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On hearts which are module categories

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Abstract. Given a torsion pair $\boldsymbol{t} = (\mathcal{T}, \mathcal{F})$ in a module category R-Mod we give necessary and sufficient conditions for the associated Happel–Reiten–Smalø t-structure in $\mathcal{D}(R)$ to have a heart $\mathcal{H}_{\boldsymbol{t}}$ which is a module category. We also study when such a pair is given by a 2-term complex of projective modules in the way described by Hoshino–Kato–Miyachi ([**HKM**]). Among other consequences, we completely identify the hereditary torsion pairs $\boldsymbol{t}$ for which $\mathcal{H}_{\boldsymbol{t}}$ is a module category in the following cases: i) when $\boldsymbol{t}$ is the left constituent of a TTF triple, showing that $\boldsymbol{t}$ need not be HKM; ii) when $\boldsymbol{t}$ is faithful; iii) when $\boldsymbol{t}$ is arbitrary and the ring R is either commutative, semi-hereditary, local, perfect or Artinian. We also give a systematic way of constructing non-tilting torsion pairs for which the heart is a module category generated by a stalk complex at zero.

1. Introduction.

Beilinson, Bernstein and Deligne [**BBD**] introduced the notion of t-structure in a triangulated category in their study of perverse sheaves on an algebraic or analytic variety. If $\mathcal{D}$ is such a triangulated category, a t-structure in $\mathcal{D}$ is a pair of full subcategories satisfying suitable axioms (see the precise definition in next section) which guarantee that their intersection is an abelian category $\mathcal{H}$, called the heart of the t-structure. This category comes with a cohomological functor $\mathcal{D} \longrightarrow \mathcal{H}$. Roughly speaking, a t-structure allows to develop an intrinsic (co)homology theory, where the homology 'spaces' are again objects of $\mathcal{D}$ itself.

In the context of bounded derived categories, Happel, Reiten and Smalø [**HRS**] associated to each torsion pair $\boldsymbol{t}$ in an abelian category $\mathcal{A}$, a t-structure in the bounded derived category $\mathcal{D}^b(\mathcal{A})$. This t-structure is actually the restriction of a t-structure in the unbounded derived category $\mathcal{D}(\mathcal{A})$, when this later category is defined. Several authors (see [**CGM**], [**CMT**], [**MT**], [**CG**]) have dealt with the problem of deciding when its heart $\mathcal{H}_{\boldsymbol{t}}$ is a Grothendieck or module category. When $\mathcal{A} = \mathcal{G}$ is a Grothendieck category, after recent work by the authors (see [**PS**] and [**PS1**]), the condition that $\mathcal{H}_{\boldsymbol{t}}$ be a Grothendieck category is well understood. Indeed, $\mathcal{H}_{\boldsymbol{t}}$ is a Grothendieck category if, and

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only if, the torsionfree class of the pair is closed under taking direct limits in $\mathcal{G}$ (see [**PS**, Theorem 4.9] and [**PS1**, Theorem 1.2]).

The situation when $\mathcal{H}_{\boldsymbol{t}}$ is a module category is far less understood, even in the case when $\mathcal{A} = R$-Mod is a module category. The problem has been tackled, from different perspectives, in [**HKM**], [**CGM**], [**CMT**] and [**MT**]. In the second of these references, the authors show that an abelian category with a classical 1-tilting object is equivalent to $\mathcal{H}_{\boldsymbol{t}}$, for some faithful torsion pair $\boldsymbol{t}$ in a module category. Since a classical tilting object defines an equivalence between the derived categories of the ambient abelian category and of the endomorphism ring of the object, faithful torsion pairs in module categories became natural candidates to study when the heart is a module category. In [**CMT**] the authors pursued this line and gave necessary and sufficient conditions for a faithful torsion pair in a module category to have a modular heart. In the earlier paper [**HKM**], the authors had associated a pair of subcategories $(\mathcal{X}(P^\bullet), \mathcal{Y}(P^\bullet))$ of R-Mod to a 2-term complex $P^\bullet$ of finitely generated projective modules. Then they gave necessary and sufficient conditions for the pair to be a torsion pair, in which case the corresponding heart was a module category. In [**MT**], for a given torsion pair $\boldsymbol{t}$ in R-Mod, the authors compared the conditions that the heart be a module category with the condition that $\boldsymbol{t}$ be a torsion pair as in [**HKM**]. In particular, they proved that if $\boldsymbol{t}$ is faithful then both conditions were equivalent.

In the present paper, given any torsion pair $\boldsymbol{t}$ in a module category R-Mod, we give necessary and sufficient conditions for the heart $\mathcal{H}_{\boldsymbol{t}}$ to be a module category and, simultaneously, compare this property with that of $\boldsymbol{t}$ being an HKM torsion pair (see next section for all the pertinent definitions of the terms that we use in this introduction). When tackled in full generality, the conditions that appear tend to be rather technical, but a deeper look in particular cases gives more precise information on the torsion pair. Roughly speaking, when one assumes that $\boldsymbol{t}$ is hereditary one falls into the world of TTF triples, while if one assumes that the torsion class is closed under taking products in R-Mod, then one enters the world of classical tilting torsion pairs.

The following is a list of the main results, all of them given for a torsion pair $\boldsymbol{t} = (\mathcal{T}, \mathcal{F})$ in R-Mod:

1. (Part of Theorem 4.1) If $\boldsymbol{t}$ is hereditary and $\mathcal{H}_{\boldsymbol{t}}$ is a module category, then $\boldsymbol{t}' = (\mathcal{T} \cap R/t(R)\text{-Mod}, \mathcal{F})$ is the right constituent torsion pair of a TTF triple in $R/t(R)$-Mod. When $\boldsymbol{t}$ is bounded, it is itself the right constituent pair of a TTF triple in R-Mod.
2. (Corollary 5.2) $\mathcal{H}_{\boldsymbol{t}}$ has a progenerator which is a stalk complex $V[0]$ if, and only if, $\boldsymbol{t}$ is the torsion pair associated to a finitely presented quasi-tilted R-module V such that $\operatorname{Ext}^2_R(V, ?)_{|\mathcal{F}} = 0$ and $\mathcal{T}$ cogenerates $\mathcal{F}$. There is a systematic way (see Theorem 6.2) of constructing non-tilting modules V satisfying this property.
3. (Part of Proposition 5.7) If $\boldsymbol{t}$ is hereditary and the left constituent pair of a TTF triple, then $\mathcal{H}_{\boldsymbol{t}}$ is a module category if, and only if, there is a finitely generated projective module P such that $\mathcal{T} = \operatorname{Gen}(P)$. In general, $\boldsymbol{t}$ need not be HKM.
4. (Part of Theorem 6.1) If $\mathcal{T}$ is closed under taking products in R-Mod and $\mathcal{H}_{\boldsymbol{t}}$ is a module category, then there is a finitely presented module V such that $\mathcal{T} = \operatorname{Gen}(V)$ and V is classical 1-tilting over $R/\boldsymbol{a}$, where $\boldsymbol{a} = \operatorname{ann}_R(V)$. Moreover, the torsion pair $\boldsymbol{t}' = (\operatorname{Gen}(V), \mathcal{F} \cap R/\boldsymbol{a}\text{-Mod})$ in $R/\boldsymbol{a}$-Mod has a heart which is a module category and

embeds faithfully in $\mathcal{H}_{\boldsymbol{t}}$.

5. (Theorem 7.1) Suppose that $\boldsymbol{t}$ is the right constituent pair of the TTF triple $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ in R-Mod defined by the idempotent ideal $\boldsymbol{a}$. Under fairly general hypotheses, the heart $\mathcal{H}_{\boldsymbol{t}}$ is a module category if, and only if, $\boldsymbol{a}$ is finitely generated on the left and there is a finitely generated projective R-module P such that:
   (a) $P/\boldsymbol{a}P$ is a progenerator of $R/\boldsymbol{a}$-Mod;
   (b) There is an exact sequence $0 \to F \longrightarrow C \longrightarrow \boldsymbol{a}P \to 0$, with C finitely generated module in $\mathcal{C}$, such that $\mathrm{Ext}^1_R(C, ?)_{|\mathcal{F}} = 0$ and C generates $\mathcal{C} \cap \mathcal{F}$.
6. If $\boldsymbol{t}$ is the right constituent of the TTF triple defined by a finitely generated projective module whose trace in R is finitely generated, then $\mathcal{H}_{\boldsymbol{t}}$ is a module category (Corollary 7.2). Under fairly general hypotheses, the converse is also true for arbitrary faithful hereditary torsion pairs (Corollary 7.7).
7. For the following classes of rings, all hereditary torsion pairs whose heart is a module category are identified: commutative (Corollary 4.3), semihereditary (Proposition 5.9), local, perfect and Artinian (Corollary 7.5).

The organization of the paper goes as follows. Section 2 gives the preliminaries that are needed and the terminology which is used in the paper. Section 3 is devoted to giving necessary and sufficient conditions on an arbitrary torsion pair $\boldsymbol{t}$ in R-Mod for its heart to be a module category and also for it to be an HKM pair. In Section 4 we assume that $\boldsymbol{t}$ is hereditary and show how TTF triples appear naturally. In Section 5, we give necessary and sufficient conditions for $\mathcal{H}_{\boldsymbol{t}}$ to have a progenerator which is a sum of stalk complexes. In Section 6 we assume that the torsion class is closed under taking products and show that the modular condition on $\mathcal{H}_{\boldsymbol{t}}$ naturally leads to classical tilting torsion pairs. In Section 7, we assume that $\boldsymbol{t}$ is the right constituent torsion pair of a TTF triple, and give necessary and sufficient conditions for $\mathcal{H}_{\boldsymbol{t}}$ to be a module category and for $\boldsymbol{t}$ to be an HKM pair. We end the paper with a final section of illustrative examples.

2. Terminology and preliminaries.

In this paper all rings are supposed to be associative with unit and their modules will be always unital modules. Unless otherwise stated, 'module' will mean 'left module' and if R is a ring, we shall denote by R-Mod and Mod-R (=R^{op}-Mod) its categories of left and right modules, respectively. A *module category* is any one which is equivalent to R-Mod, for some ring R.

The concepts that we shall introduce in this section are mainly applied to the case of module categories, but sometimes we will use them in the most general context of Grothendieck categories and is in this context that we introduce them. Let then $\mathcal{G}$ be a Grothendieck category all throughout this section.

A *torsion pair* in $\mathcal{G}$ is a pair $\boldsymbol{t} = (\mathcal{T}, \mathcal{F})$ of full subcategories satisfying the following two conditions:

- $\mathrm{Hom}_{\mathcal{G}}(T, F) = 0$, for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$;
- For each object X of $\mathcal{G}$ there is an exact sequence $0 \to T_X \longrightarrow X \longrightarrow F_X \to 0$, where $T_X \in \mathcal{T}$ and $F_X \in \mathcal{F}$.

In such case the objects T_X and F_X are uniquely determined, up to isomorphism, and the assignment $X \rightsquigarrow T_X$ (resp. $X \rightsquigarrow F_X$) underlies a functor $t : \mathcal{G} \longrightarrow \mathcal{T}$ (resp. $(1 : t) : \mathcal{G} \longrightarrow \mathcal{F}$) which is right (resp. left) adjoint to the inclusion functor $\mathcal{T} \hookrightarrow \mathcal{G}$ (resp. $\mathcal{F} \hookrightarrow \mathcal{G}$). We will frequently write $X/t(X)$ to denote $(1 : t)(X)$. The composition $\mathcal{G} \xrightarrow{t} \mathcal{T} \hookrightarrow \mathcal{G}$, which we will still denote by t, is called the *torsion radical* associated to $\boldsymbol{t}$. We call $\mathcal{T}$ and $\mathcal{F}$ the *torsion class* and *torsionfree class* of the pair, respectively. For each class $\mathcal{X}$ of objects, we will put $\mathcal{X}^{\perp} = \{M \in \mathcal{G} : \mathrm{Hom}_{\mathcal{G}}(X, M) = 0, \text{ for all } X \in \mathcal{X}\}$ and ${}^{\perp}\mathcal{X} = \{M \in \mathcal{G} : \mathrm{Hom}_{\mathcal{G}}(M, X) = 0, \text{ for all } X \in \mathcal{X}\}$. If $\boldsymbol{t}$ is a torsion pair as above, then $\mathcal{T} = {}^{\perp}\mathcal{F}$ and $\mathcal{F} = \mathcal{T}^{\perp}$. The torsion pair is called *hereditary* when $\mathcal{T}$ is closed under taking subobjects in $\mathcal{G}$. It is called *split* when $t(X)$ is a direct summand of X, for each object X of $\mathcal{G}$. If R is a ring and $\mathcal{G} = R$-Mod, we will say that $\boldsymbol{t}$ is *faithful* when $R \in \mathcal{F}$.

A class $\mathcal{T} \subseteq \mathcal{G}$ is a *TTF* (=*torsion-torsionfree*) *class* when it is both a torsion and a torsionfree class in $\mathcal{G}$. Each triple of the form $(\mathcal{C}, \mathcal{T}, \mathcal{F}) = ({}^{\perp}\mathcal{T}, \mathcal{T}, \mathcal{T}^{\perp})$, for some TTF class $\mathcal{T}$, will be called a *TTF triple* and the two torsion pairs $(\mathcal{C}, \mathcal{T})$ and $(\mathcal{T}, \mathcal{F})$ will be called the *left constituent pair* and *right constituent pair* of the TTF triple. The TTF triple is called *left* (*resp. right*) *split* when its left (resp. right) constituent torsion pair is split. It is called *centrally split* when both constituent torsion pairs are split. When $\mathcal{G} = R$-Mod, it is well-known (see [**S**, Chapter VI]) that $\mathcal{T}$ is a TTF class if, and only if, there is a (unique) idempotent two-sided ideal $\boldsymbol{a}$ of R such that $\mathcal{T}$ consists of the R-modules T such that $\boldsymbol{a}T = 0$. Moreover, the torsion radical c with respect to $(\mathcal{C}, \mathcal{T})$ assigns to each module M the submodule $c(M) = \boldsymbol{a}M$. In particular, we have $\mathcal{C} = \mathrm{Gen}(\boldsymbol{a}) = \{C \in R\text{-Mod} : \boldsymbol{a}C = C\}$. When P is projective R-module, $\mathcal{T} = \mathrm{Ker}(\mathrm{Hom}_R(P, ?))$ is a TTF class and $\mathrm{Gen}(P) = {}^{\perp}\mathcal{T}$. The corresponding idempotent ideal is the trace of P in R.

Given any additive category $\mathcal{A}$ with coproducts, an object X of $\mathcal{A}$ is called *compact* when the functor $\mathrm{Hom}_{\mathcal{A}}(X, ?) : \mathcal{A} \longrightarrow \mathrm{Ab}$ preserves coproducts. Recall that if R is a ring, then the compact objects of its derived category $\mathcal{D}(R)$ are the complexes which are quasi-isomorphic to bounded complexes of finitely generated projective modules (see [**R**]).

Let X and V be objects of $\mathcal{G}$. We say that X is *V-generated* (*resp. V-presented*) when there is an epimorphism $V^{(I)} \twoheadrightarrow X$ (resp. an exact sequence $V^{(J)} \longrightarrow V^{(I)} \longrightarrow X \rightarrow 0$), for some sets I and J. We will denote by $\mathrm{Gen}(V)$ and $\mathrm{Pres}(V)$ the classes of V-generated and V-presented objects, respectively. The object X always contains a largest V-generated subobject, namely, $\mathrm{tr}_V(X) = \sum_{f \in \mathrm{Hom}_{\mathcal{G}}(V,X)} \mathrm{Im}(f)$. It is called the *trace of V in X*. As a sort of dual concept, given a class $\mathcal{S}$ of objects of $\mathcal{G}$, the *reject of $\mathcal{S}$ in X* is $\mathrm{Rej}_{\mathcal{S}}(X) = \bigcap_{S \in \mathcal{S}, f \in \mathrm{Hom}_{\mathcal{G}}(X,S)} \mathrm{Ker}(f)$ (note that, even though $\mathcal{S}$ is a class, the intersection ranges over a set of subobjects of X). We say that X is *V-subgenerated* when it is isomorphic to a subobject of a V-generated object. The class of V-subgenerated objects will be denoted by $\overline{\mathrm{Gen}}(V)$. This subcategory is itself a Grothendieck category and the inclusion $\overline{\mathrm{Gen}}(V) \hookrightarrow \mathcal{G}$ is an exact functor. We will denote by $\mathrm{Add}(V)$ (resp. $\mathrm{add}(V)$) the class of objects X of $\mathcal{G}$ which are isomorphic to direct summands of coproducts (resp. finite coproducts) of copies of V.

Given any category $\mathcal{C}$, an object G is called a *generator* of $\mathcal{C}$ when the functor $\mathrm{Hom}_{\mathcal{C}}(G, ?) : \mathcal{C} \longrightarrow \mathrm{Sets}$ is faithful. When $\mathcal{C} = \mathcal{A}$ is cocomplete abelian, G is a generator exactly when $\mathrm{Gen}(G) = \mathcal{A}$ (note that the definition of $\mathrm{Gen}(V)$ is also valid in this context). An object G of $\mathcal{A}$ is called a *progenerator* when it is a compact projective generator. It is a well-known result of Gabriel and Mitchell (see [**Po**, Corollary 3.6.4]) that $\mathcal{A}$ is a module category if, and only if, it has a progenerator. We will frequently use this characterization of module categories in the paper.

Slightly diverting from the terminology of [**CDT1**] and [**CDT2**], an object V of $\mathcal{G}$ will be called *quasi-tilting* when $\mathrm{Gen}(V) = \overline{\mathrm{Gen}}(V) \cap \mathrm{Ker}(\mathrm{Ext}^1_{\mathcal{G}}(V, ?))$. When, in addition, we have that $\overline{\mathrm{Gen}}(V) = \mathcal{G}$, we will say that V is a 1-*tilting object*. That is, V is 1-tilting if, and only if, $\mathrm{Gen}(V) = \mathrm{Ker}(\mathrm{Ext}^1_{\mathcal{G}}(V, ?))$. When $\mathcal{G} = R$-Mod, a module V is 1-tilting if, and only if, it satisfies the following three properties: i) the projective dimension of V, denoted $\mathrm{pd}({}_RV)$, is ≤ 1; ii) $\mathrm{Ext}^1_R(T, T^{(I)}) = 0$, for each set I; iii) there exists and exact sequence $0 \to R \longrightarrow T^0 \longrightarrow T^1 \to 0$ in R-Mod, where $T^i \in \mathrm{Add}(T)$ for $i = 0, 1$ (see [**CT**, Proposition 1.3]).

When V is a quasi-tilting object of $\mathcal{G}$, we have that $\mathrm{Gen}(V) = \mathrm{Pres}(V)$ and that $(\mathrm{Gen}(V), \mathrm{Ker}(\mathrm{Hom}_{\mathcal{A}}(V, ?)))$ is a torsion pair in $\mathcal{G}$. In the particular case when V is 1-tilting, this pair is called the *tilting torsion pair* associated to V. A *classical quasi-tilting* (*resp. classical* 1-*tilting*) object is a quasi-tilting (resp. 1-tilting) object V such that the canonical morphism $\mathrm{Hom}_{\mathcal{G}}(V, V)^{(I)} \longrightarrow \mathrm{Hom}_{\mathcal{G}}(V, V^{(I)})$ is an isomorphism, for all sets I. By [**CDT1**, Proposition 2.1], we know that if $\mathcal{G} = R$-Mod, then a classical quasi-tilting R-module is just a finitely generated quasi-tilting module. Even more (see [**CT**, Proposition 1.3]), a classical 1-tilting R-module is just a finitely presented 1-tilting R-module.

On what concerns triangulated categories, we will follow [**N**] and [**V**] as basic texts, but if $\mathcal{D}$ is a triangulated category, we will denote by $?[1] : \mathcal{D} \longrightarrow \mathcal{D}$ the suspension functor and we will write triangles in the form $X \longrightarrow Y \longrightarrow Z \xrightarrow{+}$. A *triangulated functor* between triangulated categories is a functor which preserves triangles. Given a triangulated category $\mathcal{D}$, a *t-structure* in $\mathcal{D}$ is a pair $(\mathcal{U}, \mathcal{W})$ of full subcategories, closed under taking direct summands in $\mathcal{D}$, which satisfy the following properties:

i) $\mathrm{Hom}_{\mathcal{D}}(U, W[-1]) = 0$, for all $U \in \mathcal{U}$ and $W \in \mathcal{W}$;
ii) $\mathcal{U}[1] \subseteq \mathcal{U}$;
iii) For each $X \in Ob(\mathcal{D})$, there is a triangle $U \longrightarrow X \longrightarrow V \xrightarrow{+}$ in $\mathcal{D}$, where $U \in \mathcal{U}$ and $V \in \mathcal{W}[-1]$.

It is easy to see that in such case $\mathcal{W} = \mathcal{U}^{\perp}[1]$ and $\mathcal{U} = {}^{\perp}(\mathcal{W}[-1]) = {}^{\perp}(\mathcal{U}^{\perp})$. For this reason, we will write a t-structure as $(\mathcal{U}, \mathcal{U}^{\perp}[1])$. The full subcategory $\mathcal{H} = \mathcal{U} \cap \mathcal{W} = \mathcal{U} \cap \mathcal{U}^{\perp}[1]$ is called the *heart* of the t-structure and it is an abelian category, where the short exact sequences 'are' the triangles in $\mathcal{D}$ with their three terms in $\mathcal{H}$. In particular, one has $\mathrm{Ext}^1_{\mathcal{H}}(M, N) = \mathrm{Hom}_{\mathcal{D}}(M, N[1])$, for all objects M and N in $\mathcal{H}$ (see [**BBD**]).

We will denote by $\mathcal{C}(\mathcal{G})$, $\mathcal{K}(\mathcal{G})$ and $\mathcal{D}(\mathcal{G})$ the category of chain complexes of objects of $\mathcal{G}$, the homotopy category of $\mathcal{G}$ and the derived category of $\mathcal{G}$, respectively. In the particular case when $\mathcal{G} = R$-Mod, we will write $\mathcal{C}(R) := \mathcal{C}(R\text{-Mod})$, $\mathcal{K}(R) := \mathcal{K}(R\text{-Mod})$ and $\mathcal{D}(R) := \mathcal{D}(R\text{-Mod})$. Given a torsion pair $\boldsymbol{t} = (\mathcal{T}, \mathcal{F})$ in $\mathcal{G}$, extending to the unbounded context a construction due to Happel–Reiten–Smalø (see [**HRS**]), one gets a t-structure $(\mathcal{U}_{\boldsymbol{t}}, \mathcal{U}_{\boldsymbol{t}}^{\perp}[1]) = (\mathcal{U}_{\boldsymbol{t}}, \mathcal{W}_{\boldsymbol{t}})$ in $\mathcal{D}(\mathcal{G})$ by defining:

$$\mathcal{U}_{\boldsymbol{t}} = \{X \in \mathcal{D}^{\leq 0}(\mathcal{G}) : H^0(X) \in \mathcal{T}\}$$
$$\mathcal{W}_{\boldsymbol{t}} = \{Y \in \mathcal{D}^{\geq -1}(\mathcal{G}) : H^{-1}(Y) \in \mathcal{F}\}.$$

In this case, the heart $\mathcal{H}_{\boldsymbol{t}}$ consists of the complexes M such that $H^{-1}(M) \in \mathcal{F}$, $H^0(M) \in \mathcal{T}$ and $H^k(M) = 0$, for all $k \neq -1, 0$. We will say that $\mathcal{H}_{\boldsymbol{t}}$ is the *heart of the torsion pair* $\boldsymbol{t}$.

When $\mathcal{G} = R$-Mod, we will frequently deal with complexes $\cdots \longrightarrow 0 \longrightarrow X \xrightarrow{j} Q \xrightarrow{d} P \longrightarrow 0 \longrightarrow \cdots$, concentrated in degrees $-2, -1, 0$, such that j is a monomorphism and P, Q are projective modules. All throughout the paper such a complex will be said to be a *complex in standard form* and, without loss of generality, we will assume that X is a submodule of Q and j is the inclusion. If $\boldsymbol{t}$ is a torsion pair in R-Mod, then each object of $\mathcal{H}_{\boldsymbol{t}}$ is quasi-isomorphic to a complex in standard form. Moreover, if M and N are two complexes in standard form and they represent objects of $\mathcal{H}_{\boldsymbol{t}}$, then the canonical map $\mathrm{Hom}_{\mathcal{K}(R)}(M, N) \longrightarrow \mathrm{Hom}_{\mathcal{D}(R)}(M, N) = \mathrm{Hom}_{\mathcal{H}_{\boldsymbol{t}}}(M, N)$ is bijective. We will frequently use this fact throughout the paper.

An object T of a triangulated category with coproducts $\mathcal{D}$ will be called *classical tilting* when satisfies the following conditions: i) T is compact in $\mathcal{D}$; ii) $\mathrm{Hom}_{\mathcal{D}}(T, T[i]) = 0$, for all $i \neq 0$; and iii) if $X \in \mathcal{D}$ is an object such that $\mathrm{Hom}_{\mathcal{D}}(T[i], X) = 0$, for all $i \in \mathbb{Z}$, then $X = 0$. For instance, if T is a classical 1-tilting R-module, then $T[0]$ is a classical tilting object of $\mathcal{D}(R)$. By a well-known result of Rickard (see [**R**] and [**R2**]), two rings R and S are *derived equivalent*, i.e., have equivalent derived categories, if and only if there exists a classical tilting object T in $\mathcal{D}(R)$ such that $S \cong \mathrm{End}_{\mathcal{D}(R)}(T)^{op}$.

Let $P^\bullet : \cdots \longrightarrow 0 \longrightarrow Q \xrightarrow{d} P \longrightarrow 0 \longrightarrow \cdots$ be a complex of finitely generated projective R-modules concentrated in degrees -1 and 0. In [**HKM**], the authors associated to such a complex a pair $(\mathcal{X}(P^\bullet), \mathcal{Y}(P^\bullet))$ of full subcategories of R-Mod defined as follows, where M is an R-module:

$$M \in \mathcal{X}(P^\bullet) \Longleftrightarrow \mathrm{Hom}_{\mathcal{D}(R)}(P^\bullet, M[1]) = 0$$
$$M \in \mathcal{Y}(P^\bullet) \Longleftrightarrow \mathrm{Hom}_{\mathcal{D}(R)}(P^\bullet, M[0]) = 0.$$

Under some precise conditions (see [**HKM**, Theorem 2.10]), the pair $(\mathcal{X}(P^\bullet), \mathcal{Y}(P^\bullet))$ is a torsion pair in R-Mod. When this is the case, we shall say that $P^\bullet$ is an *HKM complex* and that $\boldsymbol{t} = (\mathcal{X}(P^\bullet), \mathcal{Y}(P^\bullet))$ is the associated *HKM torsion pair*.

For any ring R, we shall denote by $V(R)$ the additive monoid whose elements are the isoclasses of finitely generated projective R-modules, where $[P] + [Q] = [P \oplus Q]$. For each two-sided ideal $\boldsymbol{a}$ of the ring R, we have an obvious morphism of monoids $V(R) \longrightarrow V(R/\boldsymbol{a})$ taking $[P] \rightsquigarrow [P/\boldsymbol{a}P]$. This morphism need not be surjective. However, the class of rings R for which it is surjective, independently of $\boldsymbol{a}$, is very large and includes the so-called exchange rings (see [**A**, Lemma 3.2, Theorem 3.3]). This class of rings includes all rings which are Von Neumann regular modulo the Jacobson radical and which have the lifting of idempotents property with respect to this radical. In particular, it includes all *semiperfect rings*, i.e., those rings R such that $R/J(R)$ is semisimple and idempotents lift modulo $J(R)$, where $J(R)$ denotes the Jacobson radical of R. All local and all (left or right) Artinian rings, in particular all Artin algebras, are semiperfect rings.

For concepts not explicitly defined in the paper, the reader is referred to [**P**] or [**Po**] for those concerning arbitrary and abelian categories, to [**K**] and [**S**] for those concerning rings and their module categories and to [**N**] and [**V**] for those concerning triangulated categories.

3. When is the heart of a torsion pair a module category?

All throughout the paper, R will be a ring and $\boldsymbol{t} = (\mathcal{T}, \mathcal{F})$ will be a torsion pair in R-Mod. Unless otherwise stated, the letter G will denote a complex in standard form. Frequently, such a complex will satisfy some or all of the following conditions with respect to $\boldsymbol{t}$, to which we will refer as the *standard conditions* (here $V = H^0(G)$):

1. $\mathcal{T} = \mathrm{Pres}(V) \subseteq \mathrm{Ker}(\mathrm{Ext}^1_R(V, ?))$;
2. Q and P are finitely generated projective R-modules;
3. $H^{-1}(G) \in \mathcal{F}$ and $H^{-1}(G) \subseteq \mathrm{Rej}_{\mathcal{T}}(Q/X)$;
4. $\mathrm{Ext}^1_R(Q/X, ?)$ vanishes on $\mathcal{F}$;
5. there is a morphism $h : (Q/X)^{(I)} \longrightarrow R/t(R)$, for some set I, such that the cokernel of its restriction to $(H^{-1}(G))^{(I)}$ is in $\overline{\mathrm{Gen}}(V)$.

Lemma 3.1. *Let G be a complex in standard form and let M be any R-module. The following assertions hold*:

1. *There is an isomorphism* $\mathrm{Hom}_R(H^0(G), M) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}(R)}(G, M[0])$, *which is natural in M*;
2. *When we view X as a submodule of Q and j as the inclusion, there are natural in M exact sequences of abelian groups*:
   (a) $\mathrm{Hom}_R(P, M) \longrightarrow \mathrm{Hom}_R(Q/X, M) \longrightarrow \mathrm{Hom}_{\mathcal{D}(R)}(G, M[1]) \longrightarrow 0.$
   (b) $\mathrm{Hom}_R(Q, M) \longrightarrow \mathrm{Hom}_R(X, M) \longrightarrow \mathrm{Hom}_{\mathcal{D}(R)}(G, M[2]) \longrightarrow 0.$

Proof. We have triangles in $\mathcal{D}(R)$:

$$H^{-1}(G)[1] \longrightarrow G \longrightarrow H^0(G)[0] \xrightarrow{+}$$

and

$$Q/X[0] \longrightarrow P[0] \longrightarrow G \xrightarrow{+} .$$

Applying the cohomological functor $\mathrm{Hom}_{\mathcal{D}(R)}(?, M[0])$ and looking at the corresponding long exact sequences, we obtain assertions 1 and 2.a. On the other hand, one easily sees that a morphism $G \longrightarrow M[2]$ in $\mathcal{D}(R)$ is represented by an R-homomorphism $f : X \longrightarrow M$. The former morphism is the zero morphism in $\mathcal{D}(R)$ precisely when f factors through j. Then the exact sequence in 2.b follows immediately. $\square$

Lemma 3.2. *If G is a progenerator of $\mathcal{H}_{\boldsymbol{t}}$, then the following assertions hold, where $V := H^0(G)$*:

1. $\mathcal{T} = \mathrm{Gen}(V) = \mathrm{Pres}(V)$, *and hence* $\mathcal{F} = \mathrm{Ker}(\mathrm{Hom}_R(V, ?))$;
2. V *is a finitely presented* R*-module*;
3. V *is a classical quasi-tilting* R*-module.*

Proof. By hypothesis $\mathcal{H}_{\boldsymbol{t}}$ is a module category, in particular $\mathcal{H}_{\boldsymbol{t}}$ is an AB5 category, so that $\mathcal{F}$ is closed under taking direct limits in R-Mod (see [**PS**, Theorem 4.8]). On the other hand, by [**PS**, Lemma 4.1], the functor $H^0 : \mathcal{H}_{\boldsymbol{t}} \longrightarrow R\text{-Mod}$ is right exact and preserves coproducts. When applied to an exact sequence $G^{(I)} \longrightarrow G^{(J)} \longrightarrow T[0] \longrightarrow 0$ in $\mathcal{H}_{\boldsymbol{t}}$, we get that $T \in \mathrm{Pres}(V)$, for each $T \in \mathcal{T}$. We then get that $\mathcal{T} = \mathrm{Pres}(V)$, and assertion 1 follows from [**MT**, Proposition 2.2].

Without loss of generality we can assume that G is in standard form. If $(T_i)_{i \in I}$ is a direct system in $\mathcal{T}$, then $\varinjlim_{\mathcal{H}_{\boldsymbol{t}}} (T_i[0]) \cong (\varinjlim T_i)[0]$ (see [**PS**, Proposition 4.2]). We then get that $\mathrm{Hom}_R(V, ?)$ preserves direct limits of objects in $\mathcal{T}$ since G is a compact object of $\mathcal{H}_{\boldsymbol{t}}$. Let now $(M_i)_{i \in I}$ be any direct system in R-Mod. We then get that $\varinjlim t(M_i) \cong t(\varinjlim M_i)$ since $\varinjlim \mathcal{F} = \mathcal{F}$. We now have isomorphisms

$$\varinjlim \mathrm{Hom}_R(V, M_i) \xleftarrow{\sim} \varinjlim \mathrm{Hom}_R(V, t(M_i)) \xrightarrow{\sim} \mathrm{Hom}_R(V, \varinjlim t(M_i))$$
$$= \mathrm{Hom}_R(V, t(\varinjlim M_i)) \xrightarrow{\sim} \mathrm{Hom}_R(V, \varinjlim M_i) \ .$$

Then assertion 2 follows. Finally, assertion 3 follows from [**MT**, Proposition 2.4], from assertions 1 and 2 and from [**CDT1**, Proposition 2.1]. $\square$

The following result is inspired by [**CMT**, Proposition 5.9].

Lemma 3.3. *Let G be a complex in standard form. If G is a projective object of $\mathcal{H}_{\boldsymbol{t}}$ such that $\mathcal{T} = \mathrm{Gen}(V) = \mathrm{Pres}(V)$, where $V := H^0(G)$, then the following assertions hold*:

1. $(M/t(M))[1] \in \mathrm{Gen}_{\mathcal{H}_{\boldsymbol{t}}}(G)$, *for each* $M \in \overline{\mathrm{Gen}}(V)$;
2. $(R/t(R))[1] \in \mathrm{Gen}_{\mathcal{H}_{\boldsymbol{t}}}(G)$ *if, and only if, G satisfies the standard condition* 5.

Proof. With an easy adaptation, assertion 1 follows from [**CMT**, Lemma 5.8 and Proposition 5.9].

We now prove assertion 2. For the *only if* part, by hypothesis, there are a set I and an exact sequence

$$0 \longrightarrow K \longrightarrow G^{(I)} \xrightarrow{p} \frac{R}{t(R)}[1] \longrightarrow 0$$

in $\mathcal{H}_{\boldsymbol{t}}$. It is easy to see that p is represented by an R-homomorphism $p^{-1} : (Q/X)^{(I)} \longrightarrow R/t(R)$. Moreover, we have that $H^{-1}(p)$ coincides with the restriction of p^{-1} to $(H^{-1}(G))^{(I)}$. Now, we consider the long exact sequence associated to the above triangle:

$$0 \longrightarrow H^{-1}(K) \longrightarrow H^{-1}(G)^{(I)} \xrightarrow{H^{-1}(p)} \frac{R}{t(R)} \longrightarrow H^0(K) \longrightarrow V^{(I)} \longrightarrow 0$$

The result follows by putting $h = p^{-1}$ since $H^0(K) \in \mathcal{T}$.

For the *if part*, assume that the standard condition 5 holds and let Z be the cokernel of the restriction of h to $(H^{-1}(G))^{(I)}$. Clearly, we can extend h to a morphism from $G^{(I)}$ to $(R/t(R))[1]$ in $D(R)$, which we denote by $\bar{h}$. We now complete $\bar{h}$ to a triangle in $D(R)$, we get:

$$M \longrightarrow G^{(I)} \xrightarrow{\bar{h}} \frac{R}{t(R)}[1] \xrightarrow{+}$$

Using the long exact sequence of homologies, we then obtain an exact sequence in R-Mod of the form:

$$0 \longrightarrow Z \longrightarrow H^0(M) \longrightarrow V^{(\alpha)} \longrightarrow 0$$

By [**CMT**, Lemma 5.6], we get that $H^0(M) \in \overline{\mathrm{Gen}}(V)$ and then, by assertion 1, we also get that $(H^0(M)/t(H^0(M)))[1] \in \mathrm{Gen}_{\mathcal{H}_{\boldsymbol{t}}}(G)$. Consider now the following diagram commutative

$$\begin{tikzcd}
 & & & t(H^0(M))[1] \arrow[dd] \arrow[r, "+"] & {} \\
 & N \arrow[urr] \arrow[dr] & & & \\
H^{-1}(M)[2] \arrow[ur] \arrow[rr] & & M[1] \arrow[r] \arrow[dr] & H^0(M)[1] \arrow[d] \arrow[r, "+"] & {} \\
 & & & \frac{H^0(M)}{t(H^0(M))}[1] \arrow[r, "+"] & {}
\end{tikzcd}$$

Note that $N \in \mathcal{H}_{\boldsymbol{t}}[1]$. By [**BBD**], we get that $\mathrm{Coker}_{\mathcal{H}_{\boldsymbol{t}}}(\bar{h}) \cong (H^0(M)/t(H^0(M)))[1]$. Then we have the following diagram with exact row in $\mathcal{H}_{\boldsymbol{t}}$:

$$\begin{tikzcd}
 & & G^{(J)} \arrow[dl, dashed, "p'"'] \arrow[d, "p"] & \\
G^{(I)} \arrow[r, "\bar{h}"] & \frac{R}{t(R)}[1] \arrow[r] & \frac{H^0(M)}{t(H^0(M))}[1] \arrow[r] & 0
\end{tikzcd}$$

where p is an epimorphism and p' is obtained by the projectivity of $G^{(J)}$ in $\mathcal{H}_{\boldsymbol{t}}$. It follows that $(\bar{h}\ p') : G^{(I)} \coprod G^{(J)} \longrightarrow (R/t(R))[1]$ is also an epimorphism in $\mathcal{H}_{\boldsymbol{t}}$. □

We are now able to give a general criterion for $\mathcal{H}_{\boldsymbol{t}}$ to be a module category.

Theorem 3.4. *Let R be a ring and $\boldsymbol{t}$ be a torsion pair in R-*Mod*. A complex G is a progenerator of the heart $\mathcal{H}_{\boldsymbol{t}}$ if, and only if, it is quasi-isomorphic to a complex in standard form satisfying the standard conditions* 1–5*. In particular $\mathcal{H}_{\boldsymbol{t}}$ is a module category if, and only if, this latter complex exists.*

Proof. Let us assume that G is a complex in standard form which is in $\mathcal{H}_{\boldsymbol{t}}$. By Lemma 3.2, if G is a progenerator of $\mathcal{H}_{\boldsymbol{t}}$, then $V := H^0(G)$ is finitely presented. This allows us, for both implications in the proof, to assume that P is a finitely generated projective R-module.

We claim that G is a projective object in $\mathcal{H}_{\boldsymbol{t}}$ if, and only if, $\mathcal{T} \subseteq \mathrm{Ker}(\mathrm{Ext}^1_R(V, ?))$ and the standard conditions 3 and 4 hold. Indeed each object $M \in \mathcal{H}_{\boldsymbol{t}}$ fits into an exact sequence in this category

$$0 \longrightarrow H^{-1}(M)[1] \longrightarrow M \longrightarrow H^0(M)[0] \longrightarrow 0 \qquad (*)$$

Then G is projective in $\mathcal{H}_{\boldsymbol{t}}$ if, and only if, $0 = \mathrm{Ext}^1_{\mathcal{H}_{\boldsymbol{t}}}(G, T[0]) = \mathrm{Hom}_{\mathcal{D}(R)}(G, T[1])$ and $0 = \mathrm{Ext}^1_{\mathcal{H}_{\boldsymbol{t}}}(G, F[1]) = \mathrm{Hom}_{\mathcal{D}(R)}(G, F[2])$, for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$ (see [**BBD**, Rémarque 3.1.17(ii)]). By Lemma 3.1, the first equality holds if, and only if, the map $\mathrm{Hom}_R(P, T) \xrightarrow{\bar{d}^*} \mathrm{Hom}_R(Q/X, T)$ is surjective, where $\bar{d} : Q/X \longrightarrow P$ is the obvious R-homomorphism. But, in turn, this last condition is equivalent to the sum of the following two conditions, for each $T \in \mathcal{T}$:

i) Each R-homomorphism $f : Q/X \longrightarrow T$ vanishes on $H^{-1}(G) = \mathrm{Ker}(d)/X$;
ii) Each morphism $g : \mathrm{Im}(\bar{d}) = \mathrm{Im}(d) \longrightarrow T$ extends to P.

Condition i) is equivalent to the standard condition 3. On the other hand, condition ii) above is equivalent to saying that $\mathrm{Ext}^1_R(H^0(G), T) = 0$, for all $T \in \mathcal{T}$. Now, by Lemma 3.1, the equality $\mathrm{Hom}_{\mathcal{D}(R)}(G, F[2]) = 0$ holds when each R-homomorphism $g : X \longrightarrow F$ extends to Q, for all $F \in \mathcal{F}$. This is clearly equivalent to the standard condition 4.

Suppose that G is projective in $\mathcal{H}_{\boldsymbol{t}}$ or its equivalent conditions mentioned in the previous paragraph. Recall from [**PS**, Section 4] that $\mathcal{H}_{\boldsymbol{t}}$ is AB4. Applying this fact to any family of exact sequences as $(*)$, we see that G is a compact object of $\mathcal{H}_{\boldsymbol{t}}$ if, and only if, the canonical morphisms

$$\coprod_{i \in I} \mathrm{Hom}_{\mathcal{D}(R)}(G, T_i[0]) \longrightarrow \mathrm{Hom}_{\mathcal{D}(R)}\left(G, \left(\coprod_{i \in I} T_i\right)[0]\right)$$

and

$$\coprod_{i\in I}\mathrm{Hom}_{\mathcal{D}(R)}(G,F_i[1]) \longrightarrow \mathrm{Hom}_{\mathcal{D}(R)}\left(G,\left(\coprod_{i\in I}F_i\right)[1]\right)$$

are isomorphisms, for all families (T_i) in $\mathcal{T}$ and (F_i) in $\mathcal{F}$. By Lemma 3.1, we easily get that the first of these morphisms is an isomorphism precisely when V is a compact object of $\mathcal{T}$. On the other hand, by Lemma 3.1(2), the second centered homomorphism is an isomorphism whenever P and Q/X are finitely generated modules. Therefore, if G satisfies the standard conditions 1–4, then G is a compact projective object of $\mathcal{H}_{\boldsymbol{t}}$.

Suppose now that these last conditions hold. Then, due to the canonical sequence $(*)$, we know that G is a generator if, and only if, each $M\in\mathcal{T}[0]\cup\mathcal{F}[1]$ is generated by G. Note that we have an epimorphism $q:G\twoheadrightarrow V[0]$ in $\mathcal{H}_{\boldsymbol{t}}$, which implies that $\mathrm{Gen}_{\mathcal{H}_{\boldsymbol{t}}}(V[0])\subseteq\mathrm{Gen}_{\mathcal{H}_{\boldsymbol{t}}}(G)$. But the equality $\mathcal{T}=\mathrm{Gen}(V)=\mathrm{Pres}(V)$ easily gives that $\mathcal{T}[0]\subseteq\mathrm{Gen}_{\mathcal{H}_{\boldsymbol{t}}}(V[0])$. On the other hand, each $F\in\mathcal{F}$ gives rise to an exact sequence $0\longrightarrow F'\hookrightarrow (R/t(R))^{(I)}\twoheadrightarrow F\longrightarrow 0$ in R-Mod which, in turn, yields an exact sequence in $\mathcal{H}_{\boldsymbol{t}}$:

$$0\longrightarrow F'[1]\longrightarrow\left(\frac{R}{t(R)}\right)^{(I)}[1]\longrightarrow F[1]\longrightarrow 0\,.$$

Thus, G generates $\mathcal{H}_{\mathbf{t}}$ if, and only if, it generates $(R/t(R))[1]$. By Lemma 3.3, this is equivalent to the standard condition 5.

Note that the *'if' part* of the proof follows from the previous paragraphs. By Lemma 3.2 and Lemma 3.3, in order to prove the *'only if' part*, we only need to prove that if G is a complex in standard form, with P finitely generated, and it is a progenerator of $\mathcal{H}_{\boldsymbol{t}}$, then G is quasi-isomorphic to a complex satisfying the standard conditions 1–5. But Lemma 3.5 below shows that Q/X is finitely generated, which allows us to replace Q by a finitely generated projective module Q' and get a complex

$$\cdots\longrightarrow 0\longrightarrow X'\longrightarrow Q'\xrightarrow{d'} P\longrightarrow 0\longrightarrow\cdots$$

which is quasi-isomorphic to G and satisfies all the standard conditions. □

LEMMA 3.5. *Let G be a complex in standard form with P finitely generated. Suppose that G is a progenerator of $\mathcal{H}_{\boldsymbol{t}}$. Then Q/X is a finitely generated R-module.*

PROOF. We identify G with the complex

$$\cdots\longrightarrow 0\longrightarrow Q/X\xrightarrow{\bar{d}} P\longrightarrow 0\longrightarrow\cdots$$

By Lemma 3.2, we know that $V:=H^0(G)$ is a finitely presented R-module, thus $\mathrm{Im}(d)=\mathrm{Im}(\bar{d})$ is a finitely generated submodule of P, we can select a finitely generated submodule $A'<Q/X$ such that $\bar{d}(A')=\mathrm{Im}(d)$. We fix a direct system $(A_\lambda)_{\lambda\in\Lambda}$ of finitely generated submodules of Q/X, such that $A'<A_\lambda$ and $\varinjlim A_\lambda=Q/X$. For each $\lambda\in\Lambda$, we denote by G_λ the following complex:

$$G_\lambda : \quad \cdots \longrightarrow 0 \longrightarrow A_\lambda \xrightarrow{\bar{d}_{|A_\lambda}} P \longrightarrow 0 \longrightarrow \cdots$$

It is clear that $(G_\lambda)_{\lambda\in\Lambda}$ is a direct system in $\mathcal{C}(R)$ and in $\mathcal{H}_{\boldsymbol{t}}$, and that we have $\varinjlim_{\mathcal{C}(R)} G_\lambda \cong G$. By Lemma 3.2 and [**PS**, Lemma 4.4], we have that $\varinjlim_{\mathcal{H}_{\boldsymbol{t}}} G_\lambda \cong \varinjlim_{\mathcal{C}(R)} G_\lambda = G$. But, since G is a finitely presented object of $\mathcal{H}_{\boldsymbol{t}}$, the identify map $1_G : G \longrightarrow G$ factors in this category in the form $G \xrightarrow{f} G_\mu \xrightarrow{i_\mu} G$, for some $\mu \in \Lambda$. It follows that $H^{-1}(\iota_\mu)$ is an epimorphism and, therefore, it is an isomorphism. We then get a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & H^{-1}(G_\mu) \arrow[r] \arrow[d, "H^{-1}(i_\mu)"', "\wr"] & A_\mu \arrow[r] \arrow[d, hook, "i_\mu^{-1}"] & \bar{d}(A_\mu) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H^{-1}(G) \arrow[r] & \frac{Q}{X} \arrow[r, "\bar{d}"] & \mathrm{Im}(d) \arrow[r] & 0
\end{tikzcd}$$

Therefore, $A_\mu \cong Q/X$ is a finitely generated R-module. □

REMARK 3.6. If G is a complex in standard form satisfying the standard condition 2 (i.e. P and Q are finitely generated), the proof of Theorem 3.4 shows that G is a progenerator of $\mathcal{H}_{\boldsymbol{t}}$ if, and only if, G itself satisfies all standard conditions 1–5.

Our next result in this section gives a criterion for a torsion pair to be HKM:

PROPOSITION 3.7. *Let* $P^\bullet := \cdots \longrightarrow 0 \longrightarrow Q \xrightarrow{d} P \longrightarrow 0 \longrightarrow \cdots$ *be a complex of finitely generated projective modules concentrated in degrees* -1 *and* 0, *put* $V = H^0(P^\bullet)$, *and let* $\boldsymbol{t} = (\mathcal{T}, \mathcal{F})$ *be a torsion pair in* R-Mod. *The following assertions are equivalent*:

1. $P^\bullet$ *is an HKM complex such that* $\boldsymbol{t}$ *is its associated HKM torsion pair*;
2. $V \in \mathcal{X}(P^\bullet)$ *and the complex* $G := \cdots \longrightarrow 0 \longrightarrow t(\mathrm{Ker}(d)) \xhookrightarrow{j} Q \xrightarrow{d} P \longrightarrow 0 \longrightarrow \cdots$, *concentrated in degrees* $-2, -1, 0$, *is a progenerator of* $\mathcal{H}_{\boldsymbol{t}}$;
3. $P^\bullet$ *satisfies the standard conditions* 1 *and* 5, $\mathrm{Ker}(d) \subseteq \mathrm{Rej}_{\mathcal{T}}(Q)$ *and* $\mathcal{X}(P^\bullet) \subseteq \mathcal{T}$.

PROOF. Note that the standard conditions 2 and 4 are automatically satisfied by $P^\bullet$. Note also that we have an exact sequence $0 \longrightarrow t(\mathrm{Ker}(d))[1] \longrightarrow P^\bullet \longrightarrow G' \longrightarrow 0$ in $\mathcal{C}(R)$, where G' is quasi-isomorphic to G. We then get a triangle in $\mathcal{D}(R)$:

$$t(\mathrm{Ker}(d))[1] \longrightarrow P^\bullet \longrightarrow G \xrightarrow{+}$$

In particular, we get a natural isomorphism $\mathrm{Hom}_{\mathcal{D}(R)}(G, ?[0]) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}(R)}(P^\bullet, ?[0])$ and an exact sequence of functors R-Mod $\longrightarrow$ Ab:

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{D}(R)}(G, ?[1]) \longrightarrow \mathrm{Hom}_{\mathcal{D}(R)}(P^\bullet, ?[1]) \longrightarrow$$
$$\mathrm{Hom}_R(t(\mathrm{Ker}(d)), ?) \longrightarrow \mathrm{Hom}_{\mathcal{D}(R)}(G, ?[2]) \longrightarrow 0$$

1) $\Longrightarrow$ 2) We have an isomorphism $\mathrm{Hom}_{\mathcal{H}_{\boldsymbol{t}}}(G, ?) = \mathrm{Hom}_{\mathcal{D}(R)}(G, ?)_{|\mathcal{H}_{\boldsymbol{t}}} \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}(R)}(P^\bullet, ?)_{|\mathcal{H}_{\boldsymbol{t}}}$ of functors $\mathcal{H}_{\boldsymbol{t}} \longrightarrow \mathrm{Ab}$ since $\mathrm{Hom}_{\mathcal{D}(R)}(t(\mathrm{Ker}(d))[k], ?)$ vanishes on $\mathcal{H}_{\boldsymbol{t}}$, for $k = 1, 2$. It follows that the functor $\mathrm{Hom}_{\mathcal{H}_{\boldsymbol{t}}}(G, ?) : \mathcal{H}_{\boldsymbol{t}} \longrightarrow \mathrm{Ab}$ is faithful since the induced functor $\mathrm{Hom}_{\mathcal{D}(R)}(P^\bullet, ?)_{|\mathcal{H}_{\boldsymbol{t}}} : \mathcal{H}_{\boldsymbol{t}} \longrightarrow \mathrm{End}_{\mathcal{D}(R)}(P^\bullet)$-Mod is an equivalence of categories (see [**HKM**, Theorem 2.15]). Then G is a generator of $\mathcal{H}_{\boldsymbol{t}}$. But the functor $\mathrm{Hom}_{\mathcal{H}_{\boldsymbol{t}}}(G, ?) \cong \mathrm{Hom}_{\mathcal{D}(R)}(P^\bullet, ?)_{|\mathcal{H}_{\boldsymbol{t}}} : \mathcal{H}_{\boldsymbol{t}} \longrightarrow \mathrm{Ab}$ preserves coproducts since $P^\bullet$ is a compact object of $\mathcal{D}(R)$. It follows that G is a compact object of $\mathcal{H}_{\boldsymbol{t}}$.

From the initial comments of this proof and the fact that $\mathcal{T} = \mathcal{X}(P^\bullet) = \mathrm{Ker}(\mathrm{Hom}_{\mathcal{D}(R)}(P^\bullet, ?[1])_{|R\text{-Mod}})$, we get a monomorphism $\mathrm{Ext}^1_{\mathcal{H}_{\boldsymbol{t}}}(G, T[0]) = \mathrm{Hom}_{\mathcal{D}(R)}(G, T[1]) \hookrightarrow \mathrm{Hom}_{\mathcal{D}(R)}(P^\bullet, T[1]) = 0$, for each $T \in \mathcal{T}$. On the other hand, $\mathrm{Ext}^1_{\mathcal{H}_{\boldsymbol{t}}}(G, F[1]) = \mathrm{Hom}_{\mathcal{D}(R)}(G, F[2])$ is a homomorphic image of $\mathrm{Hom}_R(t(\mathrm{Ker}(d)), F) = 0$, for all $F \in \mathcal{F}$. We conclude that G is a projective object, and hence a progenerator, of $\mathcal{H}_{\boldsymbol{t}}$ since $\mathrm{Ext}^1_{\mathcal{H}_{\boldsymbol{t}}}(G, ?)$ vanishes on $\mathcal{T}[0]$ and on $\mathcal{F}[1]$.

2) $\Longrightarrow$ 1) The mentioned initial comments show that $\mathcal{Y}(P^\bullet)$ consists of the modules F such that $\mathrm{Hom}_R(H^0(P^\bullet), F) \cong \mathrm{Hom}_{\mathcal{D}(R)}(G, F[0]) = 0$. But Theorem 3.4 and its proof tell us that $H^0(G) = V$ generates $\mathcal{T}$, so that we have $\mathcal{Y}(P^\bullet) = \mathcal{F}$. On the other hand, if $F \in \mathcal{X}(P^\bullet) \cap \mathcal{Y}(P^\bullet) = \mathcal{X}(P^\bullet) \cap \mathcal{F}$ then, again, our initial comments in this proof show that $\mathrm{Hom}_{\mathcal{D}(R)}(G, F[1]) = 0$. But this implies that $F = 0$ since G is a generator of $\mathcal{H}_{\boldsymbol{t}}$. Assertion 1 follows now from [**HKM**, Theorem 2.10] and the fact that $\mathcal{Y}(P^\bullet) = \mathcal{F}$.

1), 2) $\Longrightarrow$ 3) From Theorem 3.4 and Remark 3.6 we know that the complex G satisfies all the standard conditions. It immediately follows that $P^\bullet$ satisfies the standard condition 1. As for standard condition 5, note that we have isomorphisms of functors:

$$\mathrm{Hom}_R(Q^{(J)}, ?)_{|\mathcal{F}} \xrightarrow{\sim} \mathrm{Hom}_R((Q/X)^{(J)}, ?)_{|\mathcal{F}}$$
$$\mathrm{Hom}_R(\mathrm{Ker}(d)^{(J)}, ?)_{|\mathcal{F}} \xrightarrow{\sim} \mathrm{Hom}_R(H^{-1}(G)^{(J)}, ?)_{|\mathcal{F}},$$

where $X = t(\mathrm{Ker}(d))$. Then the standard condition 5 holds for $P^\bullet$ because it holds for G. Finally, any homomorphism $f : Q \longrightarrow T$, with $T \in \mathcal{T}$, gives a morphism $P^\bullet \longrightarrow T[1]$ in $\mathcal{D}(R)$. But this is the zero morphism since $\mathcal{T} = \mathcal{X}(P^\bullet)$. This implies that f factors through d, so that $f(\mathrm{Ker}(d)) = 0$ and, hence, that $\mathrm{Ker}(d) \subseteq \mathrm{Rej}_{\mathcal{T}}(Q)$.

3) $\Longrightarrow$ 1) By Lemma 3.1, we know that $\mathcal{Y}(P^\bullet)$ consists of the modules Y such that $\mathrm{Hom}_R(V, Y) = 0$. Standard condition 1 gives then that $\mathcal{Y}(P^\bullet) = \mathcal{F}$, which implies that $\mathcal{X}(P^\bullet) \cap \mathcal{Y}(P^\bullet) = 0$ since $\mathcal{X}(P^\bullet) \subseteq \mathcal{T}$. On the other hand, the standard condition 3 says that each homomorphism $f : Q \longrightarrow V$ vanishes on $\mathrm{Ker}(d)$. It then induces an R-homomorphism $\bar{f} : \mathrm{Im}(d) \longrightarrow V$, which necessarily extends to P since $\mathrm{Ext}^1_R(V, V) = 0$. This proves that $\mathrm{Hom}_{\mathcal{D}(R)}(P^\bullet, V[1]) = 0$, thus showing that $H^0(P^\bullet) = V \in \mathcal{X}(P^\bullet)$. Then, by [**HKM**, Theorem 2.10], the pair $(\mathcal{X}(P^\bullet), \mathcal{Y}(P^\bullet))$ is a torsion pair, which is necessarily equal to $\boldsymbol{t}$. $\square$

Corollary 3.8. *Let $\boldsymbol{t} = (\mathcal{T}, \mathcal{F})$ be a torsion pair in R-Mod and let*

$$P^\bullet := \cdots \longrightarrow 0 \longrightarrow Q \xrightarrow{d} P \longrightarrow 0 \longrightarrow \cdots$$

be a complex of finitely generated projective R-modules concentrated in degrees -1 and 0. The following assertions are equivalent:

1. *$P^\bullet$ is a classical tilting complex and $\boldsymbol{t}$ is the associated HKM torsion pair.*
2. *$P^\bullet$ is a progenerator of $\mathcal{H}_{\boldsymbol{t}}$;*
3. *The complex $P^\bullet$ satisfies the standard conditions* (1, 3 *and* 5).

Proof. 1) $\Longrightarrow$ 2) It follows directly from [**HKM**, Remark 3.9 and Theorem 3.8].

2) $\Longrightarrow$ 1) Let M be an R-module and let us apply the cohomological functor $\mathrm{Hom}_{D(R)}(P^\bullet, ?)$ to the canonical triangle

$$t(M)[0] \longrightarrow M[0] \longrightarrow M/t(M)[0] \xrightarrow{+}$$

Using the fact that $P^\bullet$ is a progenerator of $\mathcal{H}_{\boldsymbol{t}}$, that $\mathrm{Hom}_{D(R)}(P^\bullet, ?[0])$ vanish on $\mathcal{F}$ and that $\mathrm{Hom}_{D(R)}(P^\bullet, ?[1])$ vanish on $\mathcal{T}$, we get that

$$M \in \mathcal{X}(P^\bullet) \Longleftrightarrow \mathrm{Hom}_{D(R)}(P^\bullet, M/t(M)[1]) = 0 \Longleftrightarrow M/t(M) = 0 \Longleftrightarrow M \in \mathcal{T},$$

and also that

$$M \in \mathcal{Y}(P^\bullet) \Longleftrightarrow \mathrm{Hom}_{D(R)}(P^\bullet, t(M)[0]) = 0 \Longleftrightarrow t(M) = 0 \Longleftrightarrow M \in \mathcal{F}.$$

We then have that $\boldsymbol{t} = (\mathcal{X}(P^\bullet), \mathcal{Y}(P^\bullet))$ and, by [**HKM**, Remark 3.9], the complex $P^\bullet$ is classical tilting.

2) $\Longleftrightarrow$ 3) is a direct consequence of Theorem 3.4 (see Remark 3.6). $\square$

Definition 1. We shall say that $\mathcal{H}_{\boldsymbol{t}}$ *has a progenerator which is a classical tilting complex* when it has a progenerator $P^\bullet$ as in Corollary 3.8.

4. The case of a hereditary torsion pair.

Suppose now that $\boldsymbol{t} = (\mathcal{T}, \mathcal{F})$ is hereditary. We will show that the condition that its heart be a module category gives more precise information than in the general case. Recall that $\boldsymbol{t}$ is called *bounded* when its associated Gabriel topology has a basis consisting of two-sided ideals (see [**S**, Chapter VI]). Equivalently, when $R/\mathrm{ann}_R(T) \in \mathcal{T}$, for each finitely generated module $T \in \mathcal{T}$.

Theorem 4.1. *Let $\boldsymbol{t} = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion pair in R-Mod and let G be a complex in the standard form, where P and Q are finitely generated projective modules and $V = H^0(G)$. The following assertions are equivalent*:

1. *G is a progenerator of $\mathcal{H}_{\boldsymbol{t}}$;*

2. *The following conditions are satisfied*:
   (a) *G satisfies the standard conditions* 1, 3 *and* 4;
   (b) *If $\boldsymbol{b} = \mathrm{ann}_R(V/t(R)V)$, then $\boldsymbol{b}/t(R)$ is an idempotent ideal of $R/t(R)$ (which is finitely generated on the left) and $R/\boldsymbol{b}$ is in $\mathcal{T}$*;
   (c) *There is a morphism $h : (Q/X)^{(J)} \longrightarrow \boldsymbol{b}/t(R)$, for some set J, such that $h_{|H^{-1}(G)^{(J)}} : H^{-1}(G)^{(J)} \longrightarrow \boldsymbol{b}/t(R)$ is an epimorphism.*

When $\boldsymbol{t}$ is bounded, the assertions are also equivalent to:

3. *There is an idempotent ideal $\boldsymbol{a}$ of R, which is finitely generated on the left, such that*:
   (a) $\mathrm{add}(V) = \mathrm{add}(R/\boldsymbol{a})$ *and $\boldsymbol{t}$ is the right constituent torsion pair of the TTF triple defined by $\boldsymbol{a}$*;
   (b) $\mathrm{Ker}(d) \subseteq X + \boldsymbol{a}Q$;
   (c) $\mathrm{Ext}^1_R(Q/X, ?)$ *vanishes on $\mathcal{F}$*;
   (d) *There is a morphism $h : (Q/X)^{(J)} \longrightarrow \boldsymbol{a}/t(\boldsymbol{a})$, for some set J, such that $h_{|H^{-1}(G)^{(J)}} : H^{-1}(G)^{(J)} \longrightarrow \boldsymbol{a}/t(\boldsymbol{a})$ is an epimorphism.*

Proof. 1) $\Longrightarrow$ 2) By Theorem 3.4 and Remark 3.6, we may assume that G satisfies the standard conditions 1–5. So we only need to check properties 2.b and 2.c. We proceed in several steps. All throughout the proof we put $\bar{R} = R/t(R)$.

Step 1: *$(\mathcal{T} \cap \bar{R}\text{-Mod}, \mathcal{F})$ is the right constituent torsion pair of a TTF triple in* -Mod: By the standard condition 5, there is a morphism $h : (Q/X)^{(I)} \longrightarrow \bar{R}$ such that if $h' = h_{|H^{-1}(G)^{(I)}}$, then $\mathrm{Coker}(h') \in \overline{\mathrm{Gen}}(V)$, where $V = H^0(G)$. But in this case $\overline{\mathrm{Gen}}(V) = \mathrm{Gen}(V) = \mathcal{T}$ since $\boldsymbol{t}$ is hereditary.

Put $\bar{\boldsymbol{b}} = \boldsymbol{b}/t(R) = \mathrm{Im}(h')$, so that $R/\boldsymbol{b} \in \mathcal{T}$. We claim that a $\bar{R}$-module T is in $\mathcal{T}$ if, and only if, $\mathrm{Hom}_R(\bar{\boldsymbol{b}}, T) = 0$. This will imply that $\mathcal{T} \cap \bar{R}$-Mod is also a torsionfree class in $\bar{R}$-Mod. For the 'only if' part of our claim, let $f : \bar{\boldsymbol{b}} \longrightarrow T$ be any morphism, where $T \in \mathcal{T}$. We then get a pushout commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \bar{\boldsymbol{b}} \arrow[r, hook, "j"] \arrow[d, "f"] & \bar{R} \arrow[r] \arrow[d, "g'"] & R/b \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & T \arrow[r, "\lambda"] & T' \arrow[r] & R/b \arrow[r] & 0
\end{tikzcd}$$

Then $T' \in \mathcal{T}$ and so $g' \circ h_{|H^{-1}(G)^{(I)}} = 0$ since $H^{-1}(G) \subseteq \mathrm{Rej}_{\mathcal{T}}(Q/X)$. But $g' \circ h_{|H^{-1}(G)^{(I)}}$ is equal to the composition $H^{-1}(G)^{(I)} \xrightarrow{h'} \bar{\boldsymbol{b}} \xrightarrow{f} T \xrightarrow{\lambda} T'$, which is then the zero map. This implies that $f = 0$ since h' is an epimorphism and λ is a monomorphism. For the 'if' part, suppose that $\mathrm{Hom}_R(\bar{\boldsymbol{b}}, T) = 0$ and fix an epimorphism $q : \bar{R}^{(J)} \twoheadrightarrow T$. Then $q(\bar{\boldsymbol{b}}^{(J)}) = 0$, which gives an induced epimorphism $\bar{q} : \bar{R}^{(J)}/\bar{\boldsymbol{b}}^{(J)} \cong (R/\boldsymbol{b})^{(J)} \twoheadrightarrow T$. It follows that $T \in \mathcal{T}$, which settles our claim.

Step 2: *The idempotent ideal of $\bar{R}$-Mod which defines the TTF triple in* -Mod *is $\bar{\boldsymbol{b}}' = \boldsymbol{b}'/t(R)$, where $\boldsymbol{b}' = \mathrm{ann}_R(V/t(R)V)$*: Let $\bar{\boldsymbol{b}}' = \boldsymbol{b}'/t(R)$ be the idempotent ideal of $\bar{R}$ which defines the TTF triple mentioned above. We then know (see [**S**, VI.8]) that $\mathrm{Gen}(\bar{\boldsymbol{b}}') = \{C \in \bar{R}\text{-Mod} : \bar{\boldsymbol{b}}'C = C\} = \{C \in \bar{R}\text{-Mod} : \mathrm{Hom}_R(C, T) = 0$, for all $T \in \mathcal{T} \cap \bar{R}\text{-Mod}\}$ and $\mathcal{T} \cap \bar{R}\text{-Mod} = \{T \in \bar{R}\text{-Mod} : \bar{\boldsymbol{b}}'T = 0\} = \mathrm{Gen}(R/\boldsymbol{b}')$. In particular, for the ideal $\boldsymbol{b}$ of R given in the first step, we have that $\bar{\boldsymbol{b}}'\bar{\boldsymbol{b}} = \bar{\boldsymbol{b}}$ and $\bar{\boldsymbol{b}}'(R/\boldsymbol{b}) = 0$. It follows that $\bar{\boldsymbol{b}}' = \bar{\boldsymbol{b}}$. We then get that $\mathrm{Gen}(V/t(R)V) = \mathcal{T} \cap \bar{R}\text{-Mod} = \mathrm{Gen}(R/\boldsymbol{b})$, from which we deduce that $\boldsymbol{b}' = \boldsymbol{b} = \mathrm{ann}_R(V/t(R)V)$.

Step 3: *Verification of properties* 2.b *and* 2.c: Except for the finite generation of $\bar{\boldsymbol{b}}$, property 2.b follows immediately from the previous steps. But $R/\boldsymbol{b}$ is finitely generated and we have an epimorphism $\bar{V}^n \twoheadrightarrow R/\boldsymbol{b}$. This epimorphism splits since both its domain and codomain are annihilated by $\boldsymbol{b}$ and $R/\boldsymbol{b}$ is projective in $R/\boldsymbol{b}$-Mod. But $\bar{V} = V/t(R)V$ is clearly a finitely presented $\bar{R}$-module. It follows that $R/\boldsymbol{b}$ is finitely presented as a left $\bar{R}$-module, which is equivalent to saying that $\bar{\boldsymbol{b}}$ is finitely generated as a left ideal of $\bar{R} = R/t(R)$. Let us fix an epimorphism $\pi : \bar{R}^{(n)} \twoheadrightarrow \bar{\boldsymbol{b}}$. Using the canonical map $h : (Q/X)^{(I)} \longrightarrow \bar{R}$ (see step 1), we obtain a morphism $g : [(Q/X)^{(I)}]^{(n)} \xrightarrow{h^{(n)}} \bar{R}^{(n)} \xrightarrow{\pi} \bar{\boldsymbol{b}}$. If $Y := H^{-1}(G)$ then we have

$$g[(Y^{(I)})^{(n)}] = \pi(\mathrm{Im}(h')^{(n)}) = \pi(\bar{\boldsymbol{b}}^{(n)}) = \pi(\bar{\boldsymbol{b}}\bar{R}^{(n)}) = \bar{\boldsymbol{b}}\bar{\boldsymbol{b}} = \bar{\boldsymbol{b}},$$

which proves 2.c.

2) $\Longrightarrow$ 1) It remains to prove that G satisfies the standard property 5. If $h : (Q/X)^{(J)} \longrightarrow \bar{\boldsymbol{b}}$ is the homomorphism given in 2.c, then h is an epimorphism and the composition $g : (Q/X)^{(J)} \longrightarrow \bar{\boldsymbol{b}} \hookrightarrow \bar{R} = R/t(R)$ has $R/\boldsymbol{b}$ as its cokernel. By property 2.b, this cokernel is in $\mathcal{T} = \mathrm{Gen}(V)$.

We assume in the rest of the proof that $\boldsymbol{t}$ is bounded.

1), 2) $\Longrightarrow$ 3) We know that $\mathcal{T} = \mathrm{Gen}(V) \subseteq \mathrm{Ker}(\mathrm{Ext}^1_R(V, ?))$ and that ${}_RV$ is finitely presented. The bounded condition of $\boldsymbol{t}$ implies that $R/\mathrm{ann}_R(V) \in \mathcal{T}$, so that $\boldsymbol{a} := \mathrm{ann}_R(V)$ annihilates all modules in $\mathcal{T}$. By [**S**, Proposition VI.6.12], we know that $\boldsymbol{a}$ is idempotent, so that $\boldsymbol{t}$ is the right constituent torsion pair of the TTF triple defined by $\boldsymbol{a}$. This allows to identify $\mathcal{T}$ with $R/\boldsymbol{a}$-Mod and, using that also $\mathcal{T} = \mathrm{Gen}(V) \subseteq \mathrm{Ker}(\mathrm{Ext}^1_R(V, ?))$, we conclude that $\mathrm{add}(V) = \mathrm{add}(R/\boldsymbol{a})$. We then get condition 3.a. We also get that $R/\boldsymbol{a}$ is a finitely presented R-module, and so $\boldsymbol{a}$ is finitely generated on the left.

The fact that G satisfies the standard conditions and that $\mathrm{Rej}_{\mathcal{T}}(M) = \boldsymbol{a}M$, for each R-module M, automatically imply that conditions 3.b and 3.c hold. Finally, following the proof of the implication 1) $\Longrightarrow$ 2), we see that the ideal $\boldsymbol{b}$ obtained in assertion 2 is identified by the properties that $\bar{\boldsymbol{b}} = \boldsymbol{b}/t(R)$ is idempotent and a $\bar{R}$-module T is in $\mathcal{T}$ if, and only if, $\boldsymbol{b}T = 0$. Then we have $\boldsymbol{b} = \boldsymbol{a} + t(R)$ and so condition 3.d follows by using the isomorphism $\boldsymbol{b}/t(R) = (\boldsymbol{a} + t(R))/t(R) \cong \boldsymbol{a}/(\boldsymbol{a} \cap t(R)) = \boldsymbol{a}/t(\boldsymbol{a})$.

3) $\Longrightarrow$ 1) Since we have $\mathrm{Rej}_{\mathcal{T}}(M) = \boldsymbol{a}M$, for each R-module M, it is easily verified that G satisfies all the standard conditions 1-5. □

COROLLARY 4.2. *If $\boldsymbol{t} = (\mathcal{T}, \mathcal{F})$ is a faithful hereditary torsion pair such that its heart $\mathcal{H}_{\boldsymbol{t}}$ is a module category, then $\boldsymbol{t}$ is the right constituent pair of a TTF triple in R-*Mod *defined by an idempotent ideal $\boldsymbol{a}$ which is finitely generated on the left.*

COROLLARY 4.3. *Let R be a commutative ring and let $\boldsymbol{t} = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion pair in R-*Mod. *The heart $\mathcal{H}_{\boldsymbol{t}}$ is a module category if, and only if, $\boldsymbol{t}$ is* (*left or right*) *constituent pair of a centrally split TTF triple in R-*Mod. *In that case $\mathcal{H}_{\boldsymbol{t}}$ is equivalent to R-*Mod.

PROOF. Since $\boldsymbol{t}$ is bounded, last theorem says that $\boldsymbol{t}$ is the right constituent torsion pair of a TTF triple in R-Mod defined by an idempotent ideal $\boldsymbol{a}$ which is finitely generated. But each finitely generated idempotent ideal of a commutative ring is generated by an idempotent element (see the proof of Lemma VI.8.6 in [**S**]). Then the TTF triple is centrally split. By Corollary 7.8 below, we have that $\mathcal{H}_{\boldsymbol{t}}$ is equivalent to R-Mod. $\square$

5. When the progenerator is a sum of stalk complexes.

Recall that if M and N are R-modules, then $\mathrm{Ext}_R^1(M,N) = \mathrm{Hom}_{\mathcal{D}(R)}(M, N[1])$ has a canonical structure of $\mathrm{End}_R(N)$-$\mathrm{End}_R(M)$-bimodule given by composition of morphisms in $\mathcal{D}(R)$. But then it has also a structure of $\mathrm{End}_R(M)^{op}$-$\mathrm{End}_R(N)^{op}$, by defining $\alpha^o \cdot \epsilon \cdot f^o = f \circ \epsilon \circ \alpha$, for all $\alpha \in \mathrm{End}_R(M)$ and $f \in \mathrm{End}_R(N)$.

It is natural to expect that the 'simplest' case in which the heart is a module category appears when the progenerator of the heart can be chosen to be a sum of stalk complexes. Our next result gives criteria for that to happen.

PROPOSITION 5.1. *The following assertions are equivalent*:

1. $\mathcal{H}_{\boldsymbol{t}}$ *has a progenerator of the form* $V[0] \oplus Y[1]$, *where* $V \in \mathcal{T}$ *and* $Y \in \mathcal{F}$;
2. *There are* R*-modules* V *and* Y *satisfying the following properties*:
   (a) V *is finitely presented and* $\mathcal{T} = \mathrm{Pres}(V) \subseteq \mathrm{Ker}(\mathrm{Ext}_R^1(V,?))$;
   (b) $\mathrm{Ext}_R^2(V,?)$ *vanishes on* $\mathcal{F}$;
   (c) Y *is a finitely generated projective* $R/t(R)$*-module which is in* ${}^{\perp}\mathcal{T}$;
   (d) *For each* $F \in \mathcal{F}$, *the module* $(F/\mathrm{tr}_Y(F))/t(F/\mathrm{tr}_Y(F))$ *embeds into a module in* $\mathcal{T}$, *where* $\mathrm{tr}_Y(F)$ *denote the trace of* Y *in* F.

In this case $\mathcal{H}_{\boldsymbol{t}}$ *is equivalent to* S-Mod, *where* $S = \begin{pmatrix} \mathrm{End}_R(Y)^{op} & 0 \\ \mathrm{Ext}_R^1(V,Y) & \mathrm{End}_R(V)^{op} \end{pmatrix}$, *when viewing* $\mathrm{Ext}_R^1(V,Y)$ *as a* $\mathrm{End}_R(V)^{op}$-$\mathrm{End}_R(Y)^{op}$*-bimodule in the usual way.*

PROOF. By [**PS**, Theorem 4.8] and by condition 2.a, all throughout the proof we can assume that $\mathcal{F}$ is closed under taking direct limits in R-Mod.

1) $\Longrightarrow$ 2) Put $G = V[0] \oplus Y[1]$. By Lemma 3.2, we get condition 2.a. On the other hand, the projective condition of $V[0]$ in $\mathcal{H}_{\boldsymbol{t}}$ implies that $0 = \mathrm{Ext}^1_{\mathcal{H}_{\boldsymbol{t}}}(V[0], F[1]) = \mathrm{Ext}_R^2(V,F)$, for all $F \in \mathcal{F}$. Then condition 2.b also holds.

The projective condition of $Y[1]$ in $\mathcal{H}_{\boldsymbol{t}}$ implies that $0 = \mathrm{Ext}^1_{\mathcal{H}_{\boldsymbol{t}}}(Y[1], F[1]) = \mathrm{Ext}_R^1(Y,F)$, for all $F \in \mathcal{F}$ and that $0 = \mathrm{Ext}^1_{\mathcal{H}_{\boldsymbol{t}}}(Y[1], T[0]) = \mathrm{Hom}_{\mathcal{D}(R)}(Y[1], T[1]) \cong \mathrm{Hom}_R(Y,T) = 0$, for all $T \in \mathcal{T}$. Then we have that $Y \in {}^{\perp}\mathcal{T}$. Moreover, if $f : (R/t(R))^{(I)} \twoheadrightarrow Y$ is any epimorphism, then f is a retraction, which implies that Y is a projective $R/t(R)$-module. The fact that Y is finitely generated follows from the compactness of $Y[1]$ in $\mathcal{H}_{\boldsymbol{t}}$ since then $\mathrm{Hom}_R(Y,?)_{|\mathcal{F}} \cong \mathrm{Hom}_{\mathcal{D}(R)}(Y[1], ?[1])_{|\mathcal{F}}$ preserves coproducts of modules in $\mathcal{F}$. We then get condition 2.c.

For each $F \in \mathcal{F}$, let us consider the canonical morphism $g : Y^{(\mathrm{Hom}_R(Y,F))} \longrightarrow F$. We then get the morphism $Y[1]^{(\mathrm{Hom}_{\mathcal{H}_{\boldsymbol{t}}}(Y[1],F[1]))} \cong Y^{(\mathrm{Hom}_R(Y,F))}[1] \xrightarrow{g[1]} F[1]$ whose image is the trace of $Y[1]$ in $F[1]$ within the category $\mathcal{H}_{\boldsymbol{t}}$. The cokernel of $g[1]$ is precisely the stalk complex $(\mathrm{Coker}(g)/t(\mathrm{Coker}(g)))[1] = ((F/\mathrm{tr}_Y(F))/t(F/\mathrm{tr}_Y(F)))[1]$.

Due to the projectivity of $Y[1]$ in $\mathcal{H}_t$, we have that $\mathrm{Hom}_R(Y, \mathrm{Coker}(g)/t(\mathrm{Coker}(g))) \cong \mathrm{Hom}_{\mathcal{H}_t}(Y[1], (\mathrm{Coker}(g)/t(\mathrm{Coker}(g)))[1]) = 0$. The fact that $V[0] \oplus Y[1]$ is a projective generator of $\mathcal{H}_t$ implies then that the canonical morphism $q : V[0]^{(\mathrm{Hom}_{\mathcal{H}_t}(V[0],(\mathrm{Coker}(g)/t(\mathrm{Coker}(g)))[1]))} \longrightarrow (\mathrm{Coker}(g)/t(\mathrm{Coker}(g)))[1]$ is an epimorphism in $\mathcal{H}_t$. We necessarily have $\mathrm{Ker}(q) = T[0]$, for some $T \in \mathcal{T}$. Condition 2.d follows then from the long exact sequence of homologies associated to the triangle

$$T[0] \longrightarrow V[0]^{(\mathrm{Hom}_{\mathcal{H}_t}(V[0],(\mathrm{Coker}(g)/t(\mathrm{Coker}(g)))[1]))} \xrightarrow{q} \frac{\mathrm{Coker}(g)}{t(\mathrm{Coker}(g))}[1] \xrightarrow{+}$$

2) $\Longrightarrow$ 1) From conditions 2.a and 2.b we deduce that $\mathrm{Ext}^1_{\mathcal{H}_t}(V[0], ?)$ vanishes on stalk complexes $T[0]$ and $F[1]$, for each $T \in \mathcal{T}$ and $F \in \mathcal{F}$. Similarly, from condition 2.c we deduce that $\mathrm{Ext}^1_{\mathcal{H}_t}(Y[1], ?)$ vanishes on all those stalk complexes. It follows that $G := V[0] \oplus Y[1]$ is a projective object of $\mathcal{H}_t$.

Knowing that G is a projective object, in order to prove that G is a generator of $\mathcal{H}_t$, we just need to prove that it generates all stalk complexes X, with $X \in \mathcal{T}[0] \cup \mathcal{F}[1]$. Note that from condition 2.a we get that $V[0]$ generates all stalk complexes $T[0]$ and, hence, that $\mathcal{T}[0] \subseteq \mathrm{Gen}_{\mathcal{H}_t}(G)$. If now we take $F \in \mathcal{F}$, then the argument in the proof of the other implication shows that the canonical morphism $Y[1]^{(\mathrm{Hom}_{\mathcal{H}_t}(Y[1],F[1]))} \cong Y^{(\mathrm{Hom}_R(Y,F))}[1] \xrightarrow{g[1]} F[1]$ has as cokernel $F'[1]$, where $F' = (F/\mathrm{tr}_Y(F))/t(F/\mathrm{tr}_Y(F))$. By hypothesis we have a monomorphism $F' \rightarrowtail T$ and, hence, an exact sequence $0 \longrightarrow F' \longrightarrow T \longrightarrow T' \longrightarrow 0$, where T and T' are in $\mathcal{T}$. We then get an exact sequence in $\mathcal{H}_t$:

$$0 \longrightarrow T[0] \longrightarrow T'[0] \longrightarrow F'[1] \longrightarrow 0$$

which shows that $F'[1]$ is generated by $V[0]$ and, hence, that $F'[1] \in \mathrm{Gen}_{\mathcal{H}_t}(G)$. But we have an exact sequence in $\mathcal{H}_t$

$$0 \longrightarrow \mathrm{Im}_{\mathcal{H}_t}(g[1]) \longrightarrow F[1] \longrightarrow F'[1] \longrightarrow 0$$

Then we have that $\mathrm{Im}_{\mathcal{H}_t}(g[1]) \in \mathrm{Gen}_{\mathcal{H}_t}(Y[1]) \subseteq \mathrm{Gen}_{\mathcal{H}_t}(G)$ and $F'[1] \in \mathrm{Gen}_{\mathcal{H}_t}(V[0]) \subseteq \mathrm{Gen}_{\mathcal{H}_t}(G)$. The projective condition of G in $\mathcal{H}_t$ proves now that also $F[1] \in \mathrm{Gen}_{\mathcal{H}_t}(G)$. Hence G is a generator of $\mathcal{H}_t$.

We finally prove that G is compact in $\mathcal{H}_t$, which is equivalent to proving that $V[0]$ and $Y[1]$ are compact in this category. For each family $(M_i)_{i \in I}$ of objects in $\mathcal{H}_t$, we have a family of exact sequences in $\mathcal{H}_t$:

$$0 \longrightarrow H^{-1}(M_i)[1] \longrightarrow M_i \longrightarrow H^0(M_i)[0] \longrightarrow 0 \qquad (i \in I)$$

Using this and the projectivity of $V[0]$ and $Y[1]$, the task is reduced to check the following facts:

i) $\mathrm{Hom}_R(Y, ?)$ preserves coproducts of modules in $\mathcal{F}$;

ii) $\mathrm{Hom}_R(V,?)$ preserves coproducts of modules in $\mathcal{T}$;
iii) $\mathrm{Ext}^1_R(V,?)$ preserves coproducts of modules in $\mathcal{F}$.

Conditions i) and ii) automatically hold since Y and V are finitely generated modules. Condition iii) follows from the fact that V is finitely presented.

The final statement of the proposition is clear, because the ring $S = \begin{pmatrix} \mathrm{End}_R(Y)^{op} & 0 \\ \mathrm{Ext}^1_R(V,Y) & \mathrm{End}_R(V)^{op} \end{pmatrix}$ is isomorphic to $\mathrm{End}_{\mathcal{H}_{\boldsymbol{t}}}(V[0] \oplus Y[1])^{op}$. $\square$

We have now the following consequences of last proposition.

Corollary 5.2. *Let V be an R-module and consider the following conditions*

1. *V is a classical 1-tilting module*;
2. $\boldsymbol{t} = (\mathrm{Pres}(V), \mathrm{Ker}(\mathrm{Hom}_R(V,?)))$ *is a torsion pair in R-*Mod *and $V[0]$ is a progenerator of $\mathcal{H}_{\boldsymbol{t}}$*;
3. *V finitely presented and satisfies the following conditions*:
   (a) $\mathcal{T} := \mathrm{Pres}(V) = \mathrm{Gen}(V) \subseteq \mathrm{Ker}(\mathrm{Ext}^1_R(V,?))$;
   (b) $\mathrm{Ext}^2_R(V,?)$ *vanishes on* $\mathcal{F} := \mathrm{Ker}(\mathrm{Hom}_R(V,?))$;
   (c) *Each module of $\mathcal{F}$ embeds into a module of $\mathcal{T}$.*

Then the implications 1) $\Longrightarrow$ 2) $\Longleftrightarrow$ 3) hold true. Moreover, when conditions 2 or 3 hold, $\boldsymbol{t}$ is also a torsion pair in the Grothendieck category $\mathcal{G} := \overline{\mathrm{Gen}}(V)$, V is a classical 1-tilting object of $\mathcal{G}$ and the canonical functor $\mathcal{D}(\mathcal{G}) \longrightarrow \mathcal{D}(R)$ gives by restriction an equivalence of categories $\mathcal{H}_{\boldsymbol{t}}(\mathcal{G}) \xrightarrow{\sim} \mathcal{H}_{\boldsymbol{t}}$, where $\mathcal{H}_{\boldsymbol{t}}(\mathcal{G})$ is the heart of the torsion pair in $\mathcal{G}$.

Proof. 1) $\Longrightarrow$ 2) is a particular case of [**PS**, Proposition 5.3].

2) $\Longrightarrow$ 3) is a direct consequence of Proposition 5.1.

3) $\Longrightarrow$ 2) We need to prove that $\mathcal{T} = \mathrm{Gen}(V)$ is closed under taking extensions in R-Mod. In that case $\boldsymbol{t} = (\mathrm{Gen}(V), \mathrm{Ker}(\mathrm{Hom}_R(V,?)))$ is a torsion pair in R-Mod and the implication will follow from Proposition 5.1. Let $0 \to T \longrightarrow M \longrightarrow T' \to 0$ is an exact sequence in R-Mod, with $T, T' \in \mathcal{T}$. We want to prove that $M \in \mathcal{T}$. By pulling back the exact sequence along an epimorphism $p : V^{(I)} \twoheadrightarrow T'$, we can assume without loss of generality that $T' = V^{(I)}$. But in this case the sequence splits since $\mathrm{Ext}^1_R(V^{(I)}, T') \cong \mathrm{Ext}^1_R(V,T)^I = 0$.

Let us prove now the final statement. By Lemma 3.2, we know that V is classical quasi-tilting. It essentially follows from the arguments in [**CDT1**, Section 2] that V is a classical 1-tilting object of $\mathcal{G} := \overline{\mathrm{Gen}}(V)$. But it also follows from something stronger that we need, namely, that the canonical map $\varphi : \mathrm{Ext}^1_{\mathcal{G}}(V,X) \longrightarrow \mathrm{Ext}^1_R(V,X)$ is an isomorphism, for each $X \in \mathcal{G}$. It is clearly injective. To prove the surjectivity, let $0 \longrightarrow X \longrightarrow M \longrightarrow V \longrightarrow 0$ $(*)$ be an exact sequence in R-Mod. Recall that the injective objects of $\mathcal{G}$ are modules in $\mathrm{Gen}(V) = \mathcal{T}$ (see [**GG**, Introduction]). This implies that we have a monomorphism $u : X \hookrightarrow T$, with $T \in \mathcal{T}$. By pushing out the sequence $(*)$ along the monomorphism u and using the fact that $\mathrm{Ext}^1_R(V,T) = 0$, we get a monomorphism $M \hookrightarrow T \oplus V$, which implies that $M \in \mathcal{G}$. Then the sequence $(*)$ lives in $\mathcal{G}$ and, hence, φ is an isomorphism.

On the other hand, the inclusion functor $\mathcal{G} \hookrightarrow R$-Mod is exact and, hence, extends to a triangulated functor $j : \mathcal{D}(\mathcal{G}) \longrightarrow \mathcal{D}(R)$, which need be neither faithful nor full, but induces by restriction a functor $\tilde{j} : \mathcal{H}_{\boldsymbol{t}}(\mathcal{G}) \longrightarrow \mathcal{H}_{\boldsymbol{t}} := \mathcal{H}_{\boldsymbol{t}}(R\text{-Mod})$. We claim that, up to natural isomorphism, the following diagram of functors is commutative, where $S = \mathrm{End}_R(V)^{op}$:

$$\begin{tikzcd}
\mathcal{H}_{\boldsymbol{t}}(\mathcal{G}) \arrow[rr, "\tilde{j}"] \arrow[dr, "\mathrm{Hom}_{D(\mathcal{G})}(V[0],?)"'] & & \mathcal{H}_{\boldsymbol{t}} \arrow[dl, "\mathrm{Hom}_{D(R)}(V[0],?)"] \\
& S\text{-Mod} &
\end{tikzcd}$$

Due to the projective condition of $V[0]$ both in $\mathcal{H}_{\boldsymbol{t}}(\mathcal{G})$ and $\mathcal{H}_{\boldsymbol{t}}$, we just need to see that the maps induced by the functor j:

$$\mathrm{Hom}_{\mathcal{G}}(V,T) \cong \mathrm{Hom}_{\mathcal{D}(G)}(V[0],T[0]) \longrightarrow \mathrm{Hom}_{\mathcal{D}(R)}(V[0],T[0]) \cong \mathrm{Hom}_R(V,T)$$

$$\mathrm{Ext}^1_{\mathcal{G}}(V,F) \cong \mathrm{Hom}_{\mathcal{D}(G)}(V[0],F[1]) \longrightarrow \mathrm{Hom}_{\mathcal{D}(R)}(V[0],F[1]) \cong \mathrm{Ext}^1_R(V,F)$$

are isomorphisms. The first one is clear and the second one has been proved in the previous paragraph. By assertion 2, the functor $\mathrm{Hom}_{\mathcal{H}_{\boldsymbol{t}}}(V[0],?) : \mathcal{H}_{\boldsymbol{t}} \longrightarrow S\text{-Mod}$ is an equivalence of categories. Since V is a classical 1-tilting object of $\mathcal{G}$, the functor $\mathrm{Hom}_{\mathcal{H}_{\boldsymbol{t}}(\mathcal{G})}(V[0],?) : \mathcal{H}_{\boldsymbol{t}}(\mathcal{G}) \longrightarrow S\text{-Mod}$ is also an equivalence (see [**PS**, Proposition 5.3]). It follows that $\tilde{j} : \mathcal{H}_{\boldsymbol{t}}(\mathcal{G}) \longrightarrow \mathcal{H}_{\boldsymbol{t}}$ is an equivalence of categories. □

The following is now very natural.

Question 5.3. Let $\boldsymbol{t} = (\mathcal{T}, \mathcal{F})$ be a torsion pair in R-Mod satisfying the equivalent conditions 2 and 3 of Corollary 5.2. Is $\boldsymbol{t}$ a classical tilting torsion pair?

Lemma 5.4. *Let V be a classical quasi-tilting R-module such that* $\mathrm{Gen}(V)$ *is closed under submodules and let $t(R)$ be the trace of V in R. An endomorphism β of V satisfies that* $\mathrm{Im}(\beta) \subseteq t(R)V$ *if, and only if, it factors through a* (*finitely generated*) *projective R-module.*

Proof. We put $\boldsymbol{t} = (\mathrm{Gen}(V), \mathrm{Ker}(\mathrm{Hom}_R(V,?)))$, which is a hereditary torsion pair. The 'if' part is clear. Conversely, suppose that $\mathrm{Im}(\beta) \subseteq t(R)V$. Let $q : V^{(\mathrm{Hom}_R(V,R))} \twoheadrightarrow t(R) = \mathrm{tr}_V(R)$, $i : t(R) \hookrightarrow R$, $\pi : R^{(V)} \twoheadrightarrow V$ and $j : t(R)V \hookrightarrow V$ be the canonical morphisms and let $\pi' : t(R)^{(V)} \twoheadrightarrow t(R)V$ be the epimorphism given by the restriction of π to $t(R)^{(V)}$. We have a commutative diagram

$$\begin{tikzcd}
V^{(\mathrm{Hom}_R(V,R)\times V)} \arrow[r, two heads, "q^{(V)}"] \arrow[dr, two heads, "\rho"] & t(R)^{(V)} \arrow[r, "i^{(V)}"] \arrow[d, two heads, "\pi'"] & R^{(V)} \arrow[d, two heads, "\pi"] \\
& t(R)V \arrow[r, hook, "j"] & V
\end{tikzcd}$$

where $\rho := \pi' \circ q^{(V)}$. We have a factorization $j \circ \tilde{\beta} = \beta$, where $\tilde{\beta} \in \mathrm{Hom}_R(V, t(R)V)$. Due to the hereditary condition of $\boldsymbol{t}$, we know that $\mathrm{Ker}(\rho) \in \mathcal{T} \subseteq \mathrm{Ker}(\mathrm{Ext}^1_R(V, ?))$, which implies $\tilde{\beta}$ factors through ρ. Fix a morphism $\gamma : V \longrightarrow V^{(\mathrm{Hom}_R(V,R) \times V)}$ such that $\tilde{\beta} = \rho \circ \gamma$. Then we have:

$$\beta = j \circ \tilde{\beta} = j \circ \rho \circ \gamma = \pi \circ i^{(V)} \circ q^{(V)} \circ \gamma,$$

so that β factors through $R^{(V)}$. □

Corollary 5.5. *Let us assume that $\boldsymbol{t} = (\mathcal{T}, \mathcal{F})$ is a hereditary torsion pair in R-Mod. The following assertions are equivalent*:

1. *$\mathcal{H}_{\boldsymbol{t}}$ has a progenerator of the form $V[0] \oplus Y[1]$, where $V \in \mathcal{T}$ and $Y \in \mathcal{F}$*;
2. *There are R-modules V and Y satisfying the following properties*:
   (a) *V is finitely presented and $\mathcal{T} = \mathrm{Pres}(V) \subseteq \mathrm{Ker}(\mathrm{Ext}^1_R(V, ?))$*;
   (b) *$\mathrm{Ext}^2_R(V, ?)$ vanishes on $\mathcal{F}$*;
   (c) *Y is a finitely generated projective $R/t(R)$-module which is in ${}^{\perp}\mathcal{T}$*;
   (d) *For each $F \in \mathcal{F}$, the module $F/\mathrm{tr}_Y(F)$ is in $\mathcal{T}$, where $\mathrm{tr}_Y(F)$ denotes the trace of Y in F.*

In this case, $\boldsymbol{t}' := (\mathcal{T} \cap R/t(R)\text{-Mod}, \mathcal{F})$ is a torsion pair in $R/t(R)$-Mod which is the right constituent of a TTF triple in this category and has the property that $(V/t(R)V)[0] \oplus Y[1]$ is a progenerator of $\mathcal{H}_{\boldsymbol{t}'}$. Moreover, the forgetful functor $\mathcal{H}_{\boldsymbol{t}'} \longrightarrow \mathcal{H}_{\boldsymbol{t}}$ is faithful.

Proof. All throughout the proof we put $I = t(R)$ and $\bar{M} = M/IM$, for each R-module M. The equivalence of assertions 1 and 2 is a direct consequence of Proposition 5.1. From Theorem 4.1 and its proof, we know that $(\mathcal{T} \cap \bar{R}\text{-Mod}, \mathcal{F})$ is the right constituent torsion pair of a TTF triple $(\mathcal{C}_I, \mathcal{T}_I, \mathcal{F})$ in $\bar{R}$-Mod. Moreover, by property 2.c, the class $\mathrm{Ker}(\mathrm{Hom}_{\bar{R}}(Y, ?))$ contains $\mathcal{T} \cap \bar{R}$-Mod and is closed under taking quotients. Using property 2.d, it then follows that the inclusion $\mathrm{Ker}(\mathrm{Hom}_{\bar{R}}(Y, ?)) \subseteq \mathcal{T} \cap \bar{R}$-Mod also holds, which implies that $\mathcal{C}_I = \mathrm{Gen}(Y)$. If now $\boldsymbol{a}$ is the two-sided ideal of R given by the equality $\bar{\boldsymbol{a}} = \boldsymbol{a}/I = \mathrm{tr}_Y(R/I)$, then $\bar{\boldsymbol{a}}$ is the idempotent ideal of $\bar{R}$ which defines the TTF triple and, by the proof of Theorem 4.1, we know that $\boldsymbol{a} = \mathrm{ann}_R(\bar{V})$ and that $\mathrm{add}(\bar{V}) = \mathrm{add}(R/\boldsymbol{a})$, so that $\bar{V}$ is a progenerator of $R/\boldsymbol{a}$-Mod.

Now the $\bar{R}$-modules $\bar{V}$ and Y satisfy the conditions 2.a, 2.c and 2.d with respect to the torsion pair $\boldsymbol{t}' = (\mathcal{T}_I, \mathcal{F})$ of $\bar{R}$-Mod. On the other hand, $\boldsymbol{t}$ and $\boldsymbol{t}'$ are hereditary torsion pairs in R-Mod and $\bar{R}$-Mod, respectively. Then, for each $F \in \mathcal{F}$, the injective envelope $E(F)$ in R-Mod is also in $\mathcal{F}$ (see [**S**, Proposition VI.3.2]). In particular, we have that $E(F) \in \bar{R}$-Mod, so that $E(F)$ is also the injective envelope of F as a $\bar{R}$-module and, hence, the first cosyzygy $\Omega^{-1}(F)$ is the same in R-Mod and $\bar{R}$-Mod. In order to check condition 2.b for $\bar{V}$, we need to check that $\mathrm{Ext}^1_{\bar{R}}(\bar{V}, \Omega^{-1}(F)) = 0$. But, using condition 2.b for V, our needed goal will follow from something stronger that we will prove. Namely, that if $p = p_V : V \twoheadrightarrow \bar{V}$ is the canonical projection, then the composition

$$\varphi : \mathrm{Ext}^1_{\bar{R}}(\bar{V}, M) \xrightarrow{can} \mathrm{Ext}^1_R(\bar{V}, M) \xrightarrow{\mathrm{Ext}^1_R(p,M)} \mathrm{Ext}^1_R(V, M)$$

is a monomorphism, for all $M \in \bar{R}$-Mod.

Let $0 \longrightarrow M \xrightarrow{j} N \xrightarrow{q} \bar{V} \longrightarrow 0$ be an exact sequence in $\bar{R}$-Mod which represents an element of $\mathrm{Ker}(\varphi)$. Then the projection $p : V \twoheadrightarrow \bar{V}$ factors through q. Fixing a morphism $g : V \longrightarrow N$ such that $q \circ g = p$ and taking into account that $IN = 0$, we get a morphism $\bar{g} : \bar{V} \longrightarrow N$ which is a section for q.

In order to prove the final assertion, with the notation of the previous lemma, consider the following composition of morphisms of abelian groups, where $F \in \mathcal{F}$

$$\mathrm{Ext}^1_R(j \circ \rho, F) : \mathrm{Ext}^1_R(V, F) \xrightarrow{\mathrm{Ext}^1_R(j,F)} \mathrm{Ext}^1_R(t(R)V, F)$$
$$\xrightarrow{\mathrm{Ext}^1_R(\rho,F)} \mathrm{Ext}^1_R(V^{(\mathrm{Hom}_R(V,R)\times V)}, F).$$

We have that $\mathrm{Ext}^1_R(\rho, F)$ is a monomorphism, because $\mathrm{Ker}(\rho) \in \mathcal{T}$ and hence $\mathrm{Hom}_R(\mathrm{Ker}(\rho), F) = 0$. But $\mathrm{Ext}^1_R(j \circ \rho, F) = 0$ since $j \circ \rho$ factors through a projective R-module. We then get that $\mathrm{Ext}^1_R(j, F)$ is the zero map, for each $F \in \mathcal{F}$. By considering the canonical exact sequence $0 \longrightarrow t(R)V \overset{j}{\hookrightarrow} V \twoheadrightarrow \bar{V} \longrightarrow 0$ and applying to it the long exact sequence of $\mathrm{Ext}(?, F)$, we get:

$$0 = \mathrm{Hom}_R(t(R)V, F) \longrightarrow \mathrm{Ext}^1_R(\bar{V}, F) \longrightarrow \mathrm{Ext}^1_R(V, F) \xrightarrow{0} \mathrm{Ext}^1_R(t(R)V, F)$$

which proves that $\mathrm{Ext}^1_R(\bar{V}, F) \cong \mathrm{Ext}^1_R(V, F)$, for each $F \in \mathcal{F}$. Moreover, by the two previous paragraphs, we get that the map $\mathrm{Ext}^1_{\bar{R}}(\bar{V}, F) \xrightarrow{can} \mathrm{Ext}^1_R(\bar{V}, F)$ is a monomorphism.

Let us put $\bar{G} := \bar{V}[0] \oplus Y[1]$. We claim that the map $\mathrm{Hom}_{\mathcal{H}_{t'}}(\bar{G}, M) \longrightarrow \mathrm{Hom}_{\mathcal{H}_t}(\bar{G}, M)$ is injective, for all $M \in \mathcal{H}_{t'}$. Bearing in mind that we have isomorphisms of abelian groups

$$\mathrm{Hom}_{\mathcal{H}_{t'}}(Y[1], M) \cong \mathrm{Hom}_{\bar{R}}(Y, H^{-1}(M)) = \mathrm{Hom}_R(Y, H^{-1}(M)) \cong \mathrm{Hom}_{\mathcal{H}_t}(Y[1], M),$$

our task reduces to check that the canonical map $\mathrm{Hom}_{\mathcal{H}_{t'}}(\bar{V}[0], M) \longrightarrow \mathrm{Hom}_{\mathcal{H}_t}(\bar{V}[0], M)$ is injective. But we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^1_{\bar{R}}(\bar{V}, H^{-1}(M)) \arrow[r] \arrow[d, hook, "can"] & \mathrm{Hom}_{\mathcal{H}_{t'}}(\bar{V}[0], M) \arrow[r] \arrow[d] & \mathrm{Hom}_{\bar{R}}(\bar{V}, H^0(M)) \arrow[r] \arrow[d, "\wr"] & \mathrm{Ext}^2_{\bar{R}}(\bar{V}, F) = 0 \\
0 \arrow[r] & \mathrm{Ext}^1_R(\bar{V}, H^{-1}(M)) \arrow[r] & \mathrm{Hom}_{\mathcal{H}_t}(\bar{V}[0], M) \arrow[r] & \mathrm{Hom}_R(\bar{V}, H^0(M)) \arrow[r] & \mathrm{Ext}^2_R(\bar{V}, F)
\end{tikzcd}$$

The right vertical arrow is an isomorphism since $H^0(M)$ is a $\bar{R}$-module, and the left vertical arrow is a monomorphism. It then follows that the central vertical arrow is a monomorphism, as desired.

Let us fix any object $M \in \mathcal{H}_{t'}$ and consider the full subcategory $\mathcal{C}_M$ of $\mathcal{H}_{t'}$ consisting of the objects N such that the canonical map $\mathrm{Hom}_{\mathcal{H}_{t'}}(N, M) \longrightarrow \mathrm{Hom}_{\mathcal{H}_t}(N, M)$ is a monomorphism. This subcategory is closed under taking coproducts and cokernels and, by the previous paragraph, it contains $\bar{G}$. We then have $\mathcal{C}_M = \mathcal{H}_{\boldsymbol{t}'}$ and, since this is true for any $M \in \mathcal{H}_{\boldsymbol{t}'}$, we conclude that the forgetful functor $\mathcal{H}_{\boldsymbol{t}'} \longrightarrow \mathcal{H}_{\boldsymbol{t}}$ is faithful. $\square$

REMARK 5.6. It can be easily derived from the proof of Corollary 5.5 that the functor $\mathcal{H}_{\boldsymbol{t}'} \longrightarrow \mathcal{H}_{\boldsymbol{t}}$ is full if, and only if, each exact sequence $0 \longrightarrow Y \longrightarrow M \longrightarrow V/IV \longrightarrow 0$ in R-Mod satisfies that $IM = 0$.

PROPOSITION 5.7. *Let $\boldsymbol{t} = (\mathcal{T}, \mathcal{F})$ be hereditary and suppose that it is the left constituent torsion pair of a TTF triple in R-*Mod. *Then $\mathcal{H}_{\boldsymbol{t}}$ is a module category if, and only if, there is a finitely generated projective R-module P such that $\mathcal{T} = \mathrm{Gen}(P)$. In such case, the following assertions hold*:

1. *$\boldsymbol{t}$ is HKM if, and only if, there is a finitely generated projective R-module Q' such that* $\mathrm{Hom}_R(Q', P) = 0$ *and* $\mathrm{add}(Q'/t(Q')) = \mathrm{add}(R/t(R))$. *In general, $\boldsymbol{t}$ need not be an HKM torsion pair*;
2. *$\boldsymbol{t}$ is the right constituent of a TTF triple in R-*Mod *if, and only if, P is finitely generated over its endomorphism ring.*

PROOF. 'If' part: Let P be a finitely generated projective R-module such that $\mathcal{T} = \mathrm{Gen}(P)$. We will check that $V = P$ and $Y = R/t(R)$ satisfy conditions 2.a–d of Corollary 5.5. All these properties are trivially satisfied, except the fact that $Y \in {}^{\perp}\mathcal{T}$. For that, we consider the TTF triple $(\mathcal{T}, \mathcal{F}, \mathcal{F}^{\perp})$. By [**S**, Lemma VI.8.3], we know that $\mathcal{T} \subseteq \mathcal{F}^{\perp}$. It particular, $Y = R/t(R) \in \mathcal{F} = {}^{\perp}(\mathcal{F}^{\perp}) \subseteq {}^{\perp}\mathcal{T}$.

'Only if' part: Let $\boldsymbol{a}$ be the idempotent ideal which defines the TTF triple, so that $\mathcal{T} = \{T \in R\text{-Mod} : \boldsymbol{a}T = T\}$. By Theorem 3.4, we have a progenerator $G := \cdots \longrightarrow 0 \longrightarrow X \xrightarrow{j} Q \xrightarrow{d} P \longrightarrow 0 \longrightarrow \cdots$, where P and Q are finitely generated projective and $\mathcal{T} = \mathrm{Gen}(V)$, where $V = H^0(G)$. We then have $\boldsymbol{a}M = t(M) = \mathrm{tr}_V(M)$, for each R-module M. In particular, we have $\boldsymbol{a} = t(R) = \mathrm{tr}_V(R)$ and, by applying Lemma 5.4 to the identity $1_V : V \longrightarrow V$, we conclude that V is a finitely generated projective module.

We next prove assertions 1 and 2:

1) If Q' exists, then the complex $P^{\bullet} := \cdots \longrightarrow 0 \longrightarrow Q' \xrightarrow{0} P \longrightarrow 0 \longrightarrow \cdots$, concentrated in degrees -1 and 0, satisfies assertion 2 of Proposition 3.7 since we know that $P[0] \oplus (R/t(R))[1]$ is a progenerator of $\mathcal{H}_{\boldsymbol{t}}$.

Conversely, suppose that $\boldsymbol{t}$ is HKM and let $P^{\bullet} := \cdots \longrightarrow 0 \longrightarrow Q \xrightarrow{d} P' \longrightarrow 0 \longrightarrow \cdots$ be an HKM complex whose associated torsion pair is $\boldsymbol{t}$. Then, by Proposition 3.7, we know that the complex

$$G := \cdots \longrightarrow 0 \longrightarrow t(\mathrm{Ker}(d)) \longrightarrow Q \xrightarrow{d} P' \longrightarrow 0 \longrightarrow \cdots,$$

concentrated in degrees $-2, -1, 0$, is a progenerator of $\mathcal{H}_{\boldsymbol{t}}$. We then have that $\mathrm{add}_{\mathcal{H}_{\boldsymbol{t}}}(G) = \mathrm{add}_{\mathcal{H}_{\boldsymbol{t}}}(P[0] \oplus (R/t(R))[1])$. In particular, we get that $V := H^0(G)$ is a projective module and, hence, also $\mathrm{Im}(d)$ is projective. It follows that, up to isomorphism in the category $\mathcal{C}(R)$, we can rewrite G as

$$\cdots \longrightarrow 0 \longrightarrow t(Q') \xrightarrow{\binom{0}{\iota}} \mathrm{Im}(d) \oplus Q' \xrightarrow{\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)} \mathrm{Im}(d) \oplus H^0(G) \longrightarrow 0 \longrightarrow \cdots,$$

where $\iota : t(Q') \hookrightarrow Q'$ is the inclusion. This in turn implies that $P^\bullet$ is isomorphic in $\mathcal{C}(R)$ to the complex

$$\cdots \longrightarrow 0 \longrightarrow \mathrm{Im}(d) \oplus Q' \xrightarrow{\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)} \mathrm{Im}(d) \oplus H^0(G) \longrightarrow 0 \longrightarrow \cdots.$$

The fact that $\mathrm{add}(V) = \mathrm{add}(H^0(G)) = \mathrm{add}(H^0(P[0] \oplus (R/t(R))[1])) = \mathrm{add}(P)$ and $V \in \mathcal{X}(P^\bullet)$ implies that $\mathrm{Hom}_R(Q', P) = 0$. Moreover, we have $\mathrm{add}(Q'/t(Q')) = \mathrm{add}(H^{-1}(G)) = \mathrm{add}(H^{-1}(P[0] \oplus (R/t(R))[1])) = \mathrm{add}(R/t(R))$.

In order to show that, in general, the pair $\boldsymbol{t}$ need not be HKM, we consider a field K, an infinite dimensional K-vector space P and view it as left module over $R = \mathrm{End}_K(P)$. It is well-known that P is a faithful simple projective R-module, so that $\mathcal{T} = \mathrm{Add}({}_RP) = \mathrm{Gen}({}_RP)$ is closed under taking submodules and, hence, $\boldsymbol{t}$ is hereditary. However the faithful condition of ${}_RP$ implies that each projective R-module embeds in a direct product of copies of P. Then it does not exists a finitely generated projective R-module Q' such that $\mathrm{Hom}_R(Q', P) = 0$ and $\mathrm{add}(Q'/t(Q')) = \mathrm{add}(R/t(R))$. Hence $\boldsymbol{t}$ is not HKM.

2) $\boldsymbol{t}$ is the right constituent pair of a TTF triple if, and only if, $\mathcal{T} = \mathrm{Gen}(P)$ is closed under taking products in R-Mod. But this is equivalent to saying that each product of copies of P is in $\mathrm{Gen}(P)$. By [**CM**, Lemma, Section 1]), this happens exactly when P is finitely generated over its endomorphism ring. □

Recall that a ring is left *semihereditary* when its finitely generated left ideals are projective.

EXAMPLE 5.8. Let $\boldsymbol{a}$ be an idempotent two-sided ideal of R, let $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ be the associated TTF triple in R-Mod and let $\boldsymbol{t} = (\mathcal{T}, \mathcal{F})$ be its right constituent torsion pair. The following assertions are equivalent:

1. $(R/\boldsymbol{a})[0] \oplus (\boldsymbol{a}/t(\boldsymbol{a}))[1]$ is a progenerator of $\mathcal{H}_{\boldsymbol{t}}$;
2. $\mathcal{H}_{\boldsymbol{t}}$ has a progenerator of the form $V[0] \oplus Y[1]$, with $V \in \mathcal{T}$ and $Y \in \mathcal{F}$;
3. $\boldsymbol{a}$ is finitely generated on the left and $\mathrm{Ext}_R^2(R/\boldsymbol{a}, ?)$ vanishes on $\mathcal{F}$.

In particular, if R is left semi-hereditary and $\boldsymbol{t}$ is the right constituent pair of a TTF triple in R-Mod, then $\mathcal{H}_{\boldsymbol{t}}$ is a module category if, and only if, the associated idempotent ideal is finitely generated on the left.

PROOF. 1) $\Longrightarrow$ 2) is clear.

2) $\Longrightarrow$ 3) By Lemma 3.2, we know that V is finitely presented and $\mathcal{T} = \mathrm{Gen}(V) \subseteq \mathrm{Ker}(\mathrm{Ext}_R^1(V, ?))$. But we also have that $\mathcal{T} = \{T \in R\text{-Mod} : \boldsymbol{a}T = 0\} \cong R/\boldsymbol{a}\text{-Mod}$. We then get that V is a finitely presented generator of $R/\boldsymbol{a}$-Mod such that $\mathrm{Ext}_{R/\boldsymbol{a}}^1(V, ?) = 0$. That is, V a progenerator of $R/\boldsymbol{a}$-Mod, which implies that $\mathrm{add}_{R\text{-Mod}}(R/\boldsymbol{a}) = \mathrm{add}_{R\text{-Mod}}(V)$. Then $R/\boldsymbol{a}$ is a finitely presented left R-module and, hence, $\boldsymbol{a}$ is finitely generated as a left ideal. The fact that $\mathrm{Ext}_R^2(R/\boldsymbol{a}, ?)$ vanishes on $\mathcal{F}$ follows from the fact that, by Corollary 5.5, we know that $\mathrm{Ext}_R^2(V, ?)$ vanishes on $\mathcal{F}$.

3) $\Longrightarrow$ 1) We take $V = R/\boldsymbol{a}$ and $Y = \boldsymbol{a}/t(\boldsymbol{a})$. Then conditions 2.a, 2.b and 2.d of Corollary 5.5 hold since $F/\mathrm{tr}_Y(F)$ is in $\mathcal{T}$, for all $F \in \mathcal{F}$. We just need to prove that Y is a projective $R/t(R)$-module since it is clearly in ${}^{\perp}\mathcal{T} = \mathcal{C}$. Let $0 \to K \hookrightarrow Q \xrightarrow{q} \boldsymbol{a} \to 0$ be an exact sequence, with Q a finitely generated projective R-module. The canonical projection $K \twoheadrightarrow K/t(K)$ extends to Q since $\mathrm{Ext}^1_R(\boldsymbol{a}, K/t(K)) \cong \mathrm{Ext}^2_R(R/\boldsymbol{a}, K/t(K)) = 0$. It follows that the canonical monomorphism $\iota : K/t(K) \longrightarrow Q/t(Q)$ splits. But its cokernel is $Q/(K + t(Q)) \cong (Q/K)/((K + t(Q))/K) \cong \boldsymbol{a}/q(t(Q))$. It follows that this latter one is a projective $R/t(R)$-module, which implies that it is in $\mathcal{F}$ when viewed as an R-module. But then $t(\boldsymbol{a})/q(t(Q)) \in \mathcal{T} \cap \mathcal{F} = 0$. Therefore we have $q(t(Q)) = t(\boldsymbol{a})$ and $\boldsymbol{a}/t(\boldsymbol{a})$ is projective as a left $R/t(R)$-module. $\square$

We are now able to give a second significative class of rings for which we can identify all hereditary torsion pairs whose heart is a module category.

Proposition 5.9. *Let R be a left semihereditary ring and let V be a finitely presented quasi-tilting R-module whose associated torsion pair $\boldsymbol{t} = (\mathrm{Gen}(V), \mathrm{Ker}(\mathrm{Hom}_R(V, ?)))$ is hereditary. The following assertions are equivalent*:

1. *If $\boldsymbol{a} = \mathrm{ann}_R(V/t(R)V)$ then $\boldsymbol{a}/t(R)$ is an idempotent ideal of $R/t(R)$, which is finitely generated on the left, and there is a monomorphism $R/\boldsymbol{a} \hookrightarrow (V/t(R)V)^{(n)}$, for some natural number n.*
2. *The heart $\mathcal{H}_{\boldsymbol{t}}$ is a module category.*

In this case $\mathcal{H}_{\boldsymbol{t}}$ is equivalent to S-Mod, where $S = \begin{pmatrix} \mathrm{End}_R(\boldsymbol{a}/t(R))^{op} & 0 \\ \mathrm{Ext}^1_R(V, \boldsymbol{a}/t(R)) & \mathrm{End}_R(V)^{op} \end{pmatrix}$.

Proof. 1) $\Longrightarrow$ 2) Put $\bar{M} = M/t(R)M$, for each R-module M. Note that $\mathcal{T} \cap \bar{R}\text{-Mod} = \mathrm{Gen}(\bar{V})$ and that $\mathrm{ann}_{\bar{R}}(\bar{V}) = \bar{\boldsymbol{a}}$. We then get that $\mathrm{Hom}_R(\bar{\boldsymbol{a}}, T) = 0$, for each $T \in \mathcal{T}$. Indeed if $f : \bar{\boldsymbol{a}} \longrightarrow T$ is any R-homomorphism, then $\mathrm{Im}(f) \in \mathcal{T} \cap \bar{R}\text{-Mod}$ and the induced morphism $\bar{f} : \bar{\boldsymbol{a}} \longrightarrow \mathrm{Im}(f)$ is a morphism in $\bar{R}$-Mod such that $\bar{f}(\bar{\boldsymbol{a}}) = \bar{f}(\bar{\boldsymbol{a}}^2) = \bar{\boldsymbol{a}}\mathrm{Im}(f) = 0$. We then have that $\bar{\boldsymbol{a}}$ is in ${}^{\perp}\mathcal{T}$.

On the other hand, since $\bar{\boldsymbol{a}}$ is finitely generated on the left, we have a finitely generated left ideal $\boldsymbol{a}'$ of R contained in $\boldsymbol{a}$ such that the canonical composition $\boldsymbol{a}' \hookrightarrow \boldsymbol{a} \twoheadrightarrow \bar{\boldsymbol{a}}$ is an epimorphism. We then get that $\boldsymbol{a}'/t(\boldsymbol{a}') = \boldsymbol{a}'/(\boldsymbol{a}' \cap t(R)) \cong \bar{\boldsymbol{a}}$ and, since $\boldsymbol{a}'$ is projective, we conclude that $\bar{\boldsymbol{a}}$ is a finitely generated projective left $\bar{R}$-module.

Note now that V and $Y := \bar{\boldsymbol{a}}$ satisfy all conditions 2.a–c of Corollary 5.5. Moreover if $F \in \mathcal{F}$ then $F/\boldsymbol{a}F$ is generated by $R/\boldsymbol{a}$ and, due to our hypotheses, we know that $F/\boldsymbol{a}F$ is in $\mathcal{T}$, so that also property 2.d of that corollary holds. Then $\mathcal{H}_{\boldsymbol{t}}$ is a module category, actually equivalent to S-Mod (see Proposition 5.1).

2) $\Longrightarrow$ 1) Let G be a complex as in Theorem 4.1, which is then a progenerator of $\mathcal{H}_{\boldsymbol{t}}$. The fact that $\mathrm{Im}(d)$ is projective easily implies that G is isomorphic to $H^0(G)[0] \oplus H^{-1}(G)[1]$ in $\mathcal{H}_{\boldsymbol{t}}$. Putting $V = H^0(G)$ and $Y = H^{-1}(G)$ for simplicity, Corollary 5.5 and its proof show that $\boldsymbol{t}' = (\mathcal{T} \cap \bar{R}\text{-Mod}, \mathcal{F})$ is the right constituent torsion pair of a TTF triple in $\bar{R}$-Mod defined by the idempotent ideal $\bar{\boldsymbol{a}} = \mathrm{ann}_{\bar{R}}(\bar{V})$, which is finitely generated on the left. Then we have $\bar{\boldsymbol{a}} = \boldsymbol{a}/t(R)$, where $\boldsymbol{a} = \mathrm{ann}_R(V/t(R)V)$. Moreover, we have $R/\boldsymbol{a}\text{-Mod} = \mathcal{T} \cap \bar{R}\text{-Mod} = \mathrm{Gen}(\bar{V})$, so that $R/\boldsymbol{a} \in \mathrm{add}(\bar{V})$. $\square$

6. When the torsion class is closed under taking products.

Our next result shows that if the torsion class is closed under taking products in R-Mod, then classical tilting theory appears quite naturally.

THEOREM 6.1. *Let $\boldsymbol{t} = (\mathcal{T}, \mathcal{F})$ be a torsion pair in R-Mod. The following assertions are equivalent:*

1. *$\mathcal{T}$ is closed under taking products in R-Mod and the heart $\mathcal{H}_{\boldsymbol{t}}$ is a module category;*
2. *$\mathcal{T} = \mathrm{Gen}(V)$, where V is a module which is classical 1-tilting over $R/\mathrm{ann}_R(V)$ and admits a finitely generated projective presentation $Q \xrightarrow{d} P \longrightarrow V \longrightarrow 0$ in R-Mod and a submodule $X \subseteq \mathrm{Ker}(d)$ such that:*
   (a) *$\mathrm{Ker(d)}/X \in \mathcal{F}$ and $\mathrm{Ker}(d) \subseteq X + \boldsymbol{a}Q$, where $\boldsymbol{a} := \mathrm{ann}_R(V)$;*
   (b) *$\mathrm{Ext}^1_R(Q/X, ?)$ vanishes on $\mathcal{F}$;*
   (c) *There is a R-homomorphism $h : (Q/X)^{(I)} \longrightarrow R/t(R)$, for some set I, such that $h((\mathrm{Ker}(d)/X)^{(I)}) = (\boldsymbol{a} + t(R))/t(R)$.*

In this case $\boldsymbol{t}' = (\mathrm{Gen}(V), \mathcal{F} \cap R/\boldsymbol{a}\text{-Mod})$ is a classical tilting torsion pair in $R/\boldsymbol{a}$-Mod and the forgetful functor $\mathcal{H}_{\boldsymbol{t}'} \longrightarrow \mathcal{H}_{\boldsymbol{t}}$ is faithful.

PROOF. 2) $\Longrightarrow$ 1) The classical 1-tilting condition of V implies that $\mathcal{T}$ consists of the $R/\boldsymbol{a}$-modules T such that $\mathrm{Ext}^1_{R/\boldsymbol{a}}(V, T) = 0$. This class is clearly closed under taking products. We next consider the complex

$$G := \qquad \cdots \longrightarrow 0 \longrightarrow X \hookrightarrow Q \xrightarrow{d} P \longrightarrow 0 \longrightarrow \cdots$$

concentrated in degrees $-2, -1, 0$. By condition 2.a and by the equality $\mathcal{T} = \mathrm{Gen}(V)$, we have that $G \in \mathcal{H}_{\boldsymbol{t}}$. We shall check that G satisfies the standard conditions 1–5. We immediately derive the standard conditions 2, 3 and 4. On the other hand, our condition 2.c implies that $\mathrm{Coker}(h_{|(\mathrm{Ker}(d)/X)^{(I)}})$ is isomorphic to $R/(\boldsymbol{a} + t(R))$ and, hence, it is in $R/\boldsymbol{a}$-Mod. But we have that $R/\boldsymbol{a}\text{-Mod} = \overline{\mathrm{Gen}}(V)$ since $R/\boldsymbol{a} \in \overline{\mathrm{Gen}}(V)$ due to the 1-tilting condition of V over $R/\boldsymbol{a}$. Then the standard condition 5 holds. It remains to prove that $\mathcal{T} \subseteq \mathrm{Ker}(\mathrm{Ext}^1_R(V, ?))$. Let

$$0 \longrightarrow T \longrightarrow M \longrightarrow V \longrightarrow 0$$

be any exact sequence in R-Mod, with $T \in \mathcal{T}$. Since $\mathcal{T} = \mathrm{Gen}(V)$ is closed under taking extensions in R-Mod, we get that $M \in \mathcal{T}$ and, hence, the exact sequence lives in $R/\boldsymbol{a}$-Mod. But then it splits since we have that $\mathcal{T} = \mathrm{Ker}(\mathrm{Ext}^1_{R/\boldsymbol{a}}(V, ?))$.

1) $\Longrightarrow$ 2) Let G be a complex in standard form satisfying the standard conditions 1–5, which is then a progenerator of $\mathcal{H}_{\boldsymbol{t}}$ (see Theorem 3.4), and let us put $V := H^0(G)$ and $\boldsymbol{a} = \mathrm{ann}_R(V)$. There is an obvious monomorphism $R/\boldsymbol{a} \longrightarrow V^V$ and, by hypothesis, we have that $V^V \in \mathcal{T} = \mathrm{Gen}(V)$. We then get that $\overline{\mathrm{Gen}}(V) = R/\boldsymbol{a}$-Mod. Moreover, from Lemma 3.2 and [**CMT**, Proposition 3.2] we get that V is a classical 1-tilting $R/\boldsymbol{a}$-module. On the other hand, the equality $\overline{\mathrm{Gen}}(V) = R/\boldsymbol{a}$-Mod implies that $\mathrm{Rej}_{\mathcal{T}}(M) = \mathrm{Rej}_{R/\boldsymbol{a}\text{-Mod}}(M) = \boldsymbol{a}M$. Then conditions 2.a and 2.b follow directly.

It just remains to check condition 2.c. To do that, consider the morphism $h : (Q/X)^{(I)} \longrightarrow R/t(R)$ in the standard condition 5 and put $h' := h_{|(\mathrm{Ker}(d)/X)^{(I)}}$. The fact that $\mathrm{Coker}(h')$ is in $\overline{\mathrm{Gen}}(V) = R/\boldsymbol{a}$-Mod is equivalent to saying that $\boldsymbol{a}(R/t(R)) \subseteq \mathrm{Im}(h')$. But, by the already proved condition 2.a, we know that $\mathrm{Im}(h') \subseteq h(\boldsymbol{a}(Q/X)^{(I)}) = \boldsymbol{a}\mathrm{Im}(h) \subseteq \boldsymbol{a}(R/t(R))$. We then get that $\mathrm{Im}(h') = \boldsymbol{a}(R/t(R)) = (\boldsymbol{a} + t(R))/t(R)$.

Finally, it is clear that $\boldsymbol{t'} = (\mathcal{T}, \mathcal{F} \cap R/\boldsymbol{a}\text{-Mod})$ is a classical tilting torsion pair of $R/\boldsymbol{a}$-Mod. If $j : \mathcal{H}_{\boldsymbol{t'}} \longrightarrow \mathcal{H}_{\boldsymbol{t}}$ is the forgetful functor then, arguing as in the final part of the proof of Corollary 5.5, in order to prove that j is faithful, we just need to check that the canonical map $\mathrm{Hom}_{\mathcal{H}_{\boldsymbol{t'}}}(V[0], M) \longrightarrow \mathrm{Hom}_{\mathcal{H}_{\boldsymbol{t}}}(V[0], M)$ is injective, for all $M \in \mathcal{H}_{\boldsymbol{t'}}$. Similar as there, this in turn reduces to check that the canonical map

$$\mathrm{Ext}^1_{R/\boldsymbol{a}}(V, F) \cong \mathrm{Hom}_{\mathcal{H}_{\boldsymbol{t'}}}(V[0], F[1]) \longrightarrow \mathrm{Hom}_{\mathcal{H}_{\boldsymbol{t}}}(V[0], F[1]) \cong \mathrm{Ext}^1_R(V, F)$$

is injective, for all $F \in \mathcal{F} \cap R/\boldsymbol{a}$-Mod. But this is clear. $\square$

Let A be any ring. Note that if V is a non-projective classical 1-tilting A-module, then (see [**Mi**]) V is also a classical tilting right S-module, where $S = \mathrm{End}({}_AV)^{op}$, such that the canonical algebra morphism $A \longrightarrow \mathrm{End}(V_S)$ is an isomorphism. Due to the tilting theorem, we then know that $(\mathrm{Ker}(? \otimes_A V), \mathrm{Ker}(\mathrm{Tor}_1^A(?, V)))$ is a torsion pair in Mod-A. If we had $\mathrm{Ker}(? \otimes_A V) = 0$ we would have that $\mathrm{Tor}_1^A(?, V) = 0$, and hence V would be a flat left A-module, which is a contradiction (see [**L**, Corollaire 1.3]). Then there is a right A-module $X \neq 0$ such that $X \otimes_A V = 0 \neq \mathrm{Tor}_1^A(X, V)$. Considering an epimorphism $X \twoheadrightarrow X'$, with X'_A simple, and replacing X by X', we can even choose X to be a simple right A-module.

Recall that if A is a ring and M is an A-bimodule, then the *trivial extension of A by M*, denoted $A \rtimes M$, is the ring whose underlying A-bimodule is $A \oplus M$ and the multiplication is given by $(a, m) \cdot (a', m') = (aa', am' + ma')$.

We can now give a systematic way of constructing negative answers to Question 5.3.

Theorem 6.2. *Let A be a finite dimensional algebra over an algebraically closed field K, let V be a classical 1-tilting left A-module such that $\mathrm{Hom}_A(V, A) = 0$, let X be a simple right A-module such that $X \otimes_A V = 0$ and let us consider the trivial extension $R = A \rtimes M$, where $M = V \otimes_K X$. Viewing V as a left R-module annihilated by $0 \rtimes M$, the pair $\boldsymbol{t} = (\mathrm{Gen}(V), \mathrm{Ker}(\mathrm{Hom}_R(V, ?)))$ is a non-tilting torsion pair in R-Mod such that $V[0]$ is a progenerator of $\mathcal{H}_{\boldsymbol{t}}$.*

Proof. All throughout the proof, for any two-sided ideal $\boldsymbol{a}$ of a ring R, we view $R/\boldsymbol{a}$-modules as R-modules annihilated by $\boldsymbol{a}$. Note that if M is any such module and we apply $? \otimes_R M : \text{Mod-}R \longrightarrow \mathrm{Ab}$ to the canonical exact sequence $0 \to \boldsymbol{a} \hookrightarrow R \longrightarrow R/\boldsymbol{a} \to 0$, then we get an isomorphism $R \otimes_R M \xrightarrow{\cong} R/\boldsymbol{a} \otimes_R M$, which implies that the canonical morphism $\mathrm{Tor}_1^R(R/\boldsymbol{a}, M) \longrightarrow \boldsymbol{a} \otimes_R M$ is an isomorphism.

Bearing in mind that X is simple in Mod-A, if $0 \to Q' \xrightarrow{u} P' \longrightarrow V \to 0$ is the minimal projective resolution of ${}_AV$, then the map $1_X \otimes u : X \otimes_A Q' \longrightarrow X \otimes_A P'$ is the zero map. This implies that we have isomorphisms of vector spaces $X \otimes_A P' \cong X \otimes_A V = 0$ and $\mathrm{Tor}_1^A(X, V) \cong X \otimes_A Q'$.

Note that we have an isomorphism ${}_AM \cong {}_AV$ in A-Mod because, due to the algebraically closed condition of K, the simple right A-module X is one-dimensional over K. As a right A-module, M_A is in $\mathrm{add}(X_A)$. Moreover, we have $M \otimes_A V \cong V \otimes_K X \otimes_A V = 0$. We shall prove the following facts:

i) $\mathrm{ann}_R(V) = 0 \rtimes M =: \boldsymbol{a}$ and this ideal is the trace of V in R;
ii) $\boldsymbol{a} \otimes_R V = 0$;
iii) $\mathcal{T} := \mathrm{Gen}(V)$ is closed under taking extensions, and hence a torsion class, in R-Mod;
iv) $\mathrm{Ker}(\mathrm{Hom}_A(V, ?)) = \mathrm{Ker}(\mathrm{Hom}_R(V, ?))$ and, hence, $\boldsymbol{t} := (\mathrm{Gen}(V), \mathrm{Ker}(\mathrm{Hom}_R(V, ?)))$ is a torsion pair in R-Mod which lives in A-Mod;
v) There is a finitely generated projective presentation of ${}_RV$

$$Q \xrightarrow{d} P \twoheadrightarrow V \to 0$$

such that $\mathrm{Ker}(d)$ is a non-projective R-module in $\mathcal{T}$.

Suppose that all these facts have been proved. Then $\boldsymbol{t}$ is a torsion pair in R-Mod whose torsion class is closed under taking products. We claim that V satisfies all conditions of assertion 2 in Theorem 6.1, by taking $X = \mathrm{Ker}(d)$. The only nontrivial things to check are conditions 2.b and 2.c in that assertion. Condition 2.b follows by applying the exact sequence of $\mathrm{Ext}_R(?, F)$, with $F \in \mathcal{F}$, to the short exact sequence $0 \to \mathrm{Ker}(d) \hookrightarrow Q \xrightarrow{d} \mathrm{Imd}(d) \to 0$. On the other hand, $t(R)$ is the trace of V in R and, by fact i), we know that $t(R) = \boldsymbol{a}$. Then condition 2.c of assertion 2 in Theorem 6.1 also holds, simply by taking as h the zero map. Looking at the proof of implication 2) $\Longrightarrow$ 1) in that theorem, we see that the complex

$$G := \cdots \longrightarrow 0 \longrightarrow \mathrm{Ker}(d) \hookrightarrow Q \xrightarrow{d} P \longrightarrow 0 \longrightarrow \cdots,$$

concentrated in degrees $-2, -1, 0$, is a progenerator of $\mathcal{H}_{\boldsymbol{t}}$. But we have an isomorphism $G \cong V[0]$ in $\mathcal{H}_{\boldsymbol{t}}$. Therefore $V[0]$ is a progenerator of $\mathcal{H}_{\boldsymbol{t}}$. Note that this torsion pair $\boldsymbol{t}$ in R-Mod is not tilting, because the projective dimension of ${}_RV$ is > 1 (see [**C**, Section 2]).

We now pass to prove the facts i)–v) in the list above:

i) By definition of the R-module structure on V, we have $(a, m)v = av$, for each $(a, m) \in A \rtimes M$ and each $v \in V$. Then $\mathrm{ann}_R(V) = \mathrm{ann}_A(V) \rtimes M$. But $\mathrm{ann}_A(V) = 0$ due to the 1-tilting condition of ${}_AV$. On the other hand, if $f : V \longrightarrow R$ is a morphism in R-Mod, then we have two K-linear maps $g : V \longrightarrow A$ and $h : V \longrightarrow M$ such that $f(v) = (g(v), h(v)) \in A \rtimes M = R$, for all $v \in V$. Direct computation shows that g is a morphism in A-Mod, and hence $g = 0$. Then one immediately sees that $h \in \mathrm{Hom}_A(V, M)$ and, since ${}_AM$ is in $\mathrm{Gen}({}_AV)$, we conclude that $t(R) = 0 \rtimes M = \boldsymbol{a}$.

ii) Since $\boldsymbol{a}^2 = 0$ and $\boldsymbol{a}V = 0$, we have an equality $\boldsymbol{a} \otimes_R V = \boldsymbol{a} \otimes_{R/\boldsymbol{a}} V = \boldsymbol{a} \otimes_A V$. But, as a right A-module, we have that $\boldsymbol{a}_A \cong M_A$. It follows that $\boldsymbol{a} \otimes_A V \cong M \otimes_A V = 0$.

iii) Consider any exact sequence $0 \to T \longrightarrow N \longrightarrow T' \to 0$ $(*)$ in R-Mod, with $T, T' \in \mathrm{Gen}(V) = \mathcal{T}$. Taking an epimorphism $p : V^{(I)} \twoheadrightarrow T'$ and taking the image of the last exact sequence by the morphism $\mathrm{Ext}_R^1(p, T) : \mathrm{Ext}_R^1(T', T) \longrightarrow \mathrm{Ext}_R^1(V^{(I)}, T)$, we easily reduce the problem to the case when $T' = V^{(I)}$. We now apply the functor $R/\boldsymbol{a}\otimes_R ? : R\text{-Mod} \longrightarrow R/\boldsymbol{a}\text{-Mod}$ to the sequence $(*)$, with $T' = V^{(I)}$, and use the fact that, by the initial paragraph of this proof, we have $\mathrm{Tor}_1^R(R/\boldsymbol{a}, V^{(I)}) \cong \boldsymbol{a} \otimes_R V^{(I)} = 0$. We then get a commutative diagram with exact rows, where the vertical arrows are the canonical maps:

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r] \arrow[d, "\wr"] & N \arrow[r] \arrow[d] & V^{(I)} \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & \frac{R}{\boldsymbol{a}} \otimes_R T \arrow[r] & \frac{R}{\boldsymbol{a}} \otimes_R N \arrow[r] & \frac{R}{\boldsymbol{a}} \otimes_R V^{(I)} \arrow[r] & 0
\end{tikzcd}$$

The central vertical arrow $N \longrightarrow R/\boldsymbol{a} \otimes_R N \cong N/\boldsymbol{a}N$ is then an isomorphism. This implies that $\boldsymbol{a}N = 0$ and, hence, the sequence $(*)$ above lives in A-Mod and $N \in \mathcal{T}$.

iv) If $F \in \mathrm{Ker}(\mathrm{Hom}_R(V, ?))$ then $t(R)F = \mathrm{tr}_V(R)F = 0$. By fact i), we get that $\boldsymbol{a}F = 0$. Then F is an A-module, and hence $\mathrm{Ker}(\mathrm{Hom}_R(V, ?)) \subseteq \mathrm{Ker}(\mathrm{Hom}_A(V, ?))$. The converse inclusion is obvious.

v) The multiplication map $\mu : R \otimes_A V \longrightarrow V$ is surjective. Moreover, we have an isomorphism of K-vector spaces

$$R \otimes_A V \cong (A \oplus M) \otimes_A V \cong V \oplus (M \otimes_A V) = V \oplus 0 \cong V.$$

Since V is a finite dimensional K-vector space we get that μ is an isomorphism of left R-modules.

Let $0 \to Q' \xrightarrow{d'} P' \longrightarrow V \to 0$ be a finitely generated projective presentation of V in A-Mod. Using the previous paragraph, we then get a finitely generated projective presentation of V in R-Mod:

$$R \otimes_A Q' \xrightarrow{1\otimes d'} R \otimes_A P' \longrightarrow V \to 0.$$

Then we have isomorphisms of K-vector spaces

$$\mathrm{Ker}(1 \otimes d') = \mathrm{Tor}_1^A(R, V) \cong \mathrm{Tor}_1^A(A \oplus M, V) \cong \mathrm{Tor}_1^A(M, V).$$

It is easy to deduce from this that $\boldsymbol{a}\mathrm{Ker}(1 \otimes d') = 0$. Then we can view $\mathrm{Ker}(1 \otimes d')$ as a left A-module isomorphic to $\mathrm{Tor}_1^A(M, V)$. Since $M = V \otimes_K X$ we get that $\mathrm{Tor}_1^A(M, V) \cong V \otimes_K \mathrm{Tor}_1^A(X, V)$ which is nonzero and isomorphic to $V \otimes_K (X \otimes_A Q') \in \mathrm{add}({}_AV)$ due to the first paragraph of this proof. It follows that $\mathrm{Ker}(1 \otimes d')$ is a nonzero left R-module in $\mathrm{add}({}_RV) = \mathrm{add}({}_AV)$.

We finally prove that $W := \mathrm{Ker}(1 \otimes d')$ is not a projective in R-Mod. If it were so, we would have that $W = \mathrm{tr}_V(R)W$. By fact i), we would get that $\boldsymbol{a}W = W$, which would imply that $W = 0$ since $\boldsymbol{a}^2 = 0$. $\square$

7. Torsion pairs which are right constituents of TTF triples.

As shown in Theorem 4.1 and Corollaries 4.2 and 5.5, hereditary torsion pairs which are the right constituent of a TTF triple appear quite naturally when studying the modular condition of the heart. In this section we fix an idempotent ideal $\boldsymbol{a}$ of R and its associated TTF triple $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ and want to study when the pair $\boldsymbol{t} = (\mathcal{T}, \mathcal{F})$ has the property that its heart $\mathcal{H}_{\boldsymbol{t}}$ is a module category. When this is the case, by Theorem 4.1, we know that $\boldsymbol{a}$ is finitely generated on the left.

Our next result, very important in the sequel, shows that the conditions for $\mathcal{H}_{\boldsymbol{t}}$ to be a module category get rather simplified if we assume that the monoid morphism $V(R) \longrightarrow V(R/\boldsymbol{a})$ (see Section 2) is an epimorphism.

THEOREM 7.1. *Let $\boldsymbol{a}$ be an idempotent ideal of the ring R which is finitely generated on the left, and let $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ be the associated TTF triple. Consider the following assertions for $\boldsymbol{t} = (\mathcal{T}, \mathcal{F})$:*

1. *There is a finitely generated projective R-module P satisfying the following conditions:*
   (a) *$P/\boldsymbol{a}P$ is a (pro)generator of $R/\boldsymbol{a}$-*Mod;
   (b) *There is an exact sequence $0 \longrightarrow F \longrightarrow C \xrightarrow{q} \boldsymbol{a}P \longrightarrow 0$ in R-*Mod*, where $F \in \mathcal{F}$ and C is a finitely generated module which is in $\mathcal{C} \cap \mathrm{Ker}(\mathrm{Ext}^1_R(?, \mathcal{F}))$ and generates $\mathcal{C} \cap \mathcal{F}$.*
2. *The heart $\mathcal{H}_{\boldsymbol{t}}$ is a module category.*

Then 1) implies 2) and, in such a case, if $j : \boldsymbol{a}P \hookrightarrow P$ is the inclusion, then the complex concentrated in degrees $-1, 0$

$$G' := \cdots \longrightarrow 0 \longrightarrow C \oplus \frac{C}{t(C)} \xrightarrow{(\,jq\ 0\,)} P \longrightarrow 0 \longrightarrow \cdots$$

is a progenerator of $\mathcal{H}_{\boldsymbol{t}}$.

When the monoid morphism $V(R) \longrightarrow V(R/\boldsymbol{a})$ is surjective, the implication 2) $\Longrightarrow$ 1) is also true.

PROOF. 1) $\Longrightarrow$ 2) Fix an exact sequence $0 \longrightarrow F \longrightarrow C \xrightarrow{q} \boldsymbol{a}P \longrightarrow 0$ as indicated in condition 1.b. Taking $F' \in \mathcal{F}$ arbitrary, applying the long exact sequence of $\mathrm{Ext}^i_R(?, F')$ $(i \geq 0)$ to the sequence $0 \longrightarrow t(C) \hookrightarrow C \xrightarrow{pr.} C/t(C) \longrightarrow 0$ and using condition 1.b, we get that $\mathrm{Ext}^1_R(C/t(C), ?)_{|\mathcal{F}} = 0$. But then any epimorphism $(R/t(R))^n \twoheadrightarrow C/t(C)$ splits, which implies that $U := C/t(C)$ is a finitely generated projective $R/t(R)$-module which is in $\mathcal{C}$. Moreover it generates $\mathcal{C} \cap \mathcal{F}$ since so does C.

Let $\pi_U : Q_U \twoheadrightarrow U$ and $\pi_C : Q_C \twoheadrightarrow C$ be two epimorphisms from finitely generated projective modules, whose respective kernels are denoted by K_U and K_C. We will prove that the following complex in standard form, which is clearly quasi-isomorphic to G', is a progenerator of $\mathcal{H}_{\boldsymbol{t}}$.

$$G : \quad \cdots \longrightarrow 0 \longrightarrow K_U \oplus K_C \hookrightarrow Q_U \oplus Q_C \xrightarrow{(\,0\ jq\pi_C\,)} P \longrightarrow 0 \longrightarrow \cdots .$$

We have that $H^0(G) = P/\boldsymbol{a}P \in \mathcal{T}$ and, by an appropriate use of ker-coker lemma, we get an exact sequence $0 \longrightarrow K_C \hookrightarrow \mathrm{Ker}(q\pi_C) \longrightarrow F \longrightarrow 0$. It then follows that

$$H^{-1}(G) = \frac{\mathrm{Ker}((0 \quad jq\pi_C))}{K_U \oplus K_C} = \frac{Q_U \oplus \mathrm{Ker}(q\pi_C)}{K_U \oplus K_C} \cong U \oplus F,$$

which is in $\mathcal{F}$. This proves that G is an object of $\mathcal{H}_{\boldsymbol{t}}$. We next check all conditions 3.a–d of Theorem 4.1. Clearly, condition 3.a in that theorem is a consequence of our condition 1.a. Note next that $(K_U \oplus K_C) + \boldsymbol{a}(Q_U \oplus Q_C) = Q_U \oplus Q_C$ because $U \cong Q_U/K_U$ and $C \cong Q_C/K_C$ are both in $\mathcal{C} = \{X \in R\text{-Mod} : \boldsymbol{a}X = X\}$. Then condition 3.b of the mentioned theorem is automatic, as so is condition 3.c since $(Q_U \oplus Q_C)/(K_U \oplus K_C) \cong U \oplus C$.

Put now $X = K_U \oplus K_C$ and $Q = Q_U \oplus Q_C$ as in the standard notation. Using the fact that U generates $\mathcal{C} \cap \mathcal{F}$, fix an epimorphism $p : U^{(J)} \longrightarrow \boldsymbol{a}/t(\boldsymbol{a})$. Identifying $Q/X = U \oplus C$, we clearly have that $(p\ 0) : (Q/X)^{(J)} = U^{(J)} \oplus C^{(J)} \longrightarrow \boldsymbol{a}/t(\boldsymbol{a})$ is a homomorphism whose restriction to $H^{-1}(G)^{(J)} = U^{(J)} \oplus F^{(J)}$ is an epimorphism. Then also condition 3.d of Theorem 4.1 holds.

2) $\Longrightarrow$ 1) (Assuming that the monoid morphism $V(R) \longrightarrow V(R/\boldsymbol{a})$ is surjective). Let G be a progenerator of $\mathcal{H}_{\boldsymbol{t}}$. By Theorem 4.1, we know that $\mathrm{add}(H^0(G)) = \mathrm{add}(R/\boldsymbol{a})$ and our extra hypothesis gives a finitely generated projective R-module P such that $P/\boldsymbol{a}P \cong H^0(G)$, so that condition 1.a holds. Fixing such a P and following the proof of Theorem 3.4, we see that we can represent G by a chain complex

$$\cdots \longrightarrow 0 \longrightarrow X \xrightarrow{j} Q \xrightarrow{d} P \longrightarrow 0 \longrightarrow \cdots$$

where Q is finitely generated projective, j is a monomorphism, $\mathrm{Im}(d) = \boldsymbol{a}P$. Then G satisfies all conditions 3.a–d of Theorem 4.1. Note that $\boldsymbol{a}P = \boldsymbol{a}^2P = \boldsymbol{a}\mathrm{Im}(d) = d(\boldsymbol{a}Q)$, which implies that $Q = \mathrm{Ker}(d) + \boldsymbol{a}Q$ and, by condition 3.b of the mentioned theorem, that $Q = X + \boldsymbol{a}Q$. That is, the module Q/X is in $\mathcal{C}$ and, by condition 3.c of that theorem, we also have that $Q/X \in \mathrm{Ker}(\mathrm{Ext}^1_R(?, \mathcal{F}))$. The exact sequence needed for our condition 1.b is then $0 \longrightarrow H^{-1}(G) \hookrightarrow Q/X \xrightarrow{\bar{d}} \boldsymbol{a}P \longrightarrow 0$. □

We now give some applications of last theorem.

Corollary 7.2. *Let Q be a finitely generated projective R-module and let us consider the hereditary torsion pair $\boldsymbol{t} = (\mathcal{T}, \mathcal{F})$, where $\mathcal{T} = \mathrm{Ker}(\mathrm{Hom}_R(Q, ?))$. If the trace of Q in R is finitely generated on the left, then $\boldsymbol{t}$ is an HKM torsion pair and $\mathcal{H}_{\boldsymbol{t}}$ is a module category.*

Proof. We will check assertion 1 of Theorem 7.1 for the suitable choices. We have that $\mathcal{T}$ fits into a TTF triple $(\mathcal{C}, \mathcal{T}, \mathcal{F})$, where $\mathcal{C} = \mathrm{Gen}(Q) = \{T \in R\text{-Mod} : \boldsymbol{a}T = T\}$, where $\boldsymbol{a} = \mathrm{tr}_Q(R)$ (see [**S**, Proposition VI.9.4 and Corollary VI.9.5]). Taking $P = R$ in Theorem 7.1, we have an obvious epimorphism $p : Q^n \twoheadrightarrow \boldsymbol{a}$, for some integer $n > 0$. We then take $C = Q^n/t(\mathrm{Ker}(p))$ and $q : C = Q^n/t(\mathrm{Ker}(p)) \twoheadrightarrow \boldsymbol{a}$ the epimorphism defined by

p. The fact that $\mathrm{Ext}^1_R(C,?)_{|\mathcal{F}}=0$ follows by taking $F\in\mathcal{F}$ and applying the long exact sequence of $\mathrm{Ext}^i_R(?,F)$ $(i\geq 0)$ to the short exact sequence $0\to t(\mathrm{Ker}(p))\hookrightarrow Q^n\twoheadrightarrow C\to 0$.

On the other hand, the progenerator G of $\mathcal{H}_{\boldsymbol{t}}$ given in Theorem 7.1 is quasi-isomorphic to the complex

$$\cdots\longrightarrow 0\longrightarrow t(\mathrm{Ker}(p))\oplus t(Q)^{(n)}\hookrightarrow Q^{(n)}\oplus Q^{(n)}\xrightarrow{(jp\ 0)} R\longrightarrow 0\longrightarrow\cdots,$$

where $j:\boldsymbol{a}\hookrightarrow R$ is the inclusion. One readily proves now that the complex $P^{\cdot}:\cdots\longrightarrow 0\longrightarrow Q^{(n)}\oplus Q^{(n)}\xrightarrow{(jp\ 0)} R\longrightarrow 0\longrightarrow\cdots$ satisfies condition 2 of Proposition 3.7. □

The easy proof of the following auxiliary result is left to the reader.

LEMMA 7.3. *Let R be a ring and $\boldsymbol{a}$ be an idempotent ideal. The following assertions hold*:

1. *If $p:P\twoheadrightarrow M$ is a projective cover and $\boldsymbol{a}M=M$, then $\boldsymbol{a}P=P$*;
2. *Suppose that R is semiperfect and let $\{e_1,\dots,e_n\}$ be a family of primitive orthogonal idempotents such that $\sum_{1\leq i\leq n}e_i=1$. If $\boldsymbol{a}$ is finitely generated on the left, then there is an idempotent element $e\in R$ (which is a sum of e_i's) such that $\boldsymbol{a}=ReR$.*

For semiperfect rings, we have the following result.

COROLLARY 7.4. *Let R be a semiperfect ring, let $\{e_1,\dots,e_n\}$ be a complete family of primitive orthogonal idempotents, and let $\boldsymbol{t}=(\mathcal{T},\mathcal{F})$ be the right constituent torsion pair of a TTF triple in R-Mod. The following assertions are equivalent*:

1. *The heart $\mathcal{H}_{\boldsymbol{t}}$ is a module category*;
2. *$\boldsymbol{t}$ is an HKM torsion pair*;
3. *There is an idempotent element $e\in R$ (which is a sum of e_i's) such that ReR is finitely generated on the left and ReR is the idempotent ideal which defines the TTF triple.*

PROOF. 1) $\Longrightarrow$ 3) By Theorem 4.1, we know that $\boldsymbol{a}$ is finitely generated on the left. Then assertion 3 follows from Lemma 7.3.

3) $\Longrightarrow$ 2) is a particular case of Corollary 7.2.

2) $\Longrightarrow$ 1) follows from [**HKM**, Theorem 3.8]. □

As a consequence of Theorem 4.1 and Corollary 7.4, we get more significative classes of rings for which we can identify all the hereditary torsion pairs whose heart is a module category.

COROLLARY 7.5. *Let $\boldsymbol{t}=(\mathcal{T},\mathcal{F})$ be a hereditary torsion pair in R-Mod and let $\mathcal{H}_{\boldsymbol{t}}$ be its heart. The following assertions hold*:

1. *If R is a local ring and $\mathcal{H}_{\boldsymbol{t}}$ is a module category, then $\boldsymbol{t}$ is either $(R\text{-Mod},0)$ or $(0,R\text{-Mod})$*;
2. *When R is right perfect, $\mathcal{H}_{\boldsymbol{t}}$ is a module category if, and only if, there is an idempotent*

element $e \in R$ *such that* $\mathcal{T} = \{T \in R\text{-Mod} : eT = 0\}$ *and* ReR *is finitely generated on the left*;
3. *If* R *is left Artinian* (*e.g. an Artin algebra*), *then* $\mathcal{H}_{\boldsymbol{t}}$ *is always module category.*

Proof. 1) By Theorem 3.4, we have a finitely presented R-module V such that $\mathcal{T} = \mathrm{Gen}(V) \subseteq \mathrm{Ker}(\mathrm{Ext}^1_R(V, ?))$. Using Theorem 4.1 and its proof, we get that $\boldsymbol{t}' = (\mathcal{T} \cap R/t(R)\text{-Mod}, \mathcal{F})$ is the right constituent of a TTF triple in $\bar{R}$-Mod defined by an idempotent ideal $\bar{\boldsymbol{b}} = \boldsymbol{b}/t(R)$ of $\bar{R} := R/t(R)$ which is finitely generated on the left, where $\boldsymbol{b} = \mathrm{ann}_R(V/t(R)V)$. Since $\bar{R}$ is also a local ring, and hence semiperfect, Lemma 7.3 says that $\bar{\boldsymbol{b}} = \bar{R}\bar{e}\bar{R}$, for some idempotent element $\bar{e} \in \bar{R}$, which is necessarily equal to $\bar{1}$ or 0. The fact that $\bar{R} \in \mathcal{F}$ implies that $\bar{e} = 1$, so that $\bar{b} = \bar{R}$ and $b = R = \mathrm{ann}_R(V/t(R)V)$, thus $V = t(R)V$ and, by Lemma 5.4, we deduce that V is projective. But all finitely generated projective modules over a local ring are free. Then we have $V = 0$ or $V = R^{(n)}$, for some set $n \in \mathbb{N}$, so that either $\boldsymbol{t} = (0, R\text{-Mod})$ or $\boldsymbol{t} = (R\text{-Mod}, 0)$.

2) Assume now that R is right perfect and that $\mathcal{H}_{\boldsymbol{t}}$ is a module category. By [**S**, Corollary VIII.6.3], we know that $\boldsymbol{t}$ is the right constituent of a TTF triple. By Theorem 4.1, the associated idempotent ideal is finitely generated on the left and, by [**S**, Corollary VIII.6.4], we know that it is of the form AeA. Conversely, if $e \in A$ is idempotent and AeA is finitely generated on the left and $\mathcal{T} = \{T \in R\text{-Mod} : eT = 0\}$, then Corollary 7.4 says that $\mathcal{H}_{\boldsymbol{t}}$ is a module category.

3) This assertion is a direct consequence of [**S**, Example VI.8.2] and Corollary 7.4 since all left ideals are finitely generated. □

Another consequence of Theorem 7.1 is the following.

Corollary 7.6. *Let* $\boldsymbol{a}$ *be an idempotent ideal of* R, *which is finitely generated on the left, let* $\boldsymbol{t} = (\mathcal{T}, \mathcal{F})$ *be the right constituent torsion pair of the associated TTF triple in* R-Mod *and suppose that* $\boldsymbol{a} \in \mathcal{F}$ *and that the monoid morphism* $\mathcal{V}(R) \longrightarrow \mathcal{V}(R/\boldsymbol{a})$ *is surjective. Consider the following assertions*:

1. $\mathcal{H}_{\boldsymbol{t}}$ *is a module category*;
2. *There is an epimorphism* $M \twoheadrightarrow \boldsymbol{a}$, *where* M *is a finitely generated projective* $R/t(R)$*-module which is in* $\mathcal{C}$.
3. $\boldsymbol{a}$ *is the trace of some finitely generated projective left* R*-module*;
4. $\boldsymbol{t}$ *is an HKM torsion pair*;
5. $\mathcal{H}_{\boldsymbol{t}}$ *has a progenerator which is a classical tilting complex.*

Then the implications 5) $\Longrightarrow$ 4) *and* 3) $\Longrightarrow$ 4) $\Longrightarrow$ 1) $\Longleftrightarrow$ 2) *hold true. When the monoid morphism* $\mathcal{V}(R) \longrightarrow \mathcal{V}(R/t(R))$ *is also surjective, all assertions are equivalent.*

Proof. 3) $\Longrightarrow$ 4) $\Longrightarrow$ 1) and 5) $\Longrightarrow$ 4) follow from Corollaries 7.2 and 3.8.

1) $\Longrightarrow$ 2) Consider the finitely generated projective module P and the exact sequence $0 \longrightarrow F \longrightarrow C \longrightarrow \boldsymbol{a}P \longrightarrow 0$ given by Theorem 7.1. It follows that $C \in \mathcal{F}$ since so do F and $\boldsymbol{a}P$. But, then, the fact that $\mathrm{Ext}^1_R(C, ?)_{|\mathcal{F}} = 0$ implies that C is a finitely generated projective $R/t(R)$-module. Since C generates $\mathcal{C} \cap \mathcal{F}$ we get an epimorphism $M := C^n \twoheadrightarrow \boldsymbol{a}$ as desired.

2) $\Longrightarrow$ 1) is a direct consequence of Theorem 7.1.

2) $\Longrightarrow$ 3), 5) (Assuming that the monoid map $V(R) \longrightarrow V(R/t(R))$ is surjective). We have a finitely generated projective R-module Q such that $M \cong Q/t(R)Q = Q/t(Q)$. But we then get $Q = \boldsymbol{a}Q \oplus t(Q)$ since $\boldsymbol{a}M = M$ and $\boldsymbol{a}Q$ is in $\mathcal{F}$. It follows that $M \cong \boldsymbol{a}Q$ is also projective as an R-module. Then we have $\mathrm{Gen}(M) = \mathrm{Gen}(\boldsymbol{a})$, so that $\boldsymbol{a} = \mathrm{tr}_M(R)$.

On the other hand, by taking $C = M$ in Theorem 7.1, we know that the complex

$$G := \cdots \longrightarrow 0 \longrightarrow M \oplus M \xrightarrow{(jq\ 0)} P \longrightarrow 0 \longrightarrow \cdots$$

concentrated in degrees -1 and 0, is a progenerator of $\mathcal{H}_{\boldsymbol{t}}$. $\square$

In [**MT**, Corollary 2.13] (see also [**CMT**, Lemma 4.1]) the authors proved that a faithful (not necessarily hereditary) torsion pair in R-Mod has a heart which is a module category if, and only if, it is an HKM torsion pair. Our next result shows that, for hereditary torsion pairs, we can be more precise.

COROLLARY 7.7. *Let R be a ring and let $\boldsymbol{t} = (\mathcal{T}, \mathcal{F})$ be a faithful hereditary torsion pair in R-*Mod*. Consider the following assertions*:

1. *There is a finitely generated projective R-module Q such that $\mathcal{T} = \mathrm{Ker}(\mathrm{Hom}_R(Q, ?))$ and the trace $\boldsymbol{a}$ of Q in R is finitely generated as a left ideal*;
2. *$\mathcal{H}_{\boldsymbol{t}}$ is a module category*;
3. *$\boldsymbol{t}$ is an HKM torsion pair*;
4. *$\mathcal{H}_{\boldsymbol{t}}$ has a progenerator which is a classical tilting complex*;
5. *There is an idempotent ideal $\boldsymbol{a}$ of R which satisfies the following properties*:
   (a) *$\boldsymbol{t}$ is the right constituent torsion pair of the TTF triple defined by $\boldsymbol{a}$*;
   (b) *there is a progenerator V of $R/\boldsymbol{a}$-*Mod *which admits a finitely generated projective resolution $Q \xrightarrow{d} P \longrightarrow V \longrightarrow 0$ in R-*Mod *satisfying the following two properties*:
      i. $\mathrm{Ker}(d) \subseteq \boldsymbol{a}Q$;
      ii. *there is a morphism $Q^{(J)} \xrightarrow{h} \boldsymbol{a}$, for some set J, such that $h_{|\mathrm{Ker}(d)^{(J)}} : \mathrm{Ker}(d)^{(J)} \longrightarrow \boldsymbol{a}$ is an epimorphism.*

Then the implications 1) $\Longrightarrow$ 2) $\Longleftrightarrow$ 3) $\Longleftrightarrow$ 4) $\Longleftrightarrow$ 5) *hold true. When the monoid morphism $V(R) \longrightarrow V(R/I)$ is surjective, for all idempotent two-sided ideals I of R, all assertions are equivalent.*

PROOF. 1) $\Longrightarrow$ 2) follows from Corollary 7.2.

2) $\Longrightarrow$ 4) follows from [**CMT**, Lemma 4.1].

4) $\Longrightarrow$ 3) is a consequence of Corollary 3.8.

3) $\Longrightarrow$ 2) follows from [**HKM**, Theorems 2.10 and 2.15].

2), 4) $\Longrightarrow$ 5) By Corollary 4.2, we know that $\boldsymbol{t}$ is the right constituent of a TTF triple in R-Mod defined by an idempotent ideal $\boldsymbol{a}$ which is finitely generated on the left. Let now fix a classical tilting complex $G := \cdots \longrightarrow 0 \longrightarrow Q \xrightarrow{d} P \longrightarrow 0 \longrightarrow \cdots$ which is a progenerator of $\mathcal{H}_{\boldsymbol{t}}$. Assertion 5 follows by taking $V = H^0(G)$ and by applying Theorem 4.1 to G.

5) $\Longrightarrow$ 3) follows by taking the complex $P^\bullet = G := \cdots \longrightarrow 0 \longrightarrow Q \xrightarrow{d} P \longrightarrow 0 \longrightarrow \cdots$, concentrated in degrees -1 and 0, and applying Corollary 3.8.

2) $\Longrightarrow$ 1) (Assuming that $V(R) \longrightarrow V(R/I)$ is surjective, for all two-sided ideals I of R). It is a consequence of Corollary 7.6. □

COROLLARY 7.8. *Let $\boldsymbol{a}$ be a two-sided idempotent ideal of R whose associated TTF triple $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ is left split and put $\boldsymbol{t} = (\mathcal{T}, \mathcal{F})$. Then $\mathcal{H}_{\boldsymbol{t}}$ is equivalent to $R/t(R) \times R/\boldsymbol{a}$-*Mod. *When the TTF triple is centrally split, then $\mathcal{H}_{\boldsymbol{t}}$ is equivalent to R-*Mod.

PROOF. In this case $\mathbf{a}$ is a direct summand of ${}_RR$, whence projective, so that Example 5.8 and Proposition 5.1 apply. Note that then $V = R/\boldsymbol{a}$ is a projective left R-module, which implies that $\mathcal{H}_{\boldsymbol{t}}$ is equivalent to S-Mod, where $S \cong \mathrm{End}_R(R/t(R))^{op} \times \mathrm{End}_R(R/\boldsymbol{a})^{op} \cong R/t(R) \times R/\boldsymbol{a}$.

When the TTF triple is centrally split, we have a central idempotent e such that $\boldsymbol{a} = Re$ and $t(R) = R(1-e)$, so that $R/\boldsymbol{a} \cong R(1-e)$ and $R/t(R) \cong Re$. The result in this case follows immediately since we have a ring isomorphism $R \cong Re \times R(1-e)$. □

8. Some examples.

We now give a few examples which illustrate the results obtained in the previous sections. All of them refer to finite dimensional algebras over a field which are given by quivers and relations. We refer the reader to [**ASS**, Chapter II] for the terminology that we use.

EXAMPLE 8.1. Let $Q_n : 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n$ $(n > 1)$ be the Dynkin quiver of type $\boldsymbol{A}$, let $R = KQ_n$ the corresponding path algebra, where K is any field, and let us take $\boldsymbol{a} = Re_nR$. If $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ is the TTF triple associated to $\boldsymbol{a}$ and $\boldsymbol{t} = (\mathcal{T}, \mathcal{F})$ is its right constituent torsion pair, then $\mathcal{H}_{\boldsymbol{t}}$ is equivalent to $K \times KQ_{n-1}$-Mod. In particular $\mathcal{D}(R)$ and $\mathcal{D}(\mathcal{H}_{\boldsymbol{t}})$ are not equivalent triangulated categories.

PROOF. We have that $\boldsymbol{a} = Re_n$ and this shows that $e_nR(1-e_n) = 0$ and that $\boldsymbol{a}$ is projective and injective as left R-module. Since R is hereditary we conclude that $\mathcal{C}$ consists of injective modules. Therefore $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ is a left split TTF triple (see also [**NS**, Theorem 3.1]).

On the other hand, we have that $t(R) = R(1-e_n) \oplus Je_n$, where J is the Jacobson radical of R. This implies that $R/t(R) \cong K$. On the other hand, we clearly have that $R/\boldsymbol{a}$ is isomorphic to the path algebra KQ_{n-1} of type $\boldsymbol{A}_{n-1}$. Then, by Corollary 7.8, we have $\mathcal{H}_{\boldsymbol{t}} \cong K \times KQ_{n-1}$-Mod. Moreover $R = KQ_n$ and $K \times KQ_{n-1}$ cannot be derived equivalent algebras because their centers (= 0-th Hochschild cohomology spaces) are not isomorphic (see [**R2**, Proposition 2.5]). □

EXAMPLE 8.2. Let K be a field and R be the K-algebra given by the following quiver and relations:

$$1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} 2 \overset{\gamma}{\underset{\delta}{\rightrightarrows}} 3 \qquad\qquad \alpha\delta,\ \beta\gamma \text{ and } \alpha\gamma\text{-}\beta\delta$$

Let $\boldsymbol{a}$ be an idempotent ideal of R, let $(\mathcal{C},\mathcal{T},\mathcal{F})$ be the associated TTF triple in R-Mod and let $\boldsymbol{t}=(\mathcal{T},\mathcal{F})$ be its right constituent torsion pair. The following facts are true:

1. If $\boldsymbol{a}=Re_1R$ then $\mathcal{H}_{\boldsymbol{t}}$ has a progenerator which is a sum of stalk complexes and $\mathcal{H}_{\boldsymbol{t}}\cong K\Gamma$-Mod, where Γ is the quiver $2 \rightrightarrows 3 \longrightarrow 1$. The algebra R is then tilted of type Γ.
2. If $\boldsymbol{a}=Re_2R$ then $\mathcal{H}_{\boldsymbol{t}}$ is equivalent to $K\times K\times K$-Mod.
3. If $\boldsymbol{a}=R(e_1+e_2)R$ then $\mathcal{H}_{\boldsymbol{t}}$ does not have a sum of stalk complexes as a progenerator and $\mathcal{H}_{\boldsymbol{t}}\cong S$-Mod, where S is the algebra given by the following quiver and relations

$$\begin{tikzcd} 2 \arrow[r, shift left=3, "\mu_1"] \arrow[r, "\mu_2"] \arrow[r, shift right=3, "\mu_3"'] & 3 \arrow[r, shift left=1.5, "\pi_1"] \arrow[r, shift right=1.5, "\pi_2"'] & 1 \end{tikzcd}$$

$$\mu_1\pi_2=\mu_3\pi_1=0;$$
$$\mu_1\pi_1=-\mu_2\pi_2$$
$$\mu_2\pi_1=\mu_3\pi_2.$$

The algebras R and S are derived equivalent and, hence, S is piecewise hereditary (i.e. derived equivalent to a hereditary algebra).

PROOF. Using a classical visualization of modules via diagrams (see, e.g., [**F**]), the indecomposable projective left R-modules can be depicted as:

$$1 \qquad \begin{tikzcd}[column sep=small] & 2 \arrow[dl, dash] \arrow[dr, dash] & \\ {}_1\alpha & & {}_1\beta \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & 3 \arrow[dl, dash] \arrow[dr, dash] & \\ {}_2\gamma \arrow[dr, dash] & & {}_2\delta \arrow[dl, dash] \\ & {}_1(\alpha\gamma=\beta\delta) & \end{tikzcd}$$

By Corollary 7.7, whenever $\boldsymbol{t}$ is faithful, if $\mathcal{H}_{\boldsymbol{t}}$ is equivalent to S-Mod, then S and R are derived equivalent. Even more, in that case $R[1]$ is a tilting object of $\mathcal{H}_{\boldsymbol{t}}$, which implies that R and S are tilting-equivalent.

1) In this case we have $\boldsymbol{a}=\mathrm{Soc}({}_RR)\cong S_1^{(4)}\cong Re_1^{(4)}$, so that $\boldsymbol{a}$ is projective in R-Mod and $\boldsymbol{t}$ is a faithful torsion pair. Then we have that $\mathrm{Ext}^2_R(R/\boldsymbol{a},?)\equiv 0$ and, by Example 5.8, we conclude that $R/\boldsymbol{a}\oplus\boldsymbol{a}[1]$ is a progenerator of $\mathcal{H}_{\boldsymbol{t}}$. It follows that $G:=R/\boldsymbol{a}\oplus S_1[1]$ is also a progenerator of $\mathcal{H}_{\boldsymbol{t}}$.

We put $A:=R/\boldsymbol{a}$, which is isomorphic to the Kronecker algebra: $2 \rightrightarrows 3$ and which we also view as a left R module annihilated by $\boldsymbol{a}$. Then $\mathcal{H}_{\boldsymbol{t}}$ is equivalent to S-Mod, where $S=\mathrm{End}_{\mathcal{H}_{\boldsymbol{t}}}(G)^{op}\cong\begin{pmatrix}\mathrm{End}_A(S_1)^{op} & 0\\ \mathrm{Ext}^1_R(A,S_1) & A\end{pmatrix}\cong\begin{pmatrix}K & 0\\ \mathrm{Ext}^1_R(A,S_1) & A\end{pmatrix}$.

Note that $A=S_2\oplus Re_3/R\alpha\gamma$ as left R-module, so that we have a vector space decomposition $\mathrm{Ext}^1_R(A,S_1)\cong\mathrm{Ext}^1_R(S_2,S_1)\oplus\mathrm{Ext}^1_R(Re_3/R\alpha\gamma,S_1)$. Let ϵ' be the element of $\mathrm{Ext}^1_R(A,S_1)$ represented by the short exact sequence

$$0 \longrightarrow S_1 \hookrightarrow Re_3 \longrightarrow \frac{Re_3}{R\alpha\gamma} \longrightarrow 0$$

The assignment $a \rightsquigarrow a\epsilon'$ gives an isomorphism of left A-modules $Ae_3 \xrightarrow{\sim} \mathrm{Ext}^1_R(A, S_1)$, so that the algebra S is isomorphic to $\left(\begin{smallmatrix} K & 0 \\ Ae_3 & A \end{smallmatrix}\right) \cong K\Gamma$, where Γ is the quiver $2 \rightrightarrows 3 \longrightarrow 1$.

2) We have that $\boldsymbol{a} = Re_2R = Re_2 \oplus Je_3$ is not projective and that $R/Re_2R \cong S_1 \oplus S_3$ in R-Mod. We then get that $\mathcal{F} = \{F \in R\text{-Mod} : \mathrm{Soc}(F) \in \mathrm{Add}(S_2)\}$, so that $\boldsymbol{t}$ is not faithful. The minimal projective resolution of Je_3 is of the form $0 \longrightarrow P_1^{(3)} \longrightarrow P_2^{(2)} \longrightarrow Je_3 \longrightarrow 0$, where $P_1 = S_1$ is simple projective. It follows that

$$\mathrm{Ext}^2_R\left(\frac{R}{\boldsymbol{a}}, F\right) \cong \mathrm{Ext}^2_R(S_1 \oplus S_3, F) \cong \mathrm{Ext}^2_R(S_3, F) \cong \mathrm{Ext}^1_R(Je_3, F) = 0,$$

for each $F \in \mathcal{F}$. By Example 5.8, we know that $(R/\boldsymbol{a})[0] \oplus (\boldsymbol{a}/t(\boldsymbol{a}))[1] \cong (S_1 \oplus S_3)[0] \oplus S_1^{(3)}[1]$ is a progenerator of $\mathcal{H}_{\boldsymbol{t}}$, which implies that $G := (S_1 \oplus S_3)[0] \oplus S_1[1]$ is also a progenerator of $\mathcal{H}_{\boldsymbol{t}}$. Since we have $\mathrm{Ext}^1_R(S_1, S_1) = 0 = \mathrm{Ext}^1_R(S_3, S_1)$, we get from Proposition 5.1 that $\mathcal{H}_{\boldsymbol{t}}$ is equivalent to S-Mod, where $S = \mathrm{End}_R(S_1)^{op} \times \mathrm{End}_R(S_1 \oplus S_3)^{op} \cong K \times K \times K$.

3) We have isomorphisms $\boldsymbol{a} = R(e_1 + e_2)R \cong Re_1 \oplus Re_2 \oplus Je_3$ and $R/\boldsymbol{a} \cong S_3$ in R-Mod, and this implies that $\mathcal{F} = \{F \in R\text{-Mod} : \mathrm{Soc}(F) \in \mathrm{Add}(S_1 \oplus S_2)\}$ and that $\mathcal{F}$ is faithful. Since $\mathrm{Ext}^2_R(S_3, S_1) \neq 0$, this time we do not have a sum of stalk complexes as a progenerator of $\mathcal{H}_{\boldsymbol{t}}$. Instead, inspired by the proof of Corollary 7.6, we consider the minimal projective resolution of $S_3 \cong R/\boldsymbol{a}$

$$0 \longrightarrow P_1^{(3)} \longrightarrow P_2^{(2)} \xrightarrow{d'} P_3 \longrightarrow S_3 \longrightarrow 0$$

and take the complex $G' := \cdots \longrightarrow 0 \longrightarrow P_2^{(2)} \xrightarrow{d'} P_3 \longrightarrow 0 \longrightarrow \cdots$, concentrated in degrees -1 and 0. Now the complex $G := G' \oplus P_1[1] \oplus P_2[1]$ satisfies all conditions in assertion 2 of Corollary 3.8, so that it is a progenerator of $\mathcal{H}_{\boldsymbol{t}}$ and $\mathcal{H}_{\boldsymbol{t}} \cong S$-Mod, where $S = \left(\begin{smallmatrix} \mathrm{End}_R(P_1 \oplus P_2)^{op} & \mathrm{Hom}_{\mathcal{D}(R)}(P_1[1] \oplus P_2[2], G') \\ \mathrm{Hom}_{\mathcal{D}(R)}(G', P_1[1] \oplus P_2[2]) & \mathrm{End}_{\mathcal{D}(R)}(G')^{op} \end{smallmatrix}\right)$.

We clearly have $\mathrm{End}_R(P_1 \oplus P_2)^{op} \cong \left(\begin{smallmatrix} \mathrm{End}_R(P_2)^{op} & 0 \\ \mathrm{Hom}_R(P_1, P_2) & \mathrm{End}_R(P_1)^{op} \end{smallmatrix}\right) \cong \left(\begin{smallmatrix} K & 0 \\ K^2 & K \end{smallmatrix}\right) =: A$, which is isomorphic to the Kronecker algebra. Moreover, the 0-homology functor defines an isomorphism $\mathrm{End}_{\mathcal{D}(R)}(G') \xrightarrow{\sim} \mathrm{End}_R(H^0(G')) \cong \mathrm{End}_R(S_3) \cong K$. On the other hand, $\mathrm{Hom}_{\mathcal{D}(R)}(G', P_1[1] \oplus P_2[1])$ is a 2-dimensional vector space, where a basis $\{\pi_1, \pi_2\}$ is induced by the two projections $P_2^{(2)} \twoheadrightarrow P_2$. Similarly, $\mathrm{Hom}_{\mathcal{D}(R)}(P_1[1] \oplus P_2[2], G')$ is a 3-dimensonal vector space with a basis $\{\mu_i : i = 1, 2, 3\}$ induced by the monomorphisms $P_1 = Re_1 \hookrightarrow P_2^{(2)}$ which map e_1 onto $(\beta, 0)$, $(\alpha, -\beta)$ and $(0, \alpha)$, respectively. Since the multiplication in S is given by anti-composition of the entries, we easily get that $\pi_i\mu_j = \mu_j \circ \pi_i = 0$, for all i, j. On the other hand, we have:

$$\mu_3\pi_1 = \pi_1 \circ \mu_3 = 0 = \pi_2 \circ \mu_1 = \mu_1\pi_2$$

$$\mu_1\pi_1 = \pi_1 \circ \mu_1 = -\pi_2 \circ \mu_2 = -\mu_2\pi_2 = \begin{pmatrix} 0 & 0 \\ \rho_\beta & 0 \end{pmatrix} : P_1 \oplus P_2 \longrightarrow P_1 \oplus P_2$$

$$\mu_2\pi_1 = \pi_1 \circ \mu_2 = \pi_3 \circ \mu_2 = \mu_2\pi_3 = \begin{pmatrix} 0 & 0 \\ \rho_\alpha & 0 \end{pmatrix} : P_1 \oplus P_2 \longrightarrow P_1 \oplus P_2$$

where $\rho_x : P_1 = Re_1 \longrightarrow P_2$ maps $a \rightsquigarrow ax$, for each $x \in P_2$. It easily follows that S is given by quivers and relations as claimed in the statement. $\square$

Given a finite quiver Q with no oriented cycle, a path p will be called a *maximal path* when its origin is a source and its terminus is a sink. We put $D = \mathrm{Hom}_K(?, KQ) = KQ - \mathrm{mod}^{op} \overset{\cong}{\longleftrightarrow} \mathrm{mod} - KQ$ to denote the usual duality between finitely generated left and right KQ-modules.

Example 8.3. Let Q be a finite connected quiver with no oriented cycles which is different from $1 \to \cdots \to n$, and let $i \in Q_0$ be a source. Let us form a new quiver $\hat{Q}$ as follows. We put $\hat{Q}_0 = Q_0$ and the arrows of $\hat{Q}$ are the arrows of Q plus an arrow $\alpha_p : t(p) \to i$, for each maximal path p in Q. Given a field K, we consider the K-algebra R with quiver $\hat{Q}$ and relations:

1. $\alpha_p\beta = 0$, for each $\beta \in Q_1$ and each maximal path p in Q;
2. $p'\alpha_p = q'\alpha_q$, whenever p' and q' are paths in Q such that $s(p') = s(q')$ and there is a path $\pi : j \to \cdots \to s(p') = s(q')$ in Q such that $\pi p' = p$ and $\pi q' = q$.

We identify KQ-Mod with the full subcategory of R-Mod consisting of the R-modules annihilated by the two-sided ideal generated by the α_p. Then $\boldsymbol{t} = (KQ{-}\mathrm{Inj}, (KQ{-}\mathrm{Inj})^{\perp})$ is a non-tilting torsion pair in R-Mod such that $D(KQ)[0]$ is a progenerator of $\mathcal{H}_{\boldsymbol{t}}$.

Proof. Let $\mathcal{M}$ be the set of maximal path in Q and consider the paths of Q as the canonical basis B of KQ. Its dual basis is denoted by B^*. Consider the assignment $\sum_{p\in\mathcal{M}} a_p\alpha_p \rightsquigarrow \left(\sum_{p\in\mathcal{M}} a_p p^*\right) \otimes \bar{e}_i$, where $a_p \in KQe_{t(p)}$ for each $p \in \mathcal{M}$ and $\bar{e}_i = e_i + e_iJ \in e_iKQ/e_iJ$ is the canonical element. Here $J = J(KQ)$ is the Jacobson radical, a basis of which is given by the paths of length > 0. This assignment defines an isomorphism of KQ-bimodules

$$\boldsymbol{a} := \sum_{p\in\mathcal{M}} R\alpha_p R = \sum_{p\in\mathcal{M}} R\alpha_p \overset{\cong}{\longrightarrow} D(KQ) \otimes_K \frac{e_iKQ}{e_iJ} =: M.$$

Moreover, it is the restriction of an algebra isomorphism $R \overset{\cong}{\longrightarrow} KQ \rtimes M$ which maps $e_i \rightsquigarrow (e_i, 0)$, $\beta \rightsquigarrow (\beta, 0)$ and $\alpha_p \rightsquigarrow (0, p^* \otimes \bar{e}_i)$, for all $i \in Q_0$, all $\beta \in Q_1$ and all $p \in \mathcal{M}$.

Our hypotheses on Q guarantee that there is no projective-injective KQ-module. Then $D(KQ)$ is a classical 1-tilting KQ-module whose associated torsion pair in KQ-Mod, namely $\boldsymbol{t} = (KQ - \mathrm{Inj}, (KQ - \mathrm{Inj})^{\perp})$, is faithful. On the other hand, the fact that i is a source and Q is connected implies that $e_iKQ/e_iJ \otimes_{KQ} D(KQ) = 0$. Then Theorem 6.2 applies, with $A = KQ$, $V = D(KQ)$ and $X = e_iKQ/e_iJ$. $\square$

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