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J. Math. Soc. Japan
Vol. 68, No. 2 (2016) pp. 637–651
doi: 10.2969/jmsj/06820637

Triple chords and strong (1, 2) homotopy

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(Received June 4, 2014)

Abstract. A triple chord ⊛ is a sub-diagram of a chord diagram that consists of a circle and finitely many chords connecting the preimages for every double point on a spherical curve. This paper describes some relationships between the number of triple chords and an equivalence relation called strong (1, 2) homotopy, which consists of the first and one kind of the second Reidemeister moves involving inverse self-tangency if the curve is given any orientation. We show that a knot projection is trivialized by strong (1, 2) homotopy, if it is a simple closed curve or a prime knot projection without 1- and 2-gons whose chord diagram does not contain any triple chords. We also discuss the relation between Shimizu's reductivity and triple chords.

1. Introduction.

Sakamoto and Taniyama [**6**] characterized the sub-chord diagrams ⊗ (cross chord) and ⊕ (H chord), embedded in a chord diagram associated with a generic plane curve, where a *chord diagram* is a circle with the preimages of each double point of the curve connected by a chord. For example, a chord diagram of a plane curve contains ⊕, if and only if the plane curve is not equivalent to any connected sum of plane curves, each of which is either the simple closed curve ○, the curve that appears similar to ∞, or a standard torus knot projection [**6**, Theorem 3.2].

This paper aims to obtain a similar characterization of the *triple chord* ⊛, stated in Theorem 1. A *knot projection* is a generic spherical curve that is a regular projection image on S^2 of a knot. For a knot projection P, a chord diagram CD_P is defined as a circle with the preimages of each double point of the knot projection connected by a chord. A knot projection is called *prime*, if it is not a simple closed curve and it is not the connected sum of two knot projections, each of which is not a simple closed curve. Let P^r be a unique knot projection with no 1- or 2-gons obtained by a finite sequence of the first and second Reidemeister moves always decreasing double points in an arbitrary manner for an arbitrary knot projection P (for the uniqueness of P^r, see [**5**], [**3**]).

Theorem 1. *If a chord diagram CD_P of a knot projection P has no triple chord ⊛, and P^r is a prime knot projection or a simple closed curve, then there exists a finite sequence consisting of local replacements* 1a *and* s2a *shown in Figure* 1 *from a simple closed curve* ○ *to P.*

2010 *Mathematics Subject Classification.* Primary 57M25; Secondary 57Q35.

Key Words and Phrases. triple chords, knot projections, spherical curves, strong (1, 2) homotopy.

The first author was partly supported by a Program of WIAS in the Program for the Enhancement of Research Universities, a Waseda University Grant for Special Research Projects (Project number: 2014K-6292) and the JSPS Japanese-German Graduate Externship.

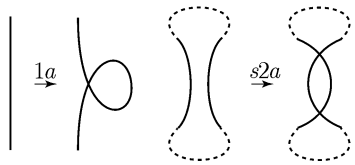


Figure 1. Local replacements $1a$ (left) and $s2a$ (right).
Dotted arcs show the connections of non-dotted arcs.

We define *strong* (1, 2) *homotopy equivalence* as follows: two knot projections P and P' are strong (1, 2) homotopy equivalent, if and only if P is related to P' by a finite sequence consisting of local replacements 1 and $s2$, as shown in Figure 2. Corollary 1 from Theorem 1 helps in understanding the relation between the triple chords and strong (1, 2) homotopy.

COROLLARY 1. *If a chord diagram of an arbitrary prime knot projection P with no* 1- *or* 2*-gons has no triple chord* $\circledast$ *or P is a simple closed curve, then P is equivalent to a simple closed curve* $\bigcirc$ *under strong* (1, 2) *homotopy.*

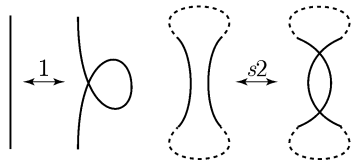


Figure 2. Local replacements 1 (left) and $s2$ (right).

The reminder of this paper contains the following sections. Section 2 states our conventions. Section 3 and Section 4 provide proofs of Theorems 1 and 2, respectively. Section 5 mentions a relation between Shimizu's reductivity of knot projections and the triple chord.

2. Preliminary.

Reidemeister moves, which are three local replacements on an arbitrary knot projection, are defined by Figure 3. It is known that there exists a finite sequence of Reidemeister moves between any two knot projections. Shown left to right in Figure 3 are the first, second, and third Reidemeister moves. There are two types of the second Reidemeister moves, local replacement, $s2$, shown in Figure 2, and $w2$, shown in Figure 4. Now, we define the notion of *reducible* and *reduced* knot projection.

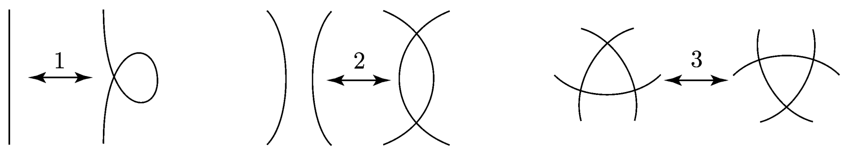


Figure 3. First (left), second (center), and third (right) Reidemeister moves.

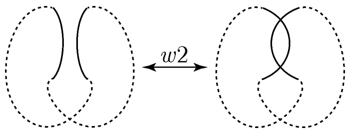


Figure 4. Local replacement $w2$. Dotted arcs show the connections of non-dotted arcs.

Definition 1 (Reducible and reduced knot projection). A knot projection P is *reducible*, if there is a double point d, called a *reducible crossing*, in P, as shown in Figure 5. If a knot projection is not reducible, it is called a *reduced knot projection.*

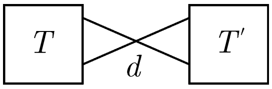


Figure 5. Reducible crossing d.

From this definition, we obtain Lemma 1, which is easy to prove and is used often throughout this paper.

Lemma 1. *An arbitrary prime knot projection with no* 1*-gons is a reduced knot projection.*

Proof. To establish the claim, it is sufficient to show that ($\star$) if an arbitrary knot projection with no 1-gons is reducible, then knot projection is non-prime. We will now show ($\star$). Let P be an arbitrary knot projection with no 1-gons. Assume that P is reducible. Then, P can be presented in Figure 5. If the two faces having the point d of P, as in Figure 5, are not 1-gons, then T and T' are not simple arcs. Thus, P is non-prime. This completes the proof. □

3. Proof of Theorem 1.

To establish Theorem 1, we prove Theorem 2.

Theorem 2. *A prime knot projection P with no* 1*- or* 2*-gons has at least one triple chord in CD_P.*

Now, we deduce Theorem 1 from Theorem 2.

Proof. Based on our assumption in Theorem 1, a knot projection P has no triple chords in CD_P. For P, we can consider P^r, the unique knot projection with no 1- or 2-gons by a finite sequence consisting of the first and second Reidemeister moves decreasing the number of double points [**5**, Theorem 2.2] or [**3**, Theorem 2.2]. By the assumption of Theorem 1, P^r is a prime knot projection with no 1- and 2-gons or a simple closed curve $\bigcirc$. Thus, by Theorem 2, $P^r = \bigcirc$.

Recover P from P^r using the sequence consisting of $1a$ and $2a$, where $1a$ (resp. $2a$) is the first (resp. second) Reidemeister move always increasing a double point (resp. double points). If at least one $2a$ in the sequence is $w2$, then there exists at least one triple chord $\circledast$ in CD_P. This is because a 2-gon raised by $w2$ can be represented as shown in

Figure 6, and the corresponding chord diagram is shown at the left of Figure 6. We can see that the point contained in both dotted arcs exists. Thus, we can find ⊛ in CD_P.

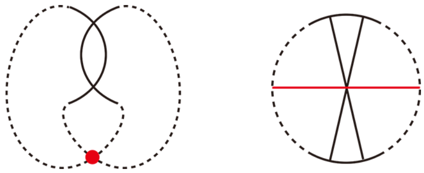


Figure 6. 2-gon appearing in $w2$ (left) and its chord diagram (right).

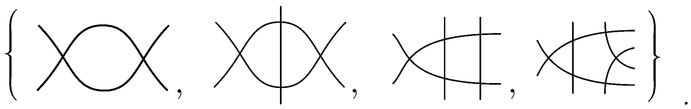
However, the existence of ⊛ contradicts the assumption that P has no triple chords in CD_P. Thus, the sequence consisting of $1a$ and $2a$ must consist of $1a$ and $s2a$. We conclude that Theorem 2 implies Theorem 1. □

In the next section, we present the proof of Theorem 2.

4. Proof of Theorem 2.

To prove Theorem 2, we first recall Fact 1. Fact 1 and its proof were obtained by A. Shimizu [**7**, Proof of Proposition 3.1]. A knot projection that is not a simple closed curve ○ is called a *non-trivial* knot projection.

Fact 1 ([**7**]). *A non-trivial reduced knot projection P contains at least one element of the following set*:



We must also check the following Lemma 2.

Lemma 2. (a): *A non-trivial knot projection with no* 1- *or* 2-*gons has at least eight* 3-*gons.* (*cf.* [**1**, Theorem 2.2].)

(b): *If a prime knot projection P with no* 1- *or* 2-*gons has at least one* 3-*gon in* {A, B, C} *in the following, then P has a triple chord in CD_P.*

Proof. (a): Let V be the number of double points (i.e., vertices), E the number of edges, and F the number of faces. Let p_k be the number of k-gons. For a non-trivial

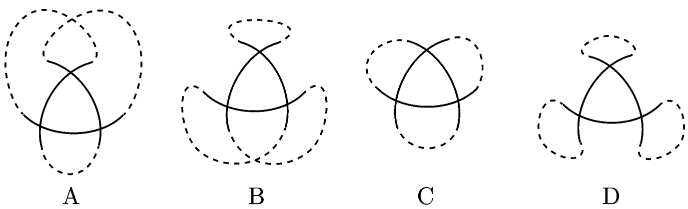


Figure 7. All types of 3-gons. Dotted arcs show the connections of arcs.

knot projection P with no 1- or 2-gons,

$$\begin{aligned}\sum_{k\geq 3} kp_k &= 2E,\\ \sum_{k\geq 3} p_k &= F.\end{aligned} \tag{1}$$

Now, we consider knot projections that are graphs on S^2 such that every vertex has four edges. Thus,

$$\begin{aligned}4V &= 2E,\\ V - E + F &= 2.\end{aligned} \tag{2}$$

Formula (2) implies $4F - 2E = 8$. Substituting formula (1) into $2E$ and F of $4F - 2E = 8$, we have

$$p_3 + \sum_{k\geq 4}(4-k)p_k = 8.$$

Then, we have $p_3 \geq 8$. This completes the proof.

(b):
- A-type 3-gon. Observe the figure of the spherical curve that contains dotted arcs and an A-type 3-gon shown in Figure 7. From the assumption, a knot projection P containing an A-type 3-gon is prime. Then, the α-part of Figure 8 must intersect at least one of the other dotted arcs. Similar to Figure 8, there exists $\circledast$ in CD_P.

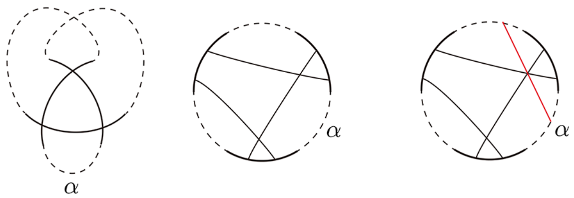


Figure 8. A-type 3-gon having the dotted arc labeled α (left), the corresponding chord diagram (center), and chord diagram with a triple chord (right).

- B-type 3-gon. Note the spherical curve P that contains dotted arcs and a B-type 3-gon shown in Figure 7. The corresponding chord diagram CD_P is shown at the right of Figure 9. In CD_P, we can find $\circledast$, since there are two dotted arcs in a B-type 3-gon that must intersect (Figure 9, left).
- C-type 3-gon. If a knot projection P contains a C-type 3-gon, then CD_P immediately has a triple chord.

The consideration of the three cases completes the proof. □

Now, we prove Theorem 2.

Proof. By Fact 1, a knot projection P, which we have considered, contains at

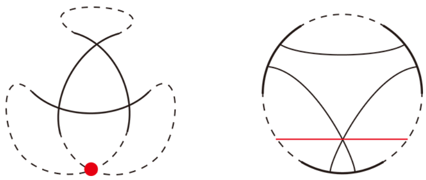


Figure 9. B-type 3-gon, which must have a marked double point, and its chord diagram.

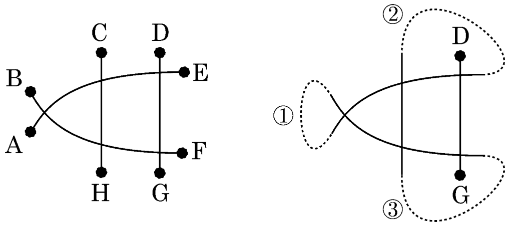


Figure 10. Third element (left) of the set of Fact 1 and the case having a D-type 3-gon (right). Dotted arcs show the connections of non-dotted arcs.

least one of the elements mentioned in Fact 1. Thus, we consider the possibilities that P contains the first, second, third, or fourth of those elements. In what follows, checking the possibility of the first (resp. second, third, or fourth) element is called *the first* (*resp. second, third, or fourth*) *element case.*

The first element case. By assumption, P has no 2-gon. Thus, there is no possibility of the existence of the first element of the set of Fact 1.

The second element case. If P has the second element (i.e., two neighboring 3-gons) from the left-hand side of the set shown in Fact 1, assume that one of the two neighboring 3-gons is D type. In this case, another 3-gon in the two neighborhood 3-gons is type B. This implies that P has at least a type A, B, or C 3-gon, from which we conclude that P has triple chords in CD_P by Lemma 2. Thus, it is sufficient to consider the two cases of the third or the fourth figure from the left-hand side in the set of Fact 1. Below, we consider these figures.

The third element case. By Lemma 2, if a knot projection contains the part shown in Figure 10, we can assume that the 3-gon is type D from that figure. Since the 3-gon is type D, dotted arcs arise as shown at the right-hand side. Thus, we distinguish the following cases in which a dotted arc contains the arc DG shown in the figure:

- Arc number 1 contains DG (Case A, B),
- Arc number 2 (or 3) contains DG (Case C, D).

In the remainder of the proof, the symbol (X, Y) (resp. $(x \sim y)$) means we connect a point X with a point Y (resp. a vertex x with a vertex y) via a route outside the fixed part of a knot projection, e.g., as seen below, Case A and Figure 11.

- Case A is defined by (A, G), (B, D), (C, E), and (F, H). See Figure 11.
- Case B is defined by (A, D), (B, G), (C, E), and (F, H). See Figure 12.
- Case C is defined by (A, B), (C, D), (E, G), and (F, H). See Figure 13.

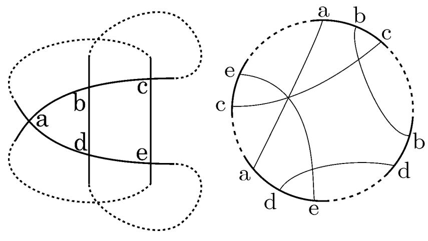


Figure 11. Case A.

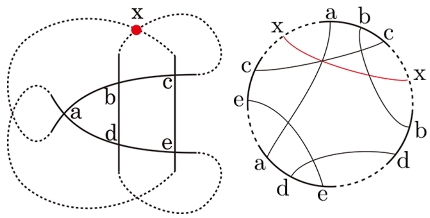


Figure 12. Case B.

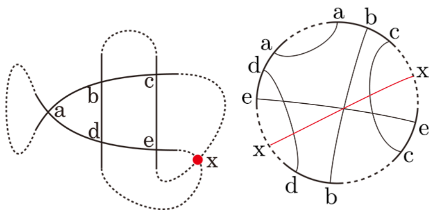


Figure 13. Case C.

- Case D is defined by (A, B), (C, G), (D, E), and (F, H) as shown in Figure 14. This knot projection P is a prime knot projection with no 1- or 2-gons; hence, P is reduced (Lemma 1). Thus, (a$\sim$a) intersects another dotted arc $(*)$. If $(*)$ is (b$\sim$e) or (d$\sim$e), P has a triple chord in CD_P. If $(*)$ is neither (b$\sim$e) nor (d$\sim$e), but is (c$\sim$c), the knot projection P and its CD_P appears as Figure 15, and thus, there exists a triple chord in CD_P.

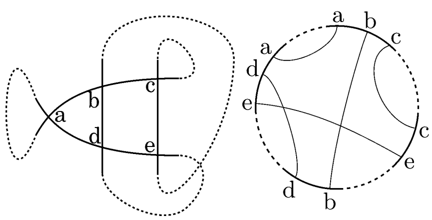


Figure 14. Case D.

In summary, if a knot projection P has the third element of the set of Fact 1, then P has a triple chord in CD_P. Then, we are left with only the case of the fourth element of the set of Fact 1.

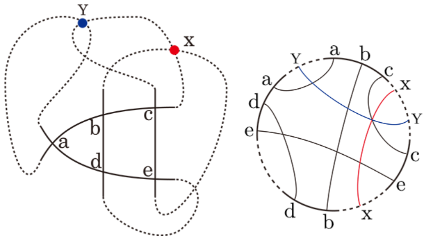


Figure 15. Instance of Case D.

The fourth element case. Start by setting the symbols for points to be connected and vertices as in Figure 16. By Lemma 2, we can fix the 3-gon in Figure 16 as type D. Then, we can draw dotted arcs as in the figure. Next, we consider the dotted arcs that contain the non-dotted arcs DG and FI. Based on this consideration, we prove the claim case by case. Since there is the symmetry between arc numbers 2 and 3, it is sufficient to consider the following four groups, each of which contains eight cases (in total, 32 cases). The points of grouping are as follows.

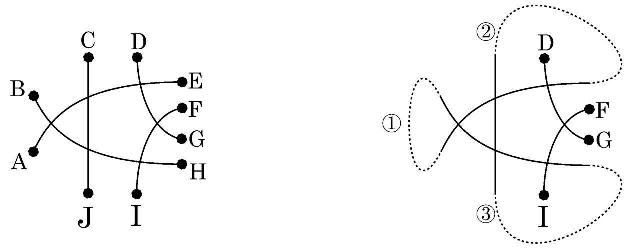
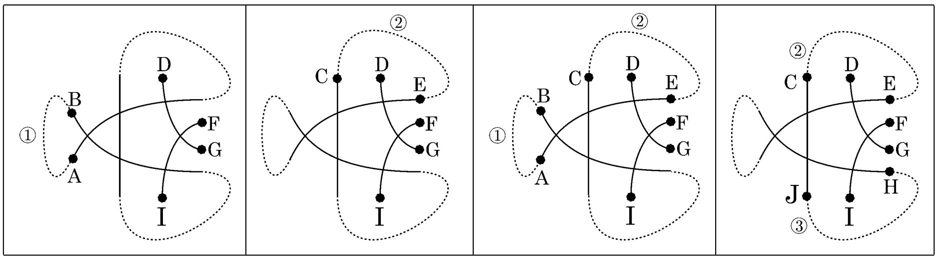


Figure 16. Fourth element (left) of the set of Fact 1 and the case having a D-type 3-gon (right). Dotted arcs show the connections of non-dotted arcs.

Table 1. Each of four groups having eight cases. Dotted arcs show the connections of arcs.



- Dotted arc number 1 contains both DG and FI (Cases 1–8). This condition fixes (C, E) and (H, J).
- Dotted arc number 2 contains both DG and FI (Cases 9–16). This condition fixes (A, B) and (H, J).
  (Replacing 2 with 3 replicates the discussion as a result of their symmetry; hence, we omit the respective cases.)
- Arc number 1 contains exactly one non-dotted arc (i.e., DG or FI), and arc number 2 contains exactly one non-dotted arc (Cases 17–24). This condition fixes (H, J).

(Replaying 2 with 3 replicates the discussion as a result of their symmetry, hence we omit the respective cases.)

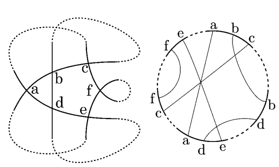
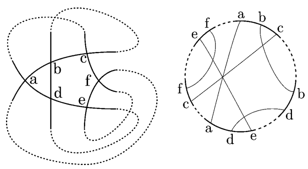
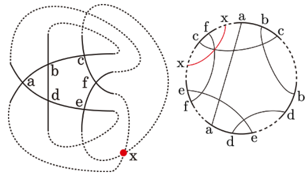
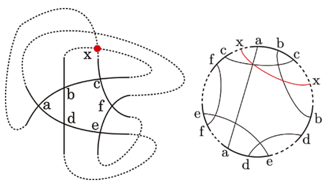
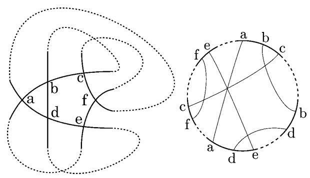
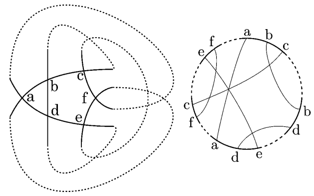
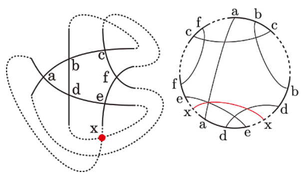
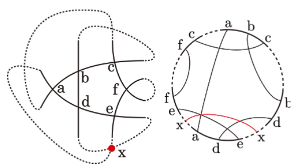
- Arc number 2 contains exactly one non-dotted arc (i.e., DG or FI) and arc number 3 contains exactly one non-dotted arc (Cases 25–32). This condition fixes (A, B).

Cases 1–8. If arc number 1 contains both two arcs DG and FI, then we can automatically fix (C, E) and (J, H) (Table 1, Cases 1–8). Table 2 shows how arcs connect, considering all possibilities. Recall that the symbol "(X, Y)" means that we connect X and Y. For every case 1–8, a knot projection P has at least one triple chord in CD_P. See Table 3.

Table 2. Method to split into Cases 1–8.

(B, D)	(G, F)(I, A) (Case 1) (G, I)(F, A) (Case 2)	(B, F)	(I, D)(G, A) (Case 3) (I, G)(D, A) (Case 4)
(B, G)	(D, F)(I, A) (Case 5) (D, I)(F, A) (Case 6)	(B, I)	(F, D)(G, A) (Case 7) (F, G)(D, A) (Case 8)

Table 3. Easy cases to prove. Cases 1–8.

Case 1	Case 2	Case 3
		
Case 4	Case 5	Case 6
		
Case 7	Case 8	
		

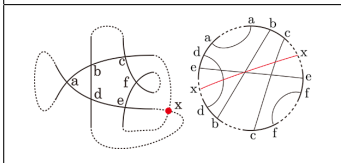
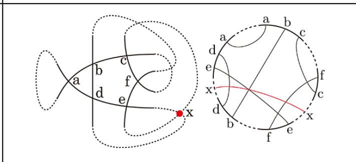
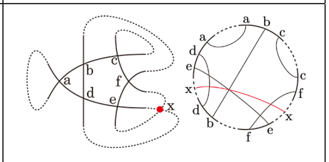
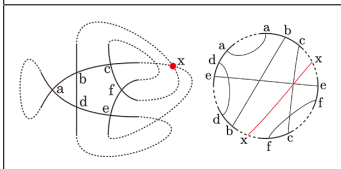
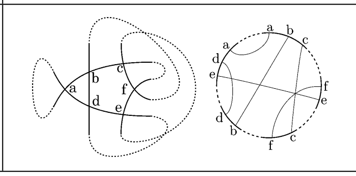
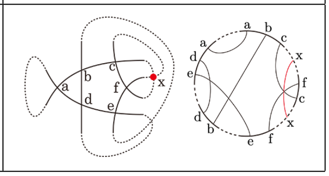
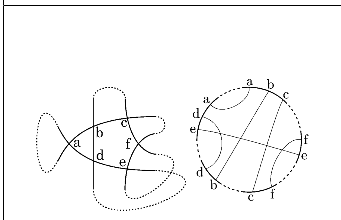
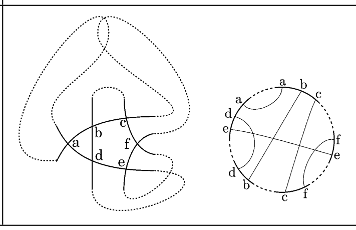
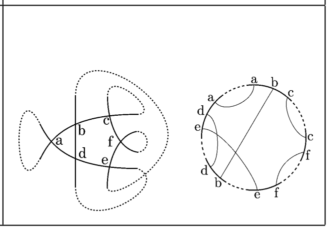
Cases 9–18. If arc number 2 contains both arcs DG and FI, we can automatically fix connections (A, B) and (H, J) (Table 1, Cases 9–16). Table 4 shows how arcs connect,

considering all possibilities. Except for Cases 10 and 16, the existence of a triple chord is directly proved by Table 5.

Table 4. Method to split into Cases 9–16.

(C, D)	(G, F)(I, E) (Case 9) (G, I)(F, E) (Case 10)	(C, F)	(I, D)(G, E) (Case 11) (I, G)(D, E) (Case 12)
(C, G)	(D, F)(I, E) (Case 13) (D, I)(F, E) (Case 14)	(C, I)	(F, D)(G, E) (Case 15) (F, G)(D, E) (Case 16)

Table 5. Cases easily proved: Case 9, Cases 11–15. Non-easy cases: Case 10 and its additional figure Case 10a, Case 16.

Case 9	Case 11	Case 12
		
Case 13	Case 14	Case 15
		
Case 10	Case 10a	Case 16
		

Case 10 (not easily proved). Observe the figure in Case 10 on the bottom line of Table 5. First, this knot projection P is a prime knot projection with no 1- or 2-gons. Thus, P is reduced (Lemma 1). Therefore, the dotted arc (a$\sim$a) must intersect at least one of the other dotted arcs. If (a$\sim$a) intersects (b$\sim$c), (d$\sim$e), or (e$\sim$f), then P has a triple chord in CD_P. Therefore, we can assume that (a$\sim$a) intersects (c$\sim$f). In this case, observe the figure in Case 10a on the bottom line of Table 5.

Next, since the knot projection we considered is a prime knot projection with no 1- or 2-gons, the dotted arc (b$\sim$c) intersects at least one of the other dotted arcs.

- If (b$\sim$c) intersects (a$\sim$a) or (c$\sim$f), then P has triple chords in CD_P.

- If (b∼c) intersects (e∼f), but not (a∼a) or (c∼f), then (e∼f) intersects (a∼a) or (c∼f). However, in each of these two cases, P has a triple chord in CD_P.
- If (b∼c) intersects (d∼e), but not (a∼a) or (c∼f), then (d∼e) intersects (a∼a) or (c∼f). However, in each of these two cases, P has a triple chord in CD_P.

Therefore, when (b∼c) intersects another dotted arc, a knot projection P that we considered has a triple chord in CD_P.

Case 16 (not easily proved). Observe the right-bottom figure of Table 5. The existence of a triple chord this case is proved in the same way as Case D, by omitting dotted 1-gons (f∼f). Compare Figure 14 with Case 16 in Table 5.

Cases 17–24. Dotted arc numbers 1 and 2 each contains an instance of DG and FI. This case implies fixing (H, J), i.e., H must connect with J (Table 1, Cases 17–24). In this case, Table 6 shows how the case is split into eight cases, and it is easy to show that each knot projection P of those cases has a triple chord in CD_P. See Table 7.

Table 6. Method to split into Cases 17–24.

(B, D)(G, A) $\begin{cases}\text{(C, F)(I, E)} & \text{(Case 17)}\\ \text{(C, I)(F, E)} & \text{(Case 18)}\end{cases}$	(B, G)(D, A) $\begin{cases}\text{(C, F)(I, E)} & \text{(Case 19)}\\ \text{(C, I)(F, E)} & \text{(Case 20)}\end{cases}$
(B, F)(I, A) $\begin{cases}\text{(C, D)(G, E)} & \text{(Case 21)}\\ \text{(C, G)(D, E)} & \text{(Case 22)}\end{cases}$	(B, I)(F, A) $\begin{cases}\text{(C, D)(G, E)} & \text{(Case 23)}\\ \text{(C, G)(D, E)} & \text{(Case 24)}\end{cases}$

Cases 25–32. Dotted arc nsumbers 2 and 3 each contains exactly one of the non-dotted arcs. Hence, we can automatically fix the connection (A, B) (Table 1, Cases 25–32). Table 8 shows how arcs connect considering all possibilities.

Case 25 (not easily proved). Observe the figure for Case 25 in the lower part of Table 9. First, since the considered knot projection P is reduced (cf. Lemma 1), the dotted arc (a∼a) intersects one of the other dotted arcs. If (a∼a) intersects (b∼c) or (d∼e), then P has a triple chord in CD_P. Therefore, we can assume that (a∼a) intersects (c∼f) (the figure Case 25a of Table 9). Here, note that the assumption that (a∼a) intersects (e∼f) is equivalent to the assumption that (a∼a) intersects (c∼f) by symmetry, hence we omit the case (a∼a) intersects (e∼f).

Next, consider Case 25a in Table 9. Since P is a prime knot projection with no 1- or 2-gons, (b∼c) must intersect one of the other dotted arcs.

- If (b∼c) intersects (a∼a) or (c∼f), then P has a triple chord in CD_P.
- If (b∼c) intersects (e∼f), then P has a triple chord in CD_P.
- If (b∼c) intersects (d∼e), but not (a∼a) or (c∼f), then (d∼e) must intersect (a∼a) or (c∼f). However, whether (d∼e) intersects (a∼a) or (c∼f), P has a triple chord in CD_P.

Thus, if (b∼c) intersects one of the other dotted arcs, a considered knot projection P has a triple chord in CD_P.

Case 28. See the bottom line of Table 9. By the assumption, the considered knot

Table 7. Cases easily proved.

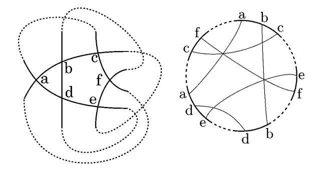
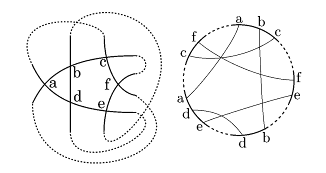
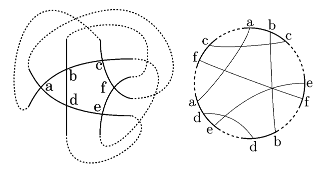
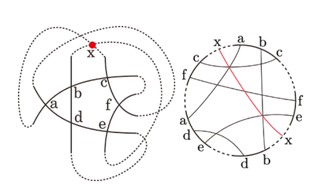
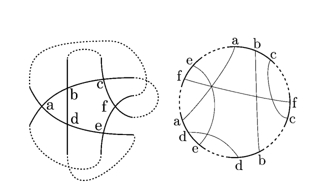
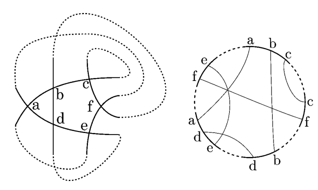
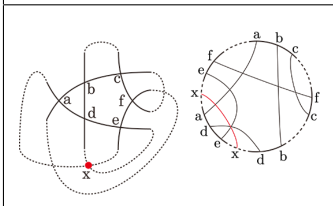
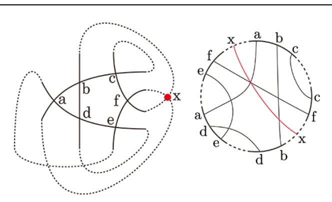
Case 17	Case 18	Case 19
		
Case 20	**Case 21**	**Case 22**
		
Case 23	**Case 24**	
		

Table 8. Method to split into Cases 25–32.

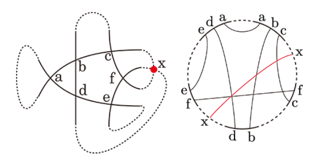
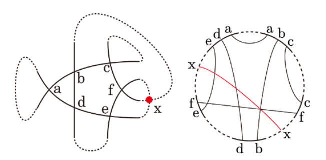
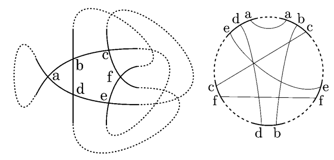
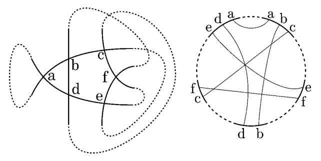
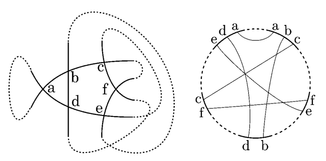
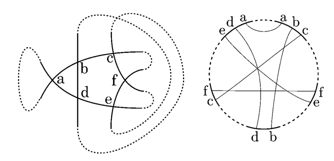
(C, D)(G, E)	(H, F)(I, J) (Case 25) (H, I)(F, J) (Case 26)	(C, G)(D, E)	(H, F)(I, J) (Case 27) (H, I)(F, J) (Case 28)
(C, F)(I, E)	(H, D)(G, J) (Case 29) (H, G)(D, J) (Case 30)	(C, I)(F, E)	(H, D)(G, J) (Case 31) (H, G)(D, J) (Case 32)

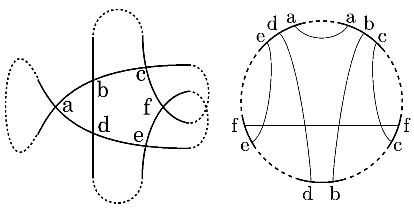
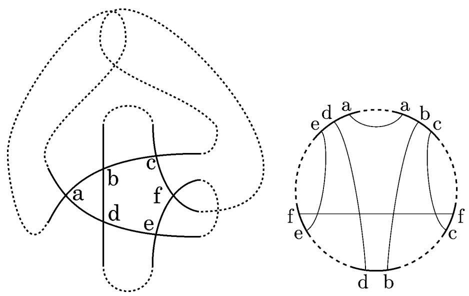
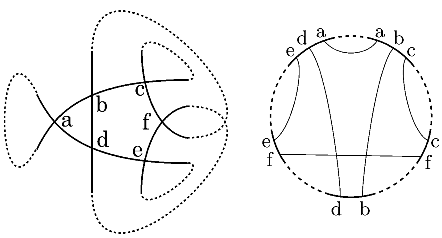
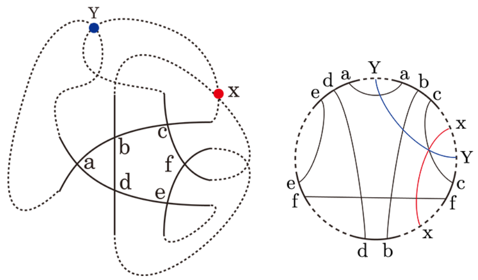
projection P is a prime knot projection with no 1- or 2-gons. Thus, P is reduced (Lemma 1), and the dotted arc (a∼a) intersects the other dotted arcs.

- If (a∼a) intersects (b∼f), then P has a triple chord in CD_P.
- If (a∼a) intersects (d∼f), then P has a triple chord in CD_P.
- If (a∼a) intersects (c∼c), but not (b∼f) or (d∼f), then P has a triple chord in CD_P, as shown in the bottom line of Table 9.
- If (a∼a) intersects (e∼e), but not (b∼f) or (d∼f), then P has a triple chord in CD_P by the same reasoning as that of (c∼c) via their symmetry between (c∼c) and (e∼e).

These 32 cases complete the proof of Theorem 2. □

Table 9. Cases easily proved: Cases 26, 27 and Cases 29–32. Non-easy cases: Case 25 and its additional figure Case 25a, Case 28.

Case 26	Case 27	Case 29
		
Case 30	Case 31	Case 32
		

Case 25	Case 25a
	
Case 28	Case 28a
	

5. Reductivity and Triple chords.

This section mentions a relation between the reductivity of a knot projection and triple chords. The reductivity is introduced by A. Shimizu [**7**, Section 1] using local replacement A^{-1} (also called A^{-1} move in this paper) that appears in [**2**].

Definition 2. The local replacement A^{-1} move at a double point is defined by Figure 17. The reductivity $r(P)$ of a knot projection is the minimal number of A^{-1}

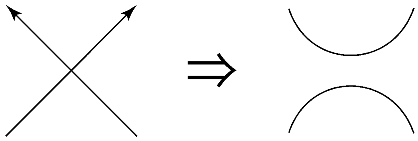


Figure 17. A^{-1} move.

moves to produce a reducible knot projection.

Remark 1. It is worthwhile mentioning the following fact here. Any non-trivial reduced knot projections are related by a finite sequence consisting of A^{-1} moves and inverse moves, where every knot projection appearing in each step in the sequence is reduced [**2**, Corollary 1.2]. Therefore, it is natural to consider the notion of reductivity [**7**].

We characterize knot projections with $r(P) = 1$.

Theorem 3. *For a non-trivial reduced knot projection P, there exists a circle with two double points as shown in Figure* 18 *if and only if* $r(P) = 1$.

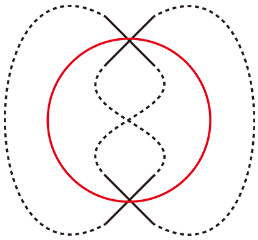


Figure 18. Circle with two double points splitting S^2 into two disks. Dotted arcs indicate the connections of branches.

Proof. • (*If part*) Assume that $r(P) = 1$. Let P' be a reducible knot projection obtained from P by applying an A^{-1} move at a double point, say a, of P, and b a reducible crossing of P'. Then it follows by definition that there exists a simple circle which intersects P with a and b only. There are four cases with respect to the connectivity among the paths at a and b as shown in Figure 19. Since P is an immersion of a single circle, we have the second and third cases.

- (*Only if part*) Assume that a reduced knot projection P has two double points as shown in Figure 18. By applying an A^{-1} move at one of the double points, we have a reducible knot projection. □

Corollary 2. *A knot projection P with $r(P) = 1$ has at least one triple chord in CD_P.*

Proof. If P satisfies $r(P) = 1$, then it has two double points, say a and b, as shown in Figure 18. Let x be a double point of P in the region surrounded by the simple circle. Then the double points a, b, and x gives a triple chord in CD_P. □

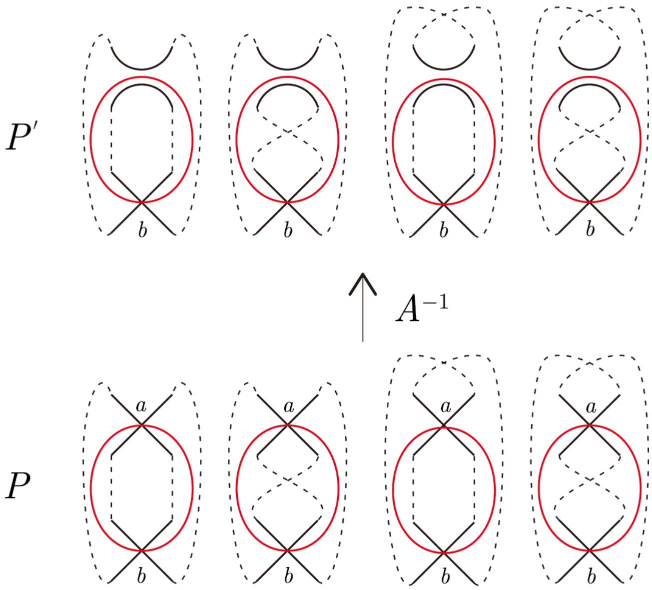


Figure 19. All the possibilities of the distributions of arcs and the circle splitting S^2 into two disks after applying one A^{-1} move (upper) and all the possibilities before applying A^{-1} (lower).

Acknowledgements. The authors would like to thank Professor Kouki Taniyama for his helpful comments.

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