2b+2c+2)\binom{n}{k}}{k^{2a+1}\binom{n+k}{k}}+8\sum_{k=1}^n\frac{H_{k-1}(2b+1,2c+1)\binom{n}{k}}{k^{2a+1}\binom{n+k}{k}}.\end{aligned}$$

Using Maple we have verified both formulas numerically for $a,b,c\le 5$ and $n\le 100$.

We now generalize Theorem 1.5(B).

COROLLARY 4.3. *Suppose $r\in\mathbb{N}_0$ and $\boldsymbol{s}=(\{2\}^{a_1},1,\dots,\{2\}^{a_r},1,\{2\}^{a_{r+1}})$ where $a_j\in\mathbb{N}_0$ for all $j\le r$ and $a_{r+1}\in\mathbb{N}$. Then we have*

$$H_n^\star(\boldsymbol{s})=-\sum_{\boldsymbol{p}\in\Pi(2a_1+1,\dots,2a_r+1,\overline{2a_{r+1}})}\mathcal{H}_n(\boldsymbol{p}).$$

PROOF. Apply Theorem 1.4 repeatedly. □

REMARK 4.4. When $r=0$ Corollary 4.3 implies [**7**, (19)]. When $r=1$ Corollary 4.3 becomes Theorem 1.5(B). When $r=2$ we get the following: for all $n,c\in\mathbb{N}$ and $a,b\in\mathbb{N}_0$

$$\begin{aligned}H_n^\star(\{2\}^a,1,\{2\}^b,1,\{2\}^c)&=-2\sum_{k=1}^n\frac{(-1)^k\binom{n}{k}}{k^{2(a+b+c)+2}\binom{n+k}{k}}-4\sum_{k=1}^n\frac{H_{k-1}(\overline{2c})\binom{n}{k}}{k^{2a+2b+2}\binom{n+k}{k}}\\&\quad-4\sum_{k=1}^n\frac{H_{k-1}(\overline{2b+1+2c})\binom{n}{k}}{k^{2a+1}\binom{n+k}{k}}-8\sum_{k=1}^n\frac{H_{k-1}(2b+1,\overline{2c})\binom{n}{k}}{k^{2a+1}\binom{n+k}{k}}.\end{aligned}\tag{23}$$

We can now prove Theorems 1.2 and 1.3 by using Corollary 4.1 and 4.3 and the following key lemma proved in [**16**].

LEMMA 4.5 ([**16**, Lemma 4.2]). *Let $d\in\mathbb{N}_0$ and let e be a real number with $e>1$. Then for all $\boldsymbol{s}\in\mathbb{D}^d$ ($\boldsymbol{s}=\emptyset$ if $d=0$) we have*

$$\lim_{n\to\infty}\sum_{k=1}^n\frac{|H_{k-1}(\boldsymbol{s})|}{k^e}\left(1-\frac{\binom{n}{k}}{\binom{n+k}{k}}\right)=0.\tag{24}$$

PROOF OF THEOREM 1.2 AND THEOREM 1.3. We observe that in Corollary 4.1

the first component $\geq 2a_1+1\geq 3$, and in Corollary 4.3 the absolute value of first component $\geq 2a_1+1\geq 3$. Therefore both theorems follow from Lemma 4.5 immediately. $\square$

REMARK 4.6. In [**24**] Yamamoto considers some algebraic structures depending on a variable t which reflect the properties of MZV and MZSV when $t=0$ and $t=1$, respectively. As he pointed out [**23**, Conjecture 4.4] the validity of the $\underline{\mathbf{2}\text{-}1}$ formula Theorem 1.2 implies that the algebra structure of MZSVs of the form $\zeta^\star(\{2\}^{a_1},1,\dots,\{2\}^{a_r},1)$ is reflected by setting $t=1/2$.

COROLLARY 4.7. *Let $a_j,b_j,c_j-3\in\mathbb{N}_0$ for all $j\geq 0$. Then*
$(\underline{\mathbf{2}\text{-}c\text{-}\mathbf{2}\text{-}1})$: *For* $\boldsymbol{s}=(\{2\}^{b_1},c_1,\{2\}^{a_1},1,\dots,\{2\}^{b_r},c_r,\{2\}^{a_r},1)$, $r\geq 1$, *we have*

$$H_n^\star(\boldsymbol{s})=\sum_{\boldsymbol{p}\in\Pi(\overline{2b_1+2},\,\{1\}^{c_1-3},\,\overline{2a_1+2},\dots,\overline{2b_r+2},\,\{1\}^{c_r-3},\,\overline{2a_r+2})}\mathcal{H}_n(\boldsymbol{p}).$$

$(\mathbf{2}\text{-}1\text{-}\underline{\mathbf{2}\text{-}c\text{-}\mathbf{2}\text{-}1})$: *For* $\boldsymbol{s}=(\{2\}^{a_0},1,\{2\}^{b_1},c_1,\{2\}^{a_1},1,\dots,\{2\}^{b_r},c_r,\{2\}^{a_r},1)$, $r\geq 0$, *we have*

$$H_n^\star(\boldsymbol{s})=\sum_{\boldsymbol{p}\in\Pi(2a_0+1,\,\overline{2b_1+2},\{1\}^{c_1-3},\,\overline{2a_1+2},\dots,\overline{2b_r+2},\,\{1\}^{c_r-3},\,\overline{2a_r+2})}\mathcal{H}_n(\boldsymbol{p}).$$

PROOF. Apply Theorem 1.4 repeatedly. $\square$

When $r=1$ we get the following: for all $n\in\mathbb{N}$ and $a,b\in\mathbb{N}_0$

$$H_n^\star(\{2\}^b,3,\{2\}^a,1)=2\sum_{k=1}^n\frac{\binom{n}{k}}{k^{2(b+a)+4}\binom{n+k}{k}}+4\sum_{k=1}^n\frac{(-1)^kH_{k-1}(\overline{2a+2})\binom{n}{k}}{k^{2b+2}\binom{n+k}{k}},\tag{25}$$

in case $(\underline{\mathbf{2}\text{-}c\text{-}\mathbf{2}\text{-}1})$, and in case $(\mathbf{2}\text{-}1\text{-}\underline{\mathbf{2}\text{-}c\text{-}\mathbf{2}\text{-}1})$

$$\begin{aligned}&H_n^\star(\{2\}^{a_1},1,\{2\}^b,3,\{2\}^{a_2},1)\\&\quad=2\sum_{k=1}^n\frac{\binom{n}{k}}{k^{2(a_1+b+a_2)+5}\binom{n+k}{k}}+4\sum_{k=1}^n\frac{H_{k-1}(2b+2a_2+4)\binom{n}{k}}{k^{2a_1+1}\binom{n+k}{k}}\\&\qquad+4\sum_{k=1}^n\frac{(-1)^kH_{k-1}(\overline{2a_2+2})\binom{n}{k}}{k^{2a_1+2b+3}\binom{n+k}{k}}+8\sum_{k=1}^n\frac{H_{k-1}(\overline{2b+2},\overline{2a_2+2})\binom{n}{k}}{k^{2a_1+1}\binom{n+k}{k}}.\end{aligned}\tag{26}$$

THEOREM 4.8. *Let $r\in\mathbb{N}$ and $a_j,b_j,c_j-3\in\mathbb{N}_0$ for all $j\geq 1$. Then*
$(\underline{\mathbf{2}\text{-}c\text{-}\mathbf{2}\text{-}1})$: *For* $\boldsymbol{s}=(\{2\}^{b_1},c_1,\{2\}^{a_1},1,\dots,\{2\}^{b_r},c_r,\{2\}^{a_r},1)$, $r\geq 1$, *we have*

$$\zeta^\star(\boldsymbol{s})=\sum_{\boldsymbol{p}\in\Pi(\overline{2b_1+2},\,\{1\}^{c_1-3},\,\overline{2a_1+2},\dots,\overline{2b_r+2},\,\{1\}^{c_r-3},\,\overline{2a_r+2})}2^{\ell(\boldsymbol{p})}\zeta(\boldsymbol{p}).\tag{27}$$

($\mathbf{2}$-1-$\underline{\mathbf{2}\text{-}c\text{-}\mathbf{2}\text{-}1}$) : *For* $\boldsymbol{s} = (\{2\}^{a_0}, 1, \{2\}^{b_1}, c_1, \{2\}^{a_1}, 1, \dots, \{2\}^{b_r}, c_r, \{2\}^{a_r}, 1)$, $r \geq 0$ *and* $a_0 \geq 1$, *we have*

$$\zeta^\star(\boldsymbol{s}) = \sum_{\boldsymbol{p}\in\Pi(2a_0+1,\, \overline{2b_1+2}, \{1\}^{c_1-3},\, \overline{2a_1+2},\dots,\overline{2b_r+2},\, \{1\}^{c_r-3},\, \overline{2a_r+2})} 2^{\ell(\boldsymbol{p})}\zeta(\boldsymbol{p}). \tag{28}$$

Proof. This follows from Corollary 4.7 and Lemma 4.5 easily. □

For example, by (25) and (26) we see that

$$\zeta^\star(\{2\}^b, 3, \{2\}^a, 1) = 2\zeta(2a+2b+4) + 4\zeta(\overline{2b+2}, \overline{2a+2}), \tag{29}$$

and

$$\begin{aligned}\zeta^\star(\{2\}^{a_1}, 1, \{2\}^b, 3, \{2\}^{a_2}, 1) &= 2\zeta(2(a_1+b+a_2)+5) + 4\zeta(2a_1+1, 2b+2a_2+4)\\ &\quad + 4\zeta(\overline{2a_1+2b+3}, \overline{2a_2+2}) + 8\zeta(2a_1+1, \overline{2b+2}, \overline{2a_2+2}).\end{aligned}$$

Corollary 4.9. *Let* $t, r \in \mathbb{N}$ *and* $a_j, b_j, c_j - 3 \in \mathbb{N}_0$ *for all* $j \geq 1$. *Then*
($\underline{\mathbf{2}\text{-}c\text{-}\mathbf{2}\text{-}1}$-$\mathbf{2}$). *For* $\boldsymbol{s} = (\{2\}^{b_1}, c_1, \{2\}^{a_1}, 1, \dots, \{2\}^{b_r}, c_r, \{2\}^{a_r}, 1, \{2\}^t)$ *we have*

$$H_n^\star(\boldsymbol{s}) = -\sum_{\boldsymbol{p}\in\Pi(\overline{2b_1+2},\, \{1\}^{c_1-3},\, \overline{2a_1+2},\dots,\overline{2b_r+2},\, \{1\}^{c_r-3},\, \overline{2a_r+2},\overline{2t})} \mathcal{H}_n(\boldsymbol{p}).$$

($\mathbf{2}$-1-$\underline{\mathbf{2}\text{-}c\text{-}\mathbf{2}\text{-}1}$-$\mathbf{2}$). *For* $\boldsymbol{s} = (\{2\}^{a_0}, 1, \{2\}^{b_1}, c_1, \{2\}^{a_1}, 1, \dots, \{2\}^{b_r}, c_r, \{2\}^{a_r}, 1, \{2\}^t)$, $r \geq 0$, *we have*

$$H_n^\star(\boldsymbol{s}) = -\sum_{\boldsymbol{p}\in\Pi(2a_0+1,\, \overline{2b_1+2},\{1\}^{c_1-3},\, \overline{2a_1+2},\dots,\overline{2b_r+2},\, \{1\}^{c_r-3},\, \overline{2a_r+2},\overline{2t})} \mathcal{H}_n(\boldsymbol{p}).$$

By taking $n \to \infty$ and using Lemma 4.5 we get immediately the following results.

Theorem 4.10. *Let* $t, r \in \mathbb{N}$ *and* $a_j, b_j, c_j - 3 \in \mathbb{N}_0$ *for all* $j \geq 1$. *Then*
($\underline{\mathbf{2}\text{-}c\text{-}\mathbf{2}\text{-}1}$-$\mathbf{2}$). *For* $\boldsymbol{s} = (\{2\}^{b_1}, c_1, \{2\}^{a_1}, 1, \dots, \{2\}^{b_r}, c_r, \{2\}^{a_r}, 1, \{2\}^t)$, $r \geq 1$, *we have*

$$\zeta^\star(\boldsymbol{s}) = -\sum_{\boldsymbol{p}\in\Pi(\overline{2b_1+2},\, \{1\}^{c_1-3},\, \overline{2a_1+2},\dots,\overline{2b_r+2},\, \{1\}^{c_r-3},\, \overline{2a_r+2},\overline{2t})} 2^{\ell(\boldsymbol{p})}\zeta(\boldsymbol{p}). \tag{30}$$

($\mathbf{2}$-1-$\underline{\mathbf{2}\text{-}c\text{-}\mathbf{2}\text{-}1}$-$\mathbf{2}$) : *For* $\boldsymbol{s} = (\{2\}^{a_0}, 1, \{2\}^{b_1}, c_1, \{2\}^{a_1}, 1, \dots, \{2\}^{b_r}, c_r, \{2\}^{a_r}, 1, \{2\}^t)$, $r \geq 0$ *and* $a_0 \geq 1$, *we have*

$$\zeta^\star(\boldsymbol{s}) = -\sum_{\boldsymbol{p}\in\Pi(2a_0+1,\, \overline{2b_1+2},\{1\}^{c_1-3},\, \overline{2a_1+2},\dots,\overline{2b_r+2},\, \{1\}^{c_r-3},\, \overline{2a_r+2},\overline{2t})} 2^{\ell(\boldsymbol{p})}\zeta(\boldsymbol{p}). \tag{31}$$

For example, taking $r = 1$ and $c_1 = 3$ we get in case ($\underline{\mathbf{2}\text{-}c\text{-}\mathbf{2}\text{-}1}$-$\mathbf{2}$)

$$\zeta^\star(\{2\}^b,3,\{2\}^a,1,\{2\}^t)=-2\zeta(\overline{2b+2a+2t+4})-4\zeta(2b+2a+4,\overline{2t})$$
$$-4\zeta(\overline{2b+2},2a+2t+2)-8\zeta(\overline{2b+2},\overline{2a+2},\overline{2t}),$$

and in case (**2**-1-<u>**2**-c-**2**-1</u>-**2**)

$$\begin{aligned}&\zeta^\star(\{2\}^{a_1},1,\{2\}^b,3,\{2\}^{a_2},1,\{2\}^t)\\&=-2\zeta(\overline{2a_1+2b+2a_2+2t+5})-4\zeta(2a_1+2b+2a_2+5,\overline{2t})\\&\quad-4\zeta(\overline{2a_1+2b+3},2a_2+2t+2)-8\zeta(\overline{2a_1+2b+3},\overline{2a_2+2},\overline{2t})\\&\quad-4\zeta(2a_1+1,\overline{2b+2a_2+2t+4})-8\zeta(2a_1+1,2b+2a_2+4,\overline{2t})\\&\quad-8\zeta(2a_1+1,\overline{2b+2},2a_2+2t+2)-16\zeta(2a_1+1,\overline{2b+2},\overline{2a_2+2},\overline{2t}).\end{aligned}$$

We have verified these formulas numerically for $a_1,a_2,a,b,t\le 2$ using EZ-face [**3**].

COROLLARY 4.11. *Let $r\in\mathbb{N}$ and $t,a_j,b_j,c_j-3\in\mathbb{N}_0$ for all $j\ge 1$. Then* (<u>**2**-1-**2**-c</u>-**2**). *For $\boldsymbol{s}=(\{2\}^{a_1},1,\{2\}^{b_1},c_1,\dots,\{2\}^{a_r},1,\{2\}^{b_r},c_r,\{2\}^t)$, we have*

$$H_n^\star(\boldsymbol{s})=-\sum_{\boldsymbol{p}\in\Pi(2a_1+1,\,\overline{2b_1+2},\,\{1\}^{c_1-3},\,\overline{2a_2+2},\,\dots,\,\overline{2a_r+2},\,\overline{2b_r+2},\,\{1\}^{c_r-3},\,2t+1)}\mathcal{H}_n(\boldsymbol{p}).$$

(**2**-c-<u>**2**-1-**2**-c</u>-**2**). *For $\boldsymbol{s}=(\{2\}^{b_1},c_1,\{2\}^{a_1},1,\dots,\{2\}^{b_r},c_r,\{2\}^{a_r},1,\{2\}^{b_{r+1}},c_{r+1},\{2\}^t)$ we have*

$$H_n^\star(\boldsymbol{s})=-\sum_{\boldsymbol{p}\in\Pi(\overline{2b_1+2},\{1\}^{c_1-3},\,\overline{2a_1+2},\dots,\overline{2a_r+2},\,\overline{2b_{r+1}+2},\,\{1\}^{c_{r+1}-3},\,2t+1)}\mathcal{H}_n(\boldsymbol{p}).$$

Setting $r=0$ in Corollary 4.11 we recover [**7**, Theorem 2.1]. When $r=1$ and $t=0$ we get the following: for all $n\in\mathbb{N}$ and $a,b\in\mathbb{N}_0$

$$\begin{aligned}H_n^\star(\{2\}^a,1,\{2\}^b,3)=&-2\sum_{k=1}^n\frac{(-1)^k\binom{n}{k}}{k^{2(a+b)+4}\binom{n+k}{k}}-4\sum_{k=1}^n\frac{H_{k-1}(\overline{2b+3})\binom{n}{k}}{k^{2a+1}\binom{n+k}{k}}\\&-4\sum_{k=1}^n\frac{H_{k-1}(1)(-1)^k\binom{n}{k}}{k^{2a+2b+3}\binom{n+k}{k}}-8\sum_{k=1}^n\frac{H_{k-1}(\overline{2b+2},1)\binom{n}{k}}{k^{2a+1}\binom{n+k}{k}}\end{aligned}$$

in case (<u>**2**-1-**2**-c</u>-**2**), and in case (**2**-c-<u>**2**-1-**2**-c</u>-**2**):

$$\begin{aligned}&H_n^\star(\{2\}^{b_1},3,\{2\}^a,1,\{2\}^{b_2},3)\\&=-2\sum_{k=1}^n\frac{(-1)^k\binom{n}{k}}{k^{2(b_1+a+b_2)+7}\binom{n+k}{k}}-4\sum_{k=1}^n\frac{H_{k-1}(\overline{2b_2+3})\binom{n}{k}}{k^{2b_1+2a+4}\binom{n+k}{k}}\end{aligned}$$

$$- 4\sum_{k=1}^{n} \frac{H_{k-1}(2a+2b_2+5)(-1)^k\binom{n}{k}}{k^{2b_1+2}\binom{n+k}{k}} - 8\sum_{k=1}^{n} \frac{H_{k-1}(\overline{2a+2},\overline{2b_2+3})(-1)^k\binom{n}{k}}{k^{2b_1+2}\binom{n+k}{k}}$$
$$- 4\sum_{k=1}^{n} \frac{H_{k-1}(1)(-1)^k\binom{n}{k}}{k^{2b_1+2a+2b_2+6}\binom{n+k}{k}} - 8\sum_{k=1}^{n} \frac{H_{k-1}(2a+2b_2+4,1)(-1)^k\binom{n}{k}}{k^{2b_1+2}\binom{n+k}{k}}$$
$$- 8\sum_{k=1}^{n} \frac{H_{k-1}(\overline{2b_2+2},1)\binom{n}{k}}{k^{2b_1+2a+4}\binom{n+k}{k}} - 16\sum_{k=1}^{n} \frac{H_{k-1}(\overline{2a+2},\overline{2b_2+2},1)(-1)^k\binom{n}{k}}{k^{2b_1+2}\binom{n+k}{k}}.$$

By taking $n \to \infty$ in Corollary 4.11 and using Lemma 4.5 we obtain

THEOREM 4.12. *Let $r \in \mathbb{N}$ and $a_j, b_j, c_j - 3 \in \mathbb{N}_0$ for all $j \geq 1$. Then*

($\underline{\mathbf{2}\text{-}1\text{-}\mathbf{2}\text{-}c}\text{-}\mathbf{2}$). *For $\boldsymbol{s} = (\{2\}^{a_1}, 1, \{2\}^{b_1}, c_1, \dots, \{2\}^{a_r}, 1, \{2\}^{b_r}, c_r, \{2\}^t)$ with $a_1 \geq 1$, we have*

$$\zeta^\star(\boldsymbol{s}) = - \sum_{\boldsymbol{p} \in \Pi(2a_1+1,\, \overline{2b_1+2},\, \{1\}^{c_1-3},\, \overline{2a_2+2}, \dots, \overline{2a_r+2},\, \overline{2b_r+2},\, \{1\}^{c_r-3},\, 2t+1)} 2^{\ell(\boldsymbol{p})} \zeta(\boldsymbol{p}). \tag{32}$$

($\mathbf{2}\text{-}c\text{-}\underline{\mathbf{2}\text{-}1\text{-}\mathbf{2}\text{-}c}\text{-}\mathbf{2}$). *For $\boldsymbol{s} = (\{2\}^{b_1}, c_1, \{2\}^{a_1}, 1, \dots, \{2\}^{b_r}, c_r, \{2\}^{a_r}, 1, \{2\}^{b_{r+1}}, c_{r+1}, \{2\}^t)$ we have*

$$\zeta^\star(\boldsymbol{s}) = - \sum_{\boldsymbol{p} \in \Pi(\overline{2b_1+2}, \{1\}^{c_1-3},\, \overline{2a_1+2}, \dots, \overline{2a_r+2},\, \overline{2b_{r+1}+2},\, \{1\}^{c_{r+1}-3},\, 2t+1)} 2^{\ell(\boldsymbol{p})} \zeta(\boldsymbol{p}). \tag{33}$$

For example, taking $r = 1$ and $t = 0$ in case ($\underline{\mathbf{2}\text{-}1\text{-}\mathbf{2}\text{-}c}\text{-}\mathbf{2}$) we get

$$\begin{aligned} \zeta^\star(\{2\}^a, 1, \{2\}^b, 3) = &-2\zeta(\overline{2a+2b+4}) - 4\zeta(1+2a, \overline{2b+3}) \\ &- 4\zeta(\overline{2a+2b+3}, 1) - 8\zeta(2a+1, \overline{2b+2}, 1). \end{aligned}$$

and in case ($\mathbf{2}\text{-}c\text{-}\underline{\mathbf{2}\text{-}1\text{-}\mathbf{2}\text{-}c}\text{-}\mathbf{2}$) we get

$$\begin{aligned} &\zeta^\star(\{2\}^{b_1}, 3, \{2\}^a, 1, \{2\}^{b_2}, 3) \\ &\quad = -2\zeta(\overline{2(b_1+a+b_2)+7}) - 4\zeta(2b_1+2a+4, \overline{2b_2+3}) \\ &\qquad - 4\zeta(\overline{2b_1+2}, 2a+2b_2+5) - 4\zeta(\overline{2b_1+2a+2b_2+6}, 1) \\ &\qquad - 8\zeta(\overline{2b_1+2}, \overline{2a+2}, \overline{2b_2+3}) - 8\zeta(\overline{2b_1+2}, 2a+2b_2+4, 1) \\ &\qquad - 8\zeta(2b_1+2a+4, \overline{2b_2+2}, 1) - 16\zeta(\overline{2b_1+2}, \overline{2a+2}, \overline{2b_2+2}, 1). \end{aligned}$$

We have numerically verified these formulas with $a, b_1, b_2 \leq 2$ using EZ-face [**3**].

5. Conjectures of Imatomi et al. on MZSV of type $\underline{\mathbf{2}\text{-}3\text{-}\mathbf{2}\text{-}1}$ and $\underline{\mathbf{2}\text{-}3\text{-}\mathbf{2}\text{-}1}\text{-}\mathbf{2}\text{-}1$.

The following Theorem 5.2 was first conjectured by Imatomi et al. [**13**, Conjectures 4.1 and 4.3]. Special cases have been proved in [**13**, Theorem 1.1] and by Tasaka and Yamamoto in [**21**]. Yamamoto proves a more precise version in [**23**]. We now give a different and concise proof using the identities we have found in the above. We begin with a lemma first.

LEMMA 5.1. *Let $n_1, \dots, n_\ell \in \mathbb{D}$ such that $|n_j|$ is even for every j. Set $m = |n_1| + \cdots + |n_\ell|$. Then*

$$\sum_{g\in\mathfrak{S}_\ell} \zeta\big(n_{g(1)}, \dots, n_{g(\ell)}\big)$$
$$= \sum_{e_1+\cdots+e_p=\ell} (-1)^{\ell-p} \prod_{s=1}^{p} (e_s-1)! \sum \zeta\Big(\bigoplus_{k\in\pi_1} n_k\Big) \dots \zeta\Big(\bigoplus_{k\in\pi_p} n_k\Big) \in \mathbb{Q}\pi^m$$

where the sum in the right is taken over all the possible unordered partitions of the set $\{1, \dots, \ell\}$ into p subsets $\pi_1, \dots, \pi_p$ with $e_1, \dots, e_p$ elements respectively.

PROOF. When all the arguments $n_1, \dots, n_\ell$ are positive the lemma becomes [**9**, Theorem 2.2]. Its proof there can be used here almost word for word. Notice that [**9**, Theorem 2.2] is re-proved as [**15**, Proposition 9.4] whose proof is different from that of [**9**] but also works here. Thus we leave the details to the interested reader. □

THEOREM 5.2. *Let r be a positive integer, and $e_1, \dots, e_{2r+1}$ nonnegative integers.*

(i) *Put $m = e_1 + \cdots + e_{2r}$. Then we have*

$$\sum_{\tau\in\mathfrak{S}_{2r}} \zeta^\star(\{2\}^{e_{\tau(1)}}, 3, \{2\}^{e_{\tau(2)}}, 1, \{2\}^{e_{\tau(3)}}, \dots, 3, \{2\}^{e_{\tau(2r)}}, 1) \in \mathbb{Q}\cdot\pi^{2m+4r}.$$

(ii) *Put $m = e_1 + \cdots + e_{2r+1}$. Then we have*

$$\sum_{\tau\in\mathfrak{S}_{2r+1}} \zeta^\star(\{2\}^{e_{\tau(1)}}, 3, \{2\}^{e_{\tau(2)}}, 1, \dots, 3, \{2\}^{e_{\tau(2n)}}, 1, \{2\}^{e_{\tau(2r+1)}+1}) \in \mathbb{Q}\cdot\pi^{2m+4r+2}.$$

PROOF. We start with (i) first. When $r = 1$ this follows quickly from (29) by shuffle relation. For general r let $a_j = e_{2j}$ and $b_j = e_{2j-1}$ for all $j \le r$ and let $A_j = 2e_j + 2$ for all $j \le 2r$. Then we can apply (27) of Theorem 4.8 to the string $\boldsymbol{s} = (\{2\}^{b_1}, 3, \{2\}^{a_1}, 1, \dots, \{2\}^{b_r}, 3, \{2\}^{a_r}, 1)$ and get

$$\zeta^\star(\boldsymbol{s}) = \sum_{\boldsymbol{p}\in\Pi(\overline{A_1},\dots,\overline{A_{2r}})} 2^{\ell(\boldsymbol{p})}\zeta(\boldsymbol{p}).$$

For any permutation $\tau \in \mathfrak{S}_{2r}$ and $\boldsymbol{s} = (\{2\}^{e_1}, 3, \{2\}^{e_2}, 1, \dots, \{2\}^{e_{2r-1}}, 3, \{2\}^{e_{2r}}, 1)$ we define

$$\boldsymbol{s}^\tau = (\{2\}^{e_{\tau(1)}}, 3, \{2\}^{e_{\tau(2)}}, 1, \dots, \{2\}^{e_{\tau(2r-1)}}, 3, \{2\}^{e_{\tau(2r)}}, 1)$$

Let $\boldsymbol{A} = (\overline{A_1}, \dots, \overline{A_{2r}})$, $\boldsymbol{A}^\tau = (\overline{A_{\tau(1)}}, \dots, \overline{A_{\tau(2r)}})$, and $P_\ell(2r)$ be the set of all partitions of $[2r] := \{1, 2, \dots, 2r\}$ into ℓ consecutive subsets. If $\lambda = (\lambda_1, \dots, \lambda_\ell) \in P_\ell(2r)$ then we set $\lambda_j(\boldsymbol{A}^\tau) = (\overline{A_{\tau(i)}})_{i\in\lambda_j}$ so that the concatenation $\bigsqcup_{j=1}^{\ell} \lambda_j(\boldsymbol{A}^\tau) = \boldsymbol{A}^\tau$. Because of the permutation we see that

$$\begin{aligned}\sum_{\tau\in\mathfrak{S}_{2r}} \zeta^\star(\boldsymbol{s}^\tau) &= \sum_{\tau\in\mathfrak{S}_{2r}} \sum_{\ell=1}^{2r} 2^\ell \sum_{\lambda\in P_\ell(2r)} \zeta\big(\oplus \lambda_1(\boldsymbol{A}^\tau), \dots, \oplus\lambda_\ell(\boldsymbol{A}^\tau)\big) \\ &= \sum_{\tau\in\mathfrak{S}_{2r}} \sum_{\ell=1}^{2r} \frac{2^\ell}{\ell!} \sum_{\lambda\in P_\ell(2r)} \sum_{g\in\mathfrak{S}_\ell} \zeta\big(\oplus \lambda_{g(1)}(\boldsymbol{A}^\tau), \dots, \oplus\lambda_{g(\ell)}(\boldsymbol{A}^\tau)\big),\end{aligned}$$

where $\oplus\boldsymbol{t}$ is the $\oplus$-sum of all the components of $\boldsymbol{t}$ for any composition $\boldsymbol{t}$. Hence Theorem 5.2(i) follows readily from the Lemma 5.1 since all A_j's are even numbers.

Theorem 5.2 (ii) follows from Theorem 4.10 in a similar fashion so we leave the details to the interested reader. □

Remark 5.3. We notice that in [**23**, Theorem 1.1] Yamamoto obtains a more precise formula by using partial sums and generating functions:

$$\begin{aligned}&\sum_{\substack{e_0,e_1,\dots,e_{2r}\ge 0\\ e_0+e_1+\cdots+e_{2r}=m}} \zeta^\star(\{2\}^{e_0}, 3, \{2\}^{e_1}, 1, \{2\}^{e_2}, 3, \dots, 3, \{2\}^{e_{2r-1}}, 1, \{2\}^{e_{2r}}) \\ &= \sum_{\substack{2i+k+u=2r\\ j+l+v=m}} (-1)^{j+k} \binom{k+l}{k}\binom{u+v}{u}\binom{2i+j}{j} \frac{\beta_{k+l}\beta_{u+v}\pi^{4r+2m}}{(2i+1)(4i+2j+1)!}, \qquad (34)\end{aligned}$$

where $\beta_n = (-1)^n(2-2^{2n})B_{2n}/(2n)!$. It is possible to modify our proof of Theorem 5.2 to give this more quantified version.

6. More Conjetures of Imatomi et al.

The following results were first conjectured by Imatomi et al. [**13**, Conjecture 4.5].

Theorem 6.1. *Let m and n be two nonnegative integers.*

(i) *We have*

$$\zeta^\star(\{2\}^n, 3, \{2\}^m, 1) + \zeta^\star(\{2\}^m, 3, \{2\}^n, 1) = \zeta^\star(\{2\}^{n+1})\zeta^\star(\{2\}^{m+1}).$$

(ii) *We have*

$$(2n+1)\zeta^\star(\{3,1\}^n, 2) = \sum_{j+k=n} \zeta^\star(\{3,1\}^j)\zeta^\star(\{2\}^{2k+1}).$$

(iii) *If $n \geq 1$ then we have*

$$\sum_{\substack{e_1+e_2+\cdots+e_{2n}=1\\ e_1,e_2,\dots,e_{2n}\geq 0}} \zeta^\star(\{2\}^{e_1},3,\{2\}^{e_2},1,\dots,\{2\}^{e_{2n-1}},3,\{2\}^{e_{2n}},1)$$
$$= \sum_{j+k=n-1} \zeta^\star(\{3,1\}^j,2)\zeta^\star(\{2\}^{2k+2}).$$

Proof. $\underline{\text{(i)}}$. This follows immediately from (5) and (29).

$\underline{\text{(ii)}}$. We notice that by taking $a_i=b_i=0$ and $c_i=3$ for all $i\leq r=j$ in Corollary 4.7($\underline{\mathbf{2}\text{-}c\text{-}\mathbf{2}\text{-}1}$) we get

$$\zeta^\star(\{3,1\}^j) = \sum_{\boldsymbol{p}_{2j}\in\Pi(\{\overline{2}\}^{2j})} 2^{\ell(\boldsymbol{p}_{2j})}\zeta(\boldsymbol{p}_{2j}).$$

All of the components a_j of $\overline{2}\circ\cdots\circ\overline{2}$ must satisfy the following sign rule:

$$a_j > 0 \text{ if and only if } 4|a_j. \tag{35}$$

On the other hand, by Corollary 4.9($\underline{\mathbf{2}\text{-}c\text{-}\mathbf{2}\text{-}1}$-$\mathbf{2}$) we have

$$\zeta^\star(\{3,1\}^n,2) = -\sum_{\boldsymbol{p}_{2n+1}\in\Pi(\{\overline{2}\}^{2n+1})} 2^{\ell(\boldsymbol{p}_{2n+1})}\zeta(\boldsymbol{p}_{2n+1}).$$

Hence by (5) we need to show that

$$(2n+1)\sum_{\boldsymbol{p}_{2n+1}\in\Pi(\{\overline{2}\}^{2n+1})} 2^{\ell(\boldsymbol{p}_{2n+1})}\zeta(\boldsymbol{p}_{2n+1})$$
$$=\sum_{j=0}^{n}\sum_{\boldsymbol{p}_{2j}\in\Pi(\{\overline{2}\}^{2j})} 2^{\ell(\boldsymbol{p}_{2j})}\zeta(\boldsymbol{p}_{2j})\cdot 2\zeta(\overline{4(n-j)+2}). \tag{36}$$

Suppose an index $\boldsymbol{p}_{2n+1}$ in $\Pi(\{\overline{2}\}^{2n+1})$ has length t ($1\leq t\leq 2n+1$) given as

$$(a_1,\dots,a_t),\quad a_i\in\mathbb{D},\ \forall i=1,\dots,t.$$

We now show that there are exactly $2^t(2n+1)$ copies of such term produced by stuffle product on the right hand side of (36). Indeed, for each $i=1,\dots,t$ the entry a_i has two possibilities:

(1). $a_i=4b_i>0$. Then for each $k=1,\dots,b_i$ we may produce such a term on the right hand side of (36) by stuffing

$$2^t\zeta(a_1,\dots,a_{i-1},\overline{4k-2},a_{i+1},\dots,a_t)$$

from $\boldsymbol{p}_{2j}$ having length t with the term $2\zeta(\overline{4(b_i-k)+2})$ at the right end of (36). Notice no shuffle is possible since $4(n-j)+2$ is not a multiple of 4. Hence these contribute to $2^{t+1}b_i = 2^{t-1}a_i$ copies of $\zeta(a_1,\dots,a_t)$.

(2). $a_i = \overline{4b_i+2}$. Then for each $k=1,\dots,b_i$ we may produce such a term on the right hand side of (36) by stuffing

$$2^t\zeta(a_1,\dots,a_{i-1},\overline{4k},a_{i+1},\dots,a_t)$$

from $\boldsymbol{p}_{2j}$ having length t with the term $2\zeta(\overline{4(b_i-k)+2})$ at the right end of (36). Further, there is exactly one possible shuffle given by

$$2^{t-1}\zeta(a_1,\dots,a_{i-1},a_{i+1},\dots,a_{t+1})\,ш\,\big\{2\zeta(\overline{4b_i+2})\big\},$$

since the index $(a_1,\dots,a_{i-1},a_{i+1},\dots,a_t)$ has only length $t-1$. Altogether these produce $2^{t+1}b_i+2^t = 2^{t-1}|a_i|$ copies of $\zeta(a_1,\dots,a_t)$.

By combining (1) and (2) we see that the right hand side of (36) produces exactly

$$\sum_{i=1}^{t} 2^{t-1}|a_i| = 2^{t-1}\cdot|(a_1,\dots,a_t)| = 2^t(2n+1)$$

copies of $\zeta(a_1,\dots,a_t)$ since the weight is $4n+2$. This proves (ii).

(iii). We use the same analysis as above and see that we need to prove the following identity:

$$\sum_{\boldsymbol{q}_{2n}} 2^{\ell(\boldsymbol{q}_{2n})}\zeta(\boldsymbol{q}_{2n}) = \sum_{j=0}^{n-1}\sum_{\boldsymbol{p}_{2j+1}\in\Pi(\{\overline{2}\}^{2j+1})} 2^{\ell(\boldsymbol{p}_{2j+1})}\zeta(\boldsymbol{p}_{2j+1})\cdot 2\zeta(\overline{4(n-j)}), \tag{37}$$

where $\boldsymbol{q}_{2n}$ runs through all indices of the form $A_1\circ\cdots\circ A_{2n}$ with one of the A_j's (say A_{j_0}) equal to $\overline{4}$ and all the other A_j's equal to $\overline{2}$. For each choice of $2^t\zeta(a_1,\dots,a_{t+1})$ with length $t+1$ from the left hand of (37), all but one of the argument components $a_1,\dots,a_{t+1}$ must satisfy the sign rule (35). The only exceptional component, say a_i, must involve a merge with the special entry $A_{j_0}=\overline{4}$. Now there are two possibilities:

(1). $a_i = 4b_i+2>0$. Then for each $k=0,\dots,b_i-1$ we may produce such a term on the right hand side of (37) by stuffing

$$2^t\zeta(a_1,\dots,a_{i-1},\overline{4k+2},a_{i+1},\dots,a_{t+1})$$

from $\boldsymbol{p}_{2j+1}$ having length $t+1$ with the term $2\zeta(\overline{4(b_i-k)})$ at the right end of (37). Notice no shuffle is possible since $4(n-j)$ is a multiple of 4. Hence these contribute to $2^{t+1}b_i$ copies of $\zeta(a_1,\dots,a_{t+1})$. On the left hand side, such a term must be produced by setting all $2b_i-1$ consecutive $\circ$'s around $A_{j_0}=\overline{4}$ to $\oplus$:

$$\dots,\underbrace{A_i\oplus A_{i+1}\oplus\cdots\oplus A_{j_0}\oplus\cdots\oplus A_\ell}_{2b_i\text{ entries}},\dots.$$

But A_{j_0} can be at any one of the $2b_i$ possible positions, thus producing $2^{t+1}b_i$ copies of $\zeta(a_1,\dots,a_{t+1})$ which match exactly the right hand side of (37).

(2). $a_i=\overline{4b_i}$. Then for each $k=1,\dots,b_i-1$ we may produce such a term on the right hand side of (37) by stuffing

$$2^t\zeta(a_1,\dots,a_{i-1},4k,a_{i+1},\dots,a_{t+1})$$

from $\boldsymbol{p}_{2j}$ having length $t+1$ with the term $2\zeta(\overline{4(b_i-k)})$ at the right end of (37). Further, there is exactly one possible shuffle given by

$$2^{t-1}\zeta(a_1,\dots,a_{i-1},a_{i+1},\dots,a_{t+1})\,ш\,\big\{2\zeta(\overline{4b_i})\big\}.$$

Hence these contribute to $2^{t+1}(b_i-1)+2^t=2^t(2b_i-1)$ copies of $\zeta(a_1,\dots,a_{t+1})$. Similar to (1), on the left hand side, such a term must be produced by setting all $2b_i-2$ consecutive $\circ$'s around A_{j_0} to $\oplus$. And A_{j_0} can be at any one of the $2b_i-1$ possible positions, thus producing $2^t(2b_i-1)$ copies of $\zeta(a_1,\dots,a_{t+1})$ which match exactly the right hand side of (37).

This concludes the proof of theorem. □

Note that Theorem 6.1(i) is the more precise version of the $n=1$ case of Theorem 5.2(i). And Theorem 6.1(iii) can be written more compactly as

$$\zeta^\star\big(\{2\}\,ш\{3,1\}^n\big)=\sum_{k=0}^n\zeta^\star(\{3,1\}^{n-k},2)\zeta^\star(\{2\}^{2k}),$$

which is the more precise version of the $m=1$ case of the following result of Kondo et al. [**14**]: For all nonnegative integers m and n we have

$$\zeta^\star(\{2\}^m\,ш\{3,1\}^n)\in\mathbb{Q}\pi^{2m+4n}.$$

The case $m=0$ case has the following precise formulation by Muneta [**17**]:

$$\zeta^\star\big(\{3,1\}^n\big)=\sum_{i=0}^n\left\{\frac{2}{(4i+2)!}\sum_{\substack{n_0+n_1=2(n-i)\\ n_0,n_1\ge 0}}(-1)^{n_1}\frac{(2^{2n_0}-2)B_{2n_0}}{(2n_0)!}\frac{(2^{2n_1}-2)B_{2n_1}}{(2n_1)!}\right\}\pi^{4n}.$$

Muneta also found precise form in case $m=1$. Of course, these are all special cases of Yamamoto's general formula (34).

7. MHS: $\underline{\mathbf{1}\text{-}c}\text{-}\mathbf{1}$ formula.

In this section we turn to MHS of the type $\underline{\mathbf{1}\text{-}c}\text{-}\mathbf{1}$ where the trailing **1** may be vacuous and the c's may be any positive integers such that $c\ge 2$ (which is different from the requirement $c\ge 3$ in the previous sections). The corresponding MZSVs diverge when the leading **1** is non-empty, however, in a sequel to this paper we will study the congruence properties of MHS where the results of this section will be utilized.

The following theorem generalizes [**7**, Theorem 2.2]. For $\boldsymbol{s}=(s_1,\dots,s_m)\in\mathbb{D}^m$ we define

$$\hbar_n(\boldsymbol{s}):=\sum_{n\geq k_1>\cdots>k_m\geq 1}\binom{n}{k_1}\prod_{j=1}^{m}\frac{\operatorname{sgn}(s_j)^{k_j}}{k_j^{|s_j|}}=\sum_{k=1}^{n}\frac{\operatorname{sgn}(s_1)^k}{k^{|s_1|}}\binom{n}{k}H_{k-1}(s_2,\dots,s_m).$$

THEOREM 7.1. *Let $r\in\mathbb{N}_0$ and $\boldsymbol{s}=(\{1\}^{a_1},c_1,\dots,\{1\}^{a_r},c_r,\{1\}^t)$ where $t,a_j\in\mathbb{N}_0$ and positive integers $c_j\geq 2$ for all $j\geq 1$. If $r=0$ then we have*

$$H_n^\star(\{1\}^t)=-\hbar_n(\bar{t}).\tag{38}$$

If $r\geq 1$ then we have

$$H_n^\star(\boldsymbol{s})=-\sum_{\boldsymbol{p}\in\Pi(\overline{a_1+1},\,\{1\}^{c_1-2},\,a_2+2,\dots,a_r+2,\,\{1\}^{c_r-2},\,t+1)}\hbar_n(\boldsymbol{p}).\tag{39}$$

PROOF. Equation (38) follows from [**22**, Lemma 5.4]. We now assume $r\geq 1$ and proceed by induction on $n+r\geq 2$ to prove (39). If $n+r=2$ then $n=1$ and it is clear that both sides in (39) are equal to 1. Assume now the theorem is true for all $n+r\leq N$ where $N\geq 2$. Suppose we have $n\geq 2$ and $n+r=N+1$. By definition

$$\begin{aligned}H_n^\star(\boldsymbol{s})=&\sum_{l=0}^{a_1}\frac{1}{n^{a_1-l}}H_{n-1}^\star(\{1\}^l,c_1,\dots,\{1\}^{a_r},c_r,\{1\}^t)\\&+\frac{1}{n^{a_1+c_1}}H_n^\star(\{1\}^{a_2},c_2,\dots,\{1\}^{a_r},c_r,\{1\}^t).\end{aligned}$$

For ease of reading we define the following index sets: for any composition $\boldsymbol{v}$ of integers

$$\begin{aligned}I(\boldsymbol{v})&=\Pi(\boldsymbol{v},\,\{1\}^{c_1-2},\,a_2+2,\,\{1\}^{c_2-2},\dots,a_r+2,\,\{1\}^{c_r-2},\,t+1),\\J&=\Pi(\overline{a_2+1},\,\{1\}^{c_2-2},\,a_3+2,\dots,a_r+2,\,\{1\}^{c_r-2},\,t+1)\end{aligned}$$

if $r\geq 2$ and $I(\boldsymbol{v})=\Pi(\boldsymbol{v},\,\{1\}^{c_1-2},t+1)$, $J=\Pi(\bar{t})$ if $r=1$. By induction assumption

$$\begin{aligned}H_n^\star(\boldsymbol{s})&=-\sum_{l=0}^{a_1}\frac{1}{n^{a_1-l}}\sum_{\boldsymbol{q}\in I(\overline{l+1})}\hbar_{n-1}(\boldsymbol{q})-\frac{1}{n^{a_1+c_1}}\sum_{\boldsymbol{p}\in J}\hbar_n(\boldsymbol{p})\\&=-\sum_{(q_1,\dots,q_m)\in I(\bar{1})}\sum_{l=0}^{a_1}\frac{1}{n^{a_1-l}}\sum_{k=1}^{n-1}\frac{\operatorname{sgn}(q_1)^k}{k^{l+|q_1|}}\binom{n-1}{k}H_{k-1}(q_2,\dots,q_m)\\&\quad-\frac{1}{n^{a_1+c_1}}\sum_{\boldsymbol{p}\in J}\hbar_n(\boldsymbol{p}).\end{aligned}$$

By changing the order of summations and using the identity

$$\sum_{l=0}^{a_1}\left(\frac{n}{k}\right)^l = \frac{1}{k^{a_1}}\cdot\frac{n^{a_1+1}-k^{a_1+1}}{n-k}$$

we see easily that

$$\begin{aligned}H_n^\star(\boldsymbol{s}) = &-\sum_{(q_1,\dots,q_m)\in I(\overline{1})}\sum_{k=1}^{n}\left(1-\frac{k^{a_1+1}}{n^{a_1+1}}\right)\frac{\operatorname{sgn}(q_1)^k}{k^{a_1+|q_1|}}\binom{n}{k}H_{k-1}(q_2,\dots,q_m)\\ &-\frac{1}{n^{a_1+c_1}}\sum_{\boldsymbol{p}\in J}\hbar_n(\boldsymbol{p}).\end{aligned}$$

Observe that the index set

$$I(\overline{1}) = \bigcup_{(p_1,\dots,p_m)\in J}\Big\{\big(\overline{1}\circ\underbrace{1\circ\cdots\circ 1}_{c_1-2\text{ times}}\circ(p_1+1),p_2,\dots,p_m\big)\Big\}. \tag{40}$$

For each $(\overline{p_1},p_2\dots,p_m)$ we can partition the set (40) into the following subsets:

$$\big\{\big(\overline{c_1+p_1},\boldsymbol{v}\big)\big\}\cup\big\{\big(\overline{j+1},\boldsymbol{y},i+p_1,\boldsymbol{v}\big)\big\},\quad \boldsymbol{v}=(p_2,\dots,p_m),$$

for $i\ge 1$, $j\ge 0$ and positive compositions $\boldsymbol{y}$ with $i+j+|\boldsymbol{y}|=c_1-1$. Thus it suffices to prove that

$$\sum_{\substack{i+j+|\boldsymbol{y}|=c_1-1,\\ i\ge1,j\ge0}}\sum_{k=1}^{n}\frac{H_{k-1}(\boldsymbol{y},i+p_1)\binom{n}{k}}{(-1)^k k^j} = \frac{1}{n^{c_1-1}}\sum_{k=1}^{n}\frac{(-1)^k\binom{n}{k}}{k^{p_1}} - \sum_{k=1}^{n}\frac{(-1)^k\binom{n}{k}}{k^{c_1-1+p_1}}. \tag{41}$$

Equation (41) follows from (7) of Lemma 2.1 when $m=1$, $A_{n,k}^{(1)}$ as in Remark 2.2, $c=c_1-1, a=p_1$, $\boldsymbol{x}=(\boldsymbol{y},i+p_1)$ and $\boldsymbol{v}=\emptyset$. This completes the proof of our theorem. $\square$

For example, when $r=1$ we recover [**7**, Theorem 2.2] and when $r=2$ we get for all $a_1,a_2,t\in\mathbb{N}_0$ and positive integers $c_1,c_2\ge 2$

$$\begin{aligned}&H_n^\star(\{1\}^{a_1},c_1,\{1\}^{a_2},c_2,\{1\}^t)\\ &\quad=-\sum_{k=1}^{n}\frac{(-1)^k\binom{n}{k}}{k^{a_1+c_1+a_2+c_2+t}}-\sum_{\substack{i_2+j_2+|\boldsymbol{x}_2|=c_2,\\ i_2,j_2\ge1}}\sum_{k=1}^{n}\frac{H_{k-1}(\boldsymbol{x}_2,i_2+t)\binom{n}{k}}{(-1)^k k^{a_1+c_1+a_2+j_2}}\\ &\qquad-\sum_{\substack{i_1+j_1+|\boldsymbol{x}_1|=c_1,\\ i_1,j_1\ge1}}\sum_{k=1}^{n}\frac{H_{k-1}(\boldsymbol{x}_1,i_1+a_2+c_2+t)\binom{n}{k}}{(-1)^k k^{a_1+j_1}}\end{aligned}$$

$$- \sum_{\substack{i_\alpha + j_\alpha + |\boldsymbol{x}_\alpha| = c_\alpha, \\ i_\alpha, j_\alpha \geq 1, \ \alpha=1,2}} \sum_{k=1}^{n} \frac{H_{k-1}(\boldsymbol{x}_1, i_1 + a_2 + j_2, \boldsymbol{x}_2, i_2 + t)\binom{n}{k}}{(-1)^k k^{a_1+j_1}}.$$

8. Concluding Remarks.

There are many recent studies on MZVs, MZSVs and even their q-analogs. Most of the MZSV relations in [**11**] and [**12**] involving special types of arguments like ours in this paper can be proved in a more straight-forward manner using our results. However, it seems that the techniques contained here are hard to generalize to deal with MZVs even though these two types of values are extremely closely related from the point of view of their algebraic structures (see [**10**], [**12**], [**18**], [**20**]). Such a generalization should help us resolve more conjectures such as those listed in [**2**, Section 7.2].

There are three more directions of research that should be of great interest. One is a theory generalizing the MHS identities obtained in this paper to truly alternating ones. We are aware of only one such instance. Setting $x = 0$ and $x = 1$ in [**22**, Lemma 5.4] we get

$$H_n^\star(\{1\}^a, \bar{1}) = \sum_{k=1}^{n} \frac{(2^k - 1)(-1)^k}{k^{a+1}} \binom{n}{k}, \quad \forall a \in \mathbb{N}_0.$$

Another direction is to establish a corresponding theory for the q-analogs multiple zeta values [**4**], [**28**]. Initial computations show it is quite a promising project, see [**6**], [**8**].

As for the third direction we notice that many MHS identities proved in this paper can be used not only to derive MZSV identities but also to prove many congruences of MHS. This idea has already been applied in [**7**] to prove one of our conjectures in [**29**]. In general, these congruences should shed more light on the unsolved [**31**, Conjecture 2.6] and the conjectures at the end of [**29**]. We plan to carry this out in a sequel to this paper.

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