

Minkowski sum of polytopes and its normality

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(Received Aug. 8, 2014)

(Revised Oct. 31, 2014)

Abstract. In this paper, we consider the normality or the integer decomposition property (IDP, for short) for Minkowski sums of integral convex polytopes. We discuss some properties on the toric rings associated with Minkowski sums of integral convex polytopes. We also study Minkowski sums of edge polytopes and give a sufficient condition for Minkowski sums of edge polytopes to have IDP.

1. Introduction.

Normality and integer decomposition property are quite important properties on not only integral convex polytopes but also polytopal affine semigroup rings and toric varieties (consult, e.g., [1], [2], [3]).

First of all, let us recall some definitions related to integral convex polytopes. Let $\mathcal{P} \subset \mathbb{R}^N$ be an *integral* convex polytope, which is a convex polytope all of whose vertices have integer coordinates. Let $L(\mathcal{P}) \subset \mathbb{Z}^N$ be the sublattice of \mathbb{Z}^N generated by the differences of the elements of $\mathcal{P} \cap \mathbb{Z}^N$, i.e., $L(\mathcal{P}) = \{\sum_{v \in \mathcal{P} \cap \mathbb{Z}^N} z_v(v - v_0) : z_v \in \mathbb{Z}\}$, where v_0 is some vertex of \mathcal{P} . For a given integer point $w \in \mathbb{Z}^N$, let $w + L(\mathcal{P}) = \{w + \alpha : \alpha \in L(\mathcal{P})\}$.

- (See [4, Definition 1.1].) We say that \mathcal{P} is *normal* if for any integer $k = 1, 2, \dots$ and $\alpha \in k\mathcal{P} \cap (kv_0 + L(\mathcal{P}))$, where $k\mathcal{P} = \{k\alpha : \alpha \in \mathcal{P}\}$ and v_0 is some integer point $v_0 \in \mathcal{P} \cap \mathbb{Z}^N$, there exist $\alpha_1, \dots, \alpha_k$ belonging to $\mathcal{P} \cap \mathbb{Z}^N$ such that $\alpha = \alpha_1 + \dots + \alpha_k$.
- We say that \mathcal{P} has the *integer decomposition property* (IDP, for short) if for any integer $k = 1, 2, \dots$ and $\alpha \in k\mathcal{P} \cap \mathbb{Z}^N$, there exist $\alpha_1, \dots, \alpha_k$ belonging to $\mathcal{P} \cap \mathbb{Z}^N$ such that $\alpha = \alpha_1 + \dots + \alpha_k$. Thus, if \mathcal{P} has IDP, then \mathcal{P} is normal. What \mathcal{P} has IDP is also said to be what \mathcal{P} is *integrally closed* (see, e.g., [4]). It is well-known that \mathcal{P} always has IDP when $\dim \mathcal{P} \leq 2$.
- For some subsets A_1, \dots, A_m of \mathbb{R}^N , let $A_1 + \dots + A_m = \{\sum_{i=1}^m a_i : a_i \in A_i, 1 \leq i \leq m\}$. This set $A_1 + \dots + A_m$ is called the *Minkowski sum* of A_1, \dots, A_m . Note that when $\mathcal{P} \subset \mathbb{R}^N$ is a convex set, the Minkowski sum $\underbrace{\mathcal{P} + \dots + \mathcal{P}}_m$ of m copies of \mathcal{P} coincides with the dilation $m\mathcal{P}$. Hence, taking Minkowski sum of convex polytopes can be regarded as a generalization of a dilation of a convex polytope.

2010 *Mathematics Subject Classification.* Primary 52B20; Secondary 14M25, 13F99.

Key Words and Phrases. Minkowski sum, normal, integer decomposition property, edge polytope.

The author is partially supported by JSPS Research Fellowship for Young Scientists and by JSPS Grant-in-Aid for Young Scientists (B) (No. 26800015).

Let K be a field and R the K -algebra $K[\mathbf{x}^\pm, t] = K[x_1, x_1^{-1}, \dots, x_N, x_N^{-1}, t]$. For $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{Z}^N$, let \mathbf{x}^α denote the Laurent monomial $x_1^{\alpha_1} \cdots x_N^{\alpha_N} \in R$. Given an integral convex polytope $\mathcal{P} \subset \mathbb{R}^N$, we define two K -algebras $K[\mathcal{P}]$ and $\mathcal{E}_K(\mathcal{P})$ as follows.

1. Let $K[\mathcal{P}] \subset R$ be the K -algebra generated by $\{\mathbf{x}^\alpha t : \alpha \in \mathcal{P} \cap \mathbb{Z}^N\}$, that is,

$$K[\mathcal{P}] = K[\mathbf{x}^\alpha t : \alpha \in \mathcal{P} \cap \mathbb{Z}^N].$$

2. Let $\mathcal{E}_K(\mathcal{P}) \subset R$ be the K -algebra defined by

$$\mathcal{E}_K(\mathcal{P}) = K[\mathbf{x}^\alpha t^n : \alpha \in n\mathcal{P} \cap \mathbb{Z}^N, n \in \mathbb{Z}_{\geq 0}].$$

These algebras are finitely generated graded K -algebras, where their grading is defined by $\deg(\mathbf{x}^\alpha t^n) = n$ for $\alpha \in n\mathcal{P} \cap \mathbb{Z}^N$. We call $K[\mathcal{P}]$ the *toric ring* of \mathcal{P} and $\mathcal{E}_K(\mathcal{P})$ the *Ehrhart ring* of \mathcal{P} . Notice that \mathcal{P} is normal if and only if so is $K[\mathcal{P}]$, and \mathcal{P} has IDP if and only if $K[\mathcal{P}] = \mathcal{E}_K(\mathcal{P})$.

In the outstanding paper [2], Bruns, Gubeladze and Trung proved the following.

THEOREM 1.1 ([2, Theorem 1.3.3]). *Let \mathcal{P} be an integral convex polytope of dimension d . Then the following hold:*

- (a) $K[n\mathcal{P}]$ is normal if $n \geq d - 1$;
- (b) $K[n\mathcal{P}]$ is Koszul if $n \geq d$;
- (c) $K[n\mathcal{P}]$ is level of a -invariant -1 if $n \geq d + 1$.

This theorem concerns the “dilation” of a polytope. Hence, it is natural to think of whether we can extend those results to the “Minkowski sum” of polytopes. In this paper, we extend Theorem 1.1 (a) and (c) to the Minkowski sum of polytopes with a slight modification. More precisely, we prove that for integral convex polytopes $\mathcal{P}_1, \dots, \mathcal{P}_m \subset \mathbb{R}^N$ of dimension d_1, \dots, d_m , respectively, $K[n_1\mathcal{P}_1 + \cdots + n_m\mathcal{P}_m]$ is normal if $n_i \geq d_i$ for each $1 \leq i \leq m$ and is level of a -invariant -1 if $n_i \geq d_i + 1$ for each $1 \leq i \leq m$ (Theorem 2.1).

We are also, in particular, interested in the following.

PROBLEM 1.2. For integral convex polytopes $\mathcal{P}_1, \dots, \mathcal{P}_m \subset \mathbb{R}^N$, when is $\mathcal{P}_1 + \cdots + \mathcal{P}_m$ normal? Or, when does $\mathcal{P}_1 + \cdots + \mathcal{P}_m$ have IDP?

Of course, Theorem 2.1 (a) gives one solution. Moreover, if $\mathcal{P}_1 = \cdots = \mathcal{P}_m$ and $m \geq \dim \mathcal{P}_1 - 1$, then $\mathcal{P}_1 + \cdots + \mathcal{P}_m = m\mathcal{P}_1$ is normal by Theorem 1.1 (a). However, other related results on normality or IDP for Minkowski sums of integral convex polytopes are not so known. Furthermore, as is shown by the following example, Minkowski sums of integral convex polytopes having IDP are not necessarily normal.

EXAMPLE 1.3 (See [5, p. 2315]). Let $\mathcal{P}_1 = \text{conv}(\{(0, 0, 0), (1, 0, 0), (0, 1, 0)\}) \subset \mathbb{R}^3$ and $\mathcal{P}_2 = \text{conv}(\{(0, 0, 0), (1, 1, 3)\}) \subset \mathbb{R}^3$. Since each of $\dim \mathcal{P}_1$ and $\dim \mathcal{P}_2$ is at most 2, each of \mathcal{P}_1 and \mathcal{P}_2 has IDP. However, we see that $\mathcal{P}_1 + \mathcal{P}_2$ is not normal. In particular, this does not have IDP.

On the other hand, it is obvious from the definition that for an integral convex polytope \mathcal{P} having IDP, $n\mathcal{P}$ always has IDP for every $n \in \mathbb{Z}_{>0}$. In addition, when we dilate an integral convex polytope of dimension d by at least $(d - 1)$ times, the dilated polytope always has IDP (see [3, Section 2.2]). In this paper, for the development of the study of IDP for Minkowski sums of integral convex polytopes, we investigate the Minkowski sum of integral convex polytopes arising from graphs, called *edge polytopes* (see Section 3). We give a sufficient condition for Minkowski sums of edge polytopes to have IDP (Theorem 3.4).

A brief organization of this paper is as follows. First, in Section 2, we prove an extended version of Theorem 1.1 (a) and (c) (Theorem 2.1, which is the main result of this paper). Next, in Section 3, we study Minkowski sums of edge polytopes. After we recall some notions and definitions on graphs, we define the edge polytope and discuss the dimension of Minkowski sum of edge polytopes (Proposition 3.1) in Section 3.1. We also give a sufficient condition for Minkowski sums of edge polytopes to have IDP (Theorem 3.4) in Section 3.2. Finally, we give some examples concerning Theorem 3.4 in Section 3.3.

2. Toric rings of Minkowski sums of polytopes.

In this section, we extend Theorem 1.1 (a) and (c) from “dilations” of integral convex polytopes to “Minkowski sums”.

The main theorem of this paper is the following.

THEOREM 2.1. *Let $\mathcal{P}_1, \dots, \mathcal{P}_m \subset \mathbb{R}^N$ be integral convex polytopes and let $d_i = \dim \mathcal{P}_i$ for $1 \leq i \leq m$. Given positive integers n_1, \dots, n_m , the following hold:*

- (a) $n_1\mathcal{P}_1 + \dots + n_m\mathcal{P}_m$ has IDP (in particular, $K[n_1\mathcal{P}_1 + \dots + n_m\mathcal{P}_m]$ is normal) if $n_i \geq d_i$ for each $1 \leq i \leq m$;
- (b) $K[n_1\mathcal{P}_1 + \dots + n_m\mathcal{P}_m]$ is level of a -invariant -1 if $n_i \geq d_i + 1$ for each $1 \leq i \leq m$.

For the proof of this theorem, we prove Lemma 2.3.

For $A \subset \mathbb{R}^N$, let $\text{int}(A)$ denote the relative interior of A with respect to the affine subspace of \mathbb{R}^N spanned by A .

LEMMA 2.2 (cf. [7, Section 1]). *Let $\mathcal{P}_1, \dots, \mathcal{P}_m \subset \mathbb{R}^N$ be convex polytopes. Then one has*

$$\text{int}(\mathcal{P}_1 + \dots + \mathcal{P}_m) = \text{int}(\mathcal{P}_1) + \dots + \text{int}(\mathcal{P}_m).$$

For the proof of this lemma, it is enough to show the case $m = 2$, that is, $\text{int}(\mathcal{P}_1) + \text{int}(\mathcal{P}_2) = \text{int}(\mathcal{P}_1 + \mathcal{P}_2)$. The inclusion $\text{int}(\mathcal{P}_1 + \mathcal{P}_2) \subset \text{int}(\mathcal{P}_1) + \text{int}(\mathcal{P}_2)$ directly follows from [7, Lemma 1.3.12] and the other follows from [7, Theorem 1.1.14].

LEMMA 2.3. *Work with the same notation as in Theorem 2.1.*

- (a) *If $n_i \geq d_i + 1$ for each i , then we have*

$$\begin{aligned}
 & (n_1\mathcal{P}_1 + \cdots + n_m\mathcal{P}_m) \cap \mathbb{Z}^N \\
 &= (d_1\mathcal{P}_1 + \cdots + d_m\mathcal{P}_m) \cap \mathbb{Z}^N + \underbrace{\sum_{i=1}^m (\mathcal{P}_i \cap \mathbb{Z}^N + \cdots + \mathcal{P}_i \cap \mathbb{Z}^N)}_{n_i - d_i}. \tag{2.1}
 \end{aligned}$$

(b) If $n_i \geq d_i + 2$ for each i , then we have

$$\begin{aligned}
 & \text{int}(n_1\mathcal{P}_1 + \cdots + n_m\mathcal{P}_m) \cap \mathbb{Z}^N \\
 &= \text{int}((d_1 + 1)\mathcal{P}_1 + \cdots + (d_m + 1)\mathcal{P}_m) \cap \mathbb{Z}^N + \underbrace{\sum_{i=1}^m (\mathcal{P}_i \cap \mathbb{Z}^N + \cdots + \mathcal{P}_i \cap \mathbb{Z}^N)}_{n_i - d_i - 1}. \tag{2.2}
 \end{aligned}$$

PROOF. (a) Let $\alpha \in (n_1\mathcal{P}_1 + \cdots + n_m\mathcal{P}_m) \cap \mathbb{Z}^N$. Then there is $w_i \in n_i\mathcal{P}_i$ for each i such that $\alpha = w_1 + \cdots + w_m$. By Carathéodory's Theorem (cf. [8, Corollary 7.1i]), there are $(d_i + 1)$ affinely independent vertices $v_0^{(i)}, v_1^{(i)}, \dots, v_{d_i}^{(i)} \in \mathcal{P}_i \cap \mathbb{Z}^N$ of \mathcal{P}_i such that $w_i = \sum_{j=0}^{d_i} r_j^{(i)} v_j^{(i)}$, where $r_j^{(i)} \geq 0$ and $\sum_{j=0}^{d_i} r_j^{(i)} = n_i$. Thus, α can be written like

$$\alpha = \sum_{j=0}^{d_1} r_j^{(1)} v_j^{(1)} + \cdots + \sum_{j=0}^{d_m} r_j^{(m)} v_j^{(m)}.$$

Since $n_i \geq d_i + 1$ and $\sum_{j=0}^{d_i} r_j^{(i)} = n_i$ for each i , there is an index k_i such that $r_{k_i}^{(i)} \geq 1$. Then α can be decomposed like $\alpha' + \beta^{(1)} + \cdots + \beta^{(m)}$, where $\alpha' \in ((n_1 - 1)\mathcal{P}_1 + \cdots + (n_m - 1)\mathcal{P}_m) \cap \mathbb{Z}^N$ and $\beta^{(i)} \in \mathcal{P}_i \cap \mathbb{Z}^N$. Since we can do this decomposition whenever $n_i \geq d_i + 1$, we conclude that α belongs to the right-hand side of (2.1). This shows one inclusion. On the other hand, another inclusion is easy to see.

(b) Let $\alpha \in \text{int}(n_1\mathcal{P}_1 + \cdots + n_m\mathcal{P}_m) \cap \mathbb{Z}^N$. By Lemma 2.2, we have $\text{int}(n_1\mathcal{P}_1 + \cdots + n_m\mathcal{P}_m) = \text{int}(n_1\mathcal{P}_1) + \cdots + \text{int}(n_m\mathcal{P}_m)$. Thus, there is $w_i \in \text{int}(n_i\mathcal{P}_i)$ for each i such that $\alpha = w_1 + \cdots + w_m$. Then there are $(d'_i + 1)$ affinely independent vertices $v_0^{(i)}, v_1^{(i)}, \dots, v_{d'_i}^{(i)} \in \mathcal{P}_i \cap \mathbb{Z}^N$ of \mathcal{P}_i such that $w_i = \sum_{j=0}^{d'_i} r_j^{(i)} v_j^{(i)}$, where $d'_i \leq d_i$, $r_j^{(i)} > 0$ and $\sum_{j=0}^{d'_i} r_j^{(i)} = n_i$. Thus, α can be written like

$$\alpha = \sum_{j=0}^{d'_1} r_j^{(1)} v_j^{(1)} + \cdots + \sum_{j=0}^{d'_m} r_j^{(m)} v_j^{(m)}.$$

Since $n_i \geq d_i + 2 \geq d'_i + 2$ and $\sum_{j=0}^{d'_i} r_j^{(i)} = n_i$ for each i , there is an index k_i such that $r_{k_i}^{(i)} > 1$. Then α can be decomposed like $\alpha' + \beta^{(1)} + \cdots + \beta^{(m)}$, where $\alpha' \in \text{int}((n_1 - 1)\mathcal{P}_1 + \cdots + (n_m - 1)\mathcal{P}_m) \cap \mathbb{Z}^N$ and $\beta^{(i)} \in \mathcal{P}_i \cap \mathbb{Z}^N$. Since we can do this decomposition whenever $n_i \geq d_i + 2$, we conclude that α belongs to the right-hand side of (2.2). This shows one inclusion. Another inclusion also follows easily. \square

PROOF OF THEOREM 2.1. (a) Let $\alpha \in n(n_1\mathcal{P}_1 + \cdots + n_m\mathcal{P}_m) \cap \mathbb{Z}^N$ with $n \geq 2$.

Then $\alpha \in (nn_1\mathcal{P}_1 + \dots + nn_m\mathcal{P}_m) \cap \mathbb{Z}^N$ and $nn_i \geq d_i + 1$ for each i . By Lemma 2.3 (a), α can be written like $\alpha' + \sum_{i=1}^m \sum_{j=1}^{nn_i-d_i} \beta_j^{(i)}$, where $\alpha' \in (d_1\mathcal{P}_1 + \dots + d_m\mathcal{P}_m) \cap \mathbb{Z}^N$ and $\beta_j^{(i)} \in \mathcal{P}_i \cap \mathbb{Z}^N$. Since $n_i \geq d_i$, it is easy to see that $\alpha' + \sum_{i=1}^m \sum_{j=1}^{nn_i-d_i} \beta_j^{(i)}$ can be decomposed into n integer points belonging to $(n_1\mathcal{P}_1 + \dots + n_m\mathcal{P}_m) \cap \mathbb{Z}^N$. This implies that $n_1\mathcal{P}_1 + \dots + n_m\mathcal{P}_m$ has IDP.

(b) It is enough to show that for any $\alpha \in n \operatorname{int}(n_1\mathcal{P}_1 + \dots + n_m\mathcal{P}_m) \cap \mathbb{Z}^N$ with $n \geq 2$, α can be written like $\alpha = \beta + \beta_1 + \dots + \beta_{n-1}$, where $\beta \in \operatorname{int}(n_1\mathcal{P}_1 + \dots + n_m\mathcal{P}_m) \cap \mathbb{Z}^N$ and $\beta_1, \dots, \beta_{n-1} \in (n_1\mathcal{P}_1 + \dots + n_m\mathcal{P}_m) \cap \mathbb{Z}^N$.

Given $n \geq 2$, let $\alpha \in n \operatorname{int}(n_1\mathcal{P}_1 + \dots + n_m\mathcal{P}_m) \cap \mathbb{Z}^N$. Then we have $n \operatorname{int}(n_1\mathcal{P}_1 + \dots + n_m\mathcal{P}_m) = \operatorname{int}(nn_1\mathcal{P}_1 + \dots + nn_m\mathcal{P}_m)$ by Lemma 2.2. Moreover, by Lemma 2.3 (b), α can be expressed like $\alpha' + \sum_{i=1}^m \sum_{j=1}^{nn_i-d_i-1} \beta_j^{(i)}$, where $\alpha' \in \operatorname{int}((d_1+1)\mathcal{P}_1 + \dots + (d_m+1)\mathcal{P}_m) \cap \mathbb{Z}^N$ and $\beta_j^{(i)} \in \mathcal{P}_i \cap \mathbb{Z}^N$. Let $\alpha'' = \alpha' + \sum_{i=1}^m \sum_{j=1}^{n_i-d_i-1} \beta_j^{(i)}$. Then we have $\alpha'' \in \operatorname{int}(n_1\mathcal{P}_1 + \dots + n_m\mathcal{P}_m) \cap \mathbb{Z}^N$ and α can be rewritten like $\alpha = \alpha'' + \sum_{k=1}^{n-1} (\gamma_1^{(k)} + \dots + \gamma_m^{(k)})$, where $\gamma_i^{(k)} \in n_i\mathcal{P}_i \cap \mathbb{Z}^N$. \square

REMARK 2.4. For Theorem 2.1 (a), if there is i such that $n_i = d_i - 1$, then Theorem 2.1 (a) is no longer true. For example, let $\mathcal{P}_1 = \operatorname{conv}(\{v_1, v_2, v_3\})$ and $\mathcal{P}_2 = \operatorname{conv}(\{v_4, v_5, v_6\})$, where

$$\begin{aligned} v_1 &= (1, 1, 0, 0, 0, 0), & v_2 &= (0, 0, 1, 1, 0, 0), & v_3 &= (0, 0, 0, 0, 1, 1), \\ v_4 &= (1, 0, 0, 0, 0, 1), & v_5 &= (0, 1, 1, 0, 0, 0) & \text{and } v_6 &= (0, 0, 0, 1, 1, 0). \end{aligned}$$

Then each of $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^6$ is of dimension 2. We consider $n\mathcal{P}_1 + \mathcal{P}_2$, where $n \geq 1$. Then we have

$$\left(\frac{6n-2}{3}v_1 + \frac{1}{3}v_2 + \frac{1}{3}v_3 \right) + \frac{2}{3}(v_4 + v_5 + v_6) = (2n, 2n, 1, 1, 1, 1) \in 2(n\mathcal{P}_1 + \mathcal{P}_2) \cap \mathbb{Z}^6.$$

Moreover, one sees that

$$\begin{aligned} (n\mathcal{P}_1 + \mathcal{P}_2) \cap \mathbb{Z}^6 &= \underbrace{(\mathcal{P}_1 \cap \mathbb{Z}^6 + \dots + \mathcal{P}_1 \cap \mathbb{Z}^6)}_n + \mathcal{P}_2 \cap \mathbb{Z}^6 \\ &\cup (\{(1, 1, 1, 1, 1, 1)\} + \underbrace{\mathcal{P}_1 \cap \mathbb{Z}^6 + \dots + \mathcal{P}_1 \cap \mathbb{Z}^6}_{n-2}) \end{aligned}$$

when $n \geq 2$. Hence, $(2n, 2n, 1, 1, 1, 1) \in 2(n\mathcal{P}_1 + \mathcal{P}_2) \cap \mathbb{Z}^6$ cannot be written as a sum of any two integer points contained in $(n\mathcal{P}_1 + \mathcal{P}_2) \cap \mathbb{Z}^6$. Namely, $n\mathcal{P}_1 + \mathcal{P}_2$ does not have IDP.

For Theorem 1.1 (b), we remain the following.

QUESTION 2.5. Work with the same notation as in Theorem 2.1. Is it true that $K[n_1\mathcal{P}_1 + \dots + n_m\mathcal{P}_m]$ is Koszul or the defining ideal (toric ideal) of $K[n_1\mathcal{P}_1 + \dots + n_m\mathcal{P}_m]$ has a quadratic Gröbner basis if $n_i \geq d_i$ for each $1 \leq i \leq m$?

3. Minkowski sum of edge polytopes.

In this section, we study IDP of Minkowski sums of edge polytopes. After fixing our notation on simple graphs, we define the edge polytope and study the dimension of Minkowski sum of edge polytopes (Proposition 3.2). We also consider the problem when the Minkowski sum of edge polytopes has IDP (Theorem 3.4). Finally, we supply some examples of graphs which show that the conditions described in Theorem 3.4 are necessary for Minkowski sums of edge polytopes to have IDP.

3.1. Dimension of Minkowski sum of edge polytopes.

Let G be a simple graph on the vertex set $V(G)$ with the edge set $E(G)$. Throughout this paper, we always assume that graphs are simple, so we omit to say “simple”. We recall several terminologies on graphs.

- A graph G is called *bipartite* if $V(G)$ can be decomposed into two non-empty subsets U and V of $V(G)$ such that $V(G) = U \cup V$, $U \cap V = \emptyset$ and every edge $\{i, j\} \in E(G)$ belongs to $U \times V$. We also call this partition $U \cup V$ the *partition* of the bipartite graph G .
- A sequence v_0, v_1, \dots, v_k of vertices in G is called a *walk* if $\{v_{i-1}, v_i\} \in E(G)$ for each $1 \leq i \leq k$. A walk is called a *path* if v_i 's are all distinct. Moreover, a walk is called a *cycle* if v_0, v_1, \dots, v_{k-1} are distinct and $v_0 = v_k$.
- The length of a walk (a cycle) v_0, v_1, \dots, v_k is defined by k . A walk in G is called *odd* (resp. *even*) if its length is odd (resp. even). It is well-known that G is bipartite if and only if G has no odd cycle.
- A subgraph of G is called *spanning* if its vertex set is equal to that of G .
- A *forest* is a graph without any cycle. A *tree* is a connected forest. Note that every forest is bipartite.
- We say that G is *2-connected* if the induced subgraph with the vertex set $V(G) \setminus \{v\}$ is still connected for any vertex v of G . A subgraph is called *2-connected component* if it is a maximal 2-connected subgraph.

Let $V(G) = \{1, \dots, d\}$. For $1 \leq i \leq d$, let e_i be the i th coordinate vectors of \mathbb{R}^d . Given an edge $e = \{i, j\} \in E(G)$, let $\rho(e) \in \mathbb{R}^d$ denote the vector $e_i + e_j$. We write \mathcal{P}_G for the convex hull of the set of integer points $\{\rho(e) : e \in E(G)\}$. We call this polytope \mathcal{P}_G the *edge polytope* of G .

In [6], Ohsugi and Hibi studied some properties on edge polytopes. For example, they obtain the dimension of an edge polytope as follows.

PROPOSITION 3.1 ([6, Proposition 1.3]). *Let G be a connected graph with d vertices. Then one has*

$$\dim \mathcal{P}_G = \begin{cases} d - 2, & \text{if } G \text{ is bipartite,} \\ d - 1, & \text{if } G \text{ is non-bipartite.} \end{cases}$$

Similar to this proposition, we discuss the dimension of the Minkowski sum of some edge polytopes.

Let G_1, \dots, G_m be graphs on the same vertex set $\{1, \dots, d\}$. Let $E(G_i)$ be the edge set of G_i for each $1 \leq i \leq m$. We denote by $G_1 + \dots + G_m$ the graph on the vertex set $\{1, \dots, d\}$ with the edge set $\bigcup_{i=1}^m E(G_i)$.

PROPOSITION 3.2. *Let G_1, \dots, G_m be connected graphs on the same vertex set $\{1, \dots, d\}$. Then one has*

$$\dim(\mathcal{P}_{G_1} + \dots + \mathcal{P}_{G_m}) = \begin{cases} d - 2, & \text{if } G_1 + \dots + G_m \text{ is bipartite,} \\ d - 1, & \text{if } G_1 + \dots + G_m \text{ is non-bipartite.} \end{cases}$$

PROOF. Let $G = G_1 + \dots + G_m$ and let $\mathcal{P} = \mathcal{P}_{G_1} + \dots + \mathcal{P}_{G_m}$. Since \mathcal{P} is contained in the hyperplane defined by $\{(x_1, \dots, x_d) \in \mathbb{R}^d : \sum_{i=1}^d x_i = 2m\}$, one has $\dim \mathcal{P} \leq d - 1$. On the other hand, since each G_i is connected, each G_i contains a spanning tree. Thus, in particular, G_1 contains $(d - 1)$ edges e_1, \dots, e_{d-1} such that $\rho(e_1), \dots, \rho(e_{d-1})$ are affinely independent. Thus $d - 2 \leq \dim \mathcal{P}_{G_1} \leq \dim \mathcal{P}$. Hence, we have $d - 2 \leq \dim \mathcal{P} \leq d - 1$.

Assume that G is bipartite. Let $U \cup V$ be the partition of G . Then we see that \mathcal{P} is contained in the hyperplane defined by $\{(x_1, \dots, x_d) \in \mathbb{R}^d : \sum_{i \in U} x_i = \sum_{j \in V} x_j\}$. This implies that $\dim \mathcal{P} \leq d - 2$. Thus we obtain $\dim \mathcal{P} = d - 2$.

Assume that G is not bipartite. If there is $1 \leq i \leq m$ such that G_i is not bipartite, then $\dim \mathcal{P} \geq \dim \mathcal{P}_{G_i} = d - 1$ by Proposition 3.1. Hence, $\dim \mathcal{P} = d - 1$. If each G_i is bipartite, since G is non-bipartite, there exists an edge $f \in \bigcup_{i=2}^m E(G_i)$ such that $G_1 \cup \{f\}$ has an odd cycle. We assume that f is an edge of G_2 . Let $U_1 \cup V_1$ be the partition of G_1 . Then $f \notin U_1 \times V_1$. Thus $f \in U_1 \times U_1$ or $f \in V_1 \times V_1$, say, $f \in U_1 \times U_1$. Since G_2 is connected, there is an edge $f' \in E(G_2)$ such that $f' \notin U_1 \times U_1$. Fix some edges $f_i \in E(G_i)$ for each $3 \leq i \leq m$ and let $v = \sum_{i=3}^m \rho(f_i)$. Let e_1, \dots, e_{d-1} be edges in G_1 forming its spanning tree. We consider the d integer points

$$\rho(e_1) + \rho(f) + v, \rho(e_2) + \rho(f) + v, \dots, \rho(e_{d-1}) + \rho(f) + v, \rho(e_1) + \rho(f') + v,$$

where each of them belongs to $\mathcal{P} \cap \mathbb{Z}^d$. Let

$$v_i = \begin{cases} (\rho(e_1) + \rho(f') + v) - (\rho(e_1) + \rho(f) + v) = \rho(f') - \rho(f), & i = 1, \\ (\rho(e_i) + \rho(f) + v) - (\rho(e_1) + \rho(f) + v) = \rho(e_i) - \rho(e_1), & i = 2, \dots, d - 1. \end{cases}$$

If there is $(r_1, \dots, r_{d-1}) \in \mathbb{R}^{d-1}$ such that $\sum_{i=1}^{d-1} r_i v_i = 0$, then $r_1(\rho(f) - \rho(f')) = \sum_{i=2}^{d-1} r_i(\rho(e_i) - \rho(e_1))$. Since each $\rho(e_i) - \rho(e_1)$ is contained in the hyperplane $\mathcal{H} = \{(x_1, \dots, x_d) \in \mathbb{R}^d : \sum_{i \in U_1} x_i = \sum_{j \in V_1} x_j = 0\} \subset \mathbb{R}^d$, so should be $r_1(\rho(f) - \rho(f'))$. However, since $f \in U_1 \times U_1$ and $f' \notin U_1 \times U_1$, $\rho(f) - \rho(f')$ is never contained in \mathcal{H} . Hence $r_1 = 0$. Moreover, since $\rho(e_1), \dots, \rho(e_{d-1})$ are affinely independent, one has $r_2 = \dots = r_{d-1} = 0$. Thus $r_1 = r_2 = \dots = r_{d-1} = 0$. This implies that v_1, \dots, v_{d-1} are linearly independent. Hence $\dim \mathcal{P} \geq d - 1$. Therefore, it follows that $\dim \mathcal{P} = d - 1$, as required. \square

3.2. A sufficient condition for Minkowski sums of edge polytopes to have IDP.

In this section, we discuss the problem when the Minkowski sum of edge polytopes has IDP. Namely, we give a partial answer for Problem 1.2 in the case of edge polytopes.

We say that a connected graph G satisfies the *odd cycle condition* if for arbitrary two odd cycles C and C' in G which have no common vertex, there exists an edge of G joining some vertex of C with some vertex of C' . For the normality or IDP of edge polytopes, the following is known.

THEOREM 3.3 ([6], see also [9]). *Let G be a connected graph. Then the following four conditions are equivalent:*

- (a) \mathcal{P}_G is normal;
- (b) \mathcal{P}_G has IDP;
- (c) \mathcal{P}_G has a unimodular covering;
- (d) G satisfies the odd cycle condition.

Note that although the equivalence of (a) and (b) is not mentioned explicitly, this equivalence is essentially obtained in the proof of [6, Theorem 2.2].

In the case of Minkowski sums of edge polytopes, although it seems difficult to obtain a necessary and sufficient condition to have IDP or to be normal, we give a sufficient condition to have IDP as follows.

THEOREM 3.4. *Let G_1 be a connected graph and assume that arbitrary two odd cycles in G_1 always have a common vertex. Let G_2 be a subgraph of G_1 (not necessarily connected). Then $\mathcal{P}_{G_1} + \mathcal{P}_{G_2}$ has IDP, and thus, this is normal.*

PROOF. Let $\mathcal{P} = \mathcal{P}_{G_1} + \mathcal{P}_{G_2}$ and fix $\alpha \in k\mathcal{P} \cap \mathbb{Z}^d$ for a given positive integer k . Then α can be written like

$$\alpha = \sum_{e \in E(G_1)} r_e \rho(e) + \sum_{e' \in E(G_2)} r'_{e'} \rho(e'),$$

where $\sum_{e \in E(G_1)} r_e = \sum_{e' \in E(G_2)} r'_{e'} = k$ and $r_e \geq 0$ (resp. $r'_{e'} \geq 0$) for each $e \in E(G_1)$ (resp. $e' \in E(G_2)$).

Let $E = \{e \in E(G_1) : r_e \notin \mathbb{Z}\}$ and $E' = \{e' \in E(G_2) : r'_{e'} \notin \mathbb{Z}\}$. If $E \cap E' \neq \emptyset$, then for each $e \in E \cap E'$, we replace r_e by $\lfloor r_e \rfloor$ and $r'_{e'}$ by $r'_{e'} + r_e - \lfloor r_e \rfloor$. Then α does not change and the number of edges in $E \cap E'$ decreases. After such replacements for all elements in $E \cap E'$, we may assume that $E \cap E' = \emptyset$. Then $\sum_{e \in E(G_1)} r_e$ becomes less than or equal to k but $\sum_{e' \in E(G_2)} r'_{e'}$ becomes more than or equal to k .

Here one has

$$\alpha = \underbrace{\sum_{e \in E(G_1)} \lfloor r_e \rfloor \rho(e) + \sum_{e' \in E(G_2)} \lfloor r'_{e'} \rfloor \rho(e')}_{\in \mathbb{Z}^d} + \underbrace{\sum_{e \in E} c_e \rho(e) + \sum_{e' \in E'} c'_{e'} \rho(e')}_{\in \mathbb{Z}^d}, \tag{3.1}$$

where $c_e = r_e - \lfloor r_e \rfloor$ and $c_{e'} = r_{e'} - \lfloor r_{e'} \rfloor$. Then $0 < c_e < 1$ (resp. $0 < c_{e'} < 1$) for each $e \in E$ (resp. $e' \in E'$). Let us consider the integer point $\beta = \sum_{e \in E} c_e \rho(e) + \sum_{e' \in E'} c_{e'} \rho(e')$. Since G_2 is a subgraph of G_1 , one has $E' \subset E(G_2) \subset E(G_1)$. Thus β belongs to $q\mathcal{P}_{G_1} \cap \mathbb{Z}^d$, where $q = \sum_{e \in E} c_e + \sum_{e' \in E'} c_{e'}$.

Now, Lemma 3.5 below guarantees that β can be written like

$$\beta = \sum_{e \in E} a_e \rho(e) + \sum_{e' \in E'} a'_{e'} \rho(e'),$$

where $a_e, a'_{e'} \in \mathbb{Z}_{\geq 0}$, $\sum_{e \in E} a_e + \sum_{e' \in E'} a'_{e'} = q$ and $\sum_{e' \in E'} a'_{e'} \geq \sum_{e' \in E'} c_{e'}$. From (3.1), one has

$$\alpha = \underbrace{\sum_{e \in E(G_1)} \lfloor r_e \rfloor \rho(e) + \sum_{e \in E} a_e \rho(e)}_{\in \mathbb{Z}_{\geq 0}(\mathcal{P}_{G_1} \cap \mathbb{Z}^d)} + \underbrace{\sum_{e' \in E(G_2)} \lfloor r_{e'} \rfloor \rho(e') + \sum_{e' \in E'} a'_{e'} \rho(e')}_{\in \mathbb{Z}_{\geq 0}(\mathcal{P}_{G_2} \cap \mathbb{Z}^d)}.$$

Hence we can rewrite α like

$$\alpha = \sum_{e \in E(G_1)} b_e \rho(e) + \sum_{e' \in E(G_2)} b'_{e'} \rho(e'),$$

where $b_e \in \mathbb{Z}_{\geq 0}$ (resp. $b'_{e'} \in \mathbb{Z}_{\geq 0}$) for each $e \in E(G_1)$ (resp. $e' \in E(G_2)$), $\sum_{e \in E(G_1)} b_e + \sum_{e' \in E(G_2)} b'_{e'} = 2k$, $\sum_{e \in E(G_1)} b_e \leq k$ and $\sum_{e' \in E(G_2)} b'_{e'} \geq k$. Since $E' \subset E(G_1)$, we obtain an expression α as above satisfying $\sum_{e \in E(G_1)} b_e = \sum_{e' \in E(G_2)} b'_{e'} = k$. This means that α can be written as a sum of k integer points in $\mathcal{P} \cap \mathbb{Z}^d$. Therefore, \mathcal{P} has IDP, as desired. \square

LEMMA 3.5. *Let G be a connected graph on the vertex set $\{1, \dots, d\}$ such that arbitrary two odd cycles in G always have a common vertex. Fix a positive integer q and let $\alpha \in q\mathcal{P}_G \cap \mathbb{Z}^d$ having an expression $\alpha = \sum_{e \in E} r_e \rho(e)$, where $E \subset E(G)$ and $0 < r_e < 1$ for each $e \in E$. Let E' be a subset of E and let $q' = \sum_{e \in E'} r_e$. Then there exist nonnegative integers a_e for $e \in E$ such that $\alpha = \sum_{e \in E} a_e \rho(e)$ satisfying $\sum_{e \in E'} a_e \geq q'$.*

PROOF. Given $\alpha \in q\mathcal{P}_G \cap \mathbb{Z}^d$ with an expression $\alpha = \sum_{e \in E} r_e \rho(e)$, where $E \subset E(G)$ and $0 < r_e < 1$ for each $e \in E$, let H be the subgraph of G whose edge set is E . Since α is an integer point but each r_e is not an integer, every vertex of H is always contained in at least two edges. Thus H contains a cycle.

The first step.

First, we claim that α can be rewritten like $\alpha = \sum_{e \in E} a_e \rho(e)$, where $a_e \in \mathbb{Z}_{\geq 0}$ for each $e \in E$, by applying the following procedures (i) and (ii).

(i) If H contains an even cycle with the edges e_1, \dots, e_{2l} , then let $\varepsilon = \min\{r_{e_i} : 1 \leq i \leq 2l\}$. Without loss of generality, we may set $r_{e_1} = \varepsilon$. We replace $r_{e_{2j-1}}$ by $r_{e_{2j-1}} - \varepsilon$ and $r_{e_{2j}}$ by $r_{e_{2j}} + \varepsilon$ for each $1 \leq j \leq l$. Then α is invariant after these replacements. On the other hand, the number of edges e with $0 < r_e < 1$ decreases at least one. If e

satisfying $1 \leq r_e < 2$ appears, then we replace r_e by $r_e - 1$ and reset α by $\alpha - \rho(e)$. We reset H by the subgraph of G whose edges $e \in E$ satisfy $0 < r_e < 1$. Then such new H also contains a cycle. We repeat this procedure until H contains no even cycle.

(ii) Assume that H contains no even cycle. Then it is easy to see that each 2-connected component of the graph is either one edge or an odd cycle. Thus H contains at least one odd cycle. If there is a 2-connected component which is one edge, then H should contain at least two odd cycles which have no common vertex, a contradiction. Moreover, if H contains only one odd cycle, then H consists of only one odd cycle. In this case, the sum of the entries of α should be odd, a contradiction to $\alpha \in \{(x_1, \dots, x_d) \in \mathbb{Z}^d : \sum_{i=1}^d x_i = 2q\}$.

Hence, all 2-connected components of H are odd cycles and H contains at least two odd cycles. By our assumption, two odd cycles in H have one common vertex and such common vertex is unique in H . Let C and C' be two odd cycles in H having a unique common vertex v , let $v = v_1, v_2, \dots, v_{2p+1}$ (resp. $v = v'_1, v'_2, \dots, v'_{2p'+1}$) be vertices of C (resp. C') and let $e_i = \{v_i, v_{i+1}\}$ (resp. $e'_i = \{v'_i, v'_{i+1}\}$) for $1 \leq i \leq 2p + 1$ (resp. $1 \leq i \leq 2p' + 1$), where $v_{2p+2} = v_1$ (resp. $v'_{2p'+2} = v'_1$). Let $\varepsilon = \min\{r_{e_i}, r'_{e'_i} : 1 \leq i \leq 2p + 1, 1 \leq i' \leq 2p' + 1\}$, say, $r_{e_1} = \varepsilon$. We replace $r_{e_{2j-1}}$ (resp. $r'_{e'_{2j'}}$) by $r_{e_{2j-1}} - \varepsilon$ (resp. $r'_{e'_{2j'}} - \varepsilon$) for $1 \leq j \leq p + 1$ (resp. $1 \leq j' \leq p'$), and $r_{e_{2\ell}}$ (resp. $r'_{e'_{2\ell'-1}}$) by $r_{e_{2\ell}} + \varepsilon$ (resp. $r'_{e'_{2\ell'-1}} + \varepsilon$) for $1 \leq \ell \leq p$ (resp. $1 \leq \ell' \leq p' + 1$). Then α is invariant after these replacements and the number of edges e with $0 < r_e < 1$ decreases at least one. If e satisfying $1 \leq r_e < 2$ appears, then we replace r_e by $r_e - 1$ and reset α by $\alpha - \rho(e)$. We reset H by the subgraph of G whose edges $e \in E$ satisfy $0 < r_e < 1$. If such new H also contains odd cycles, we repeat this until H contains no cycle.

Note that this algorithm terminates with finite procedures. After these operations (i) and (ii), we eventually obtain an expression

$$\alpha = \sum_{e \in E} a_e \rho(e), \text{ where } a_e \in \mathbb{Z}_{\geq 0}. \tag{3.2}$$

Next, we prove that if we do the above procedures (i) and (ii) more properly, then we obtain a required expression of α for any subset $E' \subset E$ with $q' = \sum_{e \in E'} r_e$. In the following second and third steps, we prove that $\sum_{e \in E'} a_e \geq q'$ by induction on the number of the above procedures (i) and (ii). Assume that we obtain an expression (3.2) with N steps.

The second step.

When $N = 1$, H consists of one even cycle or two odd cycles having a unique common vertex.

- (i) When H is one even cycle with the edges e_1, \dots, e_{2l} , since $N = 1$, each of $r_{e_1}, \dots, r_{e_{2l}}$ should be

$$r_{e_{2i}} = \varepsilon \text{ and } r_{e_{2i-1}} = 1 - \varepsilon \text{ for } 1 \leq i \leq l \text{ with some } 0 < \varepsilon < 1.$$

- (ii) When H consists of two odd cycles C and C' having a unique common vertex v ,

let $v = v_1, v_2, \dots, v_{2p+1}$ (resp. $v = v'_1, v'_2, \dots, v'_{2p'+1}$) be vertices of C (resp. C'), let $e_i = \{v_i, v_{i+1}\}$ (resp. $e'_i = \{v'_i, v'_{i+1}\}$) for $1 \leq i \leq 2p+1$ (resp. $1 \leq i \leq 2p'+1$), where $v_{2p+2} = v_1$ (resp. $v'_{2p'+2} = v'_1$) and let $\varepsilon = \min\{r_{e_i}, r'_{e'_i} : 1 \leq i \leq 2p+1, 1 \leq i' \leq 2p'+1\}$. Let, say, $r_{e_1} = \varepsilon$. Then each of $r_{e_1}, \dots, r_{e_{2p+1}}, r'_{e'_1}, \dots, r'_{e'_{2p'+1}}$ should be

$$r_{e_{2j-1}} = r'_{e'_{2j}} = \varepsilon \text{ for } 1 \leq j \leq p+1 \text{ and } 1 \leq j' \leq p'$$

and $r_{e_{2j}} = r'_{e'_{2j'-1}} = 1 - \varepsilon \text{ for } 1 \leq j \leq p \text{ and } 1 \leq j' \leq p'+1.$

In both cases, let $L_1 = \{e \in E(H) : r_e = \varepsilon\}$ and $L_2 = \{e \in E(H) : r_e = 1 - \varepsilon\}$. Let $m_1 = |L_1 \cap E'|$ and $m_2 = |L_2 \cap E'|$. If $m_1 \geq m_2$, then we replace r_e by $r_e + 1 - \varepsilon$ for each $e \in L_1$ and we also replace $r_{e'}$ by $r_{e'} - 1 + \varepsilon$ for each $e' \in L_2$. Thus we obtain

$$q' = \sum_{e \in E'} r_e = \varepsilon m_1 + (1 - \varepsilon)m_2 = m_2 + \varepsilon(m_1 - m_2) \leq m_1$$

by $0 < \varepsilon < 1$ and $m_1 \geq m_2$. If $m_2 \geq m_1$, after similar replacements, we obtain $q' = \varepsilon m_1 + (1 - \varepsilon)m_2 = m_1 + (1 - \varepsilon)(m_2 - m_1) \leq m_2$. These mean that $\sum_{e \in E'} a_e = \max\{m_1, m_2\} \geq q'$.

The third step.

Assume $N > 1$. We do the procedure (i) or (ii) as in the first step.

- (i) When H contains an even cycle with the edges e_1, \dots, e_{2l} , let $L_1 = \{e_{2i-1} : 1 \leq i \leq l\}$ and $L_2 = \{e_{2i} : 1 \leq i \leq l\}$.
- (ii) When H contains no even cycle, there are two odd cycles C and C' having a unique common vertex v . Work with the same notation as in the second step. Let $L_1 = \{e_{2i-1} : 1 \leq i \leq p+1\} \cup \{e_{2i'} : 1 \leq i' \leq p'\}$ and $L_2 = \{e_{2i} : 1 \leq i \leq p\} \cup \{e_{2i'-1} : 1 \leq i' \leq p'+1\}$.

In both cases, let $m_1 = |L_1 \cap E'|$ and $m_2 = |L_2 \cap E'|$. Assume that $m_1 \geq m_2$. (The case $m_1 \leq m_2$ can be discussed by the same manner.) Then we set $\varepsilon = \min\{r_{e'} : e' \in L_2\}$ and we replace r_e by $r_e + \varepsilon$ for each $e \in L_1$ and $r_{e'}$ by $r_{e'} - \varepsilon$ for each $e' \in L_2$. After these replacements, if there is r_e with $r_e \geq 1$ (but $r_e < 2$), then we reset r_e by $r_e - 1$. Let $E'' = \{e \in E' : r_e \text{ becomes } r_e \geq 1 \text{ after the replacements}\}$. Then $q' = \sum_{e \in E'} r_e$ changes into $q' + \varepsilon(m_1 - m_2) - |E''|$ after the replacements. By the inductive hypothesis, there exist a_e 's such that $\alpha - \sum_{e \in E''} \rho(e) = \sum_{e \in E'} a_e \rho(e)$ with $a_e \in \mathbb{Z}_{\geq 0}$ and $\sum_{e \in E'} a_e \geq q' + \varepsilon(m_1 - m_2) - |E''|$. Hence, we obtain

$$|E''| + \sum_{e \in E'} a_e \geq q' + \varepsilon(m_1 - m_2) \geq q'.$$

Since $\alpha = \sum_{e \in E''} \rho(e) + \sum_{e \in E'} a_e \rho(e)$ and $|E''| + \sum_{e \in E'} a_e \geq q'$, we obtain the required assertion. □

REMARK 3.6. The condition “arbitrary two odd cycles in G_1 always have a common vertex” in Theorem 3.4 is a stronger condition than the odd cycle condition.

3.3. Examples.

We conclude this paper by the following examples, which show that each condition described in Theorem 3.4 is necessary.

EXAMPLES 3.7. (a) The following example shows that the assumption “two odd cycles always have a common vertex” is necessary. Let G_1 and G_2 be graphs in Figure 1. Then G_2 is a subgraph of G_1 . Let $\mathcal{P} = \mathcal{P}_{G_1} + \mathcal{P}_{G_2}$. We see that

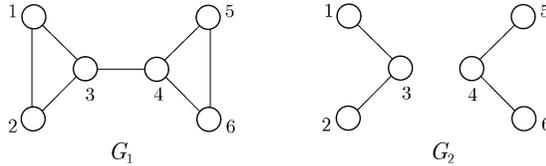


Figure 1. An example showing that our assumption is necessary.

$$\left(\frac{1}{2}\rho(\{1, 2\}) + \frac{3}{2}\rho(\{5, 6\})\right) + \left(\frac{1}{2}\rho(\{1, 3\}) + \frac{1}{2}\rho(\{2, 3\}) + \frac{1}{2}\rho(\{4, 5\}) + \frac{1}{2}\rho(\{4, 6\})\right) \\ = (1, 1, 1, 1, 2, 2) \in 2\mathcal{P} \cap \mathbb{Z}^6 = 2\mathcal{P} \cap (2v_0 + L(\mathcal{P})),$$

where $v_0 \in \mathcal{P} \cap \mathbb{Z}^6$. Since $(1, 1, 1, 1, 2, 2)$ cannot be written as any sum of two integer points in $\mathcal{P} \cap \mathbb{Z}^6$, \mathcal{P} is not normal. In particular, \mathcal{P} does not have IDP.

(b) Next, the following example shows that the assumption “ G_2 is a subgraph of G_1 ” is also necessary. Let G_1 and G_2 be graphs in Figure 2. Then each of G_1 and G_2 satisfies that two odd cycles always have a common vertex. We also see that

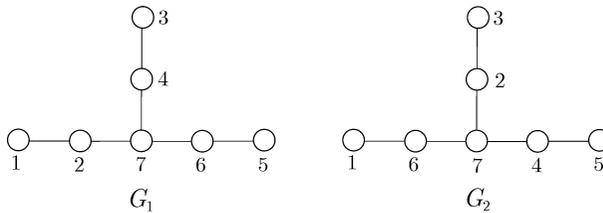


Figure 2. Other example showing that our assumption is necessary.

$$\left(\frac{4}{3}\rho(\{1, 2\}) + \frac{1}{3}\rho(\{3, 4\}) + \frac{1}{3}\rho(\{5, 6\})\right) + \left(\frac{2}{3}\rho(\{1, 6\}) + \frac{2}{3}\rho(\{2, 3\}) + \frac{2}{3}\rho(\{4, 5\})\right) \\ = (2, 2, 1, 1, 1, 1, 0) \in 2\mathcal{P} \cap \mathbb{Z}^7 = 2\mathcal{P} \cap (2v_0 + L(\mathcal{P})),$$

where $\mathcal{P} = \mathcal{P}_{G_1} + \mathcal{P}_{G_2}$ and $v_0 \in \mathcal{P} \cap \mathbb{Z}^7$. Since $(2, 2, 1, 1, 1, 1, 0)$ cannot be written as a sum of any two integer points in $\mathcal{P} \cap \mathbb{Z}^7$, \mathcal{P} is not normal, and thus, this does not have IDP.

(c) In addition, Theorem 3.4 is no longer true for the case of three graphs. Let G_1 , G_2 and G_3 be graphs in Figure 3. Then G_2 is a subgraph of G_1 and so is G_3 and G_3 is a subgraph of G_2 , too. Note that G_1 satisfies that two odd cycles have a common vertex.

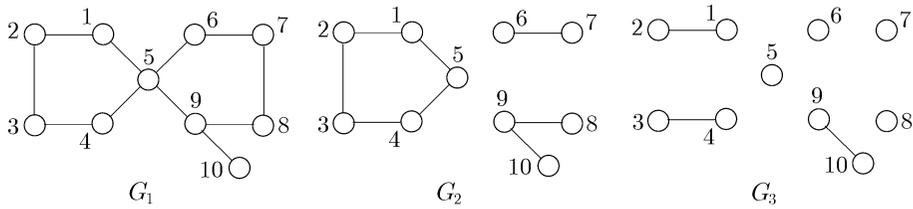


Figure 3. A counterexample for Theorem 3.4 in the case of three graphs.

We also have

$$\begin{aligned} & \left(\frac{3}{5}\rho(\{5, 6\}) + \frac{3}{5}\rho(\{7, 8\}) + \frac{3}{5}\rho(\{5, 9\}) + \frac{1}{5}\rho(\{9, 10\}) \right) \\ & + \left(\frac{2}{5}\rho(\{1, 5\}) + \frac{2}{5}\rho(\{2, 3\}) + \frac{2}{5}\rho(\{4, 5\}) + \frac{2}{5}\rho(\{6, 7\}) + \frac{2}{5}\rho(\{8, 9\}) \right) \\ & + \left(\frac{3}{5}\rho(\{1, 2\}) + \frac{3}{5}\rho(\{3, 4\}) + \frac{4}{5}\rho(\{9, 10\}) \right) \\ & = (1, 1, 1, 1, 2, 1, 1, 1, 2, 1) \in 2\mathcal{P} \cap \mathbb{Z}^{10} = 2\mathcal{P} \cap (2v_0 + L(\mathcal{P})), \end{aligned}$$

where $\mathcal{P} = \mathcal{P}_{G_1} + \mathcal{P}_{G_2} + \mathcal{P}_{G_3}$ and $v_0 \in \mathcal{P} \cap \mathbb{Z}^{10}$. One can check that $(1, 1, 1, 1, 2, 1, 1, 1, 2, 1)$ cannot be written as any sum of two integer points in $\mathcal{P} \cap \mathbb{Z}^{10}$. Thus this is not normal. In particular, this does not have IDP.

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