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First and second Hadamard variational formulae of the Green function for general domain perturbations

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Abstract. The Green function of the Laplacian with the homogeneous Dirichlet boundary condition on bounded domains is considered. The variation of the Green function with respect to domain perturbations is called the Hadamard variation. In this paper, we present a unified approach to deriving the Hadamard variation. In our approach, the classical first Hadamard variation is obtained in a natural way under a less restrictive regularity assumption on the boundary smoothness. Furthermore, the second Hadamard variational formula with respective to general domain perturbations is obtained, which is an extension of the classical result of Garabedian–Schiffer in which only normal perturbation is considered.

1. Introduction.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with the boundary $\partial\Omega$ in N-dimensional Euclidean space $\mathbb{R}^N$ $(N \geq 2)$. Let us consider the Poisson problem with the homogeneous Dirichlet boundary condition:

$$-\Delta u = f, \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1}$$

where $\Delta := \sum_{i=1}^N (\partial^2/\partial x_i^2)$ is the Laplacian and $f \in L^2(\Omega)$ is a given function. The standard theory on elliptic PDE tells us that if $\partial\Omega$ is sufficiently smooth, then the equation (1) admits a unique solution $u \in H_0^1(\Omega)$ and u is written as

$$u(x) = \int_\Omega G(x,y) f(y) dy.$$

The function $G(x,y)$ is defined on $\overline{\Omega} \times \Omega$ and is called the *Green function*.

Suppose that the domain Ω is perturbed to Ω_t in a certain way, then the Green function $G(x,y)$ is also perturbed to $G_t(x,y)$, where the parameter t, $0 < t \ll 1$ represents the magnitude of the perturbation. The variation of the Green function $G(x,y)$ defined by

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$$\delta G(x,y) := \frac{\partial}{\partial t} G_t(x,y)\bigg|_{t=0}, \quad (x,y) \in \overline{\Omega} \times \Omega$$

is called the first *Hadamard variation* with respect to the domain perturbation. Let the perturbation be expressed by $T_t x$, $x \in \overline{\Omega}$ with $T_0 x = x$. Define

$$Sx := \lim_{t \to 0} \frac{T_t x - x}{t}, \quad x \in \overline{\Omega}, \qquad \delta\rho := S \cdot n \quad \text{on } \partial\Omega,$$

where n is the unit outer normal vector on $\partial\Omega$. The explicit form of $\delta G(x,y)$ was obtained by Hadamard as

$$\delta G(x,y) = \int_{\partial\Omega} \delta\rho \, \frac{\partial}{\partial n} G(w,x) \frac{\partial}{\partial n} G(w,y) \, ds_w \tag{2}$$

under the assumption that $\partial\Omega$ is analytic. Later, it was shown that the formula holds if $\partial\Omega$ is of C^k class, where k is sufficiently large [**8**, Section 15.1]. The second Hadamard variation $\delta^2 G(x,y)$ is defined by

$$\delta^2 G(x,y) := \frac{\partial^2}{\partial t^2} G_t(x,y)\bigg|_{t=0}.$$

If the domain perturbation $T_t x$ is defined by

$$T_t x := x + \delta\rho \, n, \quad \forall x \in \partial\Omega,$$

Garabedian–Schiffer obtained an explicit form of $\delta^2 G(x,y)$ [**9**] (see Corollary 21).

The purpose of this paper is to present a unified approach to deriving the first and second Hadamard variational formulae with respect to general domain perturbations. The authors' motivation of this paper is from designing iterative numerical schemes for free boundary problems [**22**], [**27**], [**28**]. To prove the existence and uniqueness of the solutions of free boundary problems, the level-set approaches are taken by many authors. To approximate free boundaries in practical applications, however, engineers usually prefer iterative schemes (it is sometimes called *trial boundary methods*) because they are more intuitive and give sharper numerical solutions. To analyze iterative scheme for free boundary problems, we need to deal with domain perturbations and variations of quantities induced by the domain perturbations. Establishing a unified approach to the Hadamard variational formulae is the first step toward developing a theory which would provide a systematic way of analyzing a wide range of trial boundary methods for free boundary problems.

One of the characteristic of our approach is that we assume that Ω is of $C^{k,1}$ class ($k = 0, 1, 2$). In many applications, the boundary $\partial\Omega$ is not smooth and has corners. Therefore, assuming $\partial\Omega$ to be a Lipschitz boundary (of $C^{0,1}$ class) is required from the practical point of view. However, as will be seen later, further regularities of $\partial\Omega$ are needed to derive the Hadamard variational formulae with respect to general domain perturbations.

In our analysis, the main difficulty is arisen, of course, from the fact that the domain Ω is perturbed. When a domain is perturbed, functions defined by the boundary value problem are also perturbed. Although we are interested in deriving variational formulae with respect to domain perturbations, it is not so clear how those variations are defined and how their differentiabilities are proven. As long as the authors know, there are no literature in which the differentiabilities of functions with respect to domain perturbations are explained explicitly. The nontriviality of the differentiabilities is explained at the beginning of Section 2.4.

We use the two ways of specifying the domain perturbations; the Lagrangian and the Eulerian specifications. In the Lagrangian specification, the Lagrange coordinate system moves with the domain perturbation $T_t x$. Hence, there is no loss of regularity in space variables, and the differentiability of the functions is proven rather easily (Theorems 7, 15). Then, the first and second Hadamard variations are defined in the Euler coordinate system and their existences are shown in Theorem 16 using Theorem 15. The combined usage of the Lagrange and Euler coordinate systems is the other characteristic of this paper.

The main results of the paper are Theorems 17 and 20. Theorem 17 gives the first Hadamard variational formula, that is, (2), which is shown under the assumption that $\partial\Omega$ is of $C^{1,1}$ class. When $\partial\Omega$ is of $C^{2,1}$-class, next, the second Hadamard variational formula holds as in Theorem 20. This formula is represented by the volume integral of $\nabla\delta G$ and the surface integral of $\partial G/\partial n$ with the coefficient associated with the mean curvature of $\partial\Omega$. Thus, we obtain Theorem 20 as a generalization of the classical Garabedian–Schiffer formula (Corollary 21).

Recently, H. Kozono and E. Ushikoshi [**15**], E. Ushikoshi [**29**] derived the first and second Hadamard variational formulae for the Green function of the Stokes equation with respect to general domain perturbations in the smooth category. Our methods of calculation, however, are completely different from theirs. In this paper, the main tool is Liouville's Theorem (Theorems 12 and 13). Once the existences of the first and second Hadamard variations are obtained, these formulae are derived rather easily by Liouville's Theorem.[1]

We summarize the notations used in this paper. For a domain $\Omega \subset \mathbb{R}^N$ and p, $1 \le p \le \infty$, the usual Lebesgue and Sobolev spaces are denoted by $L^p(\Omega)$ and $W^{k,p}(\Omega)$, where k is a nonnegative integer. The inner product of $L^2(\Omega)$ is denoted by $(\cdot,\cdot)_\Omega$. The trace operator is denoted by $\gamma : W^{k,p}(\Omega) \to W^{k-1/p,p}(\partial\Omega)$. As usual, $W^{k,2}(\Omega)$ and $W^{k-1/2,2}(\partial\Omega)$ are denoted by $H^k(\Omega)$ and $H^{k-1/2}(\partial\Omega)$, respectively. Let $C_0^\infty(\Omega)$ be the set of functions of arbitrary times continuously differentiable functions with compact supports. The space $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. The space of distributions on Ω is denoted by $\mathcal{D}'(\Omega)$.

We denote the dual spaces and the duality pairings of the above mentioned spaces as follows:

[1] The second variational formula obtained by Ushikoshi [**29**] is expressed by a surface integral of $\partial G/\partial n$, $\partial^2 G/\partial n^2$, and $\nabla_x \delta G$. There, the curvature of $\partial\Omega$ does not appear explicitly, differently from Garabedian–Schiffer's formula and our Theorem 20.

$$H^1(\Omega)' := \text{the dual of } H^1(\Omega) \text{ with the duality pairing } \langle\cdot,\cdot\rangle_{1,\Omega},$$

$$H^{-1/2}(\partial\Omega) := \text{the dual of } H^{1/2}(\partial\Omega) \text{ with the duality pairing } \langle\cdot,\cdot\rangle_{1/2,\partial\Omega}.$$

The dual of $H_0^1(\Omega)$ is denoted by $H^{-1}(\Omega)$ as usual.

2. Preliminaries.

2.1. Elliptic regularity on Lipschitz domains.

In this section we summarize the definition of Lipschitz domain in $\mathbb{R}^N$ and their properties. See [**6**], [**12**], [**16**], [**20**] for further details. For an open set $\Omega \subset \mathbb{R}^N$, we say that its boundary $\partial\Omega$ is *Lipschitz continuous* if the following conditions are satisfied:

(1) Each $x \in \partial\Omega$ takes an open rectangle O with $x \in O$ and new orthogonal coordinates $y = (y', y_N)$, $y' = (y_1, \dots, y_{N-1})$ such that

$$O = \{y \mid |y_j| < a_j,\ 1 \le j \le N\},$$

where $a_j > 0$, $1 \le j \le N$.

(2) Let O' be the base of O defined by

$$O' = \{y' = (y_1, \dots, y_{N-1}) \mid |y_j| < a_j,\ 1 \le j \le N-1\}.$$

Then, there is a Lipschitz continuous function $\varphi : O' \to \mathbb{R}$, such that $|\varphi(y')| \le a_N/2$ for $y' \in O'$ and

$$\Omega \cap O = \{y \mid y_N > \varphi(y'),\ y \in O\},$$

$$\partial\Omega \cap O = \{y \mid y_N = \varphi(y'),\ y' \in O\}.$$

A bounded domain $\Omega \subset \mathbb{R}^N$ is called a *Lipschitz domain* if its boundary is Lipschitz continuous. In the case of $N = 2$, for example, such a domain admits corners except for cusps on the boundary.

If $\Omega \subset \mathbb{R}^N$ is a Lipschitz domain, the boundary $\partial\Omega$ has tangent spaces $T_x(\partial\Omega)$ at almost all points $x \in \partial\Omega$ by Rademacher's theorem [**6**, Theorem 3, p. 81]. Hence, the unit outer normal vector $n = (n^1, \dots, n^N)^T \in L^\infty(\partial\Omega; \mathbb{R}^N)$ exists at almost all points on $\partial\Omega$.

Theorem 1 ([**16**, Theorem 6.9.2, p. 341], [**6**, Theorem 1, p. 133]). *If $\Omega \subset \mathbb{R}^N$ is a Lipschitz domain, the trace operator*

$$H^1(\Omega)/H_0^1(\Omega) \ni v \ \mapsto\ v|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$$

is well-defined and an isomorphism. The integration by parts

$$\left(v, \frac{\partial w}{\partial x_i}\right)_\Omega = -\left(\frac{\partial v}{\partial x_i}, w\right)_\Omega + \int_{\partial\Omega} w n^i v, \quad 1 \le i \le N \tag{3}$$

is valid for $v, w \in H^1(\Omega)$.

The following property also assures several fundamental properties of $H^1(\Omega)$ for the Lipschitz domain Ω such as the Sobolev imbeddings

$$H^1(\Omega) \hookrightarrow L^{2N/(N-2)}(\Omega), \quad N > 2, \qquad H^1(\Omega) \hookrightarrow L^q(\Omega), \quad N = 2, \ 1 \le \forall q < \infty.$$

THEOREM 2 ([**6**, Theorem 3, p. 127]). *If* $\Omega \subset \mathbb{R}^N$ *is a Lipschitz domain then* $C^\infty(\overline{\Omega})$ *is dense in* $H^1(\Omega)$ *where*

$$C^\infty(\overline{\Omega}) := \{v : \overline{\Omega} \to \mathbb{R} \mid \exists \tilde{v} \in C_0^\infty(\mathbb{R}^N), \ \tilde{v}|_{\overline{\Omega}} = v\}.$$

In the following theorems the trace $\varphi\,|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$ is taken for $\varphi \in H^1(\Omega)$. Also Δv of $v \in H^1(\Omega)$ is taken in the sense of distributions. We recall that $H_0^1(\Omega) \subset H^1(\Omega)$ implies $H^1(\Omega)' \subset H^{-1}(\Omega)$. Finally we note that the inclusion $H^{1/2}(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$ is dense, which induces the other inclusion $H^{1/2}(\partial\Omega) \hookrightarrow H^{-1/2}(\partial\Omega)$.

THEOREM 3. *Let* $\Omega \subset \mathbb{R}^N$ *be a Lipschitz domain. If* $v \in H^1(\Omega)$ *admits* $\Delta v \in H^1(\Omega)'$*, then there exists* $\partial v/\partial n \in H^{-1/2}(\partial\Omega)$ *which is defined by*

$$\left\langle \varphi, \frac{\partial v}{\partial n} \right\rangle_{1/2,\partial\Omega} := (\nabla v, \nabla\varphi)_\Omega + \langle \varphi, \Delta v \rangle_{1,\Omega}, \quad \forall \varphi \in H^1(\Omega). \tag{4}$$

PROOF. We only have to show that the right-hand side of (4) is independent of the choice of the extension of $\varphi \in H^{1/2}(\partial\Omega)$ to $H^1(\Omega)$. We thus need to show that, for any $\varphi \in H^1(\Omega)$ and $\psi \in H_0^1(\Omega)$,

$$(\nabla v, \nabla\varphi)_\Omega + \langle \varphi, \Delta v \rangle_{1,\Omega} = (\nabla v, \nabla(\varphi + \psi))_\Omega + \langle \varphi + \psi, \Delta v \rangle_{1,\Omega},$$

or equivalently

$$(\nabla v, \nabla\psi)_\Omega + \langle \psi, \Delta v \rangle_{1,\Omega} = 0, \quad \forall \psi \in H_0^1(\Omega). \tag{5}$$

Recall that the distribution Δv is defined by

$$\langle \psi, \Delta v \rangle_{1,\Omega} := -(\nabla v, \nabla\psi)_\Omega, \quad \forall \psi \in C_0^\infty(\Omega). \tag{6}$$

Therefore, (5) follows from (6) because $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. $\square$

We have confirmed that if Ω is a Lipschitz domain and $v \in H^1(\Omega)$, then $v \in H^{1/2}(\partial\Omega)$ holds in the sense of trace. If $\Delta v \in H^1(\Omega)'$ furthermore, then $\partial v/\partial n \in H^{-1/2}(\partial\Omega)$ is defined by (4). In this case, it is easy to see that this mapping is continuous;

$$\left\| \frac{\partial v}{\partial n} \right\|_{H^{-1/2}(\partial\Omega)} \le C_3(\|\nabla v\|_{L^2(\Omega)} + \|\Delta v\|_{H^1(\Omega)'}).$$

However, we cannot define $v \in H^{-1/2}(\partial\Omega)$ for general $v \in L^2(\Omega)$ nor $\partial v/\partial n \in H^{-1/2}(\partial\Omega)$ for general $v \in H^1(\Omega)$.

If $v \in H^2(\Omega)$, on the contrary, we have $\nabla v \in H^1(\Omega;\mathbb{R}^N)$. With the unit outer normal vector field $n \in L^\infty(\partial\Omega;\mathbb{R}^N)$, we can define $n \cdot \nabla v \in L^2(\partial\Omega)$. We have also $\Delta v \in L^2(\Omega) \hookrightarrow H^1(\Omega)'$ and hence $\partial v/\partial n \in H^{-1/2}(\partial\Omega)$ by Theorem 3.

Theorem 4. *Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz domain and $v \in H^2(\Omega)$. Then the above $n \cdot \nabla v \in L^2(\partial\Omega)$ is identified with $\partial v/\partial n \in H^{-1/2}(\partial\Omega)$ as a distribution on $\partial\Omega$.*

Proof. Let ω be an open set which contains $\partial\Omega$. For $\varphi \in C_0^\infty(\omega)$, we consider its zero-extension and regard it as an element in $C_0^\infty(\mathbb{R}^N)$. It follows from (4), (3) and the continuous inclusion $L^2(\Omega) \hookrightarrow H^1(\Omega)'$ that

$$\begin{aligned}\left\langle \varphi, \frac{\partial v}{\partial n}\right\rangle_{1/2,\partial\Omega} &= (\nabla v, \nabla\varphi)_\Omega + \langle \varphi, \Delta v\rangle_{1,\Omega} \\ &= (\nabla v, \nabla\varphi)_\Omega + (\varphi, \Delta v)_\Omega = \langle \varphi, n\cdot\nabla v\rangle_{1/2,\partial\Omega}\end{aligned}$$

for $v \in H^2(\Omega)$. This equality proves the identity. □

2.2. Domain perturbations.

We continue to assume that $\Omega \subset \mathbb{R}^N$ is a Lipschitz domain. Let $T_t : \overline{\Omega} \to \overline{\Omega}_t$, $\Omega_t := T_t\Omega$, $|t| < \varepsilon$, $0 < \varepsilon \ll 1$, be a family of bi-Lipschitz homeomorphisms. We assume that the mapping $\overline{\Omega} \ni x \mapsto T_t x$ is once or twice differentiable with respect to t for each $x \in \overline{\Omega}$. Henceforth, we say that $\{T_t\}$ is once or twice differentiable in t in short for the latter properties.

Definition 5. The above $\{T_t\}$ is a differentiable deformation if it is differentiable in t and the mappings

$$\frac{\partial}{\partial t}T_t x,\ \frac{\partial}{\partial t}\nabla(T_t x),\ x \in \overline{\Omega}, \quad \frac{\partial}{\partial t}T_t^{-1}x,\ \frac{\partial}{\partial t}\nabla(T_t^{-1}x),\ x \in \overline{\Omega}\cap\overline{\Omega}_t \tag{7}$$

are locally uniformly bounded on $\overline{\Omega} \times I$, $I := (-\varepsilon, \varepsilon)$. It is a twice differentiable deformation if it is twice differentiable in t and the mappings, besides (7),

$$\frac{\partial^2}{\partial t^2}T_t x,\ \frac{\partial^2}{\partial t^2}\nabla(T_t x),\ x \in \overline{\Omega}, \quad \frac{\partial^2}{\partial t^2}T_t^{-1}x,\ \frac{\partial^2}{\partial t^2}\nabla(T_t^{-1}x),\ x \in \overline{\Omega}\cap\overline{\Omega}_t$$

are locally uniformly bounded on $\overline{\Omega} \times I$.

If $\{T_t\}$ is a differentiable deformation the vector field

$$S := \left.\frac{\partial T_t}{\partial t}\right|_{t=0}$$

is Lipschitz continuous on $\overline{\Omega}$. If $\{T_t\}$ is a twice-differentiable deformation the vector field

$$R := \left.\frac{\partial^2 T_t}{\partial t^2}\right|_{t=0}$$

is also a Lipschitz continuous on $\overline{\Omega}$. Then we put

$$\delta\rho := \left.\frac{\partial T_t}{\partial t}\right|_{t=0} \cdot n = S \cdot n, \qquad \delta^2\rho := \left.\frac{\partial^2 T_t}{\partial t^2}\right|_{t=0} \cdot n = R \cdot n.$$

Example 6 (Dynamical Perturbations). Let $v = v(x)$ be a Lipschitz vector field defined in a domain $\widetilde{\Omega}$ containing $\overline{\Omega}$. Given $x \in \widetilde{\Omega}$, we consider the ordinary differential equation

$$\frac{dc(t)}{dt} = v(c(t)), \qquad c(0) = x$$

which defines the integral curve $c : (-\varepsilon, \varepsilon) \to \tilde{\Omega}$, $|t| < \varepsilon$. Writing

$$T_t x := c(t),$$

we obtain a family of bi-Lipschitz homeomorphisms $T_t : \overline{\Omega} \to \overline{\Omega}_t$, $|t| \ll 1$. Then it holds that

$$S(x) = \left.\frac{\partial}{\partial t} T_t x\right|_{t=0} = v(T_t x)|_{t=0} = v(x),$$
$$R(x) = \left.\frac{\partial^2}{\partial t^2} T_t x\right|_{t=0} = \left.\frac{\partial}{\partial t} v(T_t x)\right|_{t=0} = [(v \cdot \nabla)v](x).$$

This $\{T_t\}$ is a differentiable deformation and is twice-differentiable if ∇v is furthermore Lipschitz continuous in $\tilde{\Omega}$. Then it holds that

$$T_t(x) = x + tv(x) + \frac{t^2}{2}[(v \cdot \nabla)v](x) + o(t^2)$$

uniformly in $x \in \overline{\Omega}$.

Example 7 (Normal Perturbations). If $\partial\Omega$ is $C^{1,1}$, the unit normal vector $n = n_x$ is Lipschitz continuous on $\partial\Omega$. Then we can take a bi-Lipschitz deformation of $\partial\Omega$ by

$$\Gamma_t : x + t \cdot \delta\rho(x) n_x, \quad x \in \partial\Omega$$

for $|t| \ll 1$, where $\delta\rho = \delta\rho(x)$, $x \in \partial\Omega$, is a Lipschitz continuous function. Then there is a domain $\Omega_t \subset \mathbb{R}^N$ such that $\Gamma_t = \partial\Omega_t$. There is, also, a bi-Lipschitz mapping $T_t : \overline{\Omega} \to \overline{\Omega_t}$, $|t| \ll 1$, which satisfies

$$\delta\rho = \left.\frac{\partial T_t}{\partial t}\right|_{t=0} \cdot n, \quad \delta^2\rho = \left.\frac{\partial^2 T_t}{\partial t^2}\right|_{t=0} \cdot n = 0 \quad \text{on } \partial\Omega.$$

Although this deformation has been used in the classical theory [**9**], it does not work for the general Lipschitz domain, for example, if $\partial\Omega$ has a corner.

2.3. Variational formulae of Jacobian.

We follow the standard notation of matrices $A = (a_{ij})_{i:1\downarrow N, j:1\to N}$. Thus all vectors are regarded as column vectors unless otherwise stated. The transposition of a matrix A is denoted by A^T. For a given $N \times N$ matrices $A = (a_{ij})$ and $B = (b_{ij})$, we put

$$A : B := \sum_{i,j=1}^{N} a_{ij} b_{ij}.$$

For the Lipschitz continuous vector field T on $\overline{\Omega}$, let

$$J(T) = \left(\frac{\partial y_\ell}{\partial x_k} \right)_{k,\ell}^{T}$$

be its Jacobi matrix where $y = T(x)$.

Theorem 6. *Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz domain and $\{T_t\}$ be a differentiable deformation. Then it holds that*

$$\frac{\partial}{\partial t} \det J(T_t) \bigg|_{t=0} = \nabla \cdot S, \qquad S := \frac{\partial T_t}{\partial t} \bigg|_{t=0}. \tag{8}$$

If $\{T_t\}$ is twice differentiable, furthermore, we have

$$\frac{\partial^2}{\partial t^2} \det J(T_t) \bigg|_{t=0} = \nabla \cdot R + (\nabla \cdot S)^2 - J(S)^T : J(S). \tag{9}$$

Proof. For simplicity we consider the case that $\{T_t\}$ is twice differentiable. Then both

$$S = (S^1, \ldots, S^N)^T \quad \text{and} \quad R = (R^1, \ldots, R^N)^T$$

are Lipschitz continuous. It holds that

$$J(T_t) = \begin{pmatrix} 1 + tS^1_{x_1} + \frac{1}{2}t^2 R^1_{x_1} & \cdots & tS^1_{x_N} + \frac{1}{2}t^2 R^1_{x_N} \\ \vdots & \cdots & \vdots \\ tS^N_{x_1} + \frac{1}{2}t^2 R^N_{x_1} & \cdots & 1 + tS^N_{x_N} + \frac{1}{2}t^2 R^N_{x_N} \end{pmatrix} + o(t^2)$$

and hence

$$\det J(T_t) = 1 + t\nabla \cdot S + t^2 \sum_{i<j} (S^i_{x_i} S^j_{x_j} - S^i_{x_j} S^j_{x_i}) + \frac{t^2}{2} \sum_i R^i_{x_i} + o(t^2).$$

Then we obtain (8) and (9) by

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\det J(T_t)\bigg|_{t=0} &= 2\sum_{i<j}(S^i_{x_i}S^j_{x_j}-S^i_{x_j}S^j_{x_i})+\nabla\cdot R\\ &= \sum_{i\neq j}(S^i_{x_i}S^j_{x_j}-S^i_{x_j}S^j_{x_i})+\nabla\cdot R\\ &= \sum_{i,j}(S^i_{x_i}S^j_{x_j}-S^i_{x_j}S^j_{x_i})+\nabla\cdot R\\ &= \nabla\cdot R+(\nabla\cdot S)^2-J(S)^T:J(S).\end{aligned}$$ □

2.4. An abstract theorem.

The derivation of variational formulae is hard even if we try to do it formally. Of course a mathematical justification of formal derivation is inevitable. The origin of the difficulty is that domains themselves are perturbed and it is difficult to prove the differentiability of, for example, $G_t(\cdot,t)$ on the original domain Ω. Hence we introduce the Lagrange coordinate system on which an abstract theorem is applicable.

Let V be a Hilbert space over $\mathbb{R}$ with its dual space V'. We denote their duality pairing $\langle\cdot,\cdot\rangle_V$. Let $a_t : V\times V\to\mathbb{R}$, $t\in(-\varepsilon,\varepsilon)=:I$, be a family of symmetric bilinear forms. We assume their uniform boundedness and coercivity. That is, there are $\delta>0$ and $M>0$ such that

$$a_t(u,u)\geq\delta\|u\|_V^2,\quad |a_t(u,v)|\leq M\|u\|_V\|v\|_V \tag{10}$$

for any $u,v\in V$ and $t\in I$. For a map $f:I\to V'$, we have a unique $u=u(t)\in V$, $t\in I$, such that

$$a_t(u(t),v)=\langle v,f(t)\rangle_V,\qquad \forall v\in V \tag{11}$$

by the representation theorem of Riesz.

Theorem 7. *Let the above $a_t(\cdot,\cdot)$ and $f(t)$ be strongly differentiable in t. Namely, first, there is a symmetric bilinear form denoted by $\dot a_t : V\times V\to\mathbb{R}$ such that*

$$\lim_{h\to 0}\sup_{\|u\|\leq 1,\,\|v\|\leq 1}\left|\frac{a_{t+h}(u,v)-a_t(u,v)}{h}-\dot a_t(u,v)\right|=0,\quad \forall t\in I.$$

Next, there is $\dot f=\dot f(t)\in V'$ such that

$$\lim_{h\to 0}\sup_{\|v\|\leq 1}\left|\frac{\langle v,f(t+h)\rangle_V-\langle v,f(t)\rangle_V}{h}-\langle v,\dot f(t)\rangle_V\right|=0,\quad \forall t\in I.$$

Then $u=u(t)\in V$ defined by (11) is strongly differentiable in t, that is,

$$\exists \dot{u}(t) \in V, \quad \lim_{h \to 0} \left\| \frac{u(t+h) - u(t)}{h} - \dot{u}(t) \right\|_V = 0, \quad \forall t \in I \tag{12}$$

and it holds that

$$a_t(\dot{u}(t), v) + \dot{a}_t(u(t), v) = \langle v, \dot{f}(t) \rangle_V, \quad \forall v \in V, \ t \in I. \tag{13}$$

The second strong differentiability of $u = u(t)$ arises similarly if $a_t : V \times V \to \mathbb{R}$ and $f = f(t) : I \to V'$ are twice differentiable. Then it holds that

$$a_t(\ddot{u}(t), v) + 2\dot{a}_t(\dot{u}(t), v) + \ddot{a}_t(u(t), v) = \langle v, \ddot{f}(t) \rangle_V, \quad \forall v \in V, \ t \in I, \tag{14}$$

where

$$\lim_{h \to 0} \sup_{\|u\| \le 1, \, \|v\| \le 1} \left| \frac{\dot{a}_{t+h}(u, v) - \dot{a}_t(u, v)}{h} - \ddot{a}_t(u, v) \right| = 0,$$

$$\lim_{h \to 0} \sup_{\|v\| \le 1} \left| \frac{\langle v, \dot{f}(t+h) \rangle_V - \langle v, \dot{f}(t) \rangle_V}{h} - \langle v, \ddot{f}(t) \rangle_V \right| = 0$$

and

$$\lim_{h \to 0} \left\| \frac{\dot{u}(t+h) - \dot{u}(t)}{h} - \ddot{u}(t) \right\|_V = 0.$$

To prove the strong differentiability, for example, first, we define $\dot{u}$ by (13) and then derive (12) using (10). See, for example, [**25**] for the proof.

2.5. Normal curvatures.

If $\Omega \subset \mathbb{R}^N$ is a Lipschitz domain, we can define the standard measure on $\partial\Omega$ using the local chart. Also any Lipschitz continuous function on $\partial\Omega$ takes a Lipschitz continuous extension to its neighborhood through the flow used in Section 2.2. The tangential space $T_\xi(\partial\Omega)$, furthermore, is defined for almost every $\xi \in \partial\Omega$ with the outer unit normal vector denoted by n. Using an orthonormal basis $\{s_i \mid i = 1, \ldots, N-1\}$ of $T_\xi(\partial\Omega)$, we thus obtain the frame

$$\{s_1, \ldots, s_{N-1}, n\} \tag{15}$$

almost everywhere on $\partial\Omega$.

If $\Omega \subset \mathbb{R}^N$ is a $C^{1,1}$ domain, the frame (15) on $\partial\Omega$ is Lipschitz continuous. Let Π_i, $1 \le i \le N-1$ be the plane spanned by $\{s_i, n\}$. Then we cut $\partial\Omega$ by Π_i and obtain a curve $\gamma_i = \Pi_i \cap \partial\Omega$. Taking the parametrization $x = x(s) \in \gamma_i$ with the arc-length s, we have

$$s_i = \dot{x} := \frac{\partial x}{\partial s}, \quad |s_i| = 1,$$

which implies $\dot{s}_i \cdot s_i = 0$, where $\dot{s}_i := \partial s_i/\partial s_i$. Since $\dot{s}_i \in \Pi_i$, we define the normal curvature κ_i by

$$\dot{s}_i = -\kappa_i n, \tag{16}$$

recalling n is outer with respect to Ω. On the other hand we have $n \cdot s_i = 0$ and $|n| = 1$, which result in

$$\dot{n} \cdot s_i = \kappa_i, \quad \dot{n} \cdot n = 0, \quad \dot{n} := \frac{\partial n}{\partial s_i}.$$

Therefore, we have

$$\dot{n} = \kappa_i s_i + \sum_{j \neq i} (\dot{n} \cdot s_j) s_j. \tag{17}$$

Note that if s_i is the principal direction, we have $\dot{n} \cdot s_j = 0$, $j \neq i$ and $\dot{n} \in \Pi_i$. Concerning s_j for $j \neq i$, we use

$$s_i \cdot s_j = s_j \cdot n = 0, \quad |s_j| = 1.$$

Since $\dot{s}_j$ lies on the space generated by s_i, s_j, n, it follows that

$$\dot{s}_j = -(s_j \cdot \dot{n}) n, \quad \dot{s}_j := \frac{\partial s_j}{\partial s_i} \tag{18}$$

similarly. We write the relations (16), (17), and (18) as

$$\frac{\partial s_i}{\partial s_i} = -\kappa_i n, \quad \frac{\partial n}{\partial s_i} = \kappa_i s_i + \sum_{j \neq i} \left(\frac{\partial n}{\partial s_i} \cdot s_j \right) s_j, \quad \frac{\partial s_j}{\partial s_i} = -\left(s_j \cdot \frac{\partial n}{\partial s_i} \right) n, \ j \neq i. \tag{19}$$

Next, we use a tubular neighbourhood to extend the above frame (15) to a neighbourhood of $\partial\Omega$. We have a fiber of γ on Π_i with the length parameter t, which results in the parametrization $n = n(t, s)$ and $s_i = s_i(t, s)$ satisfying

$$n' := \frac{\partial n}{\partial t} = 0.$$

Then we obtain

$$s_i' = 0$$

by $n \cdot s_i = 0$ and $s_i \cdot s_i = 1$. Using similar notations as in (19), we end up with the formula of Frenet–Serret

$$\frac{\partial n}{\partial n} = 0, \qquad \frac{\partial s_i}{\partial n} = 0, \quad 1 \leq i \leq N - 1. \tag{20}$$

Definition 8. We put

$$a \otimes b := (a_i b_j)_{ij} = a\, b^T$$

for the vectors $a = (a_i)_i$ and $b = (b_j)_j$, and also

$$\nabla n := (\nabla \otimes n)^T := \left(\frac{\partial n^j}{\partial s_k} \right)_{jk} \tag{21}$$

for $n = (n^j)_j$.

Using the frame (15) as an orthonormal system in $\mathbb{R}^N$, we have

$$(\nabla n)^T = \sum_{i=1}^{N-1} \frac{\partial n}{\partial s_i} \otimes s_i + \frac{\partial n}{\partial n} \otimes n = \sum_{i=1}^{N-1} \kappa_i s_i \otimes s_i + \sum_{i=1}^{N-1} \sum_{j \neq i} \left(\frac{\partial n}{\partial s_i} \cdot s_j \right) s_j \otimes s_i \tag{22}$$

$$(\nabla s_j)^T = \sum_{i=1}^{N-1} \frac{\partial s_j}{\partial s_i} \otimes s_i + \frac{\partial s_j}{\partial n} \otimes n = -\kappa_j n \otimes s_j - \sum_{i \neq j} \left(\frac{\partial n}{\partial s_i} \cdot s_j \right) n \otimes s_i \tag{23}$$

by (19) and (20). Then it holds that

$$\nabla \cdot n = \mathrm{tr}(\nabla n) = \sum_{i=1}^{N-1} \kappa_i. \tag{24}$$

This value is called the *mean curvature* and is independent of choice of the frame. Note that if $\{s_1, \ldots, s_{N-1}, n\}$ is a coordinate system such that s_i, $i = 1, \ldots, N-1$ is the principal direction, we have

$$(\nabla n)^T = \sum_{i=1}^{N-1} \kappa_i s_i \otimes s_i, \qquad (\nabla s_j)^T = -\kappa_j n \otimes s_j. \tag{25}$$

Definition 9. Let $\{s_1, \ldots, s_{N-1}, n\}$ is an orthonormal frame on $\partial\Omega$. If each s_i, $i = 1, \ldots, N-1$ is the principal direction, then the frame $\{s_1, \ldots, s_{N-1}, n\}$ is called the *Morse frame.*

The existence of the Morse frame is guaranteed by Morse's Lemma [**18**].

Remark. By the above notation of (21) it holds that

$$\nabla \otimes (f\boldsymbol{a}) = (\nabla f) \otimes \boldsymbol{a} + f(\nabla \otimes \boldsymbol{a}) = (\nabla f) \otimes \boldsymbol{a} + f\,(\nabla \boldsymbol{a})^T,$$

where f and $\boldsymbol{a}$ are scalar and vector fields.

Now we show the following lemma.

LEMMA 10. *If $\Omega \subset \mathbb{R}^N$ is a $C^{1,1}$-domain and $f \in C^{1,1}(\overline{\Omega})$ then it holds that*

$$\nabla^2 f = \sum_{i=1}^{N-1} (\nabla s_i)^T \frac{\partial f}{\partial s_i} + (\nabla n)^T \frac{\partial f}{\partial n} + \sum_{i,j=1}^{N-1} s_i \otimes s_j \frac{\partial^2 f}{\partial s_i \partial s_j} + \sum_{i=1}^{N-1} (s_i \otimes n + n \otimes s_i) \frac{\partial^2 f}{\partial s_i \partial n} + n \otimes n \frac{\partial^2 f}{\partial n^2} \tag{26}$$

a.e. on $\partial\Omega$, where $\nabla^2 f := (\partial^2 f / \partial x_i \partial x_j)_{ij}$ denotes the Hesse matrix of f.

PROOF. Using the orthonormal basis (15), we have

$$\begin{aligned}\nabla^2 f &= \nabla \otimes \nabla f = \nabla \otimes \left(\sum_{i=1}^{N-1} s_i \frac{\partial f}{\partial s_i} + n \frac{\partial f}{\partial n} \right) \\ &= \sum_{i=1}^{N-1} (\nabla s_i)^T \frac{\partial f}{\partial s_i} + (\nabla n)^T \frac{\partial f}{\partial n} + \sum_{i=1}^{N-1} \nabla \frac{\partial f}{\partial s_i} \otimes s_i + \nabla \frac{\partial f}{\partial n} \otimes n.\end{aligned}$$

Here, the third and the fourth terms on the right-hand side are equal to

$$\begin{aligned}\sum_{i=1}^{N-1} \nabla \left(\frac{\partial f}{\partial s_i} \right) \otimes s_i &= \sum_{i=1}^{N-1} \left(\sum_{j=1}^{N-1} s_j \frac{\partial^2 f}{\partial s_j \partial s_i} + n \frac{\partial^2 f}{\partial n \partial s_i} \right) \otimes s_i \\ &= \sum_{i,j=1}^{N-1} [s_j \otimes s_i] \frac{\partial^2 f}{\partial s_i \partial s_j} + \sum_{i=1}^{N} [n \otimes s_i] \frac{\partial^2 f}{\partial s_i \partial n}\end{aligned}$$

and

$$\begin{aligned}\nabla \left(\frac{\partial f}{\partial n} \right) \otimes n &= \left(\sum_{i=1}^{N-1} s_i \frac{\partial^2 f}{\partial s_i \partial n} + n \frac{\partial^2 f}{\partial n^2} \right) \otimes n \\ &= \sum_{i=1}^{N-1} (s_i \otimes s_j) \frac{\partial^2 f}{\partial s_i \partial n} + (n \otimes n) \frac{\partial^2 f}{\partial n^2},\end{aligned}$$

respectively. Hence it follows that

$$\begin{aligned}\nabla^2 f = \sum_{i=1}^{N-1} (\nabla s_i)^T \frac{\partial f}{\partial s_i} + (\nabla n)^T \frac{\partial f}{\partial n} + \sum_{i,j=1}^{N-1} s_i \otimes s_j \frac{\partial^2 f}{\partial s_i \partial s_j} \\ + \sum_{i=1}^{N-1} (s_i \otimes n + n \otimes s_i) \frac{\partial^2 f}{\partial s_i \partial n} + n \otimes n \frac{\partial^2 f}{\partial n^2}.\end{aligned}$$

□

The above lemma implies the following result.

Corollary 11. *If $\Omega \subset \mathbb{R}^N$ is a $C^{1,1}$-domain and $f \in C^{1,1}(\overline{\Omega})$ then it holds that*

$$\Delta f = (\nabla \cdot n)\frac{\partial f}{\partial n} + \frac{\partial^2 f}{\partial n^2} + \sum_{i=1}^{N-1} \frac{\partial^2 f}{\partial s_i^2} \tag{27}$$

a.e. on $\partial\Omega$.

Proof. Note that $\mathrm{tr}(a \otimes b) = a \cdot b$ for a, $b \in \mathbb{R}^N$. Hence,

$$\mathrm{tr}(s_k \otimes n + n \otimes s_k) = 0.$$

We have also

$$\mathrm{tr}[(\nabla n)^T] = \nabla \cdot n, \quad \mathrm{tr}(n \otimes n) = |n|^2 = 1$$

and

$$\mathrm{tr}[(\nabla s_i)^T] = -\kappa_i \mathrm{tr}[n \otimes s_i] - \sum_{i \neq j} \left(\frac{\partial n}{\partial s_i} \cdot s_j \right) \mathrm{tr}[n \otimes s_i] = 0$$

by (23). Then (27) follows. □

2.6. Liouville's theorem.

Liouville's theorem is a set of variational formulae concerning volume and surface integrals described in the Lagrange coordinate system. These formulae may be used to derive a representation of boundary integrals of the first variation of the Green function. We continue to use the notations in Section 2.2. We also put $\dot{c} = c_t$, $\ddot{c} = c_{tt}$, and so forth.

Theorem 12 (First Volume Derivative). *Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz domain and $\{T_t\}$ be a differentiable deformation. Let, furthermore, $c = c(x,t)$ be a Lipschitz continuous function in $\widetilde{\Omega} \times (-\varepsilon, \varepsilon)$, where $\widetilde{\Omega}$ is a domain containing $\overline{\Omega}$. Then it holds that*

$$\frac{d}{dt}\left(\int_{\Omega_t} c(\cdot, t) \right)\bigg|_{t=0} = \int_\Omega \dot{c}(\cdot, 0) + \int_{\partial\Omega} c(\cdot, 0)\delta\rho. \tag{28}$$

Proof. Letting $J_t(x) = J(T_t x)$, we have

$$\int_{\Omega_t} c(y,t)\, dy = \int_\Omega c(T_t x, t)\, \det J_t(x)\, dx$$

and hence

$$\frac{d}{dt}\left(\int_{\Omega_t} c(x,t)\,dx\right)\bigg|_{t=0} = \int_\Omega \Bigg(\left(\dot{c}(T_t x,t) + \nabla c(T_t x,t)\cdot\frac{\partial T_t x}{\partial t}\right)\det J_t(x) + c(T_t x,t)\frac{\partial}{\partial t}\mathrm{det}J_t(x)\bigg|_{t=0}\Bigg)\,dx$$

$$= \int_\Omega (\dot{c}(\cdot,0) + \nabla c(\cdot,0)\cdot S + c(\cdot,0)\nabla\cdot S)\,dx$$

by (8). Then (28) follows from the divergence formula. □

THEOREM 13 (First Area Derivative). *Assume that the assumptions in the previous theorem hold. If, further, $\Omega \subset \mathbb{R}^N$ is $C^{1,1}$, then we have*

$$\frac{d}{dt}\left(\int_{\partial\Omega_t} c(\cdot,t)\right)\bigg|_{t=0} = \int_{\partial\Omega} \dot{c}(\cdot,0) + \left[(\nabla\cdot n)c(\cdot,0) + \frac{\partial c(\cdot,0)}{\partial n}\right]\delta\rho\;ds. \tag{29}$$

PROOF. Given a Lipschitz continuous vector field $a = a(\cdot,t)$ in $\tilde{\Omega}\times(-\varepsilon,\varepsilon)$, we apply Theorem 12 to $c = \nabla\cdot a$. It follows that

$$\frac{d}{dt}\int_{\Omega_t} \nabla\cdot a = \int_{\Omega_t}\nabla\cdot\dot{a} + \int_{\partial\Omega_t}(\nabla\cdot a)\left[n\cdot\frac{\partial T_t}{\partial t}\right],$$

which means

$$\frac{d}{dt}\int_{\partial\Omega_t} n\cdot a = \int_{\partial\Omega_t} n\cdot\dot{a} + (\nabla\cdot a)\left[n\cdot\frac{\partial T_t}{\partial t}\right]ds.$$

Putting $a = nc$ and $t=0$, we obtain (29) by $n\cdot\dot{n} = 0$. □

3. The Dirichlet problem.

3.1. The Green function.

Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz domain. The Green function for $-\Delta$ with the homogeneous Dirichlet condition is denoted by $G = G(x,y)$, $(x,y)\in\overline{\Omega}\times\Omega$, and is defined by

$$-\Delta G(\cdot,y) = \delta(\cdot - y) \quad \text{in } \Omega, \quad G(\cdot,y)|_{\partial\Omega} = 0,$$

where $\delta = \delta(x)$ is the delta function. The fundamental solution $\Gamma(x) = \gamma(|x|)$ of $-\Delta$,

$$\gamma(r) = \begin{cases} \dfrac{1}{2\pi}\log\dfrac{1}{r}, & N=2 \\[2ex] \dfrac{1}{(N-2)\omega_N}r^{2-N}, & N\geq 3, \end{cases}$$

takes the property

$$-\Delta\Gamma(\cdot - y) = \delta(\cdot - y),$$

where ω_N denotes the $(N-1)$-dimensional surface area of the N-dimensional unit ball. Hence, the function $u = u(x)$ defined by

$$G(\cdot, y) = \Gamma(\cdot - y) + u \tag{30}$$

satisfies

$$\Delta u = 0 \quad \text{in } \Omega, \qquad u = -\Gamma(\cdot - y) \quad \text{on } \partial\Omega. \tag{31}$$

We reduce this boundary value problem of the Laplace equation to that of the Poisson equation. To this end, take open sets $\omega \subset\subset \hat{\omega}$ satisfying $\partial\Omega \subset \omega$ and $y \notin \hat{\omega}$ and $\varphi = \varphi(x) \in C_0^\infty(\mathbb{R}^N)$ such that $0 \le \varphi \le 1$ and

$$\varphi(x) = \begin{cases} 1, & x \in \omega \\ 0, & x \notin \hat{\omega}. \end{cases}$$

Then, (31) is reduced to

$$-\Delta w = g \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \partial\Omega \tag{32}$$

concerning $w = u - \varphi\Gamma(\cdot - y)$ for $g := \Delta(\varphi\Gamma(\cdot - y))$. This $g = g(x) \in C_0^\infty(\mathbb{R}^N)$ is regarded as an element of $H^{-1}(\Omega)$ by

$$\langle v, g\rangle_{1,\Omega} = -(\nabla v, \nabla(\varphi\Gamma(\cdot - y)))_\Omega, \quad v \in H_0^1(\Omega).$$

Hence, the boundary value problem (32) admits a unique solution $w \in H_0^1(\Omega)$. Thus, the solution to (31) is obtained by $u = w + \varphi\Gamma(\cdot - y) \in H^1(\Omega)$ because $\Gamma(\cdot - y)$ is smooth in $\hat{\omega}$, which implies the well-definedness of

$$0 = G(\cdot, y) \in H^{1/2}(\partial\Omega), \quad \frac{\partial G(\cdot, y)}{\partial n} \in H^{-1/2}(\partial\Omega) \tag{33}$$

by (30).

Theorem 14. *If $\Omega \subset \mathbb{R}^N$ is a Lipschitz domain and $f \in H^1(\Omega)$ is harmonic in Ω, then it holds that*

$$f(y) = -\left\langle f, \frac{\partial G(\cdot, y)}{\partial n} \right\rangle_{1/2,\partial\Omega}, \quad y \in \Omega. \tag{34}$$

Proof. Since $\Delta f = 0 \in H^1(\Omega)'$, we have

$$(\nabla f, \nabla \varphi)_\Omega = \left\langle \varphi, \frac{\partial f}{\partial n} \right\rangle_{1/2,\partial\Omega}, \quad \varphi \in H^1(\Omega) \tag{35}$$

by Theorem 3. Given $y \in \Omega$, we take

$$\varphi_\varepsilon(x) = \begin{cases} \gamma(|x-y|), & |x-y| \geq \varepsilon \\ \gamma(\varepsilon), & |x-y| < \varepsilon \end{cases}$$

for $0 < \varepsilon \ll 1$, which belongs to $H^1(\Omega)$. Then we obtain

$$\begin{aligned}(\nabla f, \nabla \varphi_\varepsilon)_\Omega &= \int_{\Omega \setminus B(y,\varepsilon)} \nabla f(x) \cdot \nabla \Gamma(x-y)\, dx \\ &= \int_{\partial\Omega} f(x) \frac{\partial \Gamma(x-y)}{\partial n_x}\, ds_x - \int_{\partial B(y,\varepsilon)} f(x) \frac{\partial \Gamma(x-y)}{\partial n_x}\, ds_x\end{aligned}$$

using the traces $f \in H^{1/2}(\partial\Omega)$ and $f \in H^{1/2}(\partial B(y,\varepsilon))$. By Weyl's lemma, $f(x)$ is smooth in Ω. By the classical argument, hence, the left-hand side of (35) for $\varphi = \varphi_\varepsilon$, that is, $(\nabla f, \nabla \varphi_\varepsilon)_\Omega$, converges to

$$\int_{\partial\Omega} f(x) \frac{\partial \Gamma(x-y)}{\partial n_x}\, ds_x + f(y)$$

as $\varepsilon \downarrow 0$. Since its right-hand side, $\langle \varphi_\varepsilon, \partial f/\partial n \rangle_{1/2,\partial\Omega}$, is independent of $0 < \varepsilon \ll 1$, we end up with

$$\left\langle \Gamma(\cdot - y), \frac{\partial f}{\partial n} \right\rangle_{1/2,\partial\Omega} = \left\langle f, \frac{\partial \Gamma(\cdot - y)}{\partial n} \right\rangle_{1/2,\partial\Omega} + f(y). \tag{36}$$

We define $u = u(x)$ by (30), and note $\Delta f = \Delta u = 0 \in H^1(\Omega)'$. Thus, it follows that

$$(\nabla u, \nabla f)_\Omega = \left\langle u, \frac{\partial f}{\partial n} \right\rangle_{1/2,\partial\Omega} = \left\langle f, \frac{\partial u}{\partial n} \right\rangle_{1/2,\partial\Omega} \tag{37}$$

by Theorem 3. Since (33) implies

$$\left\langle u + \Gamma(\cdot - y), \frac{\partial f}{\partial n} \right\rangle_{1/2,\partial\Omega} = \left\langle G(\cdot, y), \frac{\partial f}{\partial n} \right\rangle_{1/2,\partial\Omega} = 0,$$

it follows that

$$\left\langle \Gamma(\cdot - y), \frac{\partial f}{\partial n} \right\rangle_{1/2,\partial\Omega} = -\left\langle u, \frac{\partial f}{\partial n} \right\rangle_{1/2,\partial\Omega} = -\left\langle f, \frac{\partial u}{\partial n} \right\rangle_{1/2,\partial\Omega}$$

from (37). Hence we obtain

$$f(y) = -\left\langle f, \frac{\partial \Gamma(\cdot - y)}{\partial n} + \frac{\partial u}{\partial n} \right\rangle_{1/2,\partial\Omega} = -\left\langle f, \frac{\partial G(\cdot, y)}{\partial n} \right\rangle_{1/2,\partial\Omega}$$

by (36). □

3.2. Lagrange derivative.

We have two formulations of variational formulae, using Euler and Lagrange coordinate systems. In the Lagrange coordinate system, we do not have any regularity losses in space variables. The variational formulae, however, is not so clear compared with the ones in Euler coordinate system.

We recall that $\Omega \subset \mathbb{R}^N$ is a Lipschitz domain and $\{T_t\}$ is either once or twice differentiable deformation. The Green function for $-\Delta$ in Ω_t with $\cdot|_{\partial\Omega} = 0$ is denoted by $G_t = G_t(x, y)$. We fix $y \in \Omega$ to define $u = u(\cdot, t)$ by (30):

$$G_t(x, y) = \Gamma(x - y) + u(x, t), \tag{38}$$

which satisfies

$$\Delta u(\cdot, t) = 0 \quad \text{in } \Omega_t, \qquad u(\cdot, t) = -\Gamma(\cdot - y) \quad \text{on } \partial\Omega_t. \tag{39}$$

As in Section 3.1 this problem is reduced to the boundary value problem of the Poisson equation

$$-\Delta w(\cdot, t) = g \quad \text{in } \Omega_t, \qquad w(\cdot, t) = 0 \quad \text{on } \partial\Omega_t \tag{40}$$

where $g = \Delta(\varphi\Gamma(\cdot - y))$ for $\varphi \in C_0^\infty(\mathbb{R}^N)$. Given $y \in \Omega$, this $\varphi = \varphi(x)$ is taken uniformly in $|t| \ll 1$.

Theorem 15. *Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz domain and $\{T_t\}$, $t \in I = (-\varepsilon, \varepsilon)$, be a once or twice differentiable deformation. Then, according to its differentiabilities we have the existence of $(\partial u/\partial t)(T_t x, t)$ and $(\partial^2 u/\partial t^2)(T_t x, t)$ strongly in $H^1(\Omega)$.*

Proof. The weak form of (40) is

$$\int_{\Omega_t} \nabla w(\cdot, t) \cdot \nabla \varphi \, dy = \int_{\Omega_t} \varphi g \, dy, \quad \forall \varphi \in H_0^1(\Omega_t). \tag{41}$$

Given $\psi \in H_0^1(\Omega)$, we define $\varphi = \varphi(\cdot, t)$ by $\varphi(y, t) := \psi(T_t^{-1} y)$, $y \in \Omega_t$, $|t| \ll 1$. It holds that $\varphi(\cdot, t) \in H_0^1(\Omega_t)$ because $T_t : \overline{\Omega} \to \overline{\Omega_t}$ is bi-Lipschitz. Then (41) implies

$$\begin{aligned} &\int_\Omega [J_t(x)^{-1} \nabla v(x, t) \cdot J_t(x)^{-1} \nabla \psi(x)] \det J_t(x) \, dx \\ &\quad = \int_\Omega \psi(x) g(T_t x) \det J_t(x) \, dx \end{aligned}$$

for $J_t = J(T_t)$ and $v(x, t) = w(T_t x, t)$. Now we apply Theorem 7 to the case with

$V = H^1_0(\Omega)$, $f(t) = g(T_t x)\ \det J_t(\cdot)$, and

$$a_t(v, \psi) = \int_\Omega [J_t^{-1}\nabla v \cdot J_t^{-1}\nabla \psi]\ \det J_t\, dx.$$

From the dominated convergence theorem, the strong differentiability of $v(\cdot, t) \in H^1_0(\Omega)$ once or twice in t follows, which implies those of $u(T_t x, t)$ and $G_t(T_t x, y)$. $\square$

3.3. Euler derivative.

Theorem 15 guarantees the differentiability of $u = u(\cdot, t)$ in $\mathcal{D}'(\Omega)$ for any Lipschitz domain Ω. To prescribe the boundary value of $\dot{u}$, however, we require an additional regularity to $u = u(x)$.

THEOREM 16. *Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz domain and $G = G(\cdot, y)$, $y \in \Omega$, be the Green function for $-\Delta$ with the homogeneous Dirichlet boundary condition. Define $u = u(\cdot, t)$ by* (38). *Then, if $\{T_t\}$ is a differentiable deformation, the first variation*

$$\dot{u} = \delta G(\cdot, y) = \left.\frac{\partial G_t}{\partial t}(\cdot, y)\right|_{t=0}$$

of $G_t = G_t(x, y)$ exists in the sense of distributions in Ω. It belongs to $H^1(\Omega)$ if and only if $S \cdot \nabla u \in H^1(\Omega)$. If this condition is satisfied then it holds that $S \cdot \nabla G(\cdot, y) \in H^{1/2}(\partial\Omega)$ and

$$\Delta \dot{u} = 0 \quad \text{in } \Omega, \qquad \dot{u} = -S \cdot \nabla G(\cdot, y) \quad \text{on } \partial\Omega. \tag{42}$$

If $\{T_t\}$ is twice differentiable deformation then the second variation

$$\ddot{u} = \delta^2 G(\cdot, y) = \left.\frac{\partial^2}{\partial t^2} G_t(\cdot, y)\right|_{t=0}$$

of $G_t = G_t(x, y)$ exists in the sense of distributions in Ω. It belongs to $H^1(\Omega)$ if and only if

$$2S \cdot \nabla \dot{u} + R \cdot \nabla u + [\nabla^2 u] S \cdot S \in H^1(\Omega).$$

If this condition is satisfied then it holds that $H \in H^{1/2}(\partial\Omega)$ for

$$H = 2S \cdot \nabla \dot{u} + R \cdot \nabla G(\cdot, y) + [\nabla^2 G(\cdot, y)] S \cdot S, \tag{43}$$

and furthermore,

$$\Delta \ddot{u} = 0 \quad \text{in } \Omega, \qquad \ddot{u} = -H \quad \text{on } \partial\Omega.$$

PROOF. Let $J_t(x) = J(T_t x)$. Given $\varphi \in C_0^\infty(\Omega)$, we have

$$\int_\Omega u(z,t)\varphi(z)\,dz = \int_{\Omega_t} u(z,t)\varphi(z)\ dz = \int_\Omega u(T_t x,t)\varphi(T_t x)\det J_t(x)\,dx$$

for $|t| \ll 1$. Here, Theorem 15 guarantees the existence of, in particular, $(\partial u/\partial t)(T_t x,t)$ strongly in $L^2(\Omega)$. Since the existence of

$$\frac{\partial}{\partial t}[\varphi(T_t x)\det J_t(x)]$$

is immediate, we obtain the existence of

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u(z,t)\varphi(z)\,dz\bigg|_{t=0} \\
&\quad= \int_\Omega \left[\left(\frac{\partial u}{\partial t}(T_t x,t)\cdot\varphi(T_t x) + u(T_t x,t)\frac{\partial T_t x}{\partial t}\cdot\nabla\varphi(T_t x)\right)\det J_t(x)\right. \\
&\qquad\qquad \left. + u(T_t x,t)\varphi(T_t x)\frac{\partial}{\partial t}\det J_t(x)\right]\bigg|_{t=0} dx \\
&\quad= \int_\Omega \frac{\partial u}{\partial t}(T_t x,t)\bigg|_{t=0}\varphi + u(S\cdot\nabla\varphi + \varphi\nabla\cdot S)\,dx \\
&\quad= \int_\Omega \frac{\partial u}{\partial t}(T_t x,t)\bigg|_{t=0}\varphi + u\nabla\cdot(S\varphi)\,dx
\end{aligned}$$

by Theorem 6, recalling that S is a Lipschitz continuous vector field on $\overline{\Omega}$. Then it holds that

$$\frac{d}{dt}\int_\Omega u(z,t)\varphi(z)\,dz\bigg|_{t=0} = \int_\Omega\left[\frac{\partial u}{\partial t}(T_t x,t)\bigg|_{t=0} - S\nabla\cdot u\right]\varphi\,dx$$

by $u \in H^1(\Omega)$, which means the existence of $\dot{u} = \partial u/\partial t|_{t=0}$ in $\mathcal{D}'(\Omega)$ with

$$\dot{u} = \frac{\partial u}{\partial t}(T_t\cdot,t)\bigg|_{t=0} - S\cdot\nabla u.$$

The formal calculation

$$\frac{\partial u}{\partial t}(T_t x,t)\bigg|_{t=0} = \dot{u}(x) + S\cdot\nabla u \tag{44}$$

is thus justified in the context of distributions provided that the right-hand side belongs to $L^2(\Omega)$.

The strong differentiability of $u(T_t\cdot,t)$ in $H^1(\Omega)$, $t\in I$, implies that the trace of the left-hand side of (44) belongs to $H^{1/2}(\partial\Omega)$. Since $0 = G_t(T_t\cdot,y) \in H^{1/2}(\partial\Omega)$ it holds that

$$\frac{\partial u}{\partial t}(T_t\cdot, t)\bigg|_{t=0} = -\frac{\partial \Gamma}{\partial t}(T_t \cdot - y)\bigg|_{t=0} = -S\cdot\nabla\Gamma(\cdot - y) \quad \text{in } H^{1/2}(\partial\Omega).$$

Therefore, the above $\dot{u}$ is in $H^1(\Omega)$ if and only if $S\cdot\nabla u \in H^1(\Omega)$, and if this is the case it takes the boundary value

$$\dot{u} = S\cdot\nabla(\Gamma(\cdot - y) + u) = -S\cdot\nabla G(\cdot, y) \in H^{1/2}(\partial\Omega).$$

We thus obtain the first part of the theorem because $\Delta\dot{u} = 0$ in $\mathcal{D}'(\Omega)$ is obvious.

The existence of $\ddot{u} = \partial^2 u/\partial t^2|_{t=0}$ in $\mathcal{D}'(\Omega)$ is similarly proven if $\{T_t\}$ is twice differentiable. Let the symmetric tensor

$$\nabla^2 u = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)_{ij}$$

be the Hesse matrix of u. Then, we formally obtain the equality

$$\frac{\partial^2 u}{\partial t^2}(T_t\cdot, t)\bigg|_{t=0} = \ddot{u} + 2S\cdot\nabla\dot{u} + R\cdot\nabla u + [\nabla^2 u]S\cdot S. \tag{45}$$

In fact, we have, formally,

$$\frac{\partial^2 u}{\partial t^2}(T_t x, t) = \frac{\partial}{\partial t}\left\{\dot{u}(T_t x, t) + \frac{\partial T_t x}{\partial t}\cdot\nabla u(T_t x, t)\right\}$$

with

$$\frac{\partial \dot{u}}{\partial t}(T_t x, t) = \ddot{u}(T_t x, t) + \frac{\partial T_t x}{\partial t}\cdot\nabla\dot{u}(T_t x, t)$$

and

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[\frac{\partial T_t x}{\partial t}\cdot\nabla u(T_t x, t)\right] \\
&\quad = \frac{\partial^2 T_t x}{\partial t^2}\cdot\nabla u(T_t x, t) + \frac{\partial T_t x}{\partial t}\cdot\frac{\partial}{\partial t}\nabla u(T_t x, t) \\
&\quad = \frac{\partial^2 T_t x}{\partial t^2}\cdot\nabla u(T_t x, t) + \frac{\partial T_t x}{\partial t}\cdot\left\{\nabla\dot{u}(T_t x, t) + \frac{\partial T_t x}{\partial t}\nabla^2 u(T_t x, t)\right\} \\
&\quad = \frac{\partial^2 T_t x}{\partial t^2}\cdot\nabla u(T_t x, t) + \frac{\partial T_t}{\partial t}\cdot\nabla\dot{u}(T_t x, t) + [\nabla^2 u(T_t x, t)]\frac{\partial T_t x}{\partial t}\cdot\frac{\partial T_t x}{\partial t}.
\end{aligned}$$

Putting $t = 0$, we thus obtain (45). Next, we confirm that the last three terms on the right-hand side of (45) are regarded as distributions in Ω. In fact, first, S and R are Lipschitz continuous. Then $u \in H^1(\Omega)$ implies $R\cdot\nabla u \in L^2(\Omega)$. Second, the distributions $S\cdot\nabla\dot{u}$ and $[\nabla^2 u]S\cdot S$ are defined by

$$\langle \varphi, S\cdot\nabla\dot{u}\rangle_{1,\Omega} := -(\dot{u}, \nabla\cdot(S\varphi))_{\Omega}$$

$$\left\langle \varphi, \frac{\partial^2 u}{\partial x_i \partial x_j} S^i S^j \right\rangle_{1,\Omega} = -\left(\frac{\partial u}{\partial x_i}, \frac{\partial}{\partial x_j}(S^i S^j \varphi)\right)_{\Omega}, \quad \varphi \in C_0^{\infty}(\Omega)$$

using $\dot{u} \in L^2(\Omega)$ and $u \in H^1(\Omega)$. Hence the last three terms on the right-hand side of (45) are actually distributions. The final part is the justification of the derivation of (45) in the sense of distributions. This process is the same as in the case of the first variation. That is, taking $\varphi \in C_0^{\infty}(\Omega)$, we can show the existence of $(d^2/dt^2)\langle\varphi, u(\cdot,t)\rangle|_{t=0}$ with the equality

$$\begin{aligned} &\frac{d^2u}{dt^2}\int_{\Omega}(z,t)\varphi(z)\,dz\bigg|_{t=0} \\ &\quad = \int_{\Omega}\frac{\partial^2 u}{\partial t^2}(T_t x,t)\bigg|_{t=0}\varphi + 2\dot{u}\nabla\cdot(S\varphi) + \sum_{i,j}\frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_j}(S^iS^j\varphi)\,dx \end{aligned}$$

where $z = T_t x$.

To complete the proof, we regard $u(T_t x, t) = -\Gamma(T_t x - y)$ as a function of $x \in \partial\Omega$. Obviously, this function is twice differentiable in t strongly in $H^{1/2}(\partial\Omega)$, and it holds that

$$\begin{aligned} \frac{\partial\Gamma}{\partial t}(T_t\cdot - y)\bigg|_{t=0} &= S\cdot\nabla\Gamma(\cdot - y) \\ \frac{\partial^2\Gamma}{\partial t^2}(T_t\cdot - y)\bigg|_{t=0} &= R\cdot\nabla\Gamma(\cdot - y) + [\nabla^2\Gamma(\cdot - y)]S\cdot S. \end{aligned}$$

Since it is obvious that $\ddot{u}$ is harmonic in Ω we obtain the second part of the theorem similarly. □

REMARK. The differentiability of $G_t(\cdot, y)$, $y \in \Omega$ is valid in the weak topology of $L^2(\Omega)$, taking the 0-extension outside Ω_t. For this purpose we take the 0-extension of the test function $\varphi \in L^2(\Omega)$ outside Ω in the above proof. Natural boundary condition, however, will be difficult to be treated by extensions.

3.4. $C^{1,1}$-domain.

If $\Omega \subset \mathbb{R}^N$ is a $C^{1,1}$-domain there is a Lipschitz continuous frame (15) on $\partial\Omega$. In this case, the vector field n on $\partial\Omega$ has a Lipschitz extension on $\overline{\Omega}$. In fact, first, we extend it near $\partial\Omega$ using the flow described in Section 2.2 and then take a cut-off function to extend it on $\overline{\Omega}$.

For $v \in H^2(\Omega)$, the trace of the function $n\cdot\nabla v$ belongs to $H^{1/2}(\partial\Omega)$. By the identification given in Theorem 4 it thus holds that $\partial v/\partial n \in H^{1/2}(\partial\Omega)$. We have also

$$\frac{\partial v}{\partial s_i} = s_i\cdot\nabla v \in H^{1/2}(\partial\Omega)$$

because s_i, $1 \leq i \leq N-1$, are Lipschitz continuous. If $v \in H_0^1(\Omega)$, furthermore, we obtain

$$0 = \int_{\partial\Omega} (s_i \cdot \nabla)(v\varphi)\, ds = \int_{\partial\Omega} [(s_i \cdot \nabla)v]\varphi\, ds$$

for any $\varphi \in C_0^\infty(\omega)$ by $v|_{\partial\Omega} = 0$.[2] Hence it follows that $\partial v/\partial s_i = 0$, $i = 1, \ldots, N-1$, which implies

$$\begin{aligned} S \cdot \nabla v &= \left[(S \cdot n)n + \sum_{i=1}^{N-1} (S \cdot s_i)s_i \right] \cdot \nabla v \\ &= (S \cdot n)(n \cdot \nabla v) = \delta\rho \frac{\partial v}{\partial n} \in H^{1/2}(\partial\Omega), \quad \forall v \in H^2(\Omega) \cap H_0^1(\Omega), \end{aligned}$$

because $S \cdot \nabla v \in H^1(\Omega)$ by the Lipschitz continuity of S. This category of Ω is thus appropriate to represent the first variational formula in Euler coordinate system in $H^1(\Omega)$. In the following theorem, the paring $\langle \,\cdot\,, \cdot\, \rangle_{1/2,\partial\Omega}$ is identified with the L^2 inner product on $\partial\Omega$.

THEOREM 17 (Hadamard). *If $\Omega \subset \mathbb{R}^N$ is a $C^{1,1}$-domain then it holds that*

$$\dot{u} = \delta G(\cdot, y) \in H^1(\Omega).$$

This function is harmonic in Ω and we have

$$\delta G(x,y) = \left\langle \delta\rho \frac{\partial G(\cdot,y)}{\partial n}, \frac{\partial G(\cdot,x)}{\partial n} \right\rangle_{1/2,\partial\Omega}, \quad x, y \in \Omega. \tag{46}$$

PROOF. By the L^2 elliptic regularity on $C^{1,1}$-domain Ω, we have, for $y \in \Omega$,

$$u = G(\cdot, y) - \Gamma(\cdot - y) \in H^2(\Omega).$$

Then it follows that

$$n \cdot \nabla G(\cdot, y) = \frac{\partial G(\cdot, y)}{\partial n} \in H^{1/2}(\partial\Omega) \tag{47}$$

and hence

$$S \cdot \nabla G(\cdot, y) = \delta\rho \frac{\partial G(\cdot, y)}{\partial n} \in H^{1/2}(\partial\Omega). \tag{48}$$

Then we obtain (46) by Theorems 14 and 16. □

[2] Recall that ω is an open set which contains $\partial\Omega$. For $\varphi \in C_0^\infty(\omega)$, we consider its zero-extension and regard it as an element in $C_0^\infty(\mathbb{R}^N)$.

Remark. The notations of the above theorem are the same as those of [**8**], except for the direction of n.

The second variational formula in Theorem 16 contains $\dot{u}$ in the first term of H defined by (43). The following lemma is used to reduce this term to an integration on Ω.

Lemma 18. *If $\Omega \subset \mathbb{R}^N$ is a $C^{1,1}$-domain, then it holds that*

$$(\nabla \delta G(\cdot, x), \nabla \delta G(\cdot, y))_\Omega = -\left\langle \delta\rho \frac{\partial G(\cdot, x)}{\partial n}, \frac{\partial \delta G(\cdot, y)}{\partial n} \right\rangle_{1/2, \partial\Omega}, \quad x, y \in \Omega. \tag{49}$$

Proof. For $C^{1,1}$-domain Ω, we have $u \in H^2(\Omega)$, (47), (48), and

$$\dot{u} = \delta G(\cdot, y) \in H^1(\Omega).$$

Therefore, we have, for $\varphi \in H^1(\Omega)$ with $\Delta\varphi = 0$,

$$\begin{aligned}(\nabla \delta G(\cdot, y), \nabla\varphi)_\Omega = (\nabla \dot{u}, \nabla\varphi)_\Omega &= \left\langle \dot{u}, \frac{\partial\varphi}{\partial n} \right\rangle_{1/2, \partial\Omega} \\ &= -\left\langle \delta\rho \frac{\partial G(\cdot, y)}{\partial n}, \frac{\partial\varphi}{\partial n} \right\rangle_{1/2, \partial\Omega}.\end{aligned}$$

Putting $\varphi := \delta G(\cdot, x) \in H^1(\Omega)$, $x \in \Omega$, we obtain (49) with x and y exchanged. □

Lemma 18 may be proven by Theorems 12 and 13. For the formal proof, we assume that $g = g(x)$ is a smooth harmonic function in a domain $\widetilde{\Omega}$ which contains $\overline{\Omega}$. Then, we have

$$\int_{\Omega_t} \nabla G_t(\cdot, y) \cdot \nabla g = 0.$$

Theorem 12 implies

$$\int_\Omega \nabla \delta G(\cdot, y) \cdot \nabla g + \int_{\partial\Omega} [\nabla G(\cdot, y) \cdot \nabla g] \delta\rho = 0$$

and then (49) follows with $g = \delta G(\cdot, x)$ for $x \in \Omega$. This proof is formal because of the singularity of $G_t(\cdot, y) = G_t(x, y)$ at $x = y$, but we can justify it as follows.

The Second Proof of Lemma 18. First, we take a smooth function $g = g(x)$ in $\widetilde{\Omega}$ to derive

$$\int_{\Omega_t} \nabla u(\cdot, t) \cdot \nabla g + \int_{\partial\Omega_t} n \cdot [\Gamma(\cdot - y) \nabla g] = -\int_{\Omega_t} u(\cdot, t) \Delta g$$

by $G_t(\cdot, y) = u + \Gamma(\cdot - y)$ and $G_t(\cdot, y)|_{\partial\Omega} = 0$. Here $u(\cdot, t)$ is Lagrange differentiable in

$H^1(\Omega)$ and also $\dot{u} = \delta G(\cdot, y) \in H^1(\Omega)$ because Ω is of $C^{1,1}$. Then the proof of Theorems 12 and 13 are valid, and it holds that

$$\int_\Omega \nabla\dot{u}\cdot\nabla g + \int_{\partial\Omega}[\nabla u\cdot\nabla g]\delta\rho + \int_{\partial\Omega}[\nabla\Gamma(\cdot - y)\cdot\nabla g]\delta\rho = -\int_\Omega \dot{u}\Delta g$$

or

$$\int_\Omega \nabla\delta G(\cdot, y)\cdot\nabla g + \int_{\partial\Omega}\frac{\partial G(\cdot, y)}{\partial n}\frac{\partial g}{\partial n}\delta\rho = -\int_\Omega \delta G(\cdot, y)\Delta g. \tag{50}$$

Equality (50) is now extended to $g \in H^1(\Omega)$ with $\Delta g \in H^1(\Omega)'$. Here the right-hand side is replaced by $\langle\delta G(\cdot, y), \Delta g\rangle_{1,\Omega}$, that is,

$$(\nabla\delta G(\cdot, y), \nabla g)_\Omega + \left\langle \delta\rho\frac{\partial g}{\partial n}, \frac{\partial G(\cdot, y)}{\partial n}\right\rangle_{1/2,\partial\Omega} = -\langle\delta G(\cdot, y), \Delta g\rangle_{1,\Omega}.$$

Putting $g = \delta G(\cdot, x)$, $x \in \Omega$, we obtain (46) by $\Delta\delta G(\cdot, x) = 0$ in Ω. □

3.5. $C^{2,\theta}$-domain.

Suppose now that the domain Ω is of $C^{2,\theta}$, $0 < \theta < 1$. Since $\Gamma(\cdot - y)$ is C^∞ in ω if $y \notin \omega$, we have $u \in C^{2,\theta}(\overline{\Omega})$ and $G(\cdot, y) \in C^{2,\theta}(\overline{\Omega})$. In particular we obtain

$$\frac{\partial^2 G(\cdot, y)}{\partial n^2} = -(\nabla\cdot n)\frac{\partial G(\cdot, y)}{\partial n} \quad \text{on } \partial\Omega \tag{51}$$

by Corollary 11 and $\Delta u = 0$ in Ω. To simplify the notation, we introduce the gradient $\nabla_{\partial\Omega}$ on $\partial\Omega$ which is defined by

$$\nabla_{\partial\Omega} := \sum_{i=1}^{N-1} s_i\frac{\partial}{\partial s_i} = \nabla - n\frac{\partial}{\partial n}.$$

We also decompose S as

$$S = S_{\partial\Omega} + (\delta\rho)n, \qquad S_{\partial\Omega} := \sum_{i=1}^{N-1}\mu_i s_i, \quad \mu_i := S\cdot s_i, \quad \delta\rho := S\cdot n.$$

LEMMA 19. *If $\Omega \subset \mathbb{R}^N$ be a $C^{2,\theta}$-domain and $\{T_t\}$ be a twice differentiable deformation, then it holds that*

$$\begin{aligned} &S\cdot[\nabla^2 G(\cdot, y)]S \\ &\quad = 2\delta\rho(S_{\partial\Omega}\cdot\nabla_{\partial\Omega})\frac{\partial G(\cdot, y)}{\partial n} + \big(((S\cdot\nabla)n)\cdot S - (\delta\rho)^2\nabla\cdot n\big)\frac{\partial G(\cdot, y)}{\partial n} &(52)\\ &\quad = 2\delta\rho(S\cdot\nabla)\frac{\partial G(\cdot, y)}{\partial n} + \big(((S\cdot\nabla)n)\cdot S + (\delta\rho)^2\nabla\cdot n\big)\frac{\partial G(\cdot, y)}{\partial n} &(53)\end{aligned}$$

on $\partial\Omega$, where $\mu_i = S \cdot s_i$. If $\{s_1, \ldots, s_{N-1}, n\}$ is the Morse frame, then we have

$$S \cdot [\nabla^2 G(\cdot, y)]S = 2\delta\rho \sum_{i=1}^{N-1} \mu_i \frac{\partial^2 G(\cdot, y)}{\partial s_i \partial n} + \sum_{i=1}^{N-1} \kappa_i (\mu_i^2 - (\delta\rho)^2) \frac{\partial G(\cdot, y)}{\partial n}. \tag{54}$$

Proof. We have (51) and

$$G(\cdot, y) = \frac{\partial G(\cdot, y)}{\partial s_i} = \frac{\partial^2 G(\cdot, y)}{\partial s_i \partial s_j} = 0 \quad \text{on } \partial\Omega, \quad i, j = 1, 2, \ldots, N-1.$$

Hence, it follows from Lemma 10 that

$$\begin{aligned} \nabla^2 G(\cdot, y) &= \frac{\partial G(\cdot, y)}{\partial n} (\nabla n)^T + \sum_{i=1}^{N-1} (s_i \otimes n + n \otimes s_i) \frac{\partial^2 G(\cdot, y)}{\partial s_i \partial n} + n \otimes n \frac{\partial^2 G(\cdot, y)}{\partial n^2} \\ &= \frac{\partial G(\cdot, y)}{\partial n} ((\nabla n)^T - (\nabla \cdot n) n \otimes n) + \sum_{i=1}^{N-1} \frac{\partial^2 G(\cdot, y)}{\partial s_i \partial n} (s_i \otimes n + n \otimes s_i). \end{aligned} \tag{55}$$

Note that $[a \otimes b]c \cdot d = (a \cdot d)(b \cdot c)$ for $a, b, c, d \in \mathbb{R}^N$. Then, the relations

$$[n \otimes n]n \cdot n = 1, \quad [n \otimes n]s_k \cdot n = 0, \quad [n \otimes n]s_k \cdot s_\ell = 0$$

imply

$$(\nabla \cdot n) S \cdot [n \otimes n] S = (\delta\rho)^2 \sum_{i=1}^{N-1} \kappa_i = (\delta\rho)^2 (\nabla \cdot n)$$

with (24). Note that, from (22), we can rewrite $S \cdot (\nabla n)^T S$ as

$$\begin{aligned} S \cdot (\nabla n)^T S &= S \cdot \left(\sum_{i=1}^{N-1} \kappa_i \mu_i s_i + \sum_{i=1}^{N-1} \mu_i \sum_{j \neq i} \left(\frac{\partial n}{\partial s_i} \cdot s_j \right) s_j \right) \\ &= \sum_{i=1}^{N-1} \kappa_i \mu_i^2 + \sum_{i=1}^{N-1} \mu_i \sum_{j \neq i} \left(\frac{\partial n}{\partial s_i} \cdot s_j \right) \mu_j \\ &= \sum_{i=1}^{N-1} \mu_i \sum_{j=1}^{N-1} \left(\frac{\partial n}{\partial s_i} \cdot s_j \right) \mu_j = \left(\sum_{i=1}^{N-1} (S \cdot s_i) \frac{\partial n}{\partial s_i} \right) \cdot \sum_{j=1}^{N-1} (S \cdot s_j) s_j \\ &= ((S \cdot \nabla) n) \cdot S. \end{aligned}$$

Here, we used $(\partial n / \partial s_i) \cdot n = 0$ and $(\partial n / \partial s_i) \cdot s_i = \kappa_i$.

Next, we consider the case of $\{s_1, \ldots, s_{N-1}, n\}$ being the Morse frame. In this case, we have $(\partial n / \partial s_i) \cdot s_j = 0$, $i \neq j$, and

$$[(\nabla n)^T]n \cdot n = \sum_{i=1}^{N-1} \kappa_i [s_i \otimes s_i] n \cdot n = \sum_{i=1}^{N-1} \kappa_i (s_i \cdot n)^2 = 0,$$

$$[(\nabla n)^T]s_k \cdot n = \sum_{i=1}^{N-1} \kappa_i [s_i \otimes s_i] s_k \cdot n = \sum_{i=1}^{N-1} \kappa_i (s_i \cdot s_k)(s_i \cdot n) = 0$$

by (22). Also, since

$$[s_i \otimes s_i] s_k \cdot s_\ell = (s_i \cdot s_k)(s_i \cdot s_\ell) = \delta_{ik}\delta_{i\ell}$$

we have

$$[(\nabla n)^T]s_k \cdot s_\ell = \sum_{i=1}^{N-1} \kappa_i [s_i \otimes s_i] s_k \cdot s_\ell = \sum_{i=1}^{N-1} \kappa_i \delta_{ik}\delta_{i\ell} = \kappa_k \delta_{k\ell},$$

and

$$S \cdot [(\nabla n)^T]S = \sum_{k,\ell=1}^{N-1} \mu_k \mu_\ell [(\nabla n)^T] s_k \cdot s_\ell = \sum_{k,\ell=1}^{N-1} \mu_k \mu_\ell \kappa_k \delta_{k\ell} = \sum_{\ell=1}^{N-1} \mu_\ell^2 \kappa_\ell.$$

For the second term of the right-hand side of (55), we use

$$[s_k \otimes n] n \cdot n = (s_k \cdot n)|n|^2 = 0, \qquad [s_k \otimes n] s_\ell \cdot n = (n \cdot s_\ell)(s_k \cdot n) = 0$$

$$[s_k \otimes n] s_\ell \cdot s_m = (n \cdot s_\ell)(s_k \cdot s_m) = 0, \qquad [s_k \otimes n] n \cdot s_\ell = (s_k \cdot s_\ell)|n|^2 = \delta_{k\ell}$$

and

$$[n \otimes s_k] n \cdot n = [n \otimes s_k] s_\ell \cdot s_m = [n \otimes s_k] n \cdot s_\ell = 0, \quad [n \otimes s_k] s_\ell \cdot n = \delta_{k\ell}.$$

Then it follows that $S \cdot [s_i \otimes n + n \otimes s_i]S = 2(\delta\rho)\mu_i$ and

$$\begin{aligned} \sum_{i=1}^{N-1} \frac{\partial^2 G(\cdot, y)}{\partial s_i \partial n} S \cdot [s_i \otimes n + n \otimes s_i] S &= 2\delta\rho \sum_{i=1}^{N-1} (S \cdot s_i) \frac{\partial^2 G(\cdot, y)}{\partial s_i \partial n} \\ &= 2\delta\rho (S_{\partial\Omega} \cdot \nabla_{\partial\Omega}) \frac{\partial G(\cdot, y)}{\partial n}. \end{aligned}$$

Gathering all the equations, we have shown that (52) and (54) hold.

To obtain (53), we only need to see that

$$2\delta\rho (S_{\partial\Omega} \cdot \nabla_{\partial\Omega}) \frac{\partial G(\cdot, y)}{\partial n} = 2\delta\rho \left(S \cdot \nabla - \delta\rho \frac{\partial}{\partial n} \right) \frac{\partial G(\cdot, y)}{\partial n}$$

$$= 2\delta\rho(S\cdot\nabla)\frac{\partial G(\cdot,y)}{\partial n} - 2(\delta\rho)^2\frac{\partial^2 G(\cdot,y)}{\partial n^2}$$
$$= 2\delta\rho(S\cdot\nabla)\frac{\partial G(\cdot,y)}{\partial n} + 2(\delta\rho)^2(\nabla\cdot n)\frac{\partial G(\cdot,y)}{\partial n},$$

because of (51). □

3.6. $C^{2,1}$-domain.

Henceforth, we assume that $\Omega \subset \mathbb{R}^N$ is a $C^{2,1}$-domain and

$$S = \left.\frac{\partial T_t}{\partial t}\right|_{t=0} \in C^{2,1}(\overline{\Omega}, \mathbb{R}^N). \tag{56}$$

In this case it holds that $u \in W^{3,p}(\Omega)$, $1 < p < \infty$, and hence $u \in C^2(\overline{\Omega})$. Thus Lemma 10 is applicable to $f = u$.

Theorem 20. *Let $\Omega \subset \mathbb{R}^N$ be a $C^{2,1}$ domain and $\{T_t\}$ be a twice differentiable deformation satisfying (56). Then we have $\ddot{u} = \delta^2 G(\cdot,y) \in H^1(\Omega)$. This function is harmonic in Ω and it holds that*

$$\delta^2 G(x,y) = -2(\nabla\delta G(\cdot,x), \nabla\delta G(\cdot,y))_\Omega + \left\langle \chi\frac{\partial G(\cdot,x)}{\partial n}, \frac{\partial G(\cdot,y)}{\partial n}\right\rangle_{1/2,\partial\Omega} \tag{57}$$

for x, $y \in \Omega$ with the Lipschitz continuous function χ defined by

$$\chi := \delta^2\rho - ((S\cdot\nabla)S)\cdot n - (\delta\rho)^2\nabla\cdot n - (S\cdot\nabla)\delta\rho + \frac{\partial(\delta\rho)^2}{\partial n}. \tag{58}$$

If $\{s_1, \dots, s_{N-1}, n\}$ is the Morse frame then χ becomes, recalling $\mu_i := S\cdot s_i$,

$$\chi := \delta^2\rho + \sum_{i=1}^{N-1}\left\{\kappa_i(\mu_i^2 - (\delta\rho)^2) - 2\mu_i\frac{\partial\delta\rho}{\partial s_i}\right\}. \tag{59}$$

Proof. From the assumption we have $u \in H^3(\Omega)$ and also $S\cdot\nabla u \in H^2(\Omega)$ by (56). Hence it holds that

$$S\cdot\nabla G(\cdot,y) = \delta\rho\frac{\partial G(\cdot,y)}{\partial n} \in H^{3/2}(\partial\Omega)$$

which means the existence of $g \in H^2(\Omega)$ such that $g = S\cdot\nabla G(\cdot,y)$ on $\partial\Omega$. Then the elliptic regularity applied to (42) implies

$$\dot{u} = \delta G(\cdot,y) \in H^2(\Omega).$$

Thus we obtain $S\cdot\nabla\dot{u} \in H^1(\Omega)$, and therefore, we have

$$S \cdot \nabla \delta G(\cdot, y) = (S_{\partial\Omega} \cdot \nabla_{\partial\Omega}) \delta G(\cdot, y) + \delta\rho \frac{\partial \delta G(\cdot, y)}{\partial n} \quad \text{on } \partial\Omega,$$

and is in $H^{1/2}(\partial\Omega)$.

We have also $R \cdot \nabla u \in H^2(\Omega)$ and hence,

$$R \cdot \nabla G(\cdot, y) = \delta^2 \rho \, \frac{\partial G(\cdot, y)}{\partial n} \quad \text{on } \partial\Omega,$$

and is in $H^{3/2}(\partial\Omega)$. It thus follows from Theorem 16 that

$$\begin{aligned} \ddot{u} = -H = -2\bigg((S_{\partial\Omega} \cdot \nabla_{\partial\Omega}) \delta G(\cdot, y) + \delta\rho (S_{\partial\Omega} \cdot \nabla_{\partial\Omega}) \frac{\partial G(\cdot, y)}{\partial n} \bigg) - 2\delta\rho \frac{\partial \delta G(\cdot, y)}{\partial n} \\ - \big[\delta^2 \rho + ((S \cdot \nabla) n) \cdot S - (\delta\rho)^2 \nabla \cdot n \big] \frac{\partial G(\cdot, y)}{\partial n} \quad \text{on } \partial\Omega. \end{aligned}$$

Since

$$\delta G(\cdot, y) = \dot{u} = -\delta\rho \frac{\partial G(\cdot, y)}{\partial n} \quad \text{on } \partial\Omega,$$

we have

$$(S_{\partial\Omega} \cdot \nabla_{\partial\Omega}) \delta G(\cdot, y) + \delta\rho (S_{\partial\Omega} \cdot \nabla_{\partial\Omega}) \frac{\partial G(\cdot, y)}{\partial n} = -\big[(S_{\partial\Omega} \cdot \nabla_{\partial\Omega}) \delta\rho \big] \frac{\partial G(\cdot, y)}{\partial n}$$

on $\partial\Omega$. Note that we see

$$(S_{\partial\Omega} \cdot \nabla_{\partial\Omega}) \delta\rho = (S - (\delta\rho) n) \cdot \nabla \delta\rho = (S \cdot \nabla) \delta\rho - \frac{1}{2} \frac{\partial (\delta\rho)^2}{\partial n}$$

and

$$(S \cdot \nabla) \delta\rho = (S \cdot \nabla)(S \cdot n) = ((S \cdot \nabla) S) \cdot n + ((S \cdot \nabla) n) \cdot S.$$

Thus, we end up with

$$\ddot{u} = -2\delta\rho \frac{\partial \dot{u}}{\partial n} - \chi \frac{\partial G(\cdot, y)}{\partial n} \quad \text{on } \partial\Omega$$

with (58). The proof is, therefore, completed by Theorem 14 and Lemma 18. For the case of $\{s_1, \ldots, s_{N-1}, n\}$ being the Morse frame, the proof is done quite similarly from (54). □

An immediate consequence is the following.

Corollary 21 (Garabedian–Schiffer [**9**]). *Let $\{T_t\}$ be a family of normal perturbations associated with $\delta\rho = S \cdot n$. Then it holds that*

$$\delta^2 G(x,y) = -2(\nabla\delta G(\cdot,x), \nabla\delta G(\cdot,y))_\Omega - \left\langle (\nabla\cdot n)(\delta\rho)^2 \frac{\partial G(\cdot,x)}{\partial n}, \frac{\partial G(\cdot,y)}{\partial n} \right\rangle_{1/2,\partial\Omega}, \quad x,y\in\Omega \tag{60}$$

under the assumption of Theorem 20.

Proof. The formula follows from the previous theorem because it holds that $\delta^2\rho = 0$ and $\mu_i = S\cdot s_i = 0$, $i = 1,2,\dots,N-1$ in this case. □

For the case the dynamical perturbation, we have the following corollary.

Corollary 22. *Let $\{T_t\}$ be a family of dynamical perturbations associated with the vector field v. Then it holds that*

$$\delta^2 G(x,y) = -2(\nabla\delta G(\cdot,x), \nabla\delta G(\cdot,y))_\Omega - \left\langle \sigma \frac{\partial G(\cdot,x)}{\partial n}, \frac{\partial G(\cdot,y)}{\partial n} \right\rangle_{1/2,\partial\Omega},$$

$$\sigma := (v\cdot n)^2\nabla\cdot n - \frac{\partial (v\cdot n)^2}{\partial n} + (v\cdot\nabla)(v\cdot n), \quad x,y\in\Omega \tag{61}$$

under the assumption of Theorem 20.

Proof. In this case we have $(S\cdot\nabla)S = R$ so that

$$\chi = -(\delta\rho)^2\nabla\cdot n + \frac{\partial(\delta\rho)^2}{\partial n} - (S\cdot\nabla)\delta\rho.$$

Then (61) follows from (58) with $S = v$ and $\delta\rho = v\cdot n$. □

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References

[1] R. A. Adams, Sobolev Spaces, Academic Press, 1978.
[2] H. Azegami and K. Takeuchi, A smoothing method for shape optimization: traction method using the Robin condition, Intern. J. Comp. Methods, **3** (2006). 21–33.
[3] S. C. Brenner and L. R. Scott, The Mathematical Theory of Finite Element Methods, third edition, Springer-Verlag, 2007.
[4] R. Courant, Dirichlet's Principle, Conformal Mappings, and Minimal Surfaces, Springer-Verlag, 1970.
[5] M. Dauge, Elliptic Boundary Value Problems on Corner Domains, Lecture Notes in Math., **1341**, Springer-Verlag, 1988.
[6] L. C. Evans and R. F. Gariepy, Measure Theory and Fine Properties of Functions, CRC Press, 1992.
[7] H. Fujita, N. Saito and T. Suzuki, Operator Theory and Numerical Methods, North-Holland, 2001.
[8] P. R. Garabedian, Partial Differential Equations, second edition, Chelsea, 1986.
[9] P. R. Garabedian and M. Schiffer, Convexity of domain functionals, J. Anal. Math., **2** (1952–53), 281–368.

[10] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, second edition, Springer-Verlag, 1983.
[11] V. Girault and P.-A. Raviart, Finite Element Methods for Navier-Stokes Equations, Springer-Verlag, 1986.
[12] P. Grisvard, Elliptic Problems in Nonsmooth Domains, Pitman, 1985.
[13] V. A. Kondrat'ev, Boundary Problems for Elliptic Equations in Domains with Conical or Angular Points, Trans. Mosc. Math. Soc., **16** (1967), 227–313.
[14] Y. Kotera, Hadamard variational formula for general domain perturbations (in Japanese), master's thesis, Osaka University, 2013. 3.
[15] H. Kozono and E. Ushikoshi, Hadamard variational formula for the Green's function of the boundary value problem on the Stokes equations, Arch. Rational Mech. Anal., **208** (2013), 1005–1055.
[16] A. Kufner, O. John and S. Fučík, Function Spaces, Academia, Prague, 1977.
[17] N. G. Meyers, An L^p-estimate for the gradient of solutions of second order elliptic divergence equations, Ann. Scoul. Norm. Sup. Pisa, III, **17** (1963), 189–206.
[18] J. Milnor, Morse Theory, Princeton Univ. Press, 1963.
[19] S. Mizohata, The Theory of Partial Differential Equations, Cambridge Univ. Press, 1973.
[20] J. Nečas, Les Méthodes Directes en Théorie des Equations Elliptiques, Masson, 1967.
[21] J. Nečas, Introduction to the Theory of Nonlinear Elliptic Equations, Teubner, 1983.
[22] N. L. Othman, T. Suzuki and T. Tsuchiya, A combined scheme for computing numerical solutions of a free boundary problem, Inter. J. Math. Comp. Simu., **1** (2011), 53–60.
[23] N. Saito and H. Fujita, Remarks on traces of H^1-functions defined in a domain with corners, J. Math. Sci. Univ. Tokyo, **7** (2000), 325–345.
[24] C. G. Simader, On Dirichlet's Boundary Value Problem, Lecture Notes in Math., **268**, Springer-Verlag, 1972.
[25] T. Suzuki, Full-discrete finite element approximation of evolution equation $u_t + A(t)u = 0$ of parabolic type, J. Fac. Sci. Univ. Tokyo, Sec. IA, **29** (1982), 195–240.
[26] T. Suzuki and T. Senba, Applied Analysis – Mathematical Methods for Natural Science, second edition, Imperial College Press, 2011.
[27] T. Suzuki and T. Tsuchiya, Convergence analysis of trial free boundary methods for the two-dimensional filtration problem, Numer. Math., **100** (2005), 537–564.
[28] T. Suzuki and T. Tsuchiya, Weak formulation of Hadamard variation applied to the filtration problem, Japan J. Indus. Appl. Math., **28** (2011), 327–350.
[29] E. Ushikoshi, New approach to the Hadamard variational formula for the Green function of the Stokes equations, Manuscripta Math., **146** (2015), 85–106.

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