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Universal curvature identities and Euler–Lagrange formulas for Kähler manifolds

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Abstract. We relate certain universal curvature identities for Kähler manifolds to the Euler–Lagrange equations of the scalar invariants which are defined by pairing characteristic forms with powers of the Kähler form.

1. Introduction.

Throughout this paper, we shall assume that (M,g) is a compact smooth oriented Riemannian manifold of dimension $2m$. Let $d\nu_g$ be the Riemannian volume m-form. In the introduction, we will establish the notation that will enable us to state the two main results of this paper – Theorem 1.2 (which describes the symmetric 2-tensor valued universal curvature identities in the Kähler setting) and Theorem 1.3 (which gives the Euler–Lagrange equations for the scalar invariants defined by pairing characteristic forms with powers of the Kähler form in the Kähler setting). These two Theorems extend previous results from the real setting to the Kähler setting as we shall discuss subsequently in Remark 1.2.

1.1. Kähler geometry.

A *holomorphic structure* on M is an endomorphism J of the tangent bundle TM so that $J^2 = -\operatorname{id}$ and so that there exist local *holomorphic coordinate charts* $(x^1,\dots,x^m, y^1,\dots,y^m)$ covering M satisfying

$$J\partial_{x_\alpha} = \partial_{y_\alpha} \quad \text{and} \quad J\partial_{y_\alpha} = -\partial_{x_\alpha} \quad \text{for} \quad 1 \le \alpha \le m.$$

Equivalently, via the Newlander–Nirenberg Theorem [**22**], this means that the Nijenhuis tensor N_J vanishes where one defines (see [**6**]):

$$N_J(X,Y) := [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY].$$

In a system of holomorphic coordinates, we define for $1 \le \alpha \le m$:

$$\begin{aligned} &z^\alpha := x^\alpha + \sqrt{-1}y^\alpha, \\ &\partial_{z_\alpha} := \frac{1}{2}(\partial_{x_\alpha} - \sqrt{-1}\partial_{y_\alpha}), \quad \partial_{\bar z_\alpha} := \frac{1}{2}(\partial_{x_\alpha} + \sqrt{-1}\partial_{y_\alpha}), \end{aligned}$$

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$$dz^\alpha := dx^\alpha + \sqrt{-1}dy^\alpha, \qquad d\bar{z}^\alpha := dx^\alpha - \sqrt{-1}dy^\alpha.$$

Extend J to be complex linear on the complexified tangent bundle to obtain:

$$J\partial_{z_\alpha} = \sqrt{-1}\partial_{z_\alpha} \text{ and } J\partial_{\bar{z}_\alpha} = -\sqrt{-1}\partial_{\bar{z}_\alpha}.$$

We can decompose the bundles S^2M and Λ^2M of symmetric and anti-symmetric bilinear forms as $S^2M = S^2_+M \oplus S^2_-M$ and $\Lambda^2M = \Lambda^2_+M \oplus \Lambda^2_-M$ where

$$S^2_\pm M := \{h \in S^2M : J^*h = \pm h\} \text{ and } \Lambda^2_\pm M := \{h \in \Lambda^2M : J^*h = \pm h\}.$$

A symmetric bilinear form $h \in S^2_+M$ is said to be *Hermitian*; if h is Hermitian, then associated *Kähler form* $\Omega_h \in \Lambda^2_+M$ is given by setting:

$$\Omega_h(x,y) := h(x, Jy).$$

Conversely, given $\Omega \in \Lambda^2_+M$, we can recover $h = h_\Omega$ by setting $h(x,y) = \Omega(x, -Jy)$. This correspondence defines a natural isomorphism between S^2_+M and Λ^2_+M.

A triple $\mathcal{M}^m := (M, g, J)$ is said to be a *Hermitian* manifold if $g \in C^\infty(S^2_+M)$ is positive definite (and thus defines a Riemannian metric on M) and if (M, J) is a holomorphic of complex dimension m. Let $\Omega = \Omega_g$. We then have that

$$d\nu_g = \frac{1}{m!}\Omega^m. \tag{1.a}$$

A Hermitian manifold $\mathcal{M}^m$ is said to be a *Kähler manifold* if $d\Omega = 0$. Let ∇ be the Levi–Civita connection and let

$$\mathcal{R}(x,y) := \nabla_x\nabla_y - \nabla_y\nabla_x - \nabla_{[x,y]} \text{ and } R(x,y,z,w) := g(\mathcal{R}(x,y)z, w)$$

be the *curvature operator* and the *curvature tensor*, respectively. We shall also denote these tensors by $\mathcal{R}_\mathcal{M}$ and $R_\mathcal{M}$ when it is necessary to emphasize the role that $\mathcal{M}^m$ plays. If $\mathcal{M}^m$ is a Kähler manifold, then $\nabla J = 0$ and we have an additional curvature symmetry called the *Kähler identity*:

$$\mathcal{R}(x,y)J = J\mathcal{R}(x,y) \text{ i.e. } R(x,y,z,w) = R(x,y,Jz,Jw). \tag{1.b}$$

1.2. The characteristic classes and characteristic numbers.

Let $M_m(\mathbb{C})$ be the matrix algebra of all $m \times m$ complex matrices and let the associated general linear group be $GL_m(\mathbb{C}) \subset M_m(\mathbb{C})$. Let $\mathfrak{S}_m$ be the ring of polynomial maps from $M_m(\mathbb{C})$ to $\mathbb{C}$ which are invariant under the action of $GL_m(\mathbb{C})$, i.e. $\mathcal{S} \in \mathfrak{S}_m$ if and only if

$$\mathcal{S}(ABA^{-1}) = \mathcal{S}(B) \text{ for all } A \in GL_m(\mathbb{C}) \text{ and for all } B \in M_m(\mathbb{C}).$$

Define $\mathrm{Tr}_\mu \in \mathfrak{S}_m$ by setting $\mathrm{Tr}_\mu(B) := \mathrm{Tr}(B^\mu)$. We then have:

$$\mathfrak{S}_m = \mathbb{C}[\mathrm{Tr}_1, \ldots, \mathrm{Tr}_m]. \tag{1.c}$$

Let $\mathfrak{S}_{m,k} \subset \mathfrak{S}_m$ be the finite dimensional subspace of maps which are homogeneous of degree k. We may then decompose

$$\mathfrak{S}_m = \oplus_k \mathfrak{S}_{m,k}.$$

DEFINITION 1.1. Let k be a positive integer. A *partition* π of k is a decomposition of $k = n_1 + \cdots + n_\ell$ as the sum of positive integers where we order $n_1 \geq \cdots \geq n_\ell \geq 1$. Let $\rho(k)$ be the *partition function*; this is the number of distinct partitions π of k. We use Equation (1.c) to see that a basis for $\mathfrak{S}_{m,k}$ consists of all monomials of the form $\mathrm{Tr}_1^{\nu_1} \cdots \mathrm{Tr}_m^{\nu_m}$ where $\nu_1 + 2\nu_2 + \cdots + m\nu_m = k$. Consequently

$$\dim\{\mathfrak{S}_{m,k}\} = \rho(k) \text{ if } k \leq m. \tag{1.d}$$

Let $n < m$ and let $B_n \in M_n(\mathbb{C})$. Let 0_ℓ be the additive unit of $M_\ell(\mathbb{C})$. The natural map $B_n \mapsto B_n \oplus 0_{m-n}$ defines an inclusion of $M_n(\mathbb{C})$ into $M_m(\mathbb{C})$ and induces dually a *restriction map* $r_{m,n} : \mathfrak{S}_m \to \mathfrak{S}_n$ which is characterized by the identity:

$$\{r_{m,n}(\mathcal{S}_m)\}(B_n) := \mathcal{S}_m(B_n \oplus 0_{m-n}). \tag{1.e}$$

REMARK 1.1. Let $n < m$. Since the restriction map preserves the grading, $r_{m,n}$ maps $\mathfrak{S}_{m,k}$ to $\mathfrak{S}_{n,k}$. Since $\mathrm{Tr}\{B_n^i\} = \mathrm{Tr}\{(B_n \oplus 0_{m-n})^i\}$, $r_{m,n}(\mathrm{Tr}_i) = \mathrm{Tr}_i$. Thus Equation (1.c) shows that $r_{m,n}$ is always a surjective map from $\mathfrak{S}_{m,k}$ to $\mathfrak{S}_{n,k}$. Furthermore, if $n \geq k$, then $r_{m,n}$ is an isomorphism from $\mathfrak{S}_{m,k}$ to $\mathfrak{S}_{n,k}$.

Let $\mathcal{M}^m = (M, g, J)$ be a Kähler manifold. We use J to give TM a complex structure and to regard TM as a complex vector bundle; Equation (1.b) then shows that $\mathcal{R}(x, y)$ is complex linear. We regard $\mathcal{R}$ as a matrix of 2-forms. If $\mathcal{S}_{m,k} \in \mathfrak{S}_{m,k}$, then the *evaluation* on $\mathcal{R}$ yields an element

$$\mathcal{S}_{m,k}(\mathcal{R}) \in C^\infty(\Lambda^{2k} M).$$

We have that $\mathcal{S}_{m,k}(\mathcal{R})$ is a closed differential form; the corresponding element in de Rham cohomology is independent of the particular Kähler metric g on M and is called a *characteristic class*:

$$[\mathcal{S}_{m,k}(\mathcal{R})] \in H^{2k}_{\mathrm{DeR}}(M).$$

If $k = m$, then we may use the natural orientation of M and integrate over M to define a corresponding *characteristic number* which is independent of g. If the complex dimension $m = 1$, then $\dim\{\mathfrak{S}_{1,1}\} = 1$. If $\mathcal{S}_{1,1} \in \mathfrak{S}_{1,1}$, then there is a universal constant $c = c(\mathcal{S}_{1,1})$ so that

$$\int_M \mathcal{S}_{1,1}(\mathcal{R}_\mathcal{M}) = c \cdot \chi(M)$$

where $\chi(M)$ is the *Euler–Poincaré* characteristic of M. Let sign denote the Hirzebruch signature. If the complex dimension $m = 2$, then $\dim\{\mathfrak{S}_{2,2}\} = 2$. If $\mathcal{S}_{2,2} \in \mathfrak{S}_{2,2}$, then there are universal constants $c_i = c_i(\mathcal{S}_{2,2})$ so that:

$$\int_M \mathcal{S}_{2,2}(\mathcal{R}_{\mathcal{M}}) = c_1 \cdot \chi(M) + c_2 \cdot \operatorname{sign}(M).$$

Give complex projective space $\mathbb{CP}^n$ the Fubini–Study metric. If $\vec{\nu} = (\nu_1, \ldots, \nu_\ell)$, let $\mathbb{CP}^{\vec{\nu}} = \mathbb{CP}^{\nu_1} \times \cdots \times \mathbb{CP}^{\nu_\ell}$. This is a compact homogeneous Kähler manifold of complex dimension $\nu_1 + \cdots + \nu_\ell$. If $\mathcal{S}_{k,k}$ is non-trivial as an invariant polynomial, then the associated characteristic number is non-trivial. We refer to [**1**], [**10**] for the proof of:

LEMMA 1.1. *Let $0 \neq \mathcal{S}_{k,k} \in \mathfrak{S}_{k,k}$. Then there exists $\vec{\nu}$ with $k = \nu_1 + \cdots + \nu_\ell$ so*

$$\int_{\mathbb{CP}^{\vec{\nu}}} \mathcal{S}_{k,k}(\mathcal{R}_{\mathbb{CP}^{\vec{\nu}}}) d\nu_{\mathbb{CP}^{\vec{\nu}}} \neq 0.$$

1.3. Scalar valued universal curvature identities.

In the real setting,Weyl's first theorem of invariants [**23**] can be used to show that all polynomial scalar invariants in the derivatives of the metric arise from contractions of indices in the curvature tensor and its covariant derivatives. Let $\{e_i\}$ be a local orthonormal frame for a Riemannian manifold (M, g) and let R_{ijkl} be the components of the curvature tensor. Adopt the *Einstein convention* and sum over repeated indices to define:

$$\begin{aligned} &E_2 := R_{ijji}, \qquad E_4 := R_{ijji}R_{kllk} - 4R_{aija}R_{bijb} + R_{ijkl}R_{ijkl}, \text{ and} \\ &E_6 := R_{ijji}R_{kllk}R_{abba} - 12R_{ijji}R_{aija}R_{bijb} + 3R_{abba}R_{ijkl}R_{ijkl} \\ &\qquad + 24R_{aija}R_{bklb}R_{jlik} + 16R_{aija}R_{bjkb}R_{cikc} - 24R_{aija}R_{jkln}R_{lnik} \\ &\qquad + 2R_{ijkl}R_{klan}R_{anij} - 8R_{kaij}R_{inkl}R_{jlan}. \end{aligned}$$

E_2, E_4, and E_6 are universally defined scalar invariants of order $\mu = 2$, $\mu = 4$, and $\mu = 6$, respectively. They are generically non-zero in real dimension at least μ but vanish in lower dimensions; in particular, they give non-trivial *universal curvature identities* in real dimension $\mu - 1$. Modulo a suitable normalization, these are the integrals of the Chern–Gauss–Bonnet Theorem [**3**] and more generally, up to rescaling, the Pfaffian E_μ gives the only universal curvature identity of order μ vanishing identically in real dimension $\mu - 1$. This fact plays an important role in the proof of the Chern–Gauss–Bonnet theorem using heat equation methods [**8**].

DEFINITION 1.2. Let $\mathfrak{P}_m$ be the polynomial algebra in the components of $\mathcal{R}$, in the components of the covariant derivative $\nabla\mathcal{R}$, and so forth for Kähler metrics on manifolds of complex dimension m. Let $\mathfrak{P}_{m,k}^U$ be the subspace of polynomials which are homogeneous of degree $2k$ in the derivatives of the metric and which are invariant under the action of the unitary group $U(m)$.

H. Weyl's theorem on invariants of the orthogonal group [**23**] has been extended by Fukami [**7**] and Iwahori [**14**] to this setting; all such invariants arise by contractions of indices using the metric and the Kähler form. In practice, the Kähler identity means that we will not be in fact using the Kähler form to contract indices. Rather, we will contract a lower holomorphic (resp. anti-holomorphic) index against the corresponding upper holomorphic (resp. anti-holomorphic) index. The Kähler form $\Omega := -\sqrt{-1}g_{\alpha\bar\beta}dz^\alpha \wedge d\bar z^\beta$ is given by contracting upper against lower indices of the same type; it is not necessary for the frame to be unitary. We can also contract a lower holomorphic index against a corresponding lower anti-holomorphic index using the metric relative to a unitary frame. Thus, for example, the scalar curvature is given by $\tau = R_{\alpha\bar\alpha\bar\beta\beta}$ modulo a suitable normalizing constant.

Definition 1.3. Let $\mathfrak{P}^U_{m,k}$ be as defined in Definition 1.2. Let $\mathfrak{K}_{\mathfrak{P},m,k} \subset \mathfrak{P}^U_{m,k}$ be the subspace of invariant local formulas which are homogeneous of degree $2k$ in the derivatives of the metric and which vanish when restricted from complex dimension m to complex dimension $k-1$; we shall give an algebraic characterization presently in Lemma 3.1.

Elements $0 \neq \mathcal{P}_{m,k} \in \mathfrak{K}_{\mathfrak{P},m,k}$ give universal curvature identities of degree $2k$ in complex dimension $k-1$. We sum over repeated indices in a unitary frame field to define:

$$\mathcal{P}^1_{m,2} := R_{\alpha_1\bar\alpha_1\bar\alpha_3\alpha_4}R_{\alpha_2\bar\alpha_2\bar\alpha_4\alpha_3} - R_{\alpha_1\bar\alpha_2\bar\alpha_3\alpha_4}R_{\alpha_2\bar\alpha_1\bar\alpha_4\alpha_3},$$
$$\mathcal{P}^2_{m,2} := R_{\alpha_1\bar\alpha_1\bar\alpha_3\alpha_3}R_{\alpha_2\bar\alpha_2\bar\alpha_4\alpha_4} - R_{\alpha_1\bar\alpha_2\bar\alpha_3\alpha_3}R_{\alpha_2\bar\alpha_1\bar\alpha_4\alpha_4}.$$

One then has that $P^1_{m,2}$ and $P^2_{m,2}$ are generically non-zero if $m \geq 2$ but vanish identically in complex dimension $m = 1$. Thus $\mathcal{P}^1_{m,2}$ and $\mathcal{P}^2_{m,2}$ are universal curvature identities in the Kähler setting. One sees this not by using index notation but by noting that:

$$\mathcal{P}^1_{m,2} := \frac{1}{2}g(\mathrm{Tr}\{\mathcal{R}^2\}, \Omega^2) \text{ and } \mathcal{P}^2_{m,2} := \frac{1}{2}g(\mathrm{Tr}\{\mathcal{R}\}^2, \Omega^2).$$

We generalize this construction:

Definition 1.4. If $\mathcal{S}_{m,k} \in \mathfrak{S}_{m,k}$, define $\Xi_{\mathfrak{P},m,k} : \mathfrak{S}_{m,k} \to \mathfrak{P}^U_{m,k}$ by setting:

$$\Xi_{\mathfrak{P},m,k}(\mathcal{S}_{m,k}) := \frac{1}{k!}g(\mathcal{S}_{m,k}(\mathcal{R}), \Omega^k). \tag{1.f}$$

We may use Equation (1.a) to see that if $m = k$, then

$$\Xi_{\mathfrak{P},m,m}(\mathcal{S}_{m,m})d\nu_g = \mathcal{S}_{m,m}(\mathcal{R}). \tag{1.g}$$

Thus by Lemma 1.1, $\Xi_{\mathfrak{P},m,k}(\mathcal{S}_{m,k})$ is generically non-zero in complex dimension $m \geq k$ but vanishes in complex dimension $m = k-1$. Consequently, $\Xi_{\mathfrak{P},m,k}$ takes values in $\mathfrak{K}_{\mathfrak{P},m,k}$.

The following result played an important role in the proof of the Riemann–Roch formula using heat equation methods [**9**]:

THEOREM 1.1. *If $m \geq k$, then $\Xi_{\mathfrak{P},m,k}$ is an isomorphism from $\mathfrak{S}_{m,k}$ to $\mathfrak{K}_{\mathfrak{P},m,k}$. In other words, any scalar valued curvature identity of order $2k$ that is given universally by contracting indices in pairs, that is generically non-zero in complex dimension $m \geq k$, and that vanishes in complex dimension $m = k-1$ is of this form.*

1.4. Universal curvature identities which are symmetric 2-tensor valued. In the real setting, let $S^2M \subset \otimes^2 T^*M$ be the bundle of symmetric 2-cotensors and let $S_2M \subset \otimes^2 TM$ be the dual bundle; this is the bundle of symmetric 2-tensors. We can extend H. Weyl's theorem first theorem of invariants to construct polynomial invariants which are S_2M valued by contracting all but 2 indices and symmetrizing the remaining two indices. For example, we can define:

$$\begin{aligned} T_2 &:= R_{ijji}e_k \circ e_k - 2R_{ijki}e_j \circ e_k, \\ T_4 &:= -\frac{1}{4}(R_{ijji}R_{kllk} - 4R_{ijki}R_{ljkl} + R_{ijkl}R_{ijkl})e_n \circ e_n \\ &\qquad + \{R_{klni}R_{klnj} - 2R_{knik}R_{lnjl} - 2R_{iklj}R_{nkln} + R_{kllk}R_{nijn}\}e_i \circ e_j. \end{aligned}$$

The invariants T_n are generically non-zero in real dimension greater than n but vanish identically in real dimension n. The identity $T_2 = 0$ in real dimension 2 is the classical identity relating the scalar curvature and the Ricci tensor; the identity $T_4 = 0$ in real dimension 4 is the Berger–Euh–Park–Sekigawa identity [**2**], [**4**]. More generally, such invariants can be formed through the transgression of the Euler form; we refer to [**11**] for further details. We also refer to [**12**] where the pseudo-Riemannian setting is treated and to [**13**] where manifolds with boundary are treated. We note that Navarro and Navarro [**21**] have applied the theory of natural operators [**15**], [**20**] to discuss more generally p-covariant identities for any even p.

In the Kähler setting, let S_2^+M be the bundle dual to S_+^2M and let $\langle\cdot,\cdot\rangle$ denote the natural pairing between these two bundles.

DEFINITION 1.5. Let $\mathfrak{Q}_{m,k}^U$ be the space of all S_+^2 valued invariants which are homogeneous of degree $2k$ in the derivatives of the metric and which are invariant under the action of the unitary group. We consider the subspace $\mathfrak{K}_{\mathfrak{Q},m,k} \subset \mathfrak{Q}_{m,k}^U$ of invariants which vanish when restricted from complex dimension m to complex dimension k; again, we shall give an algebraic characterization presently in Lemma 3.1.

EXAMPLE 1.1. Let $\{e_\alpha\}$ be a local unitary frame field for TM (viewed as a complex vector bundle). We contract holomorphic with anti-holomorphic indices in pairs to construct the following invariant of degree 2:

$$\mathcal{Q}_{m,1} := R_{\alpha_1\bar{\alpha}_1\bar{r}_1r_1}e_{\alpha_2} \circ e_{\bar{\alpha}_2} - R_{\alpha_1\bar{\alpha}_2\bar{r}_1r_1}e_{\alpha_2} \circ e_{\bar{\alpha}_1}.$$

Similarly, we may construct invariants of degree 4:

$$
\begin{aligned}
\mathcal{Q}^1_{m,2} := {}& R_{\alpha_1\bar\alpha_1\bar\gamma_1\delta_1} R_{\alpha_2\bar\alpha_2\bar\delta_1\gamma_1} e_{\alpha_3}\circ e_{\bar\alpha_3} + R_{\alpha_1\bar\alpha_3\bar\gamma_1\delta_1} R_{\alpha_2\bar\alpha_1\bar\delta_1\gamma_1} e_{\alpha_3}\circ e_{\bar\alpha_2}\\
&+ R_{\alpha_1\bar\alpha_2\bar\gamma_1\delta_1} R_{\alpha_2\bar\alpha_3\bar\delta_1\gamma_1} e_{\alpha_3}\circ e_{\bar\alpha_1} - R_{\alpha_1\bar\alpha_1\bar\gamma_1\delta_1} R_{\alpha_2\bar\alpha_3\bar\delta_1\gamma_1} e_{\alpha_3}\circ e_{\bar\alpha_2}\\
&- R_{\alpha_1\bar\alpha_2\bar\gamma_1\delta_1} R_{\alpha_2\bar\alpha_1\bar\delta_1\gamma_1} e_{\alpha_3}\circ e_{\bar\alpha_3} - R_{\alpha_1\bar\alpha_3\bar\gamma_1\delta_1} R_{\alpha_2\bar\alpha_2\bar\delta_1\gamma_1} e_{\alpha_3}\circ e_{\bar\alpha_1},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{Q}^2_{m,2} := {}& R_{\alpha_1\bar\alpha_1\bar\sigma_1\sigma_1} R_{\alpha_2\bar\alpha_2\bar\sigma_2\sigma_2} e_{\alpha_3}\circ e_{\bar\alpha_3} + R_{\alpha_1\bar\alpha_3\bar\sigma_1\sigma_1} R_{\alpha_2\bar\alpha_1\bar\sigma_2\sigma_2} e_{\alpha_3}\circ e_{\bar\alpha_2}\\
&+ R_{\alpha_1\bar\alpha_2\bar\sigma_1\sigma_1} R_{\alpha_2\bar\alpha_3\bar\sigma_2\sigma_2} e_{\alpha_3}\circ e_{\bar\alpha_1} - R_{\alpha_1\bar\alpha_1\bar\sigma_1\sigma_1} R_{\alpha_2\bar\alpha_3\bar\sigma_2\sigma_2} e_{\alpha_3}\circ e_{\bar\alpha_2}\\
&- R_{\alpha_1\bar\alpha_2\bar\sigma_1\sigma_1} R_{\alpha_2\bar\alpha_1\bar\sigma_2\sigma_2} e_{\alpha_3}\circ e_{\bar\alpha_3} - R_{\alpha_1\bar\alpha_3\bar\sigma_1\sigma_1} R_{\alpha_2\bar\alpha_2\bar\sigma_2\sigma_2} e_{\alpha_3}\circ e_{\bar\alpha_1}.
\end{aligned}
$$

We have $\mathcal{Q}_{m,1} \in \mathfrak{K}_{\mathfrak{Q},m,1}$, $\mathcal{Q}^1_{m,2} \in \mathfrak{K}_{\mathfrak{Q},m,2}$ and $\mathcal{Q}^2_{m,2} \in \mathfrak{K}_{\mathfrak{Q},m,2}$. The invariant $\mathcal{Q}_{m,1}$ is generically non-zero in complex dimension $m \geq 2$ but vanishes in complex dimension $m = 1$; the invariants $\mathcal{Q}^1_{m,2}$ and $\mathcal{Q}^2_{m,2}$ are generically non-zero in complex dimension $m \geq 3$ but vanish in complex dimension $m = 2$. One sees this not by using the index notation but rather by expressing

$$
\begin{aligned}
\mathcal{Q}_{m,1} &= \frac{1}{2} R_{\alpha_1\bar\beta_1\bar\gamma_1\gamma_1} e_{\alpha_2}\circ e_{\bar\beta_2} g(dz^{\alpha_1}\wedge d\bar z^{\beta_1}\wedge dz^{\alpha_2}\wedge d\bar z^{\beta_2}, \Omega^2),\\
\mathcal{Q}^1_{m,2} &= \frac{1}{6} R_{\alpha_1\bar\beta_1\bar\gamma_1\delta_1} R_{\alpha_2\bar\beta_2\bar\delta_1\gamma_1} e_{\alpha_3}\circ e_{\bar\beta_3} g(e^{\alpha_1}\wedge \bar e^{\beta_1}\wedge e^{\alpha_2}\wedge \bar e^{\beta_2}\wedge e^{\alpha_3}\wedge \bar e^{\beta_3}, \Omega^3),\\
\mathcal{Q}^2_{m,2} &= \frac{1}{6} R_{\alpha_1\bar\beta_1\bar\sigma_1\sigma_1} R_{\alpha_2\bar\beta_2\bar\sigma_2\sigma_2} e_{\alpha_3}\circ e_{\bar\beta_3} g(e^{\alpha_1}\wedge \bar e^{\beta_1}\wedge e^{\alpha_2}\wedge \bar e^{\beta_2}\wedge e^{\alpha_3}\wedge \bar e^{\beta_3}, \Omega^3).
\end{aligned}
$$

We generalize this construction:

Definition 1.6. Let $\mathcal{S}_{m,k} \in \mathfrak{S}_{m,k}$. The *transgression* $\Xi_{\mathfrak{Q},m,k}(\mathcal{S}_{m,k}) \in S^+_2$ is defined by setting:

$$
\Xi_{\mathfrak{Q},m,k}(\mathcal{S}_{m,k}) := \frac{1}{(k+1)!} g(\mathcal{S}_{m,k}(\mathcal{R})\wedge e^\alpha\wedge \bar e^\beta, \Omega^{k+1}) e_\alpha\circ \bar e_\beta.
$$

Example 1.2. Adopt the notation of Example 1.1. Then $\mathcal{Q}_{m,1} = \Xi_{\mathfrak{Q},m,1}(\mathrm{Tr}_1)$. Let ρ be the Ricci tensor and let τ be the scalar curvature. We have

$$
\mathcal{Q}_{m,1} = -\frac{1}{2}\tau g + \rho.
$$

This symmetric 2-form valued tensor is generically non-zero if $m \geq 2$ but vanishes identically in complex dimension $m = 1$; this is a classic identity. Recall that $\mathrm{Tr}_k(R) = \mathrm{Tr}(R^k)$. Let $\mathcal{Q}^1_{m,2} = \Xi_{\mathfrak{Q},m,2}(\mathrm{Tr}_2)$ and $\mathcal{Q}^2_{m,2} = \Xi_{\mathfrak{Q},m,2}(\mathrm{Tr}_1^2)$. Let ρ be the Ricci tensor. Set

$$
\check R_{ij} = R_{abci}R^{abc}{}_j, \quad \check\rho_{ij} = \rho_{ai}\rho^a{}_j, \quad L_{ij} = 2R_{iabj}\rho^{ab}.
$$

We then have:

$$
\mathcal{Q}^1_{m,2} = \left(\frac{1}{2}|\rho|^2 - \frac{1}{4}|R|^2\right)g + (\check R - L(\rho)) \text{ and } \mathcal{Q}^2_{m,2} = 2\check\rho - \tau\rho - \frac{1}{2}\left(|\rho|^2 - \frac{\tau^2}{2}\right)g.
$$

The characteristic class c_1^2 corresponds to Tr_2; the formula for $\mathcal{Q}^2_{m,2}$ agrees with that given in Theorem 5.3 [**5**] for the associated Euler–Lagrange equation. Furthermore, the Euler class in real dimension 4 corresponds to $2\det(A) = \mathcal{Q}^2_{m,2} - \mathcal{Q}^1_{m,2}$. We express:

$$\mathcal{Q}^2_{m,2} - \mathcal{Q}^1_{m,2} = \frac{1}{4}\left(|R|^2 - |\rho|^2 + \frac{\tau^2}{4}\right)g - \check{R} + L(\rho) + 2\check{\rho} - \tau\rho.$$

This is the universal curvature identity discussed in [**2**], [**4**] that is generated by the Euler–Lagrange equation of this characteristic class. Note that the complex structure is not involved; this is no longer the case when we consider invariants of order 6 and higher.

The invariants of Definition 1.6 yield the universal S_2^+ valued curvature identities that we have been searching for; every S_2^+ valued invariant which is homogeneous of degree $2k$ in the derivatives of the metric and which is generically non-zero in complex dimension $m > k$ and which vanishes in complex dimension k arises in this fasion. Theorem 1.1 generalizes to this setting to become the following result which is the first major new result of this paper:

THEOREM 1.2. *If $m > k$, then map $\Xi_{\mathfrak{Q},m,k}$ of Definition 1.6 is an isomorphism from $\mathfrak{S}_{m,k}$ to $\mathfrak{K}_{\mathfrak{Q},m,k}$. This means that a S_2^+ valued curvature identity of order $2k$ which is given universally by contracting indices in pairs, which is generically non-zero in complex dimension $m > k$, and which vanishes in complex dimension $m = k$ is of this form.*

1.5. Euler–Lagrange equations.

Let $\mathcal{M}^m = (M, g, J)$ be a compact Kähler manifold. Let $\mathcal{S}_{m,k} \in \mathfrak{S}_{m,k}$ for $k \le m$. Although $\mathcal{S}_{m,k}$ determines a cohomology class, it does not determine a corresponding scalar invariant if $k < m$. We integrate the invariant of Definition 1.4 to define:

$$\{\Xi_{\mathfrak{P},m,k}(\mathcal{S}_{m,k})\}[\mathcal{M}^m] := \frac{1}{k!}\int_M g(\mathcal{S}_{m,k}(\mathcal{R}_{\mathcal{M}}), \Omega_g^k)d\nu_g. \tag{1.h}$$

If $k = m$, we use Equation (1.g) to see

$$\{\Xi_{\mathfrak{P},k,k}(\mathcal{S}_{k,k})\}[\mathcal{M}^m] = \int_M \mathcal{S}_{k,k}(\mathcal{R}_{\mathcal{M}})$$

is a characteristic number that is independent of the metric g. However, more generally, if $m > k$, then this integral depends upon the metric. Let $g_\varepsilon := g + \varepsilon h$ be a smooth 1-parameter family of Kähler metrics; such families may be obtained using the Kähler potential as we shall discuss presently in Section 2.3. We integrate by parts to obtain the corresponding *Euler–Lagrange* formula.

DEFINITION 1.7. Let $\mathcal{S}_{m,k} \in \mathfrak{S}_{m,k}$. Let $\mathcal{M}^m = (M, g, J)$ be a Kähler manifold of complex dimension m. Let $\mathcal{M}^m_\varepsilon := (M, g + \varepsilon h, J)$ be a Kähler variation. Let $\Theta_{\mathfrak{Q},m,k}\{\mathcal{S}_{m,k}\} \in S_2^+ M$ be the associated *Euler–Lagrange invariant*; it is uniquely characterized by the identity:

$$\partial_\varepsilon\{\Xi_{\mathfrak{P},m,k}(\mathcal{S}_{m,k})[\mathcal{M}^m_\varepsilon]\}\big|_{\varepsilon=0} = \int_M \langle\{\Theta_{\mathfrak{Q},m,k}(\mathcal{S}_{m,k})\}(\mathcal{R}_\mathcal{M}), h\rangle d\nu_g.$$

What is perhaps somewhat surprising is that the Euler–Lagrange formulas for $\mathcal{S}_{m,k}$ are closely related to the universal curvature identities defined by the transgression. We adopt the notation of Example 1.1 and Example 1.2. It is well known that $\mathcal{Q}_{m,1}$ is the Euler–Lagrange equation for the Gauss–Bonnet integrand. Work of [**5**] shows that universal curvature identity $\mathcal{Q}^1_{m,2}$ is the Euler–Lagrange equation for Tr_2. Similarly, work of [**2**], [**4**] shows that the universal curvature identity $\mathcal{Q}^2_{m,2} - \mathcal{Q}^1_{m,2}$ is the Euler–Lagrange equation of the Euler class. Thus $\Xi_{\mathfrak{Q},m,k} = \Theta_{\mathfrak{Q},m,k}$ if $k = 1, 2$. This is true more generally; the map from the characteristic forms to the symmetric 2-tensors given by the Euler–Lagrange equations coincides with the map given algebraically by the transgression in the Kähler setting. Let $\Theta_{\mathfrak{Q},m,k}$ be as given in Definition 1.7 and let $\Xi_{\mathfrak{Q},m,k}$ be as given in Definition 1.6. The following is the second main result of this paper:

THEOREM 1.3. *If $m > k$, then $\Theta_{\mathfrak{Q},m,k} = \Xi_{\mathfrak{Q},m,k}$. This means that if $\mathcal{S}_{m,k} \in \mathfrak{S}_{m,k}$, if $m > k$, and if $\mathcal{M}^m_\varepsilon := (M, g + \varepsilon h, J)$ is a Kähler variation, then*

$$\begin{aligned}&\partial_\varepsilon\{\Xi_{\mathfrak{P},m,k}(\mathcal{S}_{m,k})[\mathcal{M}^m_\varepsilon]\}\big|_{\varepsilon=0}\\ &\quad= \frac{1}{(k+1)!}\int_M g(\mathcal{S}_{m,k}(\mathcal{R}) \wedge e^\alpha \wedge \bar{e}^\beta, \Omega^{k+1})\langle e_\alpha \circ \bar{e}_\beta, h\rangle d\nu_\mathcal{M}.\end{aligned}$$

REMARK 1.2. A-priori, since the local invariant $\mathcal{S}_{m,k}$ involves 2^{nd} derivatives, the associated Euler–Lagrange invariant could involve the first and second covariant derivatives of the curvature tensor. The somewhat surprising fact is that this is not the case as Theorem 1.3 shows. In the real setting, one can work with the *Pfaffian*; this is the integrand of the Chern–Gauss–Bonnet formula [**3**]. Berger [**2**] conjectured that the corresponding Euler–Lagrange invariant only involved the second derivatives of the metric. This was established by Kuz'mina [**16**] and Labbi [**17**], [**18**], [**19**] (see also the discussion in [**11**]). Theorem 1.3 is the extension to the complex setting of this result.

1.6. Outline of the paper.

Fix a point of a Kähler manifold $\mathcal{M}^m$. In Section 2, we normalize the choice of the coordinate system to be the unitary group up to arbitrarily high order. In Section 3, we give an algebraic description of the space $\mathfrak{K}_{\mathfrak{P},m,k}$ (resp. $\mathfrak{K}_{\mathfrak{Q},m,k}$) from the point of the restriction map from complex dimension m to complex dimension $k-1$ (resp. k) and show that $\Xi_{\mathfrak{P},m,k}$ (resp. $\Xi_{\mathfrak{Q},m,k}$ and $\Theta_{\mathfrak{Q},m,k}$) takes values in $\mathfrak{K}_{\mathfrak{P},m,k}$ (resp. $\mathfrak{K}_{\mathfrak{Q},m,k}$). In Section 4 we discuss invariance theory. We take a slightly non-standard point of view. Weyl's first theorem of invariants [**23**] gives generators for the space of invariants of the orthogonal group; in brief, this generating set can be described in terms of contractions of indices. Fukami [**7**] and Iwahori [**14**] have extended this result to the complex setting; the generating set is formed by using both the metric and the Kähler form to contract indices. However, what is needed in our analysis is Weyl's second theorem of invariants which describes the relations among the generating set described above. This analysis does not seem to have been extended to the complex setting. Even were this to have been done, we would still need to use the Kähler identity suitably. For that reason, it seemed easiest simply to do the necessary invariance theory from scratch in a non-standard setting and we apologize in advance if this is unfamiliar. Let $\mathfrak{K}_{\mathfrak{Q},m,k}$ be as given in Definition 1.5 and let $\rho(k)$ be the partition function of Definition 1.1. The crucial estimate in this regard is given in Lemma 4.3:

$$\dim\{\mathfrak{K}_{\mathfrak{Q},m,k}\} \leq \rho(k).$$

In Section 5, we use these results of Section 3 to establish Theorem 1.1, Theorem 1.2, and Theorem 1.3.

2. Normalizing the coordinates.

In this section, we probe in a bit more detail into Kähler geometry. In Section 2.1, we introduce some basic notational conventions. In Section 2.2, we reduce the structure group to the unitary group modulo a holomorphic transformation of arbitrarily high order. In Section 2.3, we discuss Kähler potentials; this provides a way of varying the original Kähler metric that will be very useful in considering the Euler–Lagrange equations. In Section 2.4, we will use the Kähler potential to specify the jets of the metric; we shall work with a polynomial algebra in the derivatives of the metric and in this section, we show there are no hidden relations or analogues of the Bianchi identities. This will be crucial in our subsequent discussion in Section 3.

2.1. Notational conventions.

Let P be a point of a Kähler manifold $\mathcal{M}^m$. Extend the J-invariant Riemannian metric g to be a symmetric complex bilinear form. Let

$$g_{\alpha\beta} := g(\partial_{z_\alpha}, \partial_{z_\beta}), \quad g_{\bar\alpha\bar\beta} := g(\partial_{\bar z_\alpha}, \partial_{\bar z_\beta}), \quad g_{\alpha\bar\beta} := g(\partial_{z_\alpha}, \partial_{\bar z_\beta}).$$

Since g is J-invariant, we may show that $g_{\alpha\beta} = g_{\bar\alpha\bar\beta} = 0$ by computing:

$$\begin{aligned} g_{\alpha\beta} &= g(J\partial_{z_\alpha}, J\partial_{z_\beta}) = g\big(\sqrt{-1}\partial_{z_\alpha}, \sqrt{-1}\partial_{z_\beta}\big) = -g_{\alpha\beta}, \\ g_{\bar\alpha\bar\beta} &= g(J\partial_{\bar z_\alpha}, J\partial_{\bar z_\beta}) = g\big(-\sqrt{-1}\partial_{\bar z_\alpha}, -\sqrt{-1}\partial_{\bar z_\beta}\big) = -g_{\bar\alpha\bar\beta}. \end{aligned}$$

As a result, we have that:

$$\begin{aligned} \Omega(\partial_{z_\alpha}, \partial_{z_\beta}) &= g(\partial_{z_\alpha}, J\partial_{z_\beta}) = \sqrt{-1}g_{\alpha\beta} = 0, \\ \Omega(\partial_{\bar z_\alpha}, \partial_{\bar z_\beta}) &= g(\partial_{\bar z_\alpha}, J\partial_{\bar z_\beta}) = -\sqrt{-1}g_{\bar\alpha\bar\beta} = 0, \\ \Omega(\partial_{z_\alpha}, \partial_{\bar z_\beta}) &= g(\partial_{z_\alpha}, J\partial_{\bar z_\beta}) = -\sqrt{-1}g_{\alpha\bar\beta}, \\ \Omega &= -\sqrt{-1}g_{\alpha\bar\beta}dz^\alpha \wedge d\bar z^\beta. \end{aligned}$$

The equation $d\Omega = 0$ is then equivalent to the relations

$$0 = \partial_{z_\gamma} g_{\alpha\bar\beta} dz^\gamma \wedge dz^\alpha \wedge d\bar z^\beta - \partial_{\bar z_\gamma} g_{\alpha\bar\beta} dz^\alpha \wedge d\bar z^\gamma \wedge d\bar z^\beta, \quad \text{i.e.}$$
$$\partial_{z_\gamma} g_{\alpha\bar\beta} = \partial_{z_\alpha} g_{\gamma\bar\beta} \text{ and } \partial_{\bar z_\gamma} g_{\alpha\bar\beta} = \partial_{\bar z_\beta} g_{\alpha\bar\gamma}. \tag{2.a}$$

Let δ be the Kronecker symbol. Let $A := (\alpha_1, \dots, \alpha_\nu)$ be an ordered collection of indices α_i where $1 \le \alpha_i \le m$. Let $|A| = \nu$, let $z^A = z^{\alpha_1} \cdots z^{\alpha_\nu}$, and let

$$\deg_\alpha(A) := \delta_{\alpha\alpha_1} + \dots + \delta_{\alpha\alpha_\nu}$$

be the number of times the index α appears in A. Let $B = (\beta_1, \dots, \beta_\mu)$ be another collection of indices and let $\vec z = (z^1, \dots, z^m)$ be a local holomorphic system of coordinates on a Kähler manifold $\mathcal{M}^m$. Set

$$g^{\vec z}(A; B) := \{\partial_{z_{\alpha_2}} \cdots \partial_{z_{\alpha_\nu}} \partial_{\bar z_{\beta_2}} \cdots \partial_{\bar z_{\beta_\mu}}\} g_{\alpha_1\bar\beta_1}.$$

We shall often omit the superscript $\vec z$ if there is only one coordinate system under consideration. If σ and τ are permutations, let

$$A^\sigma := (\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(\nu)}) \text{ and } B^\tau := (\beta_{\tau(1)}, \dots, \beta_{\tau(\mu)}).$$

Equation (2.a) may then be differentiated to see:

$$g(A; B) = g(A^\sigma; B^\tau)$$

so the variables $g(A; B)$ are symmetric in the holomorphic indices and also in the anti-holomorphic indices; the order of the indices comprising A and comprising B plays no role. Note that

$$g^{\vec z}(B; A) = \bar g^{\vec z}(A; B).$$

2.2. Reducing the structure group to $U(m)$.

The following result will enable us to normalize the structure group of admissible coordinate transformations from the full group of holomorphic transformations to the unitary group modulo changes which vanish to arbitrarily high order at a given point P of M:

LEMMA 2.1. *Let P be a point of a Kähler manifold $\mathcal{M}^m$. Fix n.*

1. *There exist local holomorphic coordinates $(z^1, \dots, z^m)$ centered at P so that*

$$g_{\alpha\bar\beta}(P) = \delta_{\alpha\beta} \text{ and } g^{\vec z}(A; B)(P) = 0 \text{ for } |B| = 1 \text{ and } 2 \le |A| \le n. \tag{2.b}$$

2. *If $(w^1, \dots, w^m)$ is another system of local holomorphic coordinates on M which are centered at P and which satisfy the relations of Equation* (2.b)*, then we have that $z = Tw + O(|w|^{n+1})$ for some linear map $T \in U(m)$.*

Proof. Suppose that $n=1$. We use the Gram–Schmidt process to make a complex linear change of coordinates to ensure that $g_{\alpha\bar{\beta}}(P)=\delta_{\alpha\beta}$. Assertion (1) now follows; Assertion (2) is then immediate. We therefore proceed by induction and assume that $n\geq 2$. Let z be a system of coordinates normalized satisfying $g_{\alpha\bar{\beta}}(P)=\delta_{\alpha\beta}$ and $g(A;B)=0$ for $|B|=1$ and $2\leq |A|<n$ (this condition is vacuous if $n=2$). Consider the coordinate transformation:

$$w^\beta = z^\beta + \sum_{|A|=n} c_A^\beta z^A$$

where the constants c_A^β are to be chosen suitably. Set

$$\varepsilon(A) := \partial_{z_{\alpha_1}}\cdots\partial_{z_{\alpha_n}}\{z^A\}\in\mathbb{N}. \tag{2.c}$$

We sum over repeated indices to compute:

$$\begin{gathered}
\partial_{z_\alpha}=\partial_{w_\alpha}+c_A^\gamma\partial_{z_\alpha}\{z^A\}\partial_{w_\gamma},\quad \partial_{\bar{z}_\beta}=\partial_{\bar{w}_\beta}+\bar{c}_A^\gamma\partial_{\bar{z}_\beta}\{\bar{z}^A\}\partial_{\bar{w}_\gamma},\\
g(\partial_{z_\alpha},\partial_{\bar{z}_\beta})=g(\partial_{w_\alpha},\partial_{\bar{w}_\beta})+c_A^\beta\partial_{z_\alpha}\{z^A\}+\bar{c}_A^\alpha\partial_{\bar{z}_\beta}\{\bar{z}^A\}+O(|z|^n),\\
g^{\vec{z}}(A,\beta)(P)=g^{\vec{w}}(A,\beta)(P)+\varepsilon(A)\cdot c_A^\beta.
\end{gathered}$$

To ensure that $g^{\vec{w}}(A,\beta)(P)=0$ for all A,β, we solve the equations:

$$\varepsilon(A)c_A^\beta = g^{\vec{z}}(A,\beta)(P).$$

Assertion (2) now follows since the transformation is uniquely defined if we suppose $dT(P)=\mathrm{id}$. □

We use Lemma 2.1 to normalize the system of holomorphic coordinates $\vec{z}$ to arbitrarily high order henceforth; note that we also have:

$$g^{\vec{z}}(B;A)(P)=\bar{g}^{\vec{z}}(A;B)(P)=0 \text{ for } |B|=1.$$

The structure group is now the unitary group $U(m)$ and the variables $g^{\vec{z}}(A;B)$ are tensors; we shall suppress the role of the coordinate system $\vec{z}$ whenever no confusion is likely to result. If we fix $|A|=n_1\geq 2$ and $|B|=n_2\geq 2$, then $g(\cdot;\cdot)$ is a symmetric cotensor of type (n_1,n_2), i.e.

$$g(\cdot;\cdot)\in S^{n_1}(\Lambda^{1,0})\otimes S^{n_2}(\Lambda^{0,1}).$$

The Kähler identity of Equation (1.b) yields $\mathcal{R}(\partial_{z_a},\partial_{z_b})=\mathcal{R}(\partial_{\bar{z}_a},\partial_{\bar{z}_b})=0$. Let $A=(\alpha_1,\alpha_2)$ and $B=(\beta_1,\beta_2)$. We compute that:

$$
\begin{aligned}
R(\partial_{z_{\alpha_1}},\partial_{\bar z_{\beta_1}},\partial_{\bar z_{\beta_2}},\partial_{z_{\alpha_2}})(P) &= \frac{1}{2}\{\partial_{z_{\alpha_1}}\partial_{\bar z_{\beta_2}}g(\partial_{z_{\alpha_2}},\partial_{\bar z_{\beta_1}})+\partial_{z_{\alpha_2}}\partial_{\bar z_{\beta_1}}g(\partial_{z_{\alpha_1}},\partial_{\bar z_{\beta_2}})\}(P)\\
&= g(A;B)(P).
\end{aligned}
$$

A similar computation shows for $A=(\alpha_1,\alpha_2,\alpha_3)$ and $B=(\beta_1,\beta_2)$ that:

$$\nabla R(\partial_{z_{\alpha_1}},\partial_{\bar z_{\beta_1}},\partial_{\bar z_{\beta_2}},\partial_{z_{\alpha_2}};\partial_{z_{\alpha_3}})(P)=g(A;B)(P).$$

The expression of the variables $g(A;B)(P)$ in terms of covariant derivatives of curvature (and vice-versa) for larger values of $|A|$ and $|B|$ is more complicated.

2.3. The Kähler potential.

Let

$$
\begin{aligned}
dz^I &:= dz^{i_1}\wedge\cdots\wedge dz^{i_p} \text{ for } I=\{1\le i_1<\cdots<i_p\le m\},\\
d\bar z^J &:= d\bar z^{j_1}\wedge\cdots\wedge d\bar z^{j_q} \text{ for } J=\{1\le j_1<\cdots<j_q\le m\}.
\end{aligned}
$$

We set $\Lambda^{p,q}M:=\operatorname{Span}_{\mathbb{C}}\{dz^I\wedge d\bar z^J\}_{|I|=p,|J|=q}$ and decompose

$$\Lambda^n M\otimes_{\mathbb{R}}\mathbb{C}=\bigoplus_{p+q=n}\Lambda^{p,q}M.$$

Thus, for example, $\Lambda^2_+M\otimes_{\mathbb{R}}\mathbb{C}=\Lambda^{1,1}M$. Decompose $d=\partial+\bar\partial$ where

$$\partial: C^\infty(\Lambda^{p,q}M)\to C^\infty(\Lambda^{p+1,q}M) \text{ and } \bar\partial: C^\infty(\Lambda^{p,q}M)\to C^\infty(\Lambda^{p,q+1}M)$$

are defined by setting:

$$
\begin{aligned}
\partial(f_{I,J}dz^I\wedge d\bar z^J) &:= \partial_{z_\alpha}(f_{I,J})dz^\alpha\wedge dz^I\wedge d\bar z^J,\\
\bar\partial(f_{I,J}dz^I\wedge d\bar z^J) &:= \partial_{\bar z_\alpha}(f_{I,J})d\bar z^\alpha\wedge dz^I\wedge d\bar z^J.
\end{aligned}
$$

If $f\in C^\infty(M)$, define a real Hermitian symmetric bilinear form $h_f\in C^\infty(S^2_+)$ and a corresponding real anti-symmetric 2-form $\Omega_{h_f}\in C^\infty(\Lambda^2_+)$ by setting:

$$\Omega_{h_f}=-\sqrt{-1}\partial\bar\partial f=-\sqrt{-1}\frac{\partial^2 f}{\partial_{z_\alpha}\partial_{\bar z_\beta}}dz^\alpha\wedge d\bar z^\beta \text{ and } h_f=\frac{\partial^2 f}{\partial_{z_\alpha}\partial_{\bar z_\beta}}dz^\alpha\circ d\bar z^\beta.$$

We then have $d\Omega_{h_f}=0$ and, consequently, for small ε, $g+\varepsilon h_f$ is positive definite and thus a Kähler metric.

2.4. Specifying the jets of the metric at P.

The variables $\{g(A;B)\}$ are a good choice of variables since, unlike the covariant derivatives of the curvature tensor, there are no additional identities as the following result shows; we are dealing with a pure polynomial algebra and we have avoided the Bianchi identities:

Lemma 2.2. *Fix $n \geq 2$. Let constants $c(A;B) \in \mathbb{C}$ be given for $2 \leq |A| \leq n$ and $2 \leq |B| \leq n$ so that $c(A;B) = \bar{c}(B;A)$. Let P be a point of a Kähler manifold (M, g_0, J). Use Lemma* 2.1 *to normalize the coordinate system $\vec{z}$ at P so g_0 satisfies Equation* (2.b)*. Then exists a Kähler metric g on (M, J) so that $g^{\vec{z}}$ also satisfies Equation* (2.b) *and so that $g^{\vec{z}}(A;B)(P) = c(A;B)$ for $2 \leq |A| \leq n$ and $2 \leq |B| \leq n$.*

Proof. Let ϕ be a plateau function which is identically 1 for $|z| \leq 1$ and which vanishes identically for $|z| \geq 2$. Let $\phi_r(z) := \phi(z/r)$. Let $\varepsilon(\cdot)$ be the multiplicity which was defined in Equation (2.c). For r small, we define:

$$f_r(z, \bar{z}) = \sum_{|A|=2}^{n} \sum_{|B|=2}^{n} \frac{c(A;B) - g_0^{\vec{z}}(A;B)(P)}{\varepsilon(A)\varepsilon(B)} \phi_r(z, \bar{z}) z^A \bar{z}^B.$$

The function f_r is real and is supported arbitrarily close to P for r sufficiently small. We follow the discussion of Section 2.3 to define h_f. Let $g := g_0 + h_f$. Then

$$g_{\alpha\bar{\beta}} := g_{0,\alpha\bar{\beta}} + \sum_{|A|=2}^{n} \sum_{|B|=2}^{n} \frac{c(A;B) - g_0^{\vec{z}}(A;B)(P)}{\varepsilon(A)\varepsilon(B)} \partial_{z_\alpha} \partial_{\bar{z}_\beta} \{\phi_r(z, \bar{z}) z^A \bar{z}^B\}.$$

The perturbation has compact support near P; consequently, g extends smoothly to all of M. Furthermore, since $\phi_r \equiv 1$ near P,

$$g^{\vec{z}}(A;B)(P) = g_0^{\vec{z}}(A;B)(P) + c(A;B) - g_0^{\vec{z}}(A;B)(P) = c(A;B).$$

Since $|A| \geq 2$ and $|B| \geq 2$, $g^{\vec{z}}$ satisfies Equation (2.b) at P. Thus the only point remaining is to show that $g_{\alpha\bar{\beta}}$ is positive definite if the parameter r is chosen sufficiently small. Since $|A| \geq 2$ and $|B| \geq 2$, there exists a constant C so that if r is small and if $|z| \leq r$, we have:

$$\begin{aligned}
& z^A \bar{z}^B \leq Cr^4, \quad \partial_{z_\alpha}(z^A \bar{z}^B) \leq Cr^3, \quad \partial_{\bar{z}_\beta}(z^A \bar{z}^B) \leq Cr^3, \\
& \partial_{z_\alpha} \partial_{\bar{z}_\beta}(z^A \bar{z}^B) \leq Cr^2, \quad \phi_r \leq C, \quad \partial_{z_\alpha} \phi_r \leq Cr^{-1}, \\
& \partial_{\bar{z}_\beta} \phi_r \leq Cr^{-1}, \quad \partial_{z_\alpha} \partial_{\bar{z}_\beta} \phi_r \leq Cr^{-2}.
\end{aligned}$$

After possibly increasing C, we may conclude that:

$$\partial_{z_\alpha} \partial_{\bar{z}_\beta} \{\phi_r z^A \bar{z}^B\} \leq Cr^2.$$

Thus the perturbation of the original metric can be made arbitrary small in the C^0 topology as $r \to 0$ and hence g is positive definite if r is sufficiently small. □

3. The restriction map.

It is necessary to be somewhat more formal at this stage. In Section 3, we shall establish notation and make precise the notions discussed previously in Definition 1.3 and in Definition 1.5.

Definition 3.1. Let $\mathfrak{P}_m$ be the polynomial algebra in formal variables $g(A;B)$ where $2 \le |A|$ and $2 \le |B|$. Let $\mathfrak{Q}_m$ be the $\mathfrak{P}_m$ module of all $\mathcal{Q} := \mathcal{P}^{\alpha\bar\beta}\partial_{z_\alpha} \circ \partial_{\bar z_\beta}$ which are S^2_+ valued where $\mathcal{P}^{\alpha\bar\beta} \in \mathfrak{P}_m$ for $1 \le \alpha, \beta \le m$. If $\mathcal{P} \in \mathfrak{P}_m$ (resp. $\mathcal{Q} \in \mathfrak{Q}_m$), if P is a point of Kähler manifold $\mathcal{M}^m$ of complex dimension m, and if $\vec z$ is a system of local holomorphic coordinates on M centered at P satisfying the normalizations of Lemma 2.1, then there is a natural evaluation $\mathcal{P}(\mathcal{M}^m, \vec z)(P)$ (resp. $\mathcal{Q}(\mathcal{M}^m, \vec z)(P)$). We use Lemma 2.1 to see that we can specify the variables $g(A;B)$ arbitrarily and therefore we may identify the abstract element $\mathcal{P} \in \mathfrak{P}_m$ (resp. $\mathcal{Q} \in \mathfrak{Q}_m$) with the local formula it defines. If $\mathcal{P}(\mathcal{M}^m, \vec z)(P) = \mathcal{P}(\mathcal{M}^m)(P)$ (resp. $\mathcal{Q}(\mathcal{M}^m, \vec z)(P) = \mathcal{Q}(\mathcal{M}^m)(P)$) is independent of the particular system of local holomorphic coordinates $\vec z$, then we say $\mathcal{P}$ (resp. $\mathcal{Q}$) is *invariant*. Let $\mathfrak{P}^U_m$ be the subalgebra and let $\mathfrak{Q}^U_m$ the $\mathfrak{P}^U_m$ submodule of all such invariants. The choice of $\vec z$ is unique up to the action of $U(m)$. There is a natural dual action of $U(m)$ on $\mathfrak{P}_m$ and $\mathfrak{Q}_m$; $\mathfrak{P}^U_m$ and $\mathfrak{Q}^U_m$ are simply the fixed points of this action.

A typical monomial $\mathcal{A}$ of $\mathcal{P} \in \mathfrak{P}_m$ or of $\mathcal{Q} \in \mathfrak{Q}_m$ takes the form:

$$\mathcal{A} = g(A^{\mathcal{A}}_1; B^{\mathcal{A}}_1) \cdots g(A^{\mathcal{A}}_\ell; B^{\mathcal{A}}_\ell)\partial_{z_{\alpha_{\mathcal{A}}}} \circ \partial_{\bar z_{\beta_{\mathcal{A}}}},$$

where we omit the $\partial_{z_{\alpha_{\mathcal{A}}}} \circ \partial_{\bar z_{\beta_{\mathcal{A}}}}$ variables when dealing with an element of $\mathfrak{P}_m$. Let $c(\mathcal{A}, \mathcal{P})$ (resp. $c(\mathcal{A}, \mathcal{Q})$) be the coefficient of $\mathcal{A}$ in $\mathcal{P}$ (resp. $\mathcal{Q}$); we say that $\mathcal{A}$ is a monomial of $\mathcal{P}$ (resp. $\mathcal{Q}$) if $c(\mathcal{A}, \mathcal{P})$ (resp. $c(\mathcal{A}, \mathcal{Q})$) is non-zero.

Definition 3.2. We introduce a grading on $\mathfrak{P}_m$ and on $\mathfrak{Q}_m$ by defining:

$$\operatorname{ord}(g(A;B)) := |A| + |B| - 2 \text{ and } \operatorname{ord}(\mathcal{A}) = \sum_i \{|A^{\mathcal{A}}_i| + |B^{\mathcal{A}}_i| - 2\}.$$

The components of $\mathcal{R}$ have order 2; the components of $\nabla\mathcal{R}$ have order 3, and so forth. Let $T := -\operatorname{id} \in U(m)$. Then $T\mathcal{A} = (-1)^{\operatorname{ord}(\mathcal{A})}\mathcal{A}$. Thus if $\mathcal{A}$ is a monomial of an invariant polynomial $\mathcal{P}$ or $\mathcal{Q}$, then $\operatorname{ord}(\mathcal{A})$ is necessarily even. Decompose an invariant polynomial $\mathcal{P} = \mathcal{P}_0 + \mathcal{P}_1 + \cdots$ where

$$\mathcal{P}_i := \sum_{\operatorname{ord}(\mathcal{A})=2i} c(\mathcal{A}, \mathcal{P})\mathcal{A}.$$

Each $\mathcal{P}_i$ is invariant separately since $U(m)$ preserves the order. Let $\mathfrak{P}^U_{m,k}$ be the vector space of all elements of $\mathfrak{P}^U_m$ which are homogeneous of order $2k$ in the derivatives of the metric and which are invariant under the action of the unitary group $U(m)$. We define $\mathfrak{Q}^U_m$ and $\mathfrak{Q}^U_{m,k}$ similarly. We may then decompose

$$\mathfrak{P}_m^U = \oplus_k \mathfrak{P}_{m,k}^U \text{ and } \mathfrak{Q}_m^U = \oplus_k \mathfrak{Q}_{m,k}^U.$$

DEFINITION 3.3. Let $\deg_\gamma(A)$ be the number of times the index γ appears in a collection of indices A. If

$$\mathcal{A}_0 = g(A_1^{\mathcal{A}_0}; B_1^{\mathcal{A}_0}) \cdots g(A_\ell^{\mathcal{A}_0}; B_\ell^{\mathcal{A}_0}),$$

let $\operatorname{len}(\mathcal{A}_0) := \ell$ be the *length* of $\mathcal{A}_0$. Let $\deg_\gamma(\mathcal{A}_0)$ (resp. $\deg_{\bar\gamma}(\mathcal{A}_0)$) be the number of times the holomorphic index γ (resp. the anti-holomorphic index $\bar\gamma$) appears in the monomial $\mathcal{A}_0$:

$$\begin{aligned}
\deg_\gamma(\mathcal{A}_0) &= \deg_\gamma(A_1^{\mathcal{A}_0}) + \cdots + \deg_\gamma(A_\ell^{\mathcal{A}_0}),\\
\deg_{\bar\gamma}(\mathcal{A}_0) &= \deg_{\bar\gamma}(B_1^{\mathcal{A}_0}) + \cdots + \deg_{\bar\gamma}(B_\ell^{\mathcal{A}_0}).
\end{aligned}$$

Similarly, if $\mathcal{A} = \mathcal{A}_0 \partial_{z_{\alpha_{\mathcal{A}}}} \circ \partial_{\bar z_{\beta_{\mathcal{A}}}}$, set

$$\deg_\gamma(\mathcal{A}) := \deg_\gamma(\mathcal{A}_0) + \delta_{\gamma\alpha_{\mathcal{A}}} \text{ and } \deg_{\bar\gamma}(\mathcal{A}) := \deg_{\bar\gamma}(\mathcal{A}_0) + \delta_{\gamma\beta_{\mathcal{A}}}.$$

We wish to consider the space of universal scalar valued curvature identities $\mathfrak{K}_{\mathfrak{P},m,k}$ (resp. S_2^+ valued curvature identities $\mathfrak{K}_{\mathfrak{Q},m,k}$) which are homogeneous of order $2k$ in the derivatives of the metric, which are defined on a manifold of complex dimension $m \geq k$ (resp. $m \geq k+1$), and which vanish when restricted to a manifold of complex dimension $k-1$ (resp. of complex dimension k). We define these spaces algebraically as follows to give precision to the notation introduced previously in Definition 1.3 and in Definition 1.5.

DEFINITION 3.4. Define the *restriction map*

$$r_{m,\nu}\{\mathcal{A}\} := \left\{ \begin{array}{ll} \mathcal{A} & \text{if } \deg_\alpha(\mathcal{A}) = \deg_{\bar\alpha}(\mathcal{A}) = 0 \text{ for all } \alpha > \nu \\ 0 & \text{otherwise} \end{array} \right\}.$$

We note that $r_{m,\nu}\{\mathcal{A}\}$ is then a monomial in complex dimension ν so we may extend $r_{m,\nu}$ to an algebra homomorphism and to a module homomorphism, respectively:

$$r_{m,\nu} : \mathfrak{P}_{m,k}^U \to \mathfrak{P}_{\nu,k}^U \text{ and } r_{m,\nu} : \mathfrak{Q}_{m,k}^U \to \mathfrak{Q}_{\nu,k}^U.$$

There is an equivalent geometric formulation. Let $\mathcal{T}^\ell := (\mathbb{T}^\ell, g_{\mathbb{T}}, J_{\mathbb{T}})$ be the flat Kähler torus of complex dimension ℓ where $\mathbb{T}^\ell := \mathbb{R}^{2\ell}/\mathbb{Z}^{2\ell}$ is the rectangular torus of total volume 1, where $g_{\mathbb{T}}$ is the flat metric induced by the usual Euclidean metric, and where $J_{\mathbb{T}}$ is the complex structure induced from the usual complex structure obtained by identifying $\mathbb{R}^{2\ell} = \mathbb{C}^\ell$. Fix a base point Q of $\mathcal{T}^\ell$. The group of translations acts transitively on $\mathcal{T}^\ell$ so the particular base point chosen is inessential. The following Lemma gives an equivalent algebraic representation of the spaces of universal curvature identities $\mathfrak{K}_{\mathfrak{P},m,k}$ and $\mathfrak{K}_{\mathfrak{Q},m,k}$ which were discussed in Definition 1.3 and in Definition 1.5.

LEMMA 3.1. *Let $\nu < m$. Let P be a point of a Kähler manifold $\mathcal{N}^\nu$ of complex dimension ν.*

1. *If $\mathcal{P}_{m,k} \in \mathfrak{P}^U_{m,k}$, then $\mathcal{P}_{m,k}(\mathcal{N}^\nu \times \mathcal{T}^{m-\nu})(P,Q) = (r_{m,\nu}\mathcal{P}_{m,k})(\mathcal{N}^\nu)(P)$.*
2. *Let $i(P) := (P,Q)$ be the natural inclusion map of N^ν into $N^\nu \times T^{m-\nu}$. If $\mathcal{Q}_{m,k} \in \mathfrak{Q}^U_{m,k}$, then $i^*\mathcal{Q}_{m,k}(\mathcal{N}^\nu \times \mathcal{T}^{m-\nu})(P,Q) = (r_{m,\nu}\mathcal{Q}_{m,k})(\mathcal{N}^\nu)(P)$.*
3. *$\mathfrak{K}_{\mathfrak{P},m,k} = \ker(r_{m,k-1}) \cap \mathfrak{P}^U_{m,k}$ and $\mathfrak{K}_{\mathfrak{Q},m,k} = \ker(r_{m,k}) \cap \mathfrak{Q}^U_{m,k}$.*

Note: It is necessary to use the pull-back i^* in order to regard the symmetric 2-tensor $P \to \mathcal{Q}_{m,k}(\mathcal{N}^\nu \times \mathcal{T}^{m-\nu})(P,Q)$ as a symmetric 2-tensor on $\mathcal{N}^\nu$. But it is not necessary to use pull-back to regard the function $P \to \mathcal{P}_{m,k}(\mathcal{N}^\nu \times \mathcal{T}^{m-\nu})(P,Q)$ as a function on $\mathcal{N}^\nu$ so we shall omit the i^* in that setting.

PROOF. Let $\mathcal{M}^m := \mathcal{N}^\nu \times \mathcal{T}^{m-\nu}$. Any polynomial in the derivatives of the metric which involves an index greater than ν vanishes since the metric is flat on $\mathcal{T}^{m-\nu}$. Since we have restricted the symmetric 2-tensors to $\mathcal{N}^\nu$, a symmetric 2-tensor also vanishes if it contains a holomorphic (or an anti-holomorphic) index greater than ν. Assertion (1) and Assertion (2) now follow. Lemma 2.2 permits us to identify an invariant polynomial (which is an algebraic object) with the corresponding geometric formula it defines; Assertion (3) now follows. $\square$

We can now relate the restriction maps $r_{m,\nu}$ on $\mathfrak{S}_m$ of Definition 1.1 to the restriction maps $r_{m,\nu}$ on $\mathfrak{P}^U_m$ and on $\mathfrak{Q}^U_m$ of Definition 3.4:

LEMMA 3.2.

1. *Let $\Xi_{\mathfrak{P},m,k}$ be as defined in Definition 1.4.*
   (a) *If $m > \nu$, then $r_{m,\nu}\Xi_{\mathfrak{P},m,k} = \Xi_{\mathfrak{P},\nu,k}r_{m,\nu}$ on $\mathfrak{S}_{m,k}$.*
   (b) *If $m \geq k$, then $\Xi_{\mathfrak{P},m,k}\mathfrak{S}_{m,k} \subset \mathfrak{K}_{\mathfrak{P},m,k}$.*
   (c) *If $m \geq k$ and if $0 \neq \mathcal{S}_{m,k} \in \mathfrak{S}_{m,k}$, then $r_{m,k}\Xi_{\mathfrak{P},m,k}\mathcal{S}_{m,k} \neq 0$.*
2. *Let $\Xi_{\mathfrak{Q},m,k}$ be as defined in Definition 1.6.*
   (a) *If $m > \nu$, then $r_{m,\nu}\Xi_{\mathfrak{Q},m,k} = \Xi_{\mathfrak{Q},\nu,k}r_{m,\nu}$ on $\mathfrak{S}_{m,k}$.*
   (b) *If $m \geq k+1$, then $\Xi_{\mathfrak{Q},m,k}\mathfrak{S}_{m,k} \subset \mathfrak{K}_{\mathfrak{Q},m,k}$.*
   (c) *If $m \geq k+1$ and if $0 \neq \mathcal{S}_{m,k} \in \mathfrak{S}_{m,k}$, then $r_{m,k+1}\Xi_{\mathfrak{Q},m,k}\mathcal{S}_{m,k} \neq 0$.*

PROOF. Recall that $\Xi_{\mathfrak{P},m,k}(\mathcal{S}_{m,k}) = g(\mathcal{S}_{m,k}(\mathcal{R}),\Omega^k)/k!$. Assertion (1a) is now immediate. Furthermore since Ω^k vanishes on a Kähler manifold of complex dimension $k-1$, $\Xi_{\mathfrak{P},k-1,k} = 0$. By Assertion (1a), $r_{m,k-1}\Xi_{\mathfrak{P},m,k} = \Xi_{\mathfrak{P},k-1,k}r_{m,k-1} = 0$. By Lemma 3.1, $\mathfrak{K}_{\mathfrak{P},m,k} = \ker(r_{m,k-1}) \cap \mathfrak{P}^U_{m,k}$. Assertion (1b) now follows. By Remark 1.1, $r_{m,k}$ is an isomorphism from $\mathfrak{S}_{m,k}$ to $\mathfrak{S}_{k,k}$. Thus to prove Assertion (1c), it suffices to show that $\Xi_{\mathfrak{P},k,k}$ is injective from $\mathfrak{S}_{k,k}$ to $\mathfrak{P}^U_{k,k}$. We use Equation (1.a) and Definition 1.4 to see that:

$$\Xi_{\mathfrak{P},k,k}(\mathcal{S}_{k,k})(\mathcal{R})d\nu_g = \frac{1}{k!}g(\mathcal{S}_{k,k}(\mathcal{R}),\Omega^k)d\nu_g = \mathcal{S}_{k,k}(\mathcal{R}).$$

If $\mathcal{S}_{k,k} \neq 0$, we may apply Lemma 1.1 establish Assertion (1c) by choosing $\vec{\nu}$ so that

$$\int_{\mathbb{CP}^{\bar{\nu}}} \mathcal{S}_{k,k}(\mathcal{R}_{\mathbb{CP}^{\bar{\nu}}}) \neq 0.$$

Recall that $\Xi_{\mathfrak{Q},m,k}(\mathcal{S}_{m,k}) = (1/(k+1)!)g(\mathcal{S}_{m,k}(\mathcal{R}) \wedge e^\alpha \wedge \bar{e}^\beta, \Omega^{k+1})e_\alpha \circ \bar{e}_\beta$. Assertion (2a) is now immediate. Since Ω^{k+1} vanishes on a Kähler manifold of complex dimension k, $\Xi_{\mathfrak{Q},k,k} = 0$. By Assertion (2a), $r_{m,k}\Xi_{\mathfrak{Q},m,k} = \Xi_{\mathfrak{Q},k,k}r_{m,k} = 0$. By Lemma 3.1, $\mathfrak{K}_{\mathfrak{Q},m,k} = \ker(r_{m,k}) \cap \mathfrak{Q}^U_{m,k}$. Assertion (2b) now follows. By Remark 1.1, $r_{m,k}$ is an isomorphism from $\mathfrak{S}_{m,k}$ to $\mathfrak{S}_{k+1,k}$. Thus to prove Assertion (2c), we may take $m = k+1$. Let $\mathcal{M}^{k+1} := \mathcal{N}^k \times \mathcal{T}^1$ where $\mathcal{T}^1$ is the flat Kähler torus of complex dimension 1. Let w be the usual periodic complex parameter on $\mathbb{T}^1$.

$$\frac{1}{(k+1)!}\Omega^{k+1}_{\mathcal{M}} = \frac{1}{(k+1)!}(\Omega_{\mathcal{M}} + \Omega_{\mathcal{T}})^{k+1} = \frac{1}{k!}\Omega^k_{\mathcal{M}} \wedge \Omega_{\mathcal{T}},$$

$$\Xi_{\mathfrak{Q},k+1,k}(\mathcal{S}_{k+1,k})(\mathcal{M}^{k+1}) = \big\{\Xi_{\mathfrak{P},k,k}(r_{k+1,k}\mathcal{S}_{k+1,k})(\mathcal{N}^k)\big\}\partial_w \circ \partial_{\bar{w}}.$$

Because $r_{k+1,k}$ is an injective map from $\mathfrak{S}_{k+1,k}$ to $\mathfrak{S}_{k,k}$, Assertion (2c) follows from Assertion (1c). □

Lemma 3.3. *Let $\Theta_{\mathfrak{Q},m,k}$ be as defined in Definition* 1.7.

1. *If $m > \nu$, then $r_{m,\nu}\Theta_{\mathfrak{Q},m,k} = \Theta_{\mathfrak{Q},\nu,k}r_{m,\nu}$ on $\mathfrak{S}_{m,k}$.*
2. *If $m \geq k+1$, then $\Theta_{\mathfrak{Q},m,k}\mathfrak{S}_{m,k} \subset \mathfrak{K}_{\mathfrak{Q},m,k}$.*
3. *If $m \geq k+1$ and if $0 \neq \mathcal{S}_{m,k} \in \mathfrak{S}_{m,k}$, then $r_{m,k+1}\Theta_{\mathfrak{Q},m,k}\mathcal{S}_{m,k} \neq 0$.*

Proof. It is necessary to expand the category in which we are working, if only briefly. Let $\mathcal{M}^m = (M, g, J)$ be a Hermitian manifold of complex dimension m. Let ∇^g be the associated Levi–Civita connection. We average over the action of the complex structure J to define an auxiliary connection $\tilde{\nabla}^g := (-J\nabla^g J + \nabla^g)/2$ on the tangent bundle. It is immediate that $\tilde{\nabla}^g J = J\tilde{\nabla}^g$ and thus $\tilde{\nabla}^g$ is a complex connection. The associated curvature $\mathcal{R}(\tilde{\nabla}^g)$ is then a complex endomorphism and consequently $\mathcal{S}_{m,k}(\mathcal{R}(\tilde{\nabla}^g)) \in \Lambda^{2k}(M)$ is well defined and we may extend Definition 1.4, Definition 1.6, and Definition 1.7 to this setting. If $\mathcal{M}^m_\epsilon$ is a Hermitian variation, then $\Theta_{\mathfrak{Q},m,k}(\mathcal{S}_{m,k})$ is characterized by the identity:

$$\partial_\varepsilon\bigg\{\int_M \Xi_{\mathfrak{P},m,k}(\mathcal{S}_{m,k})(\mathcal{R}_{\mathcal{M}_\varepsilon})d\nu_{\mathcal{M}_\epsilon}\bigg\}\bigg|_{\varepsilon=0} = \int_M \langle\{\Theta_{\mathfrak{Q},m,k}(\mathcal{S}_{m,k})\}(\mathcal{R}_{\mathcal{M}}), h\rangle d\nu_g.$$

Let $m > \nu$. We consider a product of the form $\mathcal{M}^m_\epsilon = \mathcal{N}^\nu_\epsilon \times \mathcal{T}^{m-\nu}$ where the variation is trivial on the Kähler torus and where $\mathcal{N}^\nu_\epsilon$ is a Hermitian variation. Since $\mathcal{T}^{m-\nu}$ has unit volume, we can ignore the integral over the torus and apply Lemma 3.1 and Lemma 3.2 to compute:

$$\begin{aligned}\partial_\varepsilon\bigg\{\int_M \Xi_{\mathfrak{P},m,k}(\mathcal{S}_{m,k})(\mathcal{R}_{\mathcal{M}_\varepsilon})d\nu_{\mathcal{M}_\epsilon}\bigg\}\bigg|_{\varepsilon=0} &= \int_M \langle\{\Theta_{\mathfrak{Q},m,k}(\mathcal{S}_{m,k})\}(\mathcal{R}_{\mathcal{M}}), h\rangle d\nu_g \\ &= \int_N \langle\{r_{m,\nu}\Theta_{\mathfrak{Q},m,k}(\mathcal{S}_{m,k})\}(\mathcal{R}_{\mathcal{N}}), h\rangle d\nu_g.\end{aligned}$$

We may also compute:

$$\partial_\varepsilon\left\{\int_N \Xi_{\mathfrak{P},\nu,k}(r_{m,\nu}\mathcal{S}_{m,k})(\mathcal{R}_{\mathcal{N}_\varepsilon})d\nu_{\mathcal{N}_\epsilon}\right\}\Big|_{\varepsilon=0} = \int_N \langle\{\Theta_{\mathfrak{Q},\nu,k}(r_{m,\nu}\mathcal{S}_{m,k})\}(\mathcal{R}_{\mathcal{N}}), h\rangle d\nu_g.$$

This shows

$$0 = \int_N \langle\{r_{m,\nu}\Theta_{\mathfrak{Q},m,k}(\mathcal{S}_{m,k}) - \Theta_{\mathfrak{Q},\nu,k}(r_{m,\nu}\mathcal{S}_{m,k})\}(\mathcal{R}_{\mathcal{N}}), h\rangle d\nu_g.$$

Since it is not necessary to restrict to Kähler variations, we can complete the proof of Assertion (1) by taking h to be the dual of

$$\{r_{m,\nu}\Theta_{\mathfrak{Q},m,k}(\mathcal{S}_{m,k}) - \Theta_{\mathfrak{Q},\nu,k}(r_{m,\nu}\mathcal{S}_{m,k})\}(\mathcal{R}_{\mathcal{N}})$$

with respect to the metric g to obtain

$$0 = \int_N \|\{r_{m,\nu}\Theta_{\mathfrak{Q},m,k}(\mathcal{S}_{m,k}) - \Theta_{\mathfrak{Q},\nu,k}(r_{m,\nu}\mathcal{S}_{m,k})\}(\mathcal{R}_{\mathcal{N}})\|_g^2 d\nu_g.$$

In complex dimension k, $\Xi_{\mathfrak{P},k,k}(\mathcal{S}_{k,k})[M]$ is a characteristic number and, consequently, since we constructed complex connections, $\Xi_{\mathfrak{P},k,k}(\mathcal{S}_{k,k})[M]$ is independent of the particular Hermitian metric chosen. This shows the Euler–Lagrange Equations are trivial and thus $\Theta_{\mathfrak{Q},k,k} = 0$. Assertion (2) now follows from Assertion (1).

We return to the Kähler setting and, by Assertion (1), take $m = k+1$ in proving Assertion (3). Let $\mathcal{M}^{k+1} := \mathcal{N}^k \times \mathcal{T}^1$ where $\mathcal{T}^1$ is the flat Kähler torus of complex dimension 1. Let w be the usual periodic complex parameter on $\mathbb{T}^1$. We take a variation of the form $g_\varepsilon := g_{\mathcal{N}} + (1+\varepsilon)dw \circ d\bar{w}$. The curvature is unchanged but we have $d\nu_\varepsilon = (1+\epsilon)d\nu_{\mathcal{N}}d\nu_{\mathcal{T}}$. Consequently,

$$\Theta_{\mathfrak{Q},k+1,k}(\mathcal{S}_{k+1,k})(\mathcal{M}^{k+1}) = \{\Xi_{\mathfrak{P},k,k}(r_{k+1,k}\mathcal{S}_{k+1,k})(\mathcal{N}^k)\}\partial_w \circ \partial_{\bar{w}}$$

and Assertion (3) follows from Assertion (1c) of Lemma 3.2. □

4. The action of the unitary group.

In this section, we use unitary invariance to study the spaces $\mathfrak{P}_m^U$ and $\mathfrak{Q}_m^U$. We then examine the spaces of universal curvature identities $\mathfrak{K}_{\mathfrak{P},m,k}$ and $\mathfrak{K}_{\mathfrak{Q},m,k}$ and obtain a fundamental estimate for their dimensions.

Lemma 4.1. *Let $\mathcal{U} \in \mathfrak{P}_m^U$ or let $\mathcal{U} \in \mathfrak{Q}_m^U$. Let $\mathcal{A}$ be a monomial of $\mathcal{U}$. Express*

$$\mathcal{A} = g(A_1^{\mathcal{A}}; B_1^{\mathcal{A}}) \cdots g(A_\ell^{\mathcal{A}}; B_\ell^{\mathcal{A}})\partial_{z_{\alpha_{\mathcal{A}}}} \circ \partial_{\bar{z}_{\beta_{\mathcal{A}}}}$$

where we omit the $\partial_{z_{\alpha_{\mathcal{A}}}} \circ \partial_{\bar{z}_{\beta_{\mathcal{A}}}}$ variables if $\mathcal{A} \in \mathfrak{P}_m^U$. Set $\operatorname{len}(\mathcal{A}) = \ell$.

1. *If $1 \le \alpha \le m$, then* $\deg_\alpha(\mathcal{A}) = \deg_{\bar{\alpha}}(\mathcal{A})$.

2. *Assume that* $\deg_\alpha(\mathcal{A}) > 0$. *Fix* $\beta \neq \alpha$ *and create a monomial* $\tilde{\mathcal{A}}$ *by changing exactly one holomorphic index in* $\mathcal{A}$ $\alpha \to \beta$. *Then there is a monomial* $\mathcal{A}_1$ *of* $\mathcal{U}$ *which is different from* $\mathcal{A}$ *and which also can create* $\tilde{\mathcal{A}}$ *either by changing exactly one holomorphic index* $\alpha \to \beta$ *or by changing exactly one anti-holomorphic index* $\bar\beta \to \bar\alpha$.
3. *If* $\mathcal{U} \in \mathfrak{P}_m^U$, *then there exists a monomial* $\mathcal{A}$ *of* $\mathcal{U}$ *so* $\deg_\alpha(\mathcal{A}) = 0$ *for* $\alpha > \operatorname{len}(\mathcal{A})$.
4. *If* $\mathcal{U} \in \mathfrak{Q}_m^U$, *then there exists a monomial* $\mathcal{A}$ *of* $\mathcal{U}$ *so* $\deg_\alpha(\mathcal{A}) = 0$ *for* $\alpha > \operatorname{len}(\mathcal{A}) + 1$.

Proof. Fix $1 \le \alpha \le m$ and consider the unitary transformation:

$$T_\alpha(\partial_{z_\gamma}) := \left\{ \begin{array}{ll} e^{\sqrt{-1}\theta}\partial_{z_\gamma} & \text{if } \gamma = \alpha \\ \partial_{z_\gamma} & \text{if } \gamma \neq \alpha \end{array} \right\},$$

$$T_\alpha(\partial_{\bar z_\gamma}) := \left\{ \begin{array}{ll} e^{-\sqrt{-1}\theta}\partial_{\bar z_\gamma} & \text{if } \gamma = \alpha \\ \partial_{\bar z_\gamma} & \text{if } \gamma \neq \alpha \end{array} \right\}.$$

Then $T_\alpha\mathcal{A} = e^{\sqrt{-1}\theta\{\deg_\alpha(\mathcal{A}) - \deg_{\bar\alpha}(\mathcal{A})\}}\mathcal{A}$, so we have

$$T_\alpha\mathcal{U} = \mathcal{U} = \sum_{\mathcal{A}} c(\mathcal{A},\mathcal{U}) e^{\sqrt{-1}\theta\{\deg_\alpha(\mathcal{A}) - \deg_{\bar\alpha}(\mathcal{A})\}}\mathcal{A}.$$

As θ was arbitrary, $c(\mathcal{A},\mathcal{U}) \neq 0$ implies $\deg_\alpha(\mathcal{A}) = \deg_{\bar\alpha}(\mathcal{A})$. Assertion (1) follows.

We now prove Assertion (2). Fix indices α and β. Set:

$$\nu := \deg_\alpha(\mathcal{A}) + \deg_\beta(\mathcal{A}) = \deg_{\bar\alpha}(\mathcal{A}) + \deg_{\bar\beta}(\mathcal{A}),$$

$$\tilde{\mathcal{U}} := \sum_{\mathcal{B}:\deg_\alpha(\mathcal{B}) + \deg_\beta(\mathcal{B}) = \nu} c(\mathcal{B},\mathcal{U})\mathcal{B}.$$

Then $\tilde{\mathcal{U}}$ is invariant under the action of $U(2)$ on the indices $\{\alpha,\beta\}$ and we work with $\tilde{\mathcal{U}}$ henceforth in the proof of Assertion (2); each monomial of $\tilde{\mathcal{U}}$ is homogeneous of degree ν in $\{\alpha,\beta\}$ and also in $\{\bar\alpha,\bar\beta\}$. Let $\tilde{\mathcal{A}}$ be obtained from $\mathcal{A}$ by changing a single holomorphic index $\alpha \to \beta$. Since

$$\deg_\alpha(\tilde{\mathcal{A}}) = \deg_\alpha(\mathcal{A}) - 1 = \deg_{\bar\alpha}(\mathcal{A}) - 1 = \deg_{\bar\alpha}(\tilde{\mathcal{A}}) - 1,$$

Assertion (1) implies $\tilde{\mathcal{A}}$ is not a monomial of $\tilde{\mathcal{U}}$. Let $u, v \in \mathbb{C}$ satisfy $|u|^2 + |v|^2 = 1$. Consider the unitary transformation

$$T\partial_{z_\sigma} = \left\{ \begin{array}{ll} \partial_{z_\sigma} & \text{if } \sigma \neq \alpha,\beta \\ u\partial_{z_\alpha} + v\partial_{z_\beta} & \text{if } \sigma = \alpha \\ -\bar v\partial_{z_\alpha} + \bar u\partial_{z_\beta} & \text{if } \sigma = \beta \end{array} \right\},$$

$$T\partial_{\bar z_\sigma}=\left\{\begin{array}{ll}\partial_{\bar z_\sigma} & \text{if } \sigma\neq\alpha,\beta\\ \bar u\partial_{\bar z_\alpha}+\bar v\partial_{\bar z_\beta} & \text{if } \sigma=\alpha\\ -v\partial_{\bar z_\alpha}+u\partial_{\bar z_\beta} & \text{if } \sigma=\beta\end{array}\right\}. \tag{4.a}$$

We may expand

$$T\tilde{\mathcal{U}}=f(u,v,\bar u,\bar v)\tilde{\mathcal{A}}+\text{other terms}$$

where f is homogeneous of degree 2ν in $\{u,v,\bar u,\bar v\}$; since $T\tilde{\mathcal{U}}=\tilde{\mathcal{U}}$ and since $\tilde{\mathcal{A}}$ is not a monomial of $\tilde{\mathcal{U}}$, $f(u,v,\bar u,\bar v)=0$ for $|u|^2+|v|^2=1$. Since f is homogeneous, $f(u,v,\bar u,\bar v)$ vanishes for all (u,v) and thus is the trivial polynomial. We have $T\mathcal{A}=n_{\mathcal{A},\tilde{\mathcal{A}}}vu^{\nu-1}\bar u^{\nu}\tilde{\mathcal{A}}+\cdots$ where $n_{\mathcal{A},\tilde{\mathcal{A}}}$ is a positive integer which reflects the number of ways that $\mathcal{A}$ can transform to $\tilde{\mathcal{A}}$ by changing a single holomorphic index $\alpha\to\beta$. There must therefore be some monomial $\mathcal{A}_1$ of $\mathcal{U}$ which is different from $\mathcal{A}$ and which transforms to $\tilde{\mathcal{A}}$ to create a term involving $vu^{\nu-1}\bar u^{\nu}\tilde{\mathcal{A}}+\cdots$ and which helps to cancel the corresponding term in $T\mathcal{A}$. In view of Equation (4.a), this can only be by changing a holomorphic index $\alpha\to\beta$ or an anti-holomorphic index $\bar\beta\to\bar\alpha$. Assertion (2) now follows.

We now prove Assertions (3) and (4). We first introduce some additional notation. Choose $\nu=\nu(\mathcal{A})$ maximal among all possible rearrangements defining $\mathcal{A}$ so

$$\deg_\alpha(A_i^{\mathcal{A}})=0 \text{ for } i<\alpha \text{ and } 1\le i\le\nu.$$

If $\nu(\mathcal{A})=\ell$, go on to the next step. If $\nu<\ell$, choose $\mathcal{A}$ to be a monomial of $\mathcal{U}$ so that $\nu(\mathcal{A})$ is maximal. Amongst all such possibilities choose $\mathcal{A}$ so that $\deg_{\nu+1}(A_{\nu+1}^{\mathcal{A}})$ is maximal. Since $\nu(\mathcal{A})<\ell$, there is some index $\alpha>\nu+1$ so $\deg_\alpha(A_{\nu+1}^{\mathcal{A}})>0$. By making a coordinate permutation, we may assume $\alpha=\nu+2$. Let $\mathcal{A}=A_{\nu+1}^{\mathcal{A}}\mathcal{A}_0$. Define $A_{\nu+1}^{\tilde{\mathcal{A}}}$ by changing one holomorphic index $\nu+2$ to $\nu+1$ in $A_{\nu+1}^{\mathcal{A}}$ and let $\tilde{\mathcal{A}}=A_{\nu+1}^{\tilde{\mathcal{A}}}\mathcal{A}_0$. Apply Assertion (2) to construct a monomial $\mathcal{A}_1\neq\mathcal{A}$ of $\mathcal{U}$. There are two possibilities:

1. $\mathcal{A}_1$ transforms to $\tilde{\mathcal{A}}$ by changing a holomorphic index $\nu+2\to\nu+1$. Since $\deg_\alpha(A_1^{\mathcal{A}})=\cdots=\deg_\alpha(A_\nu^{\mathcal{A}})=0$ for $\alpha>\nu$, $A_i^{\mathcal{A}_1}=A_i^{\mathcal{A}}$ for $i\le\nu$. Since $\mathcal{A}_1\neq\mathcal{A}$, $A_{\nu+1}^{\mathcal{A}_1}\neq A_{\nu+1}^{\mathcal{A}}$. Consequently, $\nu(\mathcal{A}_1)=\nu$ and $\deg_{\nu+1}(A_{\nu+1}^{\mathcal{A}_1})>\deg_{\nu+1}(A_{\nu+1}^{\mathcal{A}})$. This contradicts the choice of $\mathcal{A}$ with $\nu(\mathcal{A})=\nu$ and $\deg_{\nu+1}(A_{\nu+1}^{\mathcal{A}})$ maximal. Thus this possibility is impossible.
2. $\mathcal{A}_1$ transforms to $\tilde{\mathcal{A}}$ by changing an anti-holomorphic index $\bar\nu\to\overline{\nu+1}$. Then we have $A_i^{\mathcal{A}_1}=A_i^{\tilde{\mathcal{A}}}$ for all i. Thus $\nu(\mathcal{A}_1)=\nu$ and $\deg_\nu(A_\nu^{\mathcal{A}_1})>\deg_\nu(A_\nu^{\mathcal{A}})$ which is impossible.

The contradiction derived above shows we may choose $\mathcal{A}$ so $\deg_\alpha(A_i^{\mathcal{A}})=0$ for $\alpha>\ell$ and $i\le\ell$. If $\mathcal{U}\in\mathfrak{P}_{m,k}^U$, then Assertion (3) follows. Suppose $\mathcal{U}\in\mathfrak{Q}_{m,k}^U$. If $\alpha_{\mathcal{A}}\le\ell+1$, then we are done. If $\alpha_{\mathcal{A}}>\ell+1$, we may interchange the index $\alpha_{\mathcal{A}}$ and the index $\ell+1$ to assume $\alpha_{\mathcal{A}}=\ell+1$. This completes the proof of Assertion (4). □

The following technical Lemma is crucial to our study of the spaces of universal curvature identities $\mathfrak{K}_{\mathfrak{P},m,k}=\ker(r_{m,k-1})\cap\mathfrak{P}_{m,k}^U$ and $\mathfrak{K}_{\mathfrak{Q},m,k}=\ker(r_{m,k})\cap\mathfrak{Q}_{m,k}^U$.

Lemma 4.2. *Let $\mathcal{U} \in \mathfrak{K}_{\mathfrak{P},m,k}$ or let $\mathcal{U} \in \mathfrak{K}_{\mathfrak{Q},m,k}$. Let*

$$\mathcal{A} = g(A_1^{\mathcal{A}}; B_1^{\mathcal{A}}) \cdots g(A_\ell^{\mathcal{A}}; B_\ell^{\mathcal{A}}) \partial_{z_{\alpha_{\mathcal{A}}}} \circ \partial_{\bar{z}_{\beta_{\mathcal{A}}}}$$

be a monomial of $\mathcal{U}$; we omit the $\partial_{z_{\alpha_{\mathcal{A}}}} \circ \partial_{\bar{z}_{\beta_{\mathcal{A}}}}$ variables if $\mathcal{U} \in \mathfrak{K}_{\mathfrak{P},m,k}$.

1. *We have that $|A_i^{\mathcal{A}}| = |B_i^{\mathcal{A}}| = 2$ and $\ell = k$.*
2. *There exists a monomial $\mathcal{A}$ of $\mathcal{U}$ satisfying:*
   (a) *For $1 \leq i \leq k$, there exists an index α_i so that $A_i^{\mathcal{A}} = (\alpha_i, \alpha_i)$.*
   (b) *$\alpha_i = i$ for $1 \leq i \leq k$.*
   (c) *If $\mathcal{U} \in \mathfrak{K}_{\mathfrak{Q},m,k}$, then $\alpha_{\mathcal{A}} = k+1$.*
   (d) *For $1 \leq i \leq k$, there exists an index β_i so that $B_i^{\mathcal{A}} = (\beta_i, \beta_i)$.*
   (e) *The indices $\{\beta_1, \ldots, \beta_k\}$ are a permutation of the indices $\{1, \ldots, k\}$.*
   (f) *If $\mathcal{U} \in \mathfrak{K}_{\mathfrak{Q},m,k}$, then $\beta_{\mathcal{A}} = k+1$.*

Proof. The length $\operatorname{len}(\mathcal{A}) = \ell$ of a monomial is unchanged by the action of $U(m)$. Decompose

$$\mathcal{U} = \mathcal{U}_1 + \mathcal{U}_2 + \cdots \text{ where } \mathcal{U}_\ell := \sum_{\operatorname{len}(\mathcal{A})=\ell} c(\mathcal{A}, \mathcal{U})\mathcal{A}.$$

Thus in proving Assertion (1), we may suppose $\mathcal{U} = \mathcal{U}_\ell$ for some ℓ. Let $\mathcal{A}$ be any monomial of $\mathcal{U}$.

1. Suppose $\mathcal{U} \in \mathfrak{K}_{\mathfrak{P},m,k} = \ker(r_{m,k-1}) \cap \mathfrak{P}_{m,k}^U$. By Lemma 4.1 (3), we can choose a monomial $\mathcal{A}$ of $\mathcal{U}$ so that no index other than $\{1, \ldots, \ell\}$ appears in $\mathcal{A}$. As $r_{m,k-1}(\mathcal{U}) = 0$, there exists an index $\alpha \geq k$ so that $\deg_\alpha(\mathcal{A}) > 0$. Consequently, $\ell \geq k$.
2. Suppose $\mathcal{U} \in \mathfrak{K}_{\mathfrak{Q},m,k} = \ker(r_{m,k}) \cap \mathfrak{Q}_{m,k}^U$. By Lemma 4.1 (4), we can choose a monomial $\mathcal{A}$ of $\mathcal{U}$ so that no index other than $\{1, \ldots, \ell+1\}$ appears in $\mathcal{A}$. Since $r_{m,k,\mathfrak{Q}}(\mathcal{U}) = 0$, there exists an index $\alpha \geq k+1$ so that $\deg_\alpha(\mathcal{A}) > 0$. This once again implies $\ell \geq k$.

Since $|A_i^{\mathcal{A}}| \geq 2$ and $|B_i^{\mathcal{A}}| \geq 2$, we may estimate:

$$2k = \operatorname{ord}(\mathcal{A}) = \sum_{i=1}^{\ell} \{|A_i^{\mathcal{A}}| + |B_i^{\mathcal{A}}| - 2\} \geq 2\ell \geq 2k.$$

Consequently, all these inequalities must have been equalities so $|A_i^{\mathcal{A}}| = |B_i^{\mathcal{A}}| = 2$ and therefore that $\mathcal{U}$ only involves the 2-jets of the metric; the covariant derivatives of the curvature tensor play no role. It also shows that $\ell = k$ so $\operatorname{len}(\mathcal{A}) = k$. Assertion (1) now follows.

We shall assume that $\mathcal{U} = \mathcal{Q} \in \mathfrak{K}_{\mathfrak{Q},m,k} = \ker(r_{m,k}) \cap \mathfrak{Q}_{m,k}^U$ as the case in which $\mathcal{U} \in \mathfrak{K}_{\mathfrak{P},m,k} = \ker(r_{m,k-1}) \cap \mathfrak{P}_{m,k}^U$ is similar. We define

$$\mathcal{Q}_{k+1,k} = \sum_{\deg_\alpha(\mathcal{A})=0 \text{ for } \alpha > k+1} c(\mathcal{U}, \mathcal{A})\mathcal{A}.$$

This is invariant under the action of $U(k+1)$ and the argument given above shows $\mathcal{Q}_{k+1,k} \neq 0$. Furthermore, every index $\{1,\dots,k+1\}$ appears in every monomial of $\mathcal{Q}_{k+1,k}$ and thus $\mathcal{Q}_{k+1,k} \in \mathfrak{K}^U_{\mathfrak{Q},k+1,k}$. Finally, every monomial of $\mathcal{Q}_{k+1,k}$ is a monomial of $\mathcal{U}$. This shows that we may assume that the complex dimension is $m = k+1$ in the proof of Assertion (2); this is the crucial case. Thus every monomial $\mathcal{A}$ of $\mathcal{Q}_{k+1,k}$ contains as holomorphic indices exactly the indices $\{1,\dots,k+1\}$ and also contains exactly these indices as anti-holmorphic indices.

We say that a holomorphic index α *touches itself in* $\mathcal{A}$ if we have $A_i^{\mathcal{A}} = (\alpha,\alpha)$ for some i. Choose a monomial $\mathcal{A}$ of $\mathcal{Q}_{k+1,k}$ so the number of holomorphic indices which touch themselves in $\mathcal{A}$ is maximal. By making a coordinate permutation, we may assume without loss of generality the indices which touch themselves holomorphically in $\mathcal{A}$ are the indices $\{1,\dots,\nu\}$. Consequently $A_i^{\mathcal{A}} = (i,i)$ for $i \leq \nu$. Suppose $\nu < k$. Both the indices $\nu+1$ and $\nu+2$ appear holomorphically in $\mathcal{A}$ since every index $\{1,\dots,k+1\}$ appears in $\mathcal{A}$. Since only one index can appear in $\partial_{z_{\alpha_{\mathcal{A}}}}$, we may assume that $A_{\nu+1}^{\mathcal{A}} = (\nu+1,\sigma)$. Furthermore, by the maximality of ν, we have $\nu+1 \neq \sigma$. Express

$$\mathcal{A} = g(1,1;\star,\star)\cdots g(\nu,\nu;\star,\star)g(\nu+1,\sigma;\star,\star)\mathcal{A}_0$$

where "$\star$" indicates indices not of interest and where $\mathcal{A}_0$ is a suitably chosen monomial. We apply Lemma 4.1 (2) to construct $\tilde{\mathcal{A}}$ by changing a single holomorphic index $\sigma \to \nu+1$:

$$\tilde{\mathcal{A}} = g(1,1;\star,\star)\cdots g(\nu,\nu;\star,\star)g(\nu+1,\nu+1;\star,\star)\mathcal{A}_0.$$

We apply Lemma 4.1 (2) to choose a monomial $\mathcal{A}_1 \neq \mathcal{A}$ of $\mathcal{Q}_{k+1,k}$. There are two possibilities:

1. If $\mathcal{A}_1$ transforms to $\tilde{\mathcal{A}}$ by changing an anti-holomorphic index $\overline{\nu+1}$ to $\bar{\sigma}$, then the holomorphic indices are unchanged and we have found a monomial $\mathcal{A}_1$ of $\mathcal{Q}_{k+1,k}$ where one more index touches itself holomorphically. This contradicts the choice of $\mathcal{A}$ such that the number of indices touching themselves holomorphically is maximal.
2. If $\mathcal{A}_1$ transforms to $\tilde{\mathcal{A}}$ by changing a holomorphic index σ to $\nu+1$, then we can not have changed $A_i^{\mathcal{A}}$ for $i \leq \nu$ since the index $\nu+1$ does not appear here. Furthermore, since $A_{\nu+1}^{\tilde{\mathcal{A}}} = (\nu+1,\nu+1)$, and since $\mathcal{A}_1 \neq \mathcal{A}$, that variable was not changed. Thus

$$\mathcal{A}_1 = g(1,1;\star,\star)\cdots g(\nu,\nu;\star,\star)g(\nu+1,\nu+1;\star,\star)\tilde{\mathcal{A}}_0$$

   and again, one more index touches itself holomorphically. This contradicts the choice of $\mathcal{A}$ such that the number of indices touching themselves holomorphically is maximal.

We have shown $\nu = k$. This establishes Assertion (2a). Since every index must in fact appear in $\mathcal{A}$, no index can touch itself holomorphically in $\mathcal{A}$ in two different variables. Thus after permuting the indices appropriately, we have that

$$\mathcal{A} = g(1,1;\star,\star)\cdots g(k,k;\star,\star)\partial_{z_{k+1}} \circ \partial_{\bar{z}_\star}.$$

This establishes Assertion (2b) and Assertion (2c).

We will use the same argument to establish the remaining assertions; the analysis is slightly more tricky since we do not want to destroy the normalizations of Assertions (2a) and (2b). Let $\mathcal{A}$ be a monomial of $\mathcal{Q}_{k+1,k}$ which satisfies the normalizations of Assertions (2a) and (2b). Let $\sigma \leq k$. Then σ appears twice holomorphically in $\mathcal{A}$ and hence by Lemma 4.1 (1) also appears anti-holomorpically in $\mathcal{A}$ twice. The index $\sigma = k+1$ appears once holomorphically in $\mathcal{A}$ and once anti-holomorphically in $\mathcal{A}$. Choose $\mathcal{A}$ so the number ν of indices which touch themselves anti-holomorphically in $\mathcal{A}$ is maximal. If $\nu = k$, then we are done. So we assume $\nu < k$ and argue for a contradiction. By permuting the indices, we may assume the indices $\bar{1}, \dots, \bar{\nu}$ touch themselves anti-holomorphically in $\mathcal{A}$ and that the index $\overline{\nu+1}$ does not touch itself anti-holomorphically in $\mathcal{A}$. Since $\overline{\nu+1}$ appears twice anti-holomorphically, it must touch some other index $\bar{x}$ anti-holomorphically. Express:

$$\mathcal{A} = g(\star, \star; \overline{\nu+1}, \bar{x})\mathcal{A}_0.$$

Change the anti-holomorphic index $\bar{x}$ to an anti-holomorphic index $\overline{\nu+1}$ to form:

$$\tilde{\mathcal{A}} = g(\star, \star; \overline{\nu+1}, \overline{\nu+1})\mathcal{A}_0.$$

We use Lemma 4.1 (2) to construct a monomial $\mathcal{A}_1$ of $\mathcal{Q}_{k+1,k}$ different from $\mathcal{A}$. If $\mathcal{A}_1$ transforms to $\tilde{\mathcal{A}}$ by changing an anti-holomorphic index $\bar{x}$ to the anti-holomorphic index $\overline{\nu+1}$, then the fact that i touches itself anti-holomorphically for $i \leq \nu$ is not spoiled and since $\mathcal{A} \neq \mathcal{A}_1$, $\overline{\nu+1}$ touches itself anti-holomorphically in $\mathcal{A}_1$. Since only the anti-holomorphic indices are changed, the normalizations of Assertions (2a) and (2b) are not affected. Thus one more index would touch itself anti-holomorphically in $\mathcal{A}_1$ than is the case in $\mathcal{A}$ and this would contradict the maximality of ν. Thus $\mathcal{A}_1$ transforms to $\tilde{\mathcal{A}}$ by changing a holomorphic index $\nu+1$ to x. This destroys the normalizations of Assertion (2a). There are several possibilities which we examine seriatim; we shall list the generic case but if the variables collapse, this plays no role. In what follows, we permit $x = y$.

Case I: The index x appears once in $\mathcal{A}$. Let $\star$ indicate a term not of interest. Let ε be either a $\partial_{z_\alpha} \circ \partial_{\bar{z}_\beta}$ variable or a $g(-,-;-,-)$ variable to have a uniform notation and to avoid multiplying the cases unduly; we shall not fuss about the number of indices in ε and thus the second $\star$ could be the empty symbol if $\epsilon(\star; \bar{\beta}, \star)$ indicates the $\partial_{z_\alpha} \circ \partial_{\bar{z}_\beta}$ variable whereas the first $\star$ could indicate two indices if $\epsilon(\star; \bar{\beta}, \star)$ denotes a $g(\star, \star; \bar{\beta}, \star)$ variable. Let $\mathcal{A}_0$ be an auxiliary monomial. We may express

$$\mathcal{A} = g(\nu+1, \nu+1; \star, \star)g(\star, \star; \overline{\nu+1}, \bar{x})\varepsilon(\star; \overline{\nu+1}, \star)\varepsilon(x, \star; \star)\mathcal{A}_0, \text{ where}$$
$$\deg_{\nu+1}(\mathcal{A}) = 2, \quad \deg_{\overline{\nu+1}}(\mathcal{A}) = 2, \quad \deg_x(\mathcal{A}) = 1, \quad \deg_{\bar{x}}(\mathcal{A}) = 1.$$

We change an anti-holomorphic index $\bar{x}$ to an anti-holomorphic index $\overline{\nu+1}$ to construct:

$$\tilde{\mathcal{A}} = g(\nu+1, \nu+1; \star, \star)g(\star, \star; \overline{\nu+1}, \overline{\nu+1})\varepsilon(\star; \overline{\nu+1}, \star)\varepsilon(x, \star; \star)\mathcal{A}_0, \text{ where}$$
$$\deg_{\nu+1}(\tilde{\mathcal{A}}) = 2, \quad \deg_{\overline{\nu+1}}(\tilde{\mathcal{A}}) = 3, \quad \deg_x(\tilde{\mathcal{A}}) = 1, \quad \deg_{\bar{x}}(\tilde{\mathcal{A}}) = 0.$$

Since $\mathcal{A}_1$ transforms to $\tilde{\mathcal{A}}$ by changing a holomorphic index $\nu+1$ to a holomorphic index x, $\deg_{\bar{x}}(\mathcal{A}_1)=0$ which is impossible since every index from 1 to $k+1$ appears in every monomial of $\mathcal{Q}_{k+1,k}$.

Case II: The index x appears twice in $\mathcal{A}$ and does not appear in $\partial_{\bar{z}_\beta}$. Then

$$\begin{aligned}
\mathcal{A} &= g(\nu+1,\nu+1;\star,\star)g(x,x;\star,\star)g(\star,\star;\overline{\nu+1},\bar{x})g(\star,\star;\bar{x},\bar{z})\\
&\quad\times\varepsilon(\star;\overline{\nu+1},\star)\varepsilon(\star;\bar{z},\star)\mathcal{A}_0,\ \text{where}\\
&\quad\deg_{\nu+1}(\mathcal{A})=2,\quad \deg_{\overline{\nu+1}}(\mathcal{A})=2,\quad \deg_x(\mathcal{A})=2,\quad \deg_{\bar{x}}(\mathcal{A})=2,\\
\tilde{\mathcal{A}} &= g(\nu+1,\nu+1;\star,\star)g(x,x;\star,\star)g(\star,\star;\overline{\nu+1},\overline{\nu+1})g(\star,\star;\bar{x},\bar{z})\\
&\quad\times\varepsilon(\star;\overline{\nu+1},\star)\varepsilon(\star;\bar{z},\star)\mathcal{A}_0,\ \text{where}\\
&\quad\deg_{\nu+1}(\tilde{\mathcal{A}})=2,\quad \deg_{\overline{\nu+1}}(\tilde{\mathcal{A}})=3,\quad \deg_x(\tilde{\mathcal{A}})=2,\quad \deg_{\bar{x}}(\tilde{\mathcal{A}})=1,\ \text{and}\\
\mathcal{A}_1 &= g(\nu+1,\nu+1;\star,\star)g(\nu+1,x;\star,\star)g(\star,\star;\overline{\nu+1},\overline{\nu+1})g(\star,\star;\bar{x},\bar{z})\\
&\quad\times\varepsilon(\star;\overline{\nu+1},\star)\varepsilon(\star;\bar{z},\star)\mathcal{A}_0,\ \text{where}\\
&\quad\deg_{\nu+1}(\mathcal{A}_1)=3,\quad \deg_{\overline{\nu+1}}(\mathcal{A}_1)=3,\quad \deg_x(\mathcal{A}_1)=1,\quad \deg_{\bar{x}}(\mathcal{A}_1)=1.
\end{aligned}$$

We permit $z=\nu+1$. We change an anti-holomorphic index $\bar{x}$ to $\bar{z}$ to create:

$$\begin{aligned}
\tilde{\mathcal{A}}_1 &= g(\nu+1,\nu+1;\star,\star)g(\nu+1,x;\star,\star)g(\star,\star;\overline{\nu+1},\overline{\nu+1})g(\star,\star;\bar{z},\bar{z})\\
&\quad\times\varepsilon(\star;\overline{\nu+1},\star)\varepsilon(\star;\bar{z},\star)\mathcal{A}_0,\ \text{where}\\
&\quad\deg_{\nu+1}(\tilde{\mathcal{A}}_1)=3,\quad \deg_{\overline{\nu+1}}(\tilde{\mathcal{A}}_1)=3,\quad \deg_x(\tilde{\mathcal{A}}_1)=1,\quad \deg_{\bar{x}}(\tilde{\mathcal{A}}_1)=0.
\end{aligned}$$

Again, we construct $\mathcal{A}_2$. If we transform $\mathcal{A}_2$ to $\tilde{\mathcal{A}}_1$ by changing a holomorphic index z to a holomorphic index x, then

$$\deg_{\nu+1}(\mathcal{A}_2)=3,\quad \deg_{\overline{\nu+1}}(\mathcal{A}_2)=3,\quad \deg_x(\mathcal{A}_2)=0,\quad \deg_{\bar{x}}(\mathcal{A}_2)=0.$$

This contradicts the fact that $\deg_x(\mathcal{A}_2)>0$. Consequently $\mathcal{A}_2$ transforms to $\tilde{\mathcal{A}}_1$ by changing an anti-holomorphic index $\bar{x}$ to an anti-holmorphic index $\bar{z}$. Since $\mathcal{A}_2\neq\mathcal{A}_1$,

$$\begin{aligned}
\mathcal{A}_2 &= g(\nu+1,\nu+1;\star,\star)g(\nu+1,x;\star,\star)g(\star,\star;\overline{\nu+1},\overline{\nu+1})g(\star,\star;\bar{z},\bar{z})\\
&\quad\times\varepsilon(\star;\overline{\nu+1},\star)\varepsilon(\star;\bar{x},\star)\mathcal{A}_0,\ \text{where}\\
&\quad\deg_{\nu+1}(\mathcal{A}_2)=3,\quad \deg_{\overline{\nu+1}}(\mathcal{A}_2)=3,\quad \deg_x(\mathcal{A}_2)=1,\quad \deg_{\bar{x}}(\mathcal{A}_2)=1.
\end{aligned}$$

We have simply interchanged the anti-holomorphic indices $\bar{x}$ and $\bar{z}$ to construct $\mathcal{A}_2$ from $\mathcal{A}_1$. We construct $\tilde{\mathcal{A}}_2$ by changing a holomorphic index $\nu+1$ to x to create:

$$\begin{aligned}\tilde{\mathcal{A}}_2 &= g(\nu+1,\nu+1;\star,\star)g(x,x;\star,\star)g(\star,\star;\overline{\nu+1},\overline{\nu+1})g(\star,\star;\bar{z},\bar{z})\\ &\quad\times \varepsilon(\star;\overline{\nu+1},\star)\varepsilon(\star;\bar{x},\star)\mathcal{A}_0,\ \text{where}\\ &\quad \deg_{\nu+1}(\tilde{\mathcal{A}}_2)=2,\quad \deg_{\overline{\nu+1}}(\tilde{\mathcal{A}}_2)=3,\quad \deg_x(\tilde{\mathcal{A}}_2)=2,\quad \deg_{\bar{x}}(\tilde{\mathcal{A}}_2)=1.\end{aligned}$$

We consider $\mathcal{A}_3$. Since $\mathcal{A}_3 \neq \mathcal{A}_2$, $\mathcal{A}_3$ does not transform to $\tilde{\mathcal{A}}_2$ by changing a holomorphic index $\nu+1$ to x. Instead, $\mathcal{A}_3$ transforms to $\tilde{\mathcal{A}}_2$ by transforming an anti-holomorphic index $\bar{x}$ to an anti-holomorphic index $\overline{\nu+1}$. There are two possibilities

$$\begin{aligned}\mathcal{A}_3 &= g(\nu+1,\nu+1;\star,\star)g(x,x;\star,\star)g(\star,\star;\overline{\nu+1},\overline{\nu+1})g(\star,\star;\bar{z},\bar{z})\\ &\quad\times \varepsilon(\star;\bar{x},\star)\varepsilon(\star;\bar{x},\star)\mathcal{A}_0,\text{or}\\ \mathcal{A}_3 &= g(\nu+1,\nu+1;\star,\star)g(x,x;\star,\star)g(\star,\star;\overline{\nu+1},\bar{x})g(\star,\star;\bar{z},\bar{z})\\ &\quad\times \varepsilon(\star;\overline{\nu+1},\star)\varepsilon(\star;\bar{x},\star)\mathcal{A}_0.\end{aligned}$$

Both these possibilities satisfy the normalization of Assertion (2a). And there is either one more anti-holomorphic or two more anti-holomorphic indices which touch themselves. This is impossible by the maximality of $\mathcal{A}$.

Case III: The index x appears twice in $\mathcal{A}$ and appears in $\partial_{\bar{z}_\beta}$. Then $\nu+1$ does not appear in $\partial_{\bar{z}_\beta}$ and hence some other variable $g(\star,\star;\overline{\nu+1},\bar{y})$ appears in $\mathcal{A}$. If $\deg_y(\mathcal{A})=1$, then Case I pertains. If $\deg_y(\mathcal{A})=2$, then Case II pertains. This final contradiction establishes the Lemma. □

4.1. The crucial estimate.

Let $\rho(k)$ be the number of partitions of k as described in Definition 1.1.

Lemma 4.3. *If $m>k$, then $\dim\{\mathfrak{K}_{\mathfrak{Q},m,k}\}\le\rho(k)$ and $\dim\{\mathfrak{K}_{\mathfrak{P},m,k}\}\le\rho(k)$.*

Proof. Let $0\neq\mathcal{Q}_{m,k}\in\mathfrak{K}_{\mathfrak{Q},m,k}$. Apply Lemma 4.2 to find a monomial $\mathcal{A}$ of $\mathcal{Q}_{m,k}$ so that

$$\mathcal{A}_\sigma = g(1,1;\bar{\sigma}(1),\bar{\sigma}(1))g(2,2;\bar{\sigma}(2),\bar{\sigma}(2))\cdots g(k,k;\bar{\sigma}(k),\bar{\sigma}(k))\partial_{z_{k+1}}\circ\partial_{\bar{z}_{k+1}}$$

where $\sigma\in\text{Perm}(k)$ is a suitably chosen permutation. Thus $\mathcal{Q}_{m,k}\neq 0$ implies $c(\mathcal{A}_\sigma,\mathcal{Q}_{m,k})\neq 0$ for some σ. Only the conjugacy class of σ in $\text{Perm}(k)$ is important and, writing the permutation σ in terms of cycles, we see that there are $\rho(k)$ such conjugacy classes; ordering the lengths of these cycles in decreasing order determines a partition π. Thus there are $\rho(k)$ monomials A_π so that $\mathcal{Q}_{m,k}\neq 0$ implies $c(\mathcal{A}_\pi)\neq 0$; the inequality $\dim\{\mathfrak{K}_{\mathfrak{Q},m,k}\}\le\rho(k)$ now follows. The proof of the inequality $\dim\{\mathfrak{K}_{\mathfrak{P},m,k}\}\le\rho(k)$ is analogous and is therefore omitted. □

5. The proofs of Theorem 1.1, Theorem 1.2, and Theorem 1.3.

5.1. The proof of Theorem 1.1 and of Theorem 1.2.

Let $m\ge k$. By Lemma 3.2, $\Xi_{\mathfrak{P},m,k}$ is a 1-1 map from $\mathfrak{S}_{m,k}$ to $\mathfrak{K}_{\mathfrak{P},m,k}$. By Equation (1.d), we have that $\dim\{\mathfrak{S}_{m,k}\}=\rho(k)$ By Lemma 4.3, $\dim\{\mathfrak{K}_{\mathfrak{P},m,k}\}\le\rho(k)$. Consequently

$$\dim\{\mathfrak{K}_{\mathfrak{P},m,k}\} = \dim\{\mathfrak{S}_{m,k}\} = \rho(k)$$

and $\Xi_{\mathfrak{P},m,k}$ is an isomorphism. This proves Theorem 1.1. The same line of argument shows that $\Xi_{\mathfrak{Q},m,k}$ is an isomorphism from $\mathfrak{S}_{m,k}$ to $\mathfrak{K}_{\mathfrak{Q},m,k}$; this establishes Theorem 1.2.

5.2. The proof of Theorem 1.3.

We must show $\Theta_{\mathfrak{Q},m,k} = \Xi_{\mathfrak{Q},m,k}$. We argue for a contradiction. Suppose to the contrary that $\Theta_{\mathfrak{Q},m,k}\mathcal{S}_{m,k} \neq \Xi_{\mathfrak{Q},m,k}\mathcal{S}_{m,k}$ for some $\mathcal{S}_{m,k} \in \mathfrak{S}_{m,k}$. We apply Lemma 3.2 and Lemma 3.3 to see

$$0 \neq r_{m,k+1}\{\Theta_{\mathfrak{Q},m,k} - \Xi_{\mathfrak{Q},m,k}\}\mathcal{S}_{m,k} = \{\Theta_{\mathfrak{Q},m,k} - \Xi_{\mathfrak{Q},m,k}\}(r_{m,k+1}\mathcal{S}_{m,k}).$$

Thus we may suppose without loss of generality that $m = k+1$. We apply the argument used to establish Lemma 3.3 (3). Let $\mathcal{M}_\epsilon^{k+1} := \mathcal{N}^k \times \mathcal{T}_\epsilon^1$ where the metric on $\mathcal{T}_\epsilon^1$ is $(1+\epsilon)dw \circ d\bar{w}$. Since the metric on $\mathcal{N}^k$ is unchanged and only the volume element on M is changing,

$$\begin{aligned}
\frac{1}{k!} g_\epsilon(\mathcal{S}_{k+1,k}(\mathcal{R}_{\mathcal{M}_\epsilon}), \Omega_\epsilon^k) &= \frac{1}{k!} g(r_{k+1,k}\mathcal{S}_{k+1,k}(\mathcal{R}_{\mathcal{N}}), \Omega_{\mathcal{N}}^k),\\
\partial_\epsilon\{g_\epsilon(\mathcal{S}_{k+1,k}(\mathcal{R}_{\mathcal{M}_\epsilon}), \Omega_\epsilon^k)\} &= 0, \\
\partial_\epsilon\{d\nu_{\mathcal{M}_\epsilon}\} &= d\nu_{\mathcal{M}} = d\nu_{\mathcal{N}} d\nu_{\mathcal{T}}.
\end{aligned} \tag{5.a}$$

Since $\mathcal{T}^1$ has volume 1, we may use Equation (5.a) to compute:

$$\begin{aligned}
&\partial_\epsilon\left\{\frac{1}{k!}\int_M g_\epsilon(\mathcal{S}_{k+1,k}(\mathcal{R}_{\mathcal{M}_\epsilon}), \Omega_\epsilon^k) d\nu_{\mathcal{M}_\epsilon}\right\}\bigg|_{\epsilon=0}\\
&\quad = \frac{1}{k!}\int_N g(r_{k+1,k}\mathcal{S}_{k+1,k}(\mathcal{R}_{\mathcal{N}}), \Omega^k) d\nu_{\mathcal{N}}.
\end{aligned} \tag{5.b}$$

Since N has complex dimension k, we have

$$\frac{1}{k!}\int_N g(r_{k+1,k}\mathcal{S}_{k+1,k}(\mathcal{R}_{\mathcal{N}}), \Omega^k) d\nu_{\mathcal{N}} = \int_N r_{k+1,k}\mathcal{S}_{k+1,k}(\mathcal{R}_{\mathcal{N}}). \tag{5.c}$$

By Lemma 3.3, $\Theta_{\mathfrak{Q},k+1,k}\mathcal{S}_{k+1,k} \in \mathfrak{K}_{\mathfrak{Q},k+1,k}$. By Theorem 1.2, $\Xi_{\mathfrak{Q},k+1,k}$ is an isomorphism from $\mathfrak{S}_{k+1,k}$ to $\mathfrak{K}_{\mathfrak{Q},k+1,k}$. Thus we may find $\tilde{\mathcal{S}}_{k+1,k} \in \mathfrak{S}_{k+1,k}$ so that we have $\Xi_{\mathfrak{Q},k+1,k}\tilde{\mathcal{S}}_{k+1,k} = \Theta_{\mathfrak{Q},k+1,k}\mathcal{S}_{k+1,k}$. Consequently:

$$\begin{aligned}
\partial_\epsilon\left\{\frac{1}{k!}\int_M g(\mathcal{S}_{k+1,k}(\mathcal{R}_{\mathcal{M}_\epsilon}), \Omega_\epsilon^k) d\nu_{\mathcal{M}_\epsilon}\right\}\bigg|_{\epsilon=0} &= \int_M \langle\Theta_{\mathfrak{Q},k+1,k}\mathcal{S}_{k+1,k}(\mathcal{R}_{\mathcal{M}}), h\rangle d\nu_g\\
&= \int_M \langle\Xi_{\mathfrak{Q},k+1,k}\tilde{\mathcal{S}}_{k+1,k}(\mathcal{R}_{\mathcal{M}}), h\rangle d\nu_g.
\end{aligned} \tag{5.d}$$

We use the definition and the argument used to establish Equation (5.c) to compute:

$$\begin{aligned}
&\int_M \langle \Xi_{\mathfrak{Q},k+1,k}\tilde{\mathcal{S}}_{k+1,k}(\mathcal{R}_{\mathcal{M}}), h\rangle d\nu_g \\
&= \frac{1}{(k+1)!}\int_M g(\tilde{\mathcal{S}}_{k+1,k}(\mathcal{R}_{\mathcal{M}})\wedge e^\alpha\wedge\bar{e}^\beta, \Omega_{\mathcal{M}}^{k+1})\langle e_\alpha\circ\bar{e}_\beta, h\rangle d\nu_g \\
&= \frac{1}{k!}\int_M g(r_{k+1,k}\tilde{\mathcal{S}}_{k+1,k}(\mathcal{R}_{\mathcal{N}}), \Omega_{\mathcal{N}}^k)d\nu_{\mathcal{N}}d\nu_{\mathcal{T}} \\
&= \int_N r_{k+1,k}\tilde{\mathcal{S}}_{k+1,k}(\mathcal{R}_{\mathcal{N}}). \qquad (5.e)
\end{aligned}$$

We use Equation (5.b), Equation (5.c), Equation (5.d), and Equation (5.e) to see

$$\int_N r_{k+1,k}\{\mathcal{S}_{k+1,k}-\tilde{\mathcal{S}}_{k+1,k}\}(\mathcal{R}_{\mathcal{N}}) = 0.$$

Since $\mathcal{N}^k$ was an arbitrary Kähler manifold of complex dimension k, we may apply Lemma 1.1 to see $r_{k+1,k}\{\mathcal{S}_{k+1,k}-\tilde{\mathcal{S}}_{k+1,k}\} = 0$. By Remark 1.1, $\mathcal{S}_{k+1,k} = \tilde{\mathcal{S}}_{k+1,k}$ and consequently $\Xi_{\mathfrak{Q},k+1,k}\mathcal{S}_{k+1,k} = \Theta_{\mathfrak{Q},k+1,k}\mathcal{S}_{k+1,k}$. This completes the proof of Theorem 1.3. □

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