J. Math. Soc. Japan
Vol. 68, No. 2 (2016) pp. 885–898
doi: 10.2969/jmsj/06820885

Hydrodynamic limit for a certain class of two-species zero-range processes

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(Received July 23, 2014)

Abstract. Großkinsky and Spohn [**5**] studied several-species zero-range processes and gave a necessary and sufficient condition for translation invariant measures to be invariant under such processes. Based on this result, they investigated the hydrodynamic limit. In this paper, we consider a certain class of two-species zero-range processes which are outside of the family treated by Großkinsky and Spohn. We prove a homogenization property for a tagged particle and apply it to derive the hydrodynamic limit under the diffusive scaling.

1. Introduction.

The hydrodynamic limit is an important problem in studies of large scale interacting particle systems and, in fact, well studied for several models. Our model is the multi-species interacting particle system. We consider a particle system, in which interaction occurs only among particles sitting at the same site. This kind of particle system is called the zero-range process. For the case of the one-species zero-range process, Kipnis and Landim [**8**] establish the hydrodynamic behavior by means of the entropy method introduced by Guo, Papanicolaou and Varadhan [**4**]. However, for the multi-species zero-range process which will discuss, the derivation of the hydrodynamic limit is difficult because we don't know the explicit forms of invariant measures for our process. Großkinsky and Spohn study the invariant measures for their multi-species zero-range process in [**5**]. In their paper, they give necessary and sufficient conditions, like a detailed balanced condition, to exist translation invariant product invariant measures for the process. We consider some jump rates of two-species zero-range processes which do not satisfy the condition given in [**5**].

Now we informally explain our two-species zero-range processes. The two-species zero-range process describes an evolution of a system of distinguishable two-species particles, say ξ-particles and ζ-particles. The system of ζ-particles follows a collection of random walks and one of ζ-particles jumps from x to $x+z$ with rate $(g(\zeta(x))/\zeta(x))p(z)$, where $\zeta(x)$ stands for the number of ζ-particles sitting at site x. On the one hand, the system of ξ-particles follows a collection of random walks and one of ξ-particles jumps from x to $x+z$ with rate $h(\zeta(x))q(z)$, provided that g and h are functions on the set of non-negative integers and p and q are transition probabilities on the set of integers. Notice that ξ-particles are independent with each other under situations that we know

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2010 *Mathematics Subject Classification.* Primary 60K35; Secondary 82C41, 92D25.
Key Words and Phrases. zero-range process, hydrodynamic limit, cross-diffusion system.
This work was supported by the Program for Leading Graduate Schools, MEXT, Japan.

the motions of ζ-particles. This special feature enables us to derive the hydrodynamic limit for ξ-particles.

The strategy to prove the hydrodynamic limit is as follows. Pick up one ξ-particle, called the tagged particle, and we will show the fluctuation limit of this tagged particle under the diffusive scaling. Because the system of the ξ-particles is a collection of random walks, the hydrodynamic limit is derived by the homogenization property of the tagged particle. The method to derive the hydrodynamic limit from a homogenization property is introduced by Faggionato in [**1**], for instance. To determine the fluctuation limit of the tagged particle, we use the techniques introduced by Jara et al. in the papers [**6**] and [**7**]. The fluctuation limit of the tagged particle is determined through the analysis of the system of ζ-particles seen from the tagged particle, which is called the environment process. We establish the fluctuation limit of the tagged particle by proving tightness and local ergodicity. However, to determine the fluctuation limit, we need rather strong assumptions on g, h, p and q, to know the form of the invariant measures of the environment process. In addition, we can show the local ergodicity only in one-dimensional setting. The necessity that the dimension equals 1 is used only in establishing local replacements, so-called local 1-block estimate and local 2-block estimate. Other necessary properties are shown in any dimensional settings.

Another motivation to treat our model is to derive the cross-diffusion system,

$$\begin{cases} \partial_t u = \Delta(H(v)u), \\ \partial_t v = \Delta\Phi(v), \end{cases}$$

from some microscopic model, where H and Φ are suitable functions. Although we don't consider creations and annihilations of particles, this system of equations is a generalization of the system of equations treated in Funaki et al. [**3**]. In view of derivations of the system of partial differential equations, we interpret ξ-particles or ζ-particles as the number of insects or the strength of pheromones respectively in our microscopic model. In this paper, as a consequence of the hydrodynamic limit, we make a success of deriving the above system for some functions H and Φ.

This article is organized as follows. In Section 2, we introduce our model and state our main results. In Section 3, we consider the tagged particle and the environment process. We also treat invariant measures for the environment process. In Section 4, we prove the hydrodynamic limit for the two-species zero-range processes from the homogenization result for tagged particles.

2. Main results.

In this section, we formulate our model more precisely and state our main results. Throughout this paper, we will use the following notations: $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, $\mathbb{R}_+ = [0, \infty)$. For a function f on some set X, denote $\sup_{x \in X} |f(x)|$ by $\|f\|_\infty$.

2.1. Two-species zero-range processes.

Now we precisely describe our model which is called the two-species zero-range process. To avoid some technicalities, we put a periodic boundary condition and all processes will be considered only in a finite time interval $[0,T]$, where $T<\infty$ is fixed. For each $N\in\mathbb{N}$, let $\mathbb{T}_N$ be the discrete one-dimensional torus $\mathbb{Z}/N\mathbb{Z}$ and $\mathbb{T}$ be the torus $\mathbb{R}/\mathbb{Z}$. Elements of $\mathbb{T}_N$ are regarded as microscopic points and denoted by symbols x or y. In contrast, elements of $\mathbb{T}$ are regarded as macroscopic points and denoted by symbols θ or $\tilde{\theta}$. Let $\mathbb{N}_0^{\mathbb{T}_N}\times\mathbb{N}_0^{\mathbb{T}_N}$ be the state space of the two-species zero-range process. $\mathbb{N}_0^{\mathbb{T}_N}$ is called a configuration space, and generic elements of $\mathbb{N}_0^{\mathbb{T}_N}$ are denoted by Greek letters ξ, ζ, η. For each $x\in\mathbb{T}_N$, $\xi(x)$ or $\zeta(x)$ expresses the number of ξ-particles or ζ-particles at site x respectively.

Let (ξ_t,ζ_t) be the two-species zero-range process, that is, (ξ_t,ζ_t) is a Markov process on $\mathbb{N}_0^{\mathbb{T}_N}\times\mathbb{N}_0^{\mathbb{T}_N}$ whose generator is given by

$$\mathbb{L}_N=\mathbb{L}_N^1+\mathbb{L}_N^2 \tag{2.1}$$

where $\mathbb{L}_N^1$ and $\mathbb{L}_N^2$ are given by

$$(\mathbb{L}_N^1 f)(\xi,\zeta)=\sum_{x\in\mathbb{T}_N}\sum_{z\in\mathbb{Z}}q(z)h(\zeta(x))\xi(x)\{f(\xi^{x,x+z},\zeta)-f(\xi,\zeta)\},$$
$$(\mathbb{L}_N^2 f)(\xi,\zeta)=\sum_{x\in\mathbb{T}_N}\sum_{z\in\mathbb{Z}}p(z)g(\zeta(x))\{f(\xi,\zeta^{x,x+z})-f(\xi,\zeta)\},$$

respectively. In the above formula, $\eta^{x,y}$ is defined by

$$\eta^{x,y}(z)=\begin{cases}\eta(x)-1 & \text{if } z=x,\\ \eta(y)+1 & \text{if } z=y,\\ \eta(z) & \text{if } z\neq x,y.\end{cases}$$

for each configuration $\eta\in\mathbb{N}_0^{\mathbb{T}_N}$.

Now we introduce basic assumptions on g, h, p and q to construct the process, although we will require further stronger assumptions later. Let g and h be non-negative functions on $\mathbb{N}_0$. We assume that $g(0)=0$, $g(k)>0$, for any $k\in\mathbb{N}$ and Lipschitz growth condition, that is, $g^*=\sup_k|g(k+1)-g(k)|$ is finite. We assume that h is bounded on $\mathbb{N}_0$. Notice that $g(\zeta(x))/\zeta(x)$ and $h(\zeta(x))$ represent the jump rates of ξ- and ζ- particles, respectively. Let p and q be finite-range, irreducible transition probabilities of random walks on $\mathbb{Z}$ and $p(0)=q(0)=0$: Finite-rangeness means that there exists $R>0$ such that $p(z)=0$ for all $|z|>R$ and irreducibility means that for any $x\in\mathbb{Z}$, there exists a path $\{0=x_0,\dots,x_n=x\}$ such that $p(x_i-x_{i-1})>0$ for any $i=1,\dots,n$. p or q determines jump size of ζ-particles or ξ-particles respectively.

This particle system has two special features. One is that the particle system $\{\zeta_t\}$ is a Markov process itself, called the zero-range process, whose generator is given by

$$(\mathcal{L}_N f)(\xi)=\sum_{x\in\mathbb{T}_N}\sum_{z\in\mathbb{Z}}p(x)g(\zeta(x))\{f(\zeta^{x,x+z})-f(\zeta)\}.$$

The other is that the interacting particle system $\{\xi_t\}$ can be thought as a collection of time-inhomogeneous random walks $\{X_t^i\}_{i=1}^{\infty}$ which jumps from site x to site $x+z$ with rate $q(z)h(\zeta_t(x))$. The following proposition shows this second feature specifically.

PROPOSITION 2.1. *Let $(X_t^1, \dots, X_t^K, \zeta_t)$ be a Markov process on $\mathbb{T}_N^K \times \mathbb{N}_0^{\mathbb{T}_N}$ generated by the generator*

$$(LF)(\boldsymbol{x}, \zeta) = \sum_{i=1}^{K} \sum_{z \in \mathbb{Z}} q(z) h(\zeta(x_i)) \{F(\boldsymbol{x} + z\boldsymbol{e}_i, \zeta) - F(\boldsymbol{x}, \zeta)\}$$
$$+ \sum_{x \in \mathbb{T}_N} \sum_{z \in \mathbb{Z}} p(z) g(\zeta(x)) \{F(\boldsymbol{x}, \zeta^{x,x+z}) - F(\boldsymbol{x}, \zeta)\},$$

where $\boldsymbol{x} = (x_1, \dots, x_K) \in \mathbb{T}_N^K$ and $\boldsymbol{e}_i = (\delta_{ij})_{j=1}^K \in \mathbb{R}^K$ is an i-directed unit vector in $\mathbb{R}^K$ for each $1 \le i \le K$. Define the process ξ_t on $\mathbb{N}_0^{\mathbb{T}_N}$ by $\xi_t(x) = \sum_i 1_{\{X_t^i = x\}}$ for each $x \in \mathbb{T}_N$. Then, the joint process (ξ_t, ζ_t) on the state space $\mathbb{N}_0^{\mathbb{T}_N} \times \mathbb{N}_0^{\mathbb{T}_N}$ is a Markov process whose generator is given by $\mathbb{L}_N$ defined in (2.1).

PROOF. By a characterization due to a martingale problem, it is enough to show that

$$f(\xi_t, \zeta_t) - f(\xi_0, \zeta_0) - \int_0^t (\mathbb{L}_N f)(\xi_s, \zeta_s) ds \tag{2.2}$$

is a martingale for any function f on $\mathbb{N}_0^{\mathbb{T}_N} \times \mathbb{N}_0^{\mathbb{T}_N}$. Denote $(X_t^1, \dots, X_t^K)$ by $\boldsymbol{X}_t$. For $\boldsymbol{x} = (x_1, \dots, x_K) \in \mathbb{T}_N^K$, define the configuration $\xi_{\boldsymbol{x}}$ by

$$(\xi_{\boldsymbol{x}})(x) = \sum_i 1_{\{x_i = x\}},$$

for each $x \in \mathbb{T}_N$. Then, we have that $\xi_t = \xi_{\boldsymbol{X}_t}$. Let f be a function on $\mathbb{N}_0^{\mathbb{T}_N} \times \mathbb{N}_0^{\mathbb{T}_N}$. Consider the function F on $\mathbb{T}_N^K \times \mathbb{N}_0^{\mathbb{T}_N}$ defined by $F(\boldsymbol{x}, \zeta) = f(\xi_{\boldsymbol{x}}, \zeta)$ and use the Itô's formula for $F(\boldsymbol{x}, \zeta)$, to see that

$$F(\boldsymbol{X}_t, \zeta_t) - F(\boldsymbol{X}_0, \zeta_0) - \int_0^t (LF)(\boldsymbol{X}_s, \zeta_s) ds \tag{2.3}$$

is a martingale. Notice that $F(\boldsymbol{X}_t, \zeta_t) = f(\xi_{\boldsymbol{X}_t}, \zeta_t) = f(\xi_t, \zeta_t)$. The integrand of the third term in (2.3) is equal to

$$(LF)(\boldsymbol{X}_s, \zeta_s) = \sum_{i=1}^{K} \sum_{z \in \mathbb{Z}} q(z) h(\zeta_s(X_s^i)) \{f(\xi_{\boldsymbol{X}_s + z\boldsymbol{e}_i}, \zeta_s) - f(\xi_{\boldsymbol{X}_s}, \zeta_s)\}$$
$$+ \sum_{x \in \mathbb{T}_N} \sum_{z \in \mathbb{Z}} p(z) g(\zeta_s(x)) \{f(\xi_{\boldsymbol{X}_s}, \zeta_s^{x,x+z}) - f(\xi_{\boldsymbol{X}_s}, \zeta_s)\}.$$

Notice that $\xi_{\boldsymbol{X}_s+z\boldsymbol{e}_i}=\xi_s^{X_s^i,X_s^i+z}$, the first sum of the right hand side can be expressed as

$$\begin{aligned}&\sum_{x\in\mathbb{T}_N}\sum_{i=1}^{K}\sum_{z\in\mathbb{Z}}q(z)h(\zeta_s(x))\{f(\xi_s^{x,x+z},\zeta_s)-f(\xi_s,\zeta_s)\}1_{\{X_s^i=x\}}\\&\quad=\sum_{x\in\mathbb{T}_N}\sum_{z\in\mathbb{Z}}q(z)h(\zeta_s(x))\xi_s(x)\{f(\xi_s^{x,x+z},\zeta_s)-f(\xi_s,\zeta_s)\}.\end{aligned}$$

Therefore the process in (2.2) is a martingale, which concludes the proof of the proposition. □

2.2. Invariant measures for zero-range processes.

The zero-range process ζ_t is one of the well-studied interacting particle systems. In this subsection, we briefly review known results on the invariant measures of the zero-range process.

Define the partition function by

$$Z(\varphi):=\sum_{k=0}^{\infty}\frac{\varphi^k}{g(k)!}$$

where $g(k)!=1$ if $k=0$ and $g(1)\dots g(k)$ if $k\geq 1$. Denote the radius of convergence of Z by φ^* and assume that

$$\lim_{\varphi\uparrow\varphi^*}Z(\varphi)=\infty. \tag{2.4}$$

For each $0\leq\varphi<\varphi^*$, define $\bar{\mu}_\varphi$ as the product measure on $\mathbb{N}_0^{\mathbb{T}_N}$ with marginals given by

$$\bar{\mu}_\varphi(\zeta:\zeta(x)=k)=\frac{1}{Z(\varphi)}\frac{\varphi^k}{g(k)!},$$

for $k\in\mathbb{N}_0$ and $x\in\mathbb{T}_N$. The next proposition is well known [**8**].

Proposition 2.2. *For each $0\leq\varphi<\varphi^*$, $\bar{\mu}_\varphi$ is invariant under the zero-range process ζ_t. In addition, if p is symmetric in the sense that $p(z)=p(-z)$ for any $z\in\mathbb{Z}$, $\bar{\mu}_\varphi$ is a reversible measure of this process.*

For each $0\leq\varphi<\varphi^*$, set $D(\varphi):=\int\zeta(0)d\bar{\mu}_\varphi$. Under the assumption (2.4) for the partition function Z, $D(\varphi)$ diverges to ∞ as φ increases to φ^*. Therefore $D:[0,\varphi^*)\to[0,\infty)$ is diffeomorphic. Denote by Φ the inverse function of D. We re-parametrize invariant measures by the density of particles, that is, for any $\rho\geq 0$, we define the measure μ_ρ by $\mu_\rho:=\bar{\mu}_{\Phi(\rho)}$. Notice that we have $\int\zeta(0)d\mu_\rho=\rho$ and $\int g(\zeta(0))d\mu_\rho=\Phi(\rho)$.

2.3. Hydrodynamic limit for two-species zero-range processes.

We now assume on the initial distributions to introduce our results. Let initial distributions μ_N^1 and μ_N^2 of ξ_t and ζ_t, respectively. These are probability measures on

$\mathbb{N}_0^{\mathbb{T}_N}$. Let u_0 and v_0 be functions in $C(\mathbb{T})$, where $C(\mathbb{T})$ stands for the family of continuous functions on $\mathbb{T}$. Assume that μ_N^1 is a product measure on $\mathbb{N}_0^{\mathbb{T}_N}$ so that random variables $\{\xi(x)\}_{x\in\mathbb{T}_N}$ are independent under μ_N^1, the expectation of $\xi(x)$ with respect to μ_N^1 is equal to $u_0(x/N)$ and $(1/N)E_{\mu_N^1}[\sum_{x\in\mathbb{T}_N}\xi^2(x)] = o(N)$. Let μ_N^2 be the local equilibrium measure $\mu_{v_0(\cdot)}^N$, that is, $\mu_{v_0(\cdot)}^N$ is the product measure on $\mathbb{N}_0^{\mathbb{T}_N}$ whose marginals are given by $\mu_{v_0(\cdot)}^N(\zeta(x)=k) = \mu_{v_0(x/N)}(\zeta(x)=k)$.

Now we consider the Markov process $\{(\xi_t^N,\zeta_t^N); t\in[0,T]\}$ with the generator $N^2\mathbb{L}_N$ and the initial distribution $\mu_N^1\times\mu_N^2$. For each $t\in[0,T]$, define the empirical measures $\pi_t^{N,1}$ and $\pi_t^{N,2}$ by

$$\pi_t^{N,1}(d\theta) = \sum_{x\in\mathbb{T}_N}\xi_t^N(x)\delta_{x/N}(d\theta),$$

$$\pi_t^{N,2}(d\theta) = \sum_{x\in\mathbb{T}_N}\zeta_t^N(x)\delta_{x/N}(d\theta),$$

respectively, where δ_θ stands for the Dirac measure which has a unit mass at $\theta\in\mathbb{T}$. Let $\mathcal{M}_+(\mathbb{T})$ be the set of all finite non-negative measures on $\mathbb{T}$.

To prove the hydrodynamic limit, we need some additional technical assumptions on g, h, p and q. The necessity of the conditions stated below will be explained in Proposition 3.1. Assume that g is nondecreasing and there exists a positive number $c>0$ such that $(k+c)h(k)+g(k) = ch(0)$ for any $k\in\mathbb{N}$ and assume that p has mean-zero and $p(z) = q(-z)$ for any $z\in\mathbb{Z}$. Define the function on $\mathbb{R}_+$ by $H(\rho) = \int((\zeta(0)+c)/(\rho+c))h(\zeta(0))d\mu_\rho$.

The next result is well known ([**8**, Chapter 5]).

THEOREM 2.3. *For any $t\in[0,T]$, $\{\pi_t^{N,2}; N\geq 1\}$ converges to a deterministic measure $v(t,\theta)d\theta$ in probability as $N\to\infty$ where $v(t,\theta)$ is a unique weak solution of the Cauchy problem*

$$\begin{cases}\partial_t v = \dfrac{\sigma^2}{2}\Delta\Phi(v),\\ v(0,\theta) = v_0(\theta).\end{cases} \tag{2.5}$$

Now we describe our main results.

THEOREM 2.4 (Hydrodynamic limit). *For any $t\in[0,T]$, $\{\pi_t^{N,1}; N\geq 1\}$ converges to $u(t,\theta)d\theta$ as $N\to\infty$ in probability where $u(t,\theta)$ is a unique weak solution of the Cauchy problem*

$$\begin{cases}\partial_t u = \dfrac{\sigma^2}{2}\Delta(H(v)u),\\ u(0,\theta) = u_0(\theta),\end{cases} \tag{2.6}$$

where $v(t,\theta)$ is the unique solution of (2.5).

Theorem 2.4 can be derived from a homogenization result for each ξ-particle. We call each ξ-particle a "tagged particle". Before giving a proof of Theorem 2.4, we shall formulate the homogenization result for tagged particles. Theorem 2.4 will be proved in Subsection 2.4. Let κ_N be the probability on $\mathbb{T}_N$ given by $\kappa_N(x) = u_0(x/N)/\sum_{y\in\mathbb{T}_N} u_0(y/N)$. If u_0 is identically equal to 0, Theorem 2.4 holds trivially, so we exclude this case. We may assume that N is large enough so that the following definitions are meaningful. Notice that the sequence $\{\kappa_N; N \geq 1\}$ converges weakly to κ, where κ is the probability measure on $\mathbb{T}$ defined by $\kappa(d\theta) = (u_0(\theta)/\int_{\mathbb{T}} u_0(\tilde{\theta})d\tilde{\theta})d\theta$.

Consider the Markov process $\{(X_t^{N,1}, X_t^{N,2}, \zeta_t^N); t \in [0,T]\}$ with the generator

$$
\begin{aligned}
N^2(LF)(x_1,x_2,\zeta) = N^2 &\sum_{z\in\mathbb{Z}} q(z)h(\zeta(x_1))\{F(x_1+z,x_2,\zeta) - F(x_1,x_2,\zeta)\} \\
&+ N^2 \sum_{z\in\mathbb{Z}} q(z)h(\zeta(x_2))\{F(x_1,x_2+z,\zeta) - F(x_1,x_2,\zeta)\} \\
&+ N^2 \sum_{x\in\mathbb{T}_N}\sum_{z\in\mathbb{Z}} p(z)g(\zeta(x))\{F(x_1,x_2,\zeta^{x,x+z}) - F(x_1,x_2,\zeta)\}
\end{aligned}
$$

and the initial distribution $\kappa_N \times \kappa_N \times \mu_N^2$.

THEOREM 2.5 (Homogenization properties). *The rescaled process* $\{(1/N)(X_t^{N,1}, X_t^{N,2}); t \in [0,T]\}$ *converges in distribution under the uniform topology to a diffusion* $\{(\theta_t^1,\theta_t^2); t\in[0,T]\}$ *as* $N\to\infty$, *where* $\{(\theta_t^1,\theta_t^2); t\in[0,T]\}$ *is a unique weak solution of the stochastic differential equation*

$$
\begin{cases}
d\theta_t^1 = \sigma\sqrt{H(v(t,\theta_t^1))}dB_t^1, \\
d\theta_t^2 = \sigma\sqrt{H(v(t,\theta_t^2))}dB_t^2, \\
(\theta_0^1,\theta_0^2) \overset{(d)}{=} \kappa\times\kappa,
\end{cases}
$$

where $v(t,\theta)$ *is the unique solution of* (2.5) *and* (B_t^1, B_t^2) *is a two-dimensional standard Brownian motion.*

PROOF. We use characterization by martingales. Notice that tagged particles $X_t^{N,1}$ and $X_t^{N,2}$ are orthogonal martingales for each $N \geq 1$ and the convergence to a diffusion for each tagged particle will be proved in Theorem 3.2. Therefore, two processes are independent in the limit and the diffusion is given by the above stochastic differential equation. □

REMARK 2.1. In Theorem 2.5 and Theorem 3.2, we consider diffusions on $\mathbb{R}$, however, all diffusions on $\mathbb{R}$ appeared in this paper are identified with diffusions on $\mathbb{T}$ in the following sense. Let θ_t be the solution of the stochastic differential equation

$$
\begin{cases}
d\theta_t = \sigma\sqrt{H(v(t,\theta_t))}dB_t, \\
\theta_0 = 0,
\end{cases}
$$

where B_t is a one-dimensional standard Brownian motion. Define the equivalent relation $x \sim y$ in $\mathbb{R}$ by $x - y \in \mathbb{Z}$. For each $x \in \mathbb{R}$, denote the equivalence class of x by $[x]$. The diffusion θ_t on $\mathbb{R}$ is identified with the diffusion $[\theta_t]$ on $\mathbb{T}$. For simplicity, we denote these diffusions by same symbol θ_t.

3. Homogenization for the tagged particle process.

In this section, we prove an invariance principle for a tagged particle X_t starting at the origin under the diffusive scaling.

3.1. Invariant measures for environment processes.

Let X_t be the position of the tagged particle at time t. The zero-range process seen from the position of the tagged particle, called the environment process, will play an important role to analyze the motion of the tagged particle. Denote the environment process η_t by $\eta_t = \tau_{X_t}\zeta_t$ where $\{\tau_x\}_{x\in\mathbb{T}_N}$ is a translation group acting on $\mathbb{N}_0^{\mathbb{T}_N}$. The environment process $\{\eta_t : t \in [0,T]\}$ is again a Markov process. The generator of the environment process is easily computed as $L_N = L_N^{env} + L_N^{tp}$, where

$$(L_N^{env} f)(\eta) = \sum_{x\in\mathbb{T}_N}\sum_{z\in\mathbb{Z}} p(x) g(\eta(x))\{f(\eta^{x,x+z}) - f(\eta)\},$$

$$(L_N^{tp} f)(\eta) = \sum_{x\in\mathbb{Z}} q(z) h(\eta(0))\{f(\tau_z\eta) - f(\eta)\}.$$

We now consider invariant measures for the environment process η_t but generally we don't know the structure of the 1-parameter family of invariant measures for the environment process η_t. Of course, μ_ρ is not invariant under the environment process η_t by the effects of translations. So we shall start from identifying some class of invariant measures. The next result gives the invariant measures of the environment process.

PROPOSITION 3.1. *If the measure ν_ρ defined by $d\nu_\rho = (G(\eta(0))/\bar{G}(\rho))d\mu_\rho$ with a function $G : \mathbb{N}_0 \to \mathbb{R}_+$ satisfying $\bar{G}(\rho) = \int G(\eta(0))d\mu_\rho < \infty$ for any $\rho \geq 0$ is invariant under the environment process, then we have that $p(z) = q(-z)$ for any $z \in \mathbb{Z}$ and there exist $a \geq 0$ and $b > 0$ s.t. $G(k) = ak + b$ and $(ak+b)h(k) + ag(k) = bh(0)$ for any $k \geq 0$.*

Conversely, if g, h, p and q satisfy that $p(z) = q(-z)$ for any $z \in \mathbb{Z}$ and $(ak+b)h(k)+ ag(k) = bh(0)$, for any $k \geq 0$ for some $a \geq 0$ and $b > 0$, then the measure ν_ρ defined by $d\nu_\rho := (G(\eta(0))/\bar{G}(\rho))d\mu_\rho$ with $G(k) = ak + b$ is invariant under the environment process.

PROOF. We first assume that there exists a function G on $\mathbb{N}_0$ such that the measure ν_ρ defined above is invariant under the process. Fix $\rho \geq 0$. Denote the generator L_N^{env} replaced p by $p^*(z) = p(-z)$ by L_N^{env*}. For a measure m on $\mathbb{N}_0^{\mathbb{T}_N}$, denote by $\langle\cdot\rangle_m$ the expectation with respect to m.

We claim that

$$\sum_{x\in\mathbb{T}_N} p(x)[g(\eta(x))\{G(\eta(0)+1)-G(\eta(0))\}+g(\eta(0))\{G(\eta(0)-1)-G(\eta(0))\}]$$
$$+q(-x)\{h(\eta(x))G(\eta(x))-h(\eta(0))G(\eta(0))\}=0, \tag{3.1}$$

for any configuration $\eta \in \mathbb{N}_0^{\mathbb{T}_N}$. Indeed, from the definition of the generator L_N, we have

$$\langle L_N f\rangle_{\nu_\rho} = \langle L_N^{env} f\rangle_{\nu_\rho} + \langle L_N^{tp} f\rangle_{\nu_\rho} \tag{3.2}$$

for any function $f : \mathbb{N}_0^{\mathbb{T}_N} \to \mathbb{R}$. The first term of the right hand side appeared in (3.2) multiplied $\bar{G}(\rho)$ is equal to

$$\langle (L_N^{env} f)(\eta)G(\eta(0))\rangle_{\mu_\rho} = \langle f(\eta)L_N^{env*}G(\eta(0))\rangle_{\mu_\rho}.$$

Note that for each $x \in \mathbb{T}_N$ and $z \in \mathbb{Z}$, if $0 \notin \{x, x+z\}$, then $G(\eta^{x,x+z}(0)) - G(\eta(0)) = 0$. Therefore $L_N^{env*}G(\eta(0))$ is computed as

$$\begin{aligned} L_N^{env*}G(\eta(0)) &= \sum_{x\in\mathbb{T}_N}\sum_{z\in\mathbb{Z}} p^*(z)g(\eta(x))\{G(\eta^{x,x+z}(0)) - G(\eta(0))\} \\ &= \sum_{z\in\mathbb{Z}} p^*(z)g(\eta(0))\{G(\eta(0)-1) - G(\eta(0)\} \\ &\quad + \sum_{x\in\mathbb{T}_N} p^*(-x)g(\eta(x))\{G(\eta(0)+1) - G(\eta(0))\} \\ &= \sum_{x\in\mathbb{T}_N} p(x)[g(\eta(x))\{G(\eta(0)+1) - G(\eta(0))\} \\ &\qquad\qquad + g(\eta(0))\{G(\eta(0)-1) - G(\eta(0))\}]. \end{aligned}$$

In the third equality we used the fact $\sum_{z\in\mathbb{Z}} p^*(z) = 1$. On the other hand, from the definition of the generator L_N^{tp}, the second term of the right hand side appeared in (3.2) multiplied $\bar{G}(\rho)$ is equal to

$$\begin{aligned} \langle L_N^{tp} f\rangle_{\nu_\rho}\bar{G}(\rho) &= \sum_{z\in\mathbb{Z}} q(z)\int h(\eta(0))\{f(\tau_z\eta) - f(\eta)\}G(\eta(0))d\mu_\rho \\ &= \sum_{z\in\mathbb{Z}} q(z)\int h(\eta(-z))f(\eta)G(\eta(-z))d\mu_\rho \\ &\quad - \sum_{z\in\mathbb{Z}} q(z)\int h(\eta(0))f(\eta)G(\eta(0))d\mu_\rho. \end{aligned}$$

In the second equality we used the fact that the measure μ_ρ is translation invariant. Hence, for N large enough, the change of variable $z' = -x$ gives that

$$\langle L_N^{tp} f\rangle_{\nu_\rho}\bar{G}(\rho) = \int f(\eta)\Big[\sum_{x\in\mathbb{T}_N} q(-x)\{h(\eta(x))G(\eta(x)) - h(\eta(0))G(\eta(0))\}\Big]d\mu_\rho.$$

Since $\langle L_N f \rangle_{\nu_\rho} = 0$ for any function $f : \mathbb{N}_0^{\mathbb{T}_N} \to \mathbb{R}$, (3.1) holds for any configuration $\eta \in \mathbb{N}_0^{\mathbb{T}_N}$.

For fixed $k, l \in \mathbb{N}_0$, consider the configuration $\eta_{k,l}$ such that $\eta_{k,l}(x) = k$, for $x \neq 0$ and $\eta_{k,l}(x) = l$, for any $x = 0$. Substituting to (3.1) the configuration $\eta_{k,k}$, we obtain that

$$g(k)\{G(k+1) - G(k)\} = g(k)\{G(k) - G(k-1)\},$$

for any $k \in \mathbb{N}_0$. Since we assume that $g(k)$ is strictly positive for any $k \in \mathbb{N}$, the function $G(k) - G(k-1)$ is a constant function in k. Therefore there exist a non-negative number a and a positive number b such that $G(k) = ak + b$ for any $k \in \mathbb{N}_0$.

Similarly, substituting to (3.1) the configuration $\eta_{k,l}$, we obtain that

$$h(k)G(k) + ag(k) = h(l)G(l) + ag(l),$$

for any $k, l \in \mathbb{N}_0$. This implies that $h(k)G(k) + ag(k)$ is identically equal to $bh(0)$. Under this relation between g and h, the equation (3.1) is equivalent to the following equation.

$$\sum_{x \in \mathbb{T}_N} \{p(x) - q(-x)\}\{h(\eta(x))G(\eta(x)) - h(\eta(0))G(\eta(0))\} = 0.$$

It is easy to see that the above equation does not hold unless $p(z) = q(-z)$ for any $z \in \mathbb{Z}$.

The converse direction follows from the first half of computations. □

Remark 3.1. Notice that the condition of the previous proposition holds for $a = 0$ if and only if h is a constant jump rate. In this case, ν_ρ is equal to μ_ρ and the motion of the tagged particle is not affected by the zero-range process, therefore invariance principle for the tagged particle follows from invariance principle for martingales.

Throughout this paper, we assume the conditions of Proposition 3.1 and to avoid a trivial case we assume that $a > 0$. Setting $c := a/b$, our conditions can be written as $(k+c)h(k) + g(k) = h(0)$ for any $k \in \mathbb{N}_0$. In this case, the invariant measure for the environment process may be defined as $d\nu_\rho = ((\eta(0) + c)/(\rho + c))d\mu_\rho$. Moreover, in terms of ν_ρ, the function H can be written as $H(\rho) = \int h(\eta(0))d\nu_\rho$.

3.2. Invariance principle for the tagged particle.

Counting the total number of translations η_t, we can recover the position of the tagged particle. More specifically, for each $z \in \mathbb{Z}$, denote by N_t^z the total number of translations in η_t of size z up to time t. Then the position of the tagged particle X_t can be expressed as

$$X_t = \sum_{z \in \mathbb{Z}} z N_t^z.$$

Notice that $M_t^z := N_t^z - \int_0^t q(z)h(\eta_s(0))ds$ is a martingale for each $z \in \mathbb{Z}$, and its quadratic variation $\langle M^z \rangle_t$ is equal to $\int_0^t q(z)h(\eta_s(0))ds$. Since jumps are not simultaneous, these martingales are orthogonal. As we assumed that $p = q^*$ and p has mean-zero,

$$X_t = \sum_{z\in\mathbb{Z}} zM_t^z + \sum_{z\in\mathbb{Z}} zq(z)\int_0^t h(\eta_s(0))ds = \sum_{z\in\mathbb{Z}} zM_t^z$$

is a martingale and its quadratic variation $\langle X\rangle_t$ is computed as

$$\langle X\rangle_t = \sigma^2 \int_0^t h(\eta_s(0))ds,$$

where σ^2 is defined by $\sum_{z\in\mathbb{Z}} z^2 p(z)$.

As in Subsection 2.3, we shall define empirical measures for the environment process. Consider the Markov process $\eta_t^N := \eta_{tN^2}$ with the initial distribution $\nu^N = \mu_N^2$. For the rescaled environment process, define the empirical measure $\{\pi_t^N; t \in [0,T]\}$ by

$$\pi_t^N(d\theta) = \frac{1}{N}\sum_{x\in\mathbb{T}_N} \eta_t^N(x)\delta_{x/N}(d\theta).$$

Let us state the scaling limit for the position of the tagged particle and the convergence of empirical measures seen from the position of the tagged particle. Recall that the definitions of functions H and v: $H(\rho) = \int h(\eta(0))d\nu_\rho$ and v is a unique weak solution of the Cauchy problem (2.5).

THEOREM 3.2. *Let x_t^N be the rescaled position of the tagged particle $(1/N)X_{tN^2}$. Then, the $\{x_t^N; t \in [0,T]\}$ converges in distribution to a diffusion $\{\theta_t; t \in [0,T]\}$ in the Skorokhod space $D([0,T],\mathbb{R})$ as $N \to \infty$ where $\{\theta_t; t \in [0,T]\}$ is a unique weak solution of the stochastic differential equation*

$$\begin{cases} d\theta_t = \sigma\sqrt{H(v(t,\theta_t))}dB_t, \\ \theta_0 = 0, \end{cases} \tag{3.3}$$

where B_t is a one-dimensional standard Brownian motion.

PROOF. We follow the proof of Theorem 2.2 in Jara et al. [**6**] and Theorem 1.4 in Jara et al. [**7**]. We briefly describe the sketch of its proof.

We first prove tightness of the extended distribution Q^N defined by the law of the process $\{(\pi_t^{N,2}, \pi_t^N, x_t^N, \langle x^N\rangle_t); t \in [0,T]\}$ on the Skorokhod space $D([0,T], \mathcal{M}_+(\mathbb{T}) \times \mathcal{M}_+(\mathbb{T}) \times \mathbb{R} \times \mathbb{R})$. Tightness of the sequence $\{Q^N\}$ enables us to capture a sub-sequential limit. See Section 3 in Jara et al. [**6**] for more details.

The remaining thing to complete the proof of Theorem 3.2 is characterizing the limit of the sequence $\{Q^N\}$. Because π_t^N is uniquely determined by the empirical measure $\pi_t^{N,2}$ and the tagged particle x_t^N and we know the convergence of the sequence $\pi_t^{N,2}$ by Theorem 2.3, we only consider the tagged particle x_t^N. Moreover, since the rescaled position of the tagged particle x_t^N is also a martingale, we have to characterize the limit of the sequence of the quadratic variation process $\langle x^N\rangle_t$.

We now have to do is to prove the following replacement estimate, so-called the local ergodicity, which enables us to determine the limit behavior of the sequence:

$$\limsup_{l\to\infty}\limsup_{\varepsilon_2\to 0}\limsup_{\varepsilon_1\to 0}\limsup_{N\to\infty}\mathbb{E}^N\left[\left|\int_0^t \{h(\eta_s(0)) - \frac{1}{\varepsilon_2 N}\sum_{x=1}^{\varepsilon_2 N}\bar{H}_l(\eta_s^{\varepsilon_1 N}(x))\}ds\right|\right] = 0.$$

Here $\eta^l(x) = \sum_{|y-x|\le l}\eta(y)/(2l+1)$, $H_l(\eta) = H(\eta^l(0))$, $\bar{H}_l(\rho) = \langle H_l\rangle_{\mu_\rho}$ and $\mathbb{E}^N$ stands the expectation with respect to the law of the process η^N. In Theorem 1.4 in the paper [**7**], they only treat the case when $p = q$ are symmetric, $h(k) = 1_{\{k=0\}}$ and $g(k) = 1_{\{k\ge 0\}}$. However, we can show the local ergodicity under our assumptions on p, q, g and h by straightforward modifications. For more details, see Jara et al. [**7**]. Establishing the local ergodicity finishes the proof of Theorem 3.2. □

4. Proof of Theorem 2.4.

In this section, we prove Theorem 2.4.

Let P^N be the distribution of the process $\{(\xi_t^N, \zeta_t^N) : t \in [0,T]\}$. The expectation with respect to the measure P^N will be denoted by E^N. For each $x, y \in \mathbb{T}_N$, denote by $P^{x,y}$ the distribution of the process $\{(X_t^{N,1}, X_t^{N,2}, \zeta_t^N); t \in [0,T]\}$ with the initial measure $\delta_x \times \delta_y \times \mu_N^2$ and the expectation with respect to the distribution $P^{x,y}$ will be denoted by $E^{x,y}$. For each $x, y \in \mathbb{T}_N$ and $t \in [0,T]$, define the transition probability by $p_t^N(x,y) = P(X_t^N = y | X_0^N = x)$. For each $\theta \in \mathbb{T}$ and $t \in [0,T]$, denote by $p_t(\theta,\cdot)$ the density of the process θ_t determined by the stochastic differential equation

$$\begin{cases} d\theta_t = \sigma\sqrt{H(v(t,\theta_t))}dB_t, \\ \theta_0 = \theta, \end{cases}$$

where B_t is a one-dimensional standard Brownian motion. The fundamental properties of the density $p_t(\theta,\cdot)$ can be found in Friedman [**2**]. Define the operators P_t^N and P_t by

$$(P_t^N G)\left(\frac{x}{N}\right) = \sum_{y\in\mathbb{T}_N} p_t^N(x,y) G\left(\frac{y}{N}\right), \quad (P_t G)(\theta) = \int_{\mathbb{T}} p_t(\theta,\tilde{\theta})G(\tilde{\theta})d\tilde{\theta},$$

for each continuous functions G on $\mathbb{T}$.

Let G be a continuous function on $\mathbb{T}$. We show that

$$P^N\left(\left|\sum_{x\in\mathbb{T}_N}\xi_t(x)G\left(\frac{x}{N}\right) - \int_{\mathbb{T}}(P_tG)(\tilde{\theta})u_0(\tilde{\theta})d\tilde{\theta}\right| > \delta\right) \to 0, \tag{4.1}$$

as $N \to \infty$ for each positive number $\delta > 0$. Indeed, from Proposition 2.1, the process ξ_t^N can be realized as a sum of properly scaled random walks $\{X_t^{N,i}\}_i$ driven by ζ_t^N. Therefore, the above probability is equal to

$$P^N\left(\left|\sum_i G\left(\frac{X_t^{N,i}}{N}\right) - \int_{\mathbb{T}} (P_t G)(\tilde{\theta}) u_0(\tilde{\theta}) d\tilde{\theta}\right| > \delta\right).$$

By the Chebyshev's inequality and the assumptions on the initial measure μ_N^1, the above probability multiplied δ^2 is bounded above by

$$\begin{aligned}
&\sum_{x,y\in\mathbb{T}_N} E^{x,y}\left[G\left(\frac{X_t^{N,1}}{N}\right)G\left(\frac{X_t^{N,2}}{N}\right)\right]u_0\left(\frac{x}{N}\right)u_0\left(\frac{y}{N}\right)\\
&\quad -2\left(\sum_{x\in\mathbb{T}_N}(P_t^N G)\left(\frac{x}{N}\right)u_0\left(\frac{x}{N}\right)\right)\left(\int_{\mathbb{T}}(P_t G)(\tilde{\theta})u_0(\tilde{\theta})d\tilde{\theta}\right)\\
&\quad +\left(\int_{\mathbb{T}}(P_t G)(\tilde{\theta})u_0(\tilde{\theta})d\tilde{\theta}\right)^2.
\end{aligned}$$

By Theorem 2.5, this converges to 0 as $N \to \infty$, which concludes the proof of (4.1).

Define $u(t,\theta)$ by $\int_{\mathbb{T}} p_t(\tilde{\theta},\theta)u_0(\tilde{\theta})d\tilde{\theta}$. Notice that from Fubini's theorem, we have

$$\int_{\mathbb{T}}(P_t G)(\tilde{\theta})u_0(\tilde{\theta})d\tilde{\theta} = \int_{\mathbb{T}}\int_{\mathbb{T}} p_t(\tilde{\theta},\theta)G(\theta)u_0(\tilde{\theta})d\theta d\tilde{\theta} = \int_{\mathbb{T}} G(\theta)u(t,\theta)d\theta.$$

An elementary application of Itô's formula shows that u is a solution of the Cauchy problem (2.5), which finishes the proof of Theorem 2.4.

Acknowledgements. I would like to express my deepest gratitude to Professor Tadahisa Funaki whose enormous support and insightful comments were invaluable during the course of my study. I would also like to thank Professor Claudio Landim and Professor Kohei Uchiyama whose meticulous comments were enormous help to me.

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