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Lindelöf theorem for harmonic mappings

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Abstract. We extend the classical Lindelöf theorem for harmonic mappings. Assume that f is an univalent harmonic mapping of the unit disk $\boldsymbol{U}$ onto a Jordan domain with C^1 boundary. Then the function $\arg(\partial_\varphi(f(z))/z)$, where $z = re^{i\varphi}$, has continuous extension to the boundary of the unit disk, under certain condition on $f|_{\boldsymbol{T}}$.

1. Introduction.

1.1. Some elementary facts from measure theory.

Let $M(\boldsymbol{T})$ be the space of complex measures in the unit circle $\boldsymbol{T}$ and let $L^p(\boldsymbol{T})$, $0 < p \leq \infty$ be the space of Lebesgue measurable functions of the finite norm

$$\|f\|_{L^p} := \left(\int_T |f(z)|^p \frac{|dz|}{2\pi}\right)^{1/\max\{p,1\}}.$$

For $\mu \in M(\boldsymbol{T})$, denote by $\|\mu\|$ its total variation. It is a norm in $M(\boldsymbol{T})$ and $M(\boldsymbol{T})$ is a Banach space. The norm is unique rotationally invariant up to a positive constant, and we normalize it in such a way that the Lebesgue measure in $\boldsymbol{T}$ has the norm equal to 1. If μ is a absolutely continuous with respect to Lebesgue measure in the unit circle, then there exists a function $F \in L^1(\boldsymbol{T})$ such that $d\mu(e^{it}) = d\mu_F(e^{it}) = F(e^{it})dt$. Thus $L^1(\boldsymbol{T}) \subset M(\boldsymbol{T})$.

Every homeomorphism F of the unit circle onto a rectifiable Jordan curve γ has a bounded variation and therefore has the first derivative $\partial_t F(e^{it})$ almost everywhere. Moreover by the Lebesgue-Radon-Nikodym theorem,

$$dF(e^{it}) = d\Lambda_a(e^{it}) + d\Lambda_s(e^{it}),$$

where Λ_a is an absolutely continuous measure with respect to Lebesgue measure on the unit disk, and Λ_s is a singular measure orthogonal to Λ_a. Then

$$d\Lambda_a(e^{it}) = \partial_t F_0(e^{it})dt,$$

where $\partial_t F_0(e^{it})$ is "absolutely continuous part" of $\partial_t F(e^{it})$.

Let g be the arc-length parametrization of a Jordan curve γ of the class C^1, and assume for simplicity that $|\gamma| = 2\pi$. Then there is a homeomorphism $\psi : \mathbf{R} \to \mathbf{R}$ with

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$\psi(2\pi) - \psi(0) = 2k\pi$, such that $F(e^{it}) = g(\psi(t))$. Then we have

$$\partial_t F(e^{it}) = g'(\psi(t))\psi'(t) = e^{i\beta(t)}\psi'(t), \tag{1}$$

for some continuous function β. Further if ψ is an increasing function, then

$$d\psi(t) = \psi_0'(t)dt + d\lambda_s(t),$$

where $\psi_0'(t) = |\partial_t F_0(e^{it})|$ is the "absolutely continuous part of ψ'", and λ_s is a positive singular measure in $[0, 2\pi]$. By $\|d\psi\|$ we denote the total variation of the function ψ on the $[0, 2\pi]$. In view of (1) we see that $\|d\psi\| = \|dF\|$, where $\|dF\|$ is the total variation of the measure dF in $\boldsymbol{T}$.

1.2. Harmonic mappings.

A mapping f is called *harmonic* in a region D if $f = u + iv$ where u and v are harmonic functions in D. If D is simply-connected, then there are two analytic functions a and b defined on D such that $f = a + \bar{b}$. Let

$$P(r,t) = \frac{1 - r^2}{2\pi(1 - 2r\cos t + r^2)}$$

denote the Poisson kernel. Let $\boldsymbol{U}$ be the unit disk in the complex plane $\mathbf{C}$ and let $\boldsymbol{T}$ be its boundary. The Poisson integral of a measure $\mu \in M(\boldsymbol{T})$ (and of a function $F \in L^1(\boldsymbol{T})$, $d\mu(e^{it}) = F(e^{it})dt$) is a harmonic function given by

$$f(z) = \mathcal{P}[\mu](z)(= \mathcal{P}[F](z)) = \int_0^{2\pi} P(r, t - \tau)d\mu(e^{it}), \tag{2}$$

where $z = re^{i\tau} \in \boldsymbol{U}$.

A function f harmonic in the disk $\boldsymbol{U} \subset \mathbf{C}$ belongs to the Hardy class $h^p = h^p(\boldsymbol{U})$ if

$$\|f\|_{h^p} := \left(\sup_{0<r<1} \int_{\boldsymbol{T}} |f(re^{it})|^p \frac{dt}{2\pi} \right)^{1/p} < \infty. \tag{3}$$

It turns out that if $f \in h^p(\boldsymbol{U})$, then there exists the finite limit

$$\lim_{r \to 1^-} f(re^{it}) = F(e^{it}) \text{ (a.e. on } \boldsymbol{T})$$

and the boundary function belongs to $L^p(\boldsymbol{T})$ for $p > 1$. Moreover for $1 < p \le \infty$

$$\|f\|_{h^p} = \|F\|_p. \tag{4}$$

For $p = 1$ there exists a complex measure $\mu \in M(\boldsymbol{T})$ such that

$$\|f\|_{h^1} = \|\mu\|. \tag{5}$$

Standard properties of the Poisson integral show that $\mathcal{P}[F]$ extends by continuity to F on $\overline{\boldsymbol{U}}$, provided that F is continuous. For these facts and standard properties of harmonic Hardy space we refer to [**2**, Chapter 6]. With the additional assumption that F is orientation-preserving homeomorphism of this circle onto a convex Jordan curve γ, $\mathcal{P}[F]$ is an orientation preserving diffeomorphism of the open unit disk. This is indeed the celebrated theorem of Choquet-Rado-Kneser ([**7**]). This theorem is not true for non-convex domains, but holds true under some additional assumptions. It has been extended in various directions (see for example [**12**], [**13**] and [**8**]). Univalent harmonic mappings of $\boldsymbol{U}$ form an interesting and much-studied class of planar maps; see [**6**], [**19**], [**20**] or the book [**7**].

The object of this paper is to study Lindelöf theorem for univalent harmonic mappings f.

1.3. The Lindelöf Theorem.

Let f map $\boldsymbol{U}$ conformally onto the inner domain of a smooth Jordan curve γ. Since the characterization of smoothness in terms of tangent does not depend on the parametrization, we may choose the conformal parametrization

$$[0, 2\pi] \ni \varphi \to f(\varphi) = f(e^{i\varphi}) \in \gamma.$$

An analytic characterization of the smoothness is given by the classical Lindelöf [**18**] theorem:

Proposition 1.1. *Let f map $\boldsymbol{U}$ conformally onto the inner domain of a Jordan curve γ. Then γ is smooth if and only if* $\arg f'(z)$ *has a continuous extension to $\boldsymbol{T}$, which we denote by $f(e^{i\varphi})$. If γ is smooth, then*

$$\arg f'(e^{i\varphi}) = \beta(\varphi) - \varphi - \frac{\pi}{2} \tag{6}$$

where $\beta(\varphi)$ stands for the tangent angle of the curve γ at the point $f(e^{i\varphi})$.

1.4. New result.

The aim of this paper is to prove the following extension of Lindelöf theorem:

Theorem 1.2. *Let $f(z) = \mathcal{P}[F](z)$ be a harmonic mapping of the unit disk, such that F is a homeomorphism of the unit circle $\boldsymbol{T}$ onto a C^1 Jordan curve γ. Assume further that there is a constant $\kappa > 0$ such that*

$$\left|\partial_\varphi F_0(e^{i\varphi})\right|^{-1} \in L^\kappa(\boldsymbol{T}), \tag{7}$$

where $\partial_\varphi F_0(e^{i\varphi})$ is the absolutely continuous part of $\partial_\varphi F(e^{i\varphi})$. Then there is a non-negative real number $R < 1$, such that the function

$$U(z) := \arg\left(\frac{1}{z}\frac{\partial}{\partial\varphi}f(z)\right), \quad z = re^{i\varphi},$$

is well defined and continuous in $R \le |z| < 1$ *and has continuous extension to* $\boldsymbol{T}$ *with*

$$U(e^{i\varphi}) = \beta(\varphi) - \varphi. \tag{8}$$

Here $\beta(\varphi)$ *is the tangent angle of* γ *at* $F(e^{i\varphi})$.

Remark 1.3. a) In order to deduce the classical Lindelöf theorem from Theorem 1.2, observe that, if $f = \mathcal{P}[F]$ is conformal, then in view of the Smirnov theorem (see e.g. [**9**]) F is absolutely continuous in $[0, 2\pi)$ treated as a function of φ. Further, the condition (7), is a priori satisfied for conformal mappings. Indeed a classical result of Warschawski ([**23**]) states that $1/\partial_\varphi F(e^{i\varphi}) \in L^\kappa(\boldsymbol{T})$ for every $\kappa > 0$. Since for $z = re^{i\varphi}$

$$\partial_\varphi f(z) = izf'(z),$$

we infer that

$$\arg(\partial_\varphi f(z)) = \frac{\pi}{2} + \varphi + \arg(f'(z)).$$

Thus on the unit circle we have

$$\arg(f'(z)) = \beta(\varphi) - \frac{\pi}{2} - \varphi,$$

which coincides with (6).

b) The following proof can be applied to merely more general situation. It is enough to assume that the function F is a *covering* of γ in order to obtain that the function

$$V(z) := \arg\left(\frac{1}{e^{i\beta(\varphi)}}\frac{\partial}{\partial\varphi}f(z)\right)$$

is well defined in some ring domain $R \le |z| < 1$ and has continuous vanishing extension in $\boldsymbol{T}$.

c) The condition (7) is implied by the Coifman-Fefferman (A_∞) condition ([**5**], [**21**, p. 168]) for the weight $\Phi(e^{i\varphi}) = |\partial_\varphi F_0(e^{i\varphi})|$, which is equivalent with Muckenhoupt (A_p) condition for some $p > 1$ that can be formulated in the following way: there exists a positive constant M such that

$$\frac{1}{\Lambda(I)}\int_I \Phi(\zeta)|d\zeta| \le M\left(\frac{1}{\Lambda(I)}\int_I \Phi^{-1/(p-1)}(\zeta)|d\zeta|\right)^{1-p}, \tag{9}$$

for all arcs $I \subset \boldsymbol{T}$.

2. The proof of main result.

Proof of Theorem 1.2. We will assume in the proof that $\kappa \ge 1$, however the proof carries out with almost no changes for $0 < \kappa < 1$. Let $z = re^{i\varphi}$. Assume without loosing of generality that $|\gamma| = 2\pi$. We follow some notation from the introduction.

Recall that

$$f(z) = \mathcal{P}[F](z) = \int_0^{2\pi} P(r, t-\varphi) F(e^{it}) dt.$$

Since $\gamma \in C^1$, its arc-length parametrization g is in C^1 and $g'(s) = e^{i\theta(s)}$, where θ is continuous in $[0, 2\pi]$, and in view of the fact $\theta(2\pi) - \theta(0) = 2\pi$, has natural extension to $\mathbf{R}$: $\theta(x + 2k\pi) =: 2k\pi + \theta(x)$, $k \in \mathbf{Z}$. Thus $g'(\psi(t)) = e^{i\theta(\psi(t))} = e^{i\beta(t)}$, where $\beta(t) = \theta(\psi(t))$ is a continuous function in $[0, 2\pi]$ (and in $\mathbf{R}$) with $\beta(2\pi) - \beta(0) = 2\pi$.

Define

$$\begin{aligned} V(re^{i\varphi}) &:= \arg(A(z) + iB(z)) \\ &= \arg\big(e^{-i\beta(\varphi)} \partial_t f(z)\big) \\ &= \arg\left(e^{-i\beta(\varphi)} \int_0^{2\pi} P(r, t-\varphi) dF(e^{it})\right) \\ &= \arg\left(\int_0^{2\pi} P(r, t-\varphi) e^{i(\beta(t)-\beta(\varphi))} d\psi(t)\right) = \arctan\frac{A}{B}, \end{aligned}$$

where

$$A = \int_0^{2\pi} P(r, t) \sin(\beta(t+\varphi) - \beta(\varphi)) d\psi(t+\varphi)$$

and

$$B = \int_0^{2\pi} P(r, t) \cos(\beta(t+\varphi) - \beta(\varphi)) d\psi(t+\varphi).$$

We will prove that V is continuous in $R \le |z| < 1$ and has continuous extension to $\boldsymbol{T}$ with

$$V(e^{i\varphi}) \equiv 0 \tag{10}$$

for certain non-negative number $R < 1$. This statement is equivalent with the main conclusion of our theorem.

Notice that we can obtain with no effort that the radial limit

$$\lim_{r\to 1} e^{-i\beta(\varphi)} \partial_t f(z) = \lim_{r\to 1} \left(e^{-i\beta(\varphi)} \int_0^{2\pi} P(r, t-\varphi) dF(e^{it})\right) = e^{-i\beta(\varphi)} \partial_\varphi F(e^{i\varphi})$$

exists for almost every φ, without the condition (7). Also we can obtain a similar statement for non-tangential limit. However to obtain non-restricted limit, we need some non-trivial approach.

Prove that $\lim_{|z|\to 1} V(z) = 0$, where the limit is unrestricted. We will prove that for given $0 < \epsilon \le 2$ there is $\delta = \delta(\epsilon)$ such that if $0 < 1 - |z| < \delta$ we have

$$\frac{|A(z)|}{|B(z)|} \le \epsilon, \quad B(z) > 0.$$

Since β is continuous, there is $\varepsilon = \varepsilon(\epsilon) > 0$ such that

$$|\sin[\beta(\varphi + t) - \beta(\varphi)]| \le \frac{\epsilon}{8\|d\psi\|_{L^1}\|1/\psi'\|_{L^\kappa}}, \quad |e^{it} - 1| \le \varepsilon. \tag{11}$$

Further, we have

$$\int_{|e^{it}-1|>\varepsilon} P(r,t)d\psi(t+\varphi) \le \frac{1-r^2}{\varepsilon^2}\|d\psi\|. \tag{12}$$

Thus

$$\begin{aligned} |A(z)| &\le \int_0^{2\pi} P(r,t)|\sin[\beta(t+\varphi) - \beta(\varphi)]|d\psi(t+\varphi) \\ &= \int_{|e^{it}-1|>\varepsilon} P(r,t)|\sin[\beta(t+\varphi) - \beta(\varphi)]|d\psi(t+\varphi) \\ &\quad + \int_{|e^{it}-1|\le\varepsilon} P(r,t)|\sin[\beta(t+\varphi) - \beta(\varphi)]|d\psi(t+\varphi) \\ &\le \frac{1-r^2}{\varepsilon^2}\|d\psi\| + \frac{\epsilon}{8\|d\psi\|(\|1/\psi'\|_{L^\kappa})^{\frac{1+\kappa}{\kappa}}}\|d\psi\|. \end{aligned} \tag{13}$$

Since β is continuous there is $\varepsilon > 0$ satisfying (11) and

$$\cos[\beta(t+\varphi) - \beta(\varphi)] > \frac{1}{2}, \quad |e^{it} - 1| \le \varepsilon. \tag{14}$$

So we have

$$\begin{aligned} &\int_{|e^{it}-1|\le\varepsilon} P(r,t)\cos[\beta(t+\varphi) - \beta(\varphi)]d\psi(t+\varphi) \\ &\quad \ge \frac{1}{2}\int_{|e^{it}-1|\le\varepsilon} P(r,t)d\psi(t+\varphi). \end{aligned}$$

To continue observe that $d\psi(t) = \psi_0'(t)dt + d\lambda_s(t)$, which implies the following inequality between measures $\psi_0'(t)dt \le d\psi(t)$ on $[0, 2\pi]$. Since $1/\psi'(t) \in L^\kappa[0, 2\pi]$, and $\psi_0'(t) = \psi'(t)$ for almost every t, then

$$\int_E \frac{1}{(\psi'(t))^\kappa} P(r,t)dt = \int_E \frac{1}{(\psi_0'(t))^\kappa} P(r,t)dt,$$

for every measurable set E. Now by using Hölder inequality with coefficients $p = \kappa + 1$ and $q = (1+\kappa)/\kappa$, we obtain

$$\begin{aligned}
&\int_{|e^{it}-1|\le\varepsilon} P(r,t)dt \\
&\quad \le \left(\int_{|e^{it}-1|\le\varepsilon} P(r,t)\psi_0'(t+\varphi)dt\right)^{\frac{\kappa}{1+\kappa}} \left(\int_{|e^{it}-1|\le\varepsilon} P(r,t)|\psi_0'(t+\varphi)|^{-\kappa}dt\right)^{\frac{1}{1+\kappa}} \\
&\quad \le \left(\int_{|e^{it}-1|\le\varepsilon} P(r,t)d\psi(t+\varphi)\right)^{\frac{\kappa}{1+\kappa}} \left(\int_{|e^{it}-1|\le\varepsilon} P(r,t)|\psi'(t+\varphi)|^{-\kappa}dt\right)^{\frac{1}{1+\kappa}} \\
&\quad \le \left(\int_{|e^{it}-1|\le\varepsilon} P(r,t)d\psi(t+\varphi)\right)^{\frac{\kappa}{1+\kappa}} \left(\int_0^{2\pi} P(r,t)|\psi'(t+\varphi)|^{-\kappa}dt\right)^{\frac{1}{1+\kappa}} \\
&\quad \le \|1/\psi'\|_{L^\kappa}^{\frac{\kappa}{1+\kappa}} \left(\int_{|e^{it}-1|\le\varepsilon} P(r,t)d\psi(t+\varphi)\right)^{\frac{\kappa}{1+\kappa}}.
\end{aligned}$$

Thus

$$\int_{|e^{it}-1|\le\varepsilon} P(r,t)d\psi(t+\varphi) \ge \frac{\left(\int_{|e^{it}-1|\le\varepsilon} P(r,t)dt\right)^{\frac{1+\kappa}{\kappa}}}{\|1/\psi'\|_{L^\kappa}}. \tag{15}$$

By (12) and (15) we obtain that

$$B(z) \ge \frac{\left(\int_{|e^{it}-1|\le\varepsilon} P(r,t)dt\right)^{\frac{1+\kappa}{\kappa}}}{\|1/\psi'\|_{L^\kappa}} - \frac{1-r^2}{\varepsilon^2}\|d\psi\|. \tag{16}$$

From (13) and (16) we obtain

$$\begin{aligned}
\frac{|A(z)|}{|B(z)|} &\le \frac{\dfrac{1-r^2}{\varepsilon^2}\|d\psi\| + \dfrac{\epsilon}{8\|d\psi\|(\|1/\psi'\|_{L^\kappa})^{\frac{1+\kappa}{\kappa}}}\|d\psi\|}{\dfrac{\left(\int_{|e^{it}-1|\le\varepsilon} P(r,t)dt\right)^{\frac{1+\kappa}{\kappa}}}{\|1/\psi'\|_{L^\kappa}} - \dfrac{1-r^2}{\varepsilon^2}\|d\psi\|} \\
&= \frac{\dfrac{1-r^2}{\varepsilon^2} + \dfrac{\epsilon}{8\|d\psi\|(\|1/\psi'\|_{L^\kappa})^{\frac{1+\kappa}{\kappa}}}}{I - \dfrac{1-r^2}{\varepsilon^2}},
\end{aligned} \tag{17}$$

where

$$I = \frac{\left(\int_{|e^{it}-1|\le\varepsilon} P(r,t)dt\right)^{\frac{1+\kappa}{\kappa}}}{\|d\psi\|\|1/\psi'\|_{L^\kappa}}.$$

Since

$$\int_{|e^{it}-1|\le\varepsilon} P(r,t)dt = \frac{2}{\pi}\arctan\left[\frac{1+r}{1-r}\frac{\varepsilon}{\sqrt{2-\varepsilon}\sqrt{2+\varepsilon}}\right]$$

it follows that there is $\rho > 0$ such that for $r > \rho$,

$$\left(\int_{|e^{it}-1|\le\varepsilon} P(r,t)dt\right)^{\frac{1+\kappa}{\kappa}} \ge \frac{1}{2}.$$

We have that for $r > \rho$

$$I \ge \frac{1}{2\|d\psi\|\|1/\psi'\|_{L^\kappa}}.$$

Hence

$$\frac{|A(z)|}{|B(z)|} \le \frac{\dfrac{\epsilon}{8\|d\psi\|\|1/\psi'\|_{L^\kappa}} + \dfrac{1-r^2}{\varepsilon^2}}{I - \dfrac{1-r^2}{\varepsilon^2}}. \tag{18}$$

Also we can assume that ρ is such that for $r > \rho$ we have

$$\frac{1-r^2}{\varepsilon^2} \le \frac{\epsilon}{8\|d\psi\|\|1/\psi'\|_{L^\kappa}}. \tag{19}$$

From (18) and (19) we obtain

$$\frac{|A(z)|}{|B(z)|} \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \tag{20}$$

if $r = |z| > \rho$.

To define R, observe that, from the previous proof we obtain that

$$B(z) > \frac{1}{2\|1/\psi'\|_{L^\kappa}} > 0$$

if $\rho < |z| < 1$, where $\rho = \rho(\epsilon)$, $0 < \epsilon \le 2$. Then we take $R = \rho(2)$, and obtain that the continuous complex function $C(z) = A(z) + iB(z)$ maps the annulus $R \le |z| < 1$ to the upper half-plane, and this means that it allows a continuous argument $\arg(C(z)) := \mathrm{Im}(\log(C(z)))$ in $R \le |z| < 1$. Since

$$\frac{1}{z}\partial_\varphi(f(re^{i\varphi})) = re^{i(\beta(\varphi)-\varphi)}C(z),$$

we obtain that

$$\arg\left(\frac{1}{z}\partial_\varphi(f(re^{i\varphi}))\right) = \beta(\varphi) - \varphi + \mathrm{Im}(\log(C(z)))$$

and this is a well-defined continuous function in $R \le |z| < 1$. □

3. Application.

By definition, K-quasiconformal mappings are orientation preserving homeomorphisms $f : \Omega \to \Omega'$ between domains $\Omega, \Omega' \subset \mathbf{C}$, contained in the Sobolev class $W^{1,2}_{loc}(\Omega)$, for which the differential matrix and its determinant are coupled in the distortion inequality,

$$|Df(z)|^2 \le K \det Df(z), \quad \text{where } |Df(z)| = \max_{|\xi|=1} |Df(z)\xi|, \tag{21}$$

for some $K \ge 1$.

Theorem 3.1. *If $f(z) = \mathcal{P}[F](z) = g(z) + \overline{h(z)}$ is a quasiconformal harmonic mapping of the unit disk onto a Jordan domain bounded by a C^1 convex curve γ, or onto a Jordan domain bounded by a C^2 curve γ, then the function*

$$U(z) := \arg\left(\frac{1}{z}\frac{\partial}{\partial\varphi}f(z)\right)$$

is well defined and smooth in $\boldsymbol{U}^* := \boldsymbol{U} \setminus \{0\}$ and has continuous extension to $\boldsymbol{T}$ with

$$U(e^{i\varphi}) = \beta(\varphi) - \varphi.$$

Here $\beta(\varphi)$ is the tangent angle of γ at $F(e^{i\varphi})$.

Proof. By [**3**, Proposition 1.6.28], every continuous function $P : \Omega \to \mathbf{C}^*$, defined in a simply-connected domain Ω has a unique continuous logarithm, $Q : \Omega \to \mathbf{C}$, satisfying the condition $Q(z_0) = w_0$. This means that $e^{Q(z)} = P(z)$, i.e., $Q(z) = \log P(z)$. Let

$$P(z) = \frac{\partial_\varphi f(z)}{z}.$$

Then

$$P(z) = ia'(z)\left(1 - \frac{\overline{zb'(z)}}{za'(z)}\right).$$

So

$$\log(P(z)) = \log(ia'(z)) - \sum_{k=1}^{\infty} \frac{m(z)^k}{k},$$

where

$$m(z) = \frac{\overline{zb'(z)}}{za'(z)}$$

satisfies the condition

$$|m(z)| \leq \frac{K-1}{K+1} < 1,$$

is a well defined in $\boldsymbol{U}^*$. Since $U(z) = \mathrm{Im}(\log(P(z)))$, it follows that $U(z)$ is well-defined smooth function in $\boldsymbol{U}^*$.

Further to deal with convex case we use a result of the author proved in [**15**]. By [**15**], we have that

$$|Df(z)| \geq \frac{1}{4}\mathrm{dist}(f(0), \gamma)$$

for $z \in \boldsymbol{U}$. As $|Df(z)| = |a'(z)| + |b'(z)|$ and $\partial_\varphi f(z) = i\big(za'(z) - \overline{zb'(z)}\big)$, it follows that

$$|\partial_\varphi f(z)| \geq |z|(|a'(z)| - |b'(z)|) \geq K|z|(|a'(z)| + |b'(z)|) \geq \frac{|z|}{4}\mathrm{dist}(f(0), \gamma).$$

Since

$$\lim_{r\to 1} \partial_\varphi f(z) = \partial_\varphi F(e^{i\varphi})$$

for almost every $\varphi \in [0, 2\pi]$, it follows that $|\partial_\varphi F(e^{i\varphi})|^{-1} \in L^\infty(\boldsymbol{T}) \subset L^1(\boldsymbol{T})$. So for $\kappa = 1$, the conditions of Theorem 1.2 are satisfied.

If γ is not convex but $\gamma \in C^2$, then by the main result of author in [**14**] we have that f is bi-Lipschitz. By using Theorem 1.2 we obtain the desired conclusion. $\square$

The canonical representation of a harmonic mapping is $f = a + \overline{b}$, $b(0) = 0$, where a and b are analytic functions in the unit disk $\boldsymbol{U}$. With the convention that $b(0) = 0$, the representation is unique. The power series expansions of a and b are denoted by

$$a(z) = \sum_{n=0}^{\infty} a_n z^n, \quad b(z) = \sum_{n=1}^{\infty} b_n z^n.$$

If f is a sense-preserving harmonic mapping $\boldsymbol{U}$ onto some other region, then by Lewy's theorem its Jacobian is strictly positive. Equivalently, the inequality $|b'(z)| < |a'(z)|$ holds for all $z \in \boldsymbol{U}$. This shows in particular that $|a'(0)| > 0$, so there is no loss of generality in assuming that $b(0) = 0$ and $a'(0) = 1$. The class of all sense-preserving harmonic mappings of the disk with $a_0 = b_0 = a_1 - 1 = 0$ will be denoted by S_H. Thus, S_H contains the standard class S of analytic univalent functions. Let S_H^0 be the class of normalized harmonic mappings $f(z) = a(z) + \overline{b(z)}$, of the unit disk into $\mathbf{C}$ satisfying the condition $f(0) = 0$ and $a'(0) - 1 = b'(0) = 0$ ([**7**, Chapter 7]). If $f \in S_H$, then

$$f_\circ(z) = \frac{f(z) - \overline{b_1 f(z)}}{1 - |b_1|^2} \in S_H^0.$$

If F is the boundary mapping of f, denote by $F_\circ$ the boundary mapping of $f_\circ$. Also if $\gamma = F(\boldsymbol{T})$ we set $\gamma_\circ = F_\circ(\boldsymbol{T})$. Then it is clear that $\gamma \in C^1$ if and only if $\gamma_\circ \in C^1$. Having these facts in mind we formulate the following application of our main result.

THEOREM 3.2. *Let $f(z)$ be a univalent harmonic mapping of the unit disk onto a Jordan domain bounded by a C^1 curve γ such that $1/\partial_\varphi F \in L^\kappa(\boldsymbol{T})$ for some $\kappa > 0$. Then the function*

$$U(z) := \arg\left(\frac{1}{z}\frac{\partial}{\partial\varphi} f(z)\right)$$

is well defined and smooth in $\boldsymbol{U}^*$ and has a continuous extension to $\partial \boldsymbol{U} = \boldsymbol{T}$ with

$$U(e^{i\varphi}) = \beta(\varphi) - \varphi.$$

Further assume as we may that $f \in S_H$ and let $f_\circ$ be its normalization mapping. Then

$$U_\circ := \arg\left(\frac{1}{z}\frac{\partial}{\partial\varphi} f_\circ(z)\right)$$

is well defined and smooth in $\boldsymbol{U}^*$ and has continuous extension to $\partial \boldsymbol{U}^* = \boldsymbol{T} \cup \{0\}$ with

$$U_\circ(e^{i\varphi}) = \beta_\circ(\varphi) - \varphi, \quad U_\circ(0) = \frac{\pi}{2}.$$

Here $\beta(\varphi)$ and $\beta_\circ(\varphi)$ are the tangent angles of γ at $F(e^{i\varphi})$ and that of $F_\circ(e^{i\varphi})$ at $\gamma_\circ$ respectively.

PROOF. Since $f = \mathcal{P}[F]$ is univalent, F has bounded variation. By following the proof of the previous theorem we obtain the statement of the theorem but for $\boldsymbol{U}^*$. In order to show that the mapping is continuous in 0, observe first that

$$\lim_{z \to 0} ia'(z)\left(1 - \frac{\overline{zb'(z)}}{za'(z)}\right) = ia'(0) = \frac{i}{2}.$$

So $\arg(U(0)) = \arg(ia'(0)) = \pi/2$. □

The following example demonstrates that the condition (7) is important. However it also suggests that a certain generalization of the main result could hold without the a priori condition (7).

EXAMPLE 3.3. Let

$$F(e^{it}) = \begin{cases} 1, & \text{if } 0 \le t \le \pi, \\ e^{2i(t-\pi)}, & \text{if } \pi \le t \le 2\pi. \end{cases}$$

By using an approximation argument and the Choquet–Rado–Kneser theorem, we conclude that $f(z) = \mathcal{P}[F](z)$ is a harmonic diffeomorphism of the unit disk onto itself. Further

$$\partial_t F(e^{it}) = \begin{cases} 0, & \text{if } 0 < t < \pi, \\ 2ie^{2i(t-\pi)}, & \text{if } \pi < t < 2\pi. \end{cases}$$

Next we have $\beta(t) = \pi/2$ if $t \in (0, \pi)$ and $\beta(t) = 2(t - \pi) + \pi/2$ for $t \in [\pi, 2\pi]$. So for $\varphi \in [0, 2\pi]$

$$e^{-i\beta(\varphi)}\partial_t F(e^{it}) = \begin{cases} 0, & \text{if } 0 < t < \pi, \\ 2e^{2i(t-\varphi)}, & \text{if } \pi < \varphi, t < 2\pi, \\ 2e^{2it}, & \text{if } \pi < t < 2\pi,\ 0 < \varphi < \pi. \end{cases}$$

Let

$$W(z) := e^{-i\beta(\varphi)}\partial_\varphi f(z). \tag{22}$$

Then for $z = re^{i\varphi}$ and $\varphi \in [\pi, 2\pi]$

$$\begin{aligned} W(z) &= e^{-i\beta(\varphi)}\mathcal{P}[\partial_t F(e^{it})](z) \\ &= 2e^{-2i\varphi}\left(f(z) - \frac{1}{2}\left(1 + \frac{2}{\pi}\arctan\left[\frac{2r\sin\varphi}{1-r^2}\right]\right)\right). \end{aligned} \tag{23}$$

Hence for $\varphi \in (\pi, 2\pi)$ we have

$$\lim_{z \to e^{i\varphi}} W(z) = 2.$$

Also from (23) it follows that for $\varphi = \pi$ or $\varphi = 2\pi$ we have that

$$\liminf_{z \to e^{i\varphi}, \mathrm{Re}(z) \le 0} W(z) \ge 1.$$

Thus

$$\lim_{z \to e^{i\varphi}, \mathrm{Re}(z) \le 0} \frac{\mathrm{Im}(W(z))}{\mathrm{Re}(W(z))} = M(e^{i\varphi}) = 0.$$

For $\varphi \in [0, \pi]$

$$W(z) = 2\int_{\pi}^{2\pi} \frac{(1-r^2)e^{2ti}}{1+r^2-2r\cos(t-\varphi)}\frac{dt}{\pi}, \tag{24}$$

thus

$$\lim_{z\to e^{i\varphi}, \mathrm{Re}(z)\geq 0} \frac{\mathrm{Im}(W(z))}{\mathrm{Re}(W(z))} = \frac{\displaystyle\int_0^{\pi} \frac{\sin(2t)}{1+\cos(t-\varphi)}dt}{\displaystyle\int_0^{\pi} \frac{\cos(2t)}{1+\cos(t-\varphi)}dt} = M(e^{i\varphi})$$

where

$$M(e^{i\varphi}) = \frac{2\big(-4\cos[\varphi] + 2\cos[2\varphi]\log\big[\cot\big[\frac{\varphi}{2}\big]\big] + \pi\sin[2\varphi]\big)}{2\pi\cos[2\varphi] - \cot\big[\frac{\varphi}{2}\big] + 8\big(1-\cos[\varphi]\log\big[\cot\big[\frac{\varphi}{2}\big]\big]\big)\sin[\varphi] - \tan\big[\frac{\varphi}{2}\big]}.$$

Thus $M(1) = 0 = M(-1)$ and $|M(e^{i\varphi})| \leq 1$. Further we prove that

$$\mathrm{Re}(W(z)) > 0. \tag{25}$$

Thus we have that

$$\lim_{z\to e^{i\varphi}} \arg(W(z)) = \arctan M(e^{i\varphi}). \tag{26}$$

After some elementary transformation we obtain

$$\mathrm{Re}(W(z)) = \int_0^{\pi/4} \frac{m}{n}dx,$$

where

$$\begin{aligned} m &= 2r(1-r^2)\sin x\sin(2x) \\ &\quad\times \big(4(r+r^3)\cos x + \sqrt{2}(1+4r^2+r^4+2r^2\cos(2x)+2r^2\cos(2y))\sin y\big) \end{aligned}$$

and

$$\begin{aligned} n &= \left(1+r^2-2r\cos\left[\frac{\pi}{4}+x+y\right]\right)\left(1+r^2+2r\sin\left[\frac{\pi}{4}-x+y\right]\right) \\ &\quad\times\left(1+r^2-2r\sin\left[\frac{\pi}{4}+x-y\right]\right)\left(1+r^2+2r\sin\left[\frac{\pi}{4}+x+y\right]\right). \end{aligned}$$

Hence $\mathrm{Re}(W(z)) > 0$ in this case.

We conclude that $V(z) = \arg(W(z))$ has a continuous extension to $\boldsymbol{T}$ but the extension does *not* vanish on the upper half-circle $\boldsymbol{T}^+$, contrary to our conclusion (10), i.e., to (8). This implies that our assumption (7) is important.

Remark 3.4. An alternative approach for the proof of Lindelöf theorem for conformal mappings and can be found in the recent monographs [**4**] and [**11**] and in the paper [**10**].

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