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L^q-L^r estimate of the Oseen flow in plane exterior domains

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Dedicated to Professor Yoshihiro Shibata on his 60th birthday

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Abstract. We consider the initial value problem for the Oseen system in plane exterior domains and study the large time behavior of solutions. For the space dimension $n \geq 3$ the theory was well developed by [**26**], [**10**] and [**11**], while 2D case has remained open because of difficulty arising from singularity like $\log\sqrt{\lambda+\alpha^2}$ of the Oseen resolvent, where λ is the resolvent parameter and α is the Oseen parameter. In this paper we derive the local energy decay of the Oseen semigroup and apply it to deduction of L^q-L^r estimates. The dependence of estimates on the Oseen parameter α is also discussed. The proof relies on detailed analysis of asymptotic structure of the fundamental solution of the Oseen resolvent with respect to both λ and α.

1. Introduction.

Let Ω be an exterior domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. We consider the Navier–Stokes system

$$\begin{cases} \partial_t u + u\cdot\nabla u = \Delta u - \nabla p, & \operatorname{div} u = 0, \\ u|_{\partial\Omega} = 0, & \\ u \to u_\infty & \text{as } |x| \to \infty \end{cases} \tag{1.1}$$

which describes the motion of a viscous incompressible fluid past an obstacle $\mathbb{R}^2\setminus\Omega$ (rigid body) that moves with translational velocity $-u_\infty$, where $u(x,t) = (u_1,u_2)^T$ and $p(x,t)$ respectively denote unknown velocity and pressure of the fluid, while $u_\infty \in \mathbb{R}^2\setminus\{0\}$ is a given uniform velocity. Because of the Stokes paradox, we do need to consider the problem around u_∞, so that the Oseen linearization works well as an approximation of the Navier–Stokes system. Since the Navier–Stokes system is rotationally invariant, without loss of generality, one may take

$$u_\infty = -2\alpha e_1 \qquad \text{with } \alpha > 0 \text{ (Oseen parameter)}, \quad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then, by denoting $u - u_\infty$ by the same symbol u, (1.1) is reduced to

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$$\begin{cases} \partial_t u + u\cdot\nabla u = \Delta u + 2\alpha\partial_1 u - \nabla p, & \text{div } u = 0, \\ u|_{\partial\Omega} = 2\alpha e_1, & \\ u \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{1.2}$$

where $\partial_1 = \partial_{x_1}$. It is an open question to clarify the large time behavior of solutions to the initial value problem for (1.2) even when $\alpha > 0$ is small enough. Toward better understanding of this problem, it is important to study: (i) steady flows with fine decay/summability for $|x| \to \infty$; and (ii) decay properties of solutions to the Oseen initial value problem for $t \to \infty$. Concerning the first issue (i), it was proved by Finn and Smith [**14**], [**15**], [**34**] and, later on, by Galdi [**17**], [**18**] that if α is nonzero but sufficiently small, then (1.2) admits a steady flow (called a physically reasonable solution) $u(x) = (u_1, u_2)^T$ that satisfies $u(x) = O(|x|^{-1/2})$ as $|x| \to \infty$ and exhibits a parabolic wake region behind the body like the Oseen fundamenatal solution. To be precise, such an anisotropic decay structure with wake is found only for u_1, while u_2 has no wake; as a consequence, we have

$$u_1 \in L^q(\Omega) \quad \text{for } \forall q > 3; \qquad u_2 \in L^r(\Omega) \quad \text{for } \forall r > 2. \tag{1.3}$$

So far, the stability/instablity of this flow is unsolved, while we know the stability of physically reasonable solutions in 3D exterior domains as long as they are small, see Shibata [**32**] and the references therein. The difficulty in 2D is due to less summability (1.3), that is not enough to show the stablity.

In the present paper we study the second issue (ii) above, that is, the large time behavior of solutions to the initial value problem for the Oseen system

$$\begin{cases} \partial_t u - \Delta u - 2\alpha\partial_1 u + \nabla p = 0, & \text{div } u = 0 \quad \text{in } \Omega\times(0,\infty), \\ u|_{\partial\Omega} = 0, & \\ u \to 0 & \text{as } |x| \to \infty, \\ u(\cdot, 0) = f. & \end{cases} \tag{1.4}$$

Besides the motivation mentioned above, the linear analysis is of interest in itself. We consider (1.4) in the standard solenoidal Lebesgue space $L^q_\sigma(\Omega), 1 < q < \infty$, on which we are able to show the generation of analytic semigroup (the Oseen semigroup) that provides a solution operator $f \mapsto u(t)$, see section 2. Our aim is to show the L^q-L^r estimates (with $n = 2$)

$$\|u(t)\|_r \le C\, t^{-(n/q-n/r)/2}\|f\|_q \qquad (1 < q \le r \le \infty,\ q \ne \infty), \tag{1.5}$$

$$\|\nabla u(t)\|_r \le C\, t^{-(n/q-n/r)/2-1/2}\|f\|_q \qquad (1 < q \le r \le n) \tag{1.6}$$

for $t > 0$, where $n \ge 2$ is the space dimension, $\|\cdot\|_q$ stands for the norm of $L^q(\Omega)$, $\nabla = (\partial_1, \partial_2)$ and $\partial_j = \partial_{x_j}$. For the Stokes flow (case $\alpha = 0$), these estimates were deduced by Iwashita [**25**] ($n \ge 3$), Dan and Shibata [**8**], [**9**] ($n = 2$), and Maremonti and Solonnikov [**28**] ($n \ge 2$). We cannot avoid the restriction $r \le n$ for (1.6), see [**28**] and

[**23**] (while it is not clear whether the same restriction is essential for the case $\alpha > 0$). As for the Oseen flow (case $\alpha > 0$), (1.5) and (1.6) were established by Kobayashi and Shibata [**26**] ($n = 3$) and Enomoto and Shibata [**10**], [**11**] ($n \geq 3$), except the case of plane exterior domains, where the constant $C > 0$ above can be taken uniformly with respect to small $\alpha > 0$; that is, for each $M > 0$, we have $C = C(M; \Omega, q, r)$ provided $\alpha \in (0, M]$. This is important in the proof of stability of 3D steady flows as an application of (1.5)–(1.6), see [**32**].

Our main result is Theorem 2.3 in the next section. As a special case $r = q$, we see that the Oseen semigroup is uniformly bounded in $L^q_\sigma(\Omega)$, which has not been known until now. The large time behavior of the semigroup is definitely related to the asymptotic behavior of the resolvent for $\lambda \to 0$, where λ is the resolvent parameter. Look at the fundamental solution $E^\alpha_\lambda(x)$ of the Oseen resolvent in the whole plane $\mathbb{R}^2$. It is of the form (4.6) and involves the modified Bessel functions of the second kind. A typical leading profile for small (λ, α) possesses the logarithmic singularity such as $\log \sqrt{\lambda + \alpha^2}$ unlike 3D case. To be precise, the fundamental solution has even worse terms in the sense that the derivative $\partial_\lambda E^\alpha_\lambda(x)$ is a bit more singular (than $\partial_\lambda \log \sqrt{\lambda + \alpha^2}$) and that the interaction between λ and α is more complicated (than $\log \sqrt{\lambda + \alpha^2}$). Actually, those terms arise from the pressure and it does not seem to be easy to control both parameters, λ and α. In analyzing the fundamental solution, it would be fine if we could do so on the Fourier side as in Kobayashi and Shibata [**26**] for 3D case, however, we face another difficulty for small $|\xi|$; indeed, several integrals in their analysis [**26**] do not converge near $\xi = 0$ for 2D case. To get around this difficulty, in this paper, we make full use of the asymptotic expansion of the modified Bessel functions to find the asymptotic behavior of the fundamental solution for small (λ, α). In doing so, we concentrate ourselves on the analysis for $\lambda \in \overline{\mathbb{C}_+} = \{\mathrm{Re}\ \lambda \geq 0\}$ with small $|\lambda|$. The asymptotic behavior along the imaginary axis is of particular importance to justify the representation formula, see (6.1), of the semigroup. The structure (4.19) of the fundamental solution also plays a key role.

Besides the analysis of the fundamental solution mentioned above, as in [**25**], [**26**], [**10**], [**22**] and [**24**], the essential step for the proof of L^q-L^r estimate is to show local energy decay properties of the semigroup in $\Omega_R = \Omega \cap B_R = \{x \in \Omega; |x| < R\}$, see Theorem 2.1 and Theorem 2.2, by means of spectral analysis; indeed, we adopt their argument in principle. In order to analyze the regularity of the resolvent near $\lambda = 0$, a parametrix of the resolvent is constructed with use of the Oseen resolvents in $\mathbb{R}^2$ and in a bounded domain near the obstacle $\mathbb{R}^2 \setminus \Omega$ by a cut-off technique, however, the standard compactness argument provides us little information about the dependence of the resolvent on λ and α. So we have to reconstruct the resolvent especially near $\lambda = 0$. From this point of view, a key ingredient of the proof is Proposition 6.2 on some estimate of the remaining term arising from cut-off procedure for the case $\lambda = 0$ (purely Oseen system) by use of the structure of the fundamental solution $E^\alpha_0(x)$. This structure tells us that we have the singularity like $\log \alpha$ only in the degenerate part. A method based on this observation was developed by Dan and Shibata [**8**], however, we need further analysis with the aid of consideration of steady Stokes system, see Lemma 6.3. For the proof of local energy decay properties, we employ a relation between the modulus of continuity of a function and the rate of decay of its inverse Fourier transform, see Lemma 7.3.

Unfortunately, our main result does not provide desirable situation in which the constant C in (1.5)–(1.6) is independent of $\alpha \in (0, M]$. In [**8**] Dan and Shibata gave the following remarkable insight into the case $\alpha = 0$: So far as the behavior in the bounded domain $\Omega_R = \Omega \cap B_R$ is concerned, the Stokes resolvent goes to the Stokes flow as $\lambda \to 0$ in spite of the singularity like $\log \sqrt{\lambda}$ of the fundamental solution. This is because the total net force of the Stokes resolvent exerted by the fluid to the obstacle $\mathbb{R}^2 \setminus \Omega$ tends to zero as $\lambda \to 0$ and, thus, the limit Stokes flow can be bounded at space infinity (the Stokes paradox disappears). The similarity can be observed for the Oseen flow ($\lambda = 0$) for $\alpha \to 0$, see [**17**, Chapter VII] and [**30**]. In order that our L^q-L^r estimate covers the case of the Stokes semigroup ($\alpha = 0$) discussed in [**8**], very probably, we have to find analysis for small (λ, α), which includes those phonomena. I believe that such an improvement should be done in the future.

In the next section we give three theorems: the first two are concerned with local energy decay and then the third one is L^q-L^r estimate. In section 3 we prepare some estimate of the semigroup in the whole plane $\mathbb{R}^2$. Section 4 is the most important part, in which we construct the fundamental solution of the Oseen resolvent and carry out analysis of its asymptotic structure. In section 5 we consider the Oseen resolvent system in a bounded domain. Section 6 is devoted to construction of a parametrix of the resolvent in exterior domains. In section 7 we investigate the regularity of the resolvent near $\lambda = 0$ to prove Theorem 2.1 and Theorem 2.2 on local energy decay estimates. We complete the proof of Theorem 2.3 in the final section.

2. Results.

We start with introducing notation. The following subsets of the complex plane $\mathbb{C}$ are often used:

$$\mathbb{C}_+ = \{\lambda \in \mathbb{C};\ \mathrm{Re}\ \lambda > 0\}, \qquad \Sigma_\Lambda = \{\lambda \in \overline{\mathbb{C}_+} \setminus \{0\};\ |\lambda| \le \Lambda\} \tag{2.1}$$

for $\Lambda > 0$. Given a domain $D \subset \mathbb{R}^2$, $1 \le q \le \infty$ and integer $k \ge 0$, we denote by $L^q(D)$ and by $W^{k,q}(D)$ the standard Lebesgue and Sobolev spaces, respectively. For the exterior domain Ω under consideration, we simply write the norm $\|\cdot\|_q = \|\cdot\|_{L^q(\Omega)}$. For Banach spaces X and Y we denote by $\mathcal{L}(X;Y)$ the set of all bounded linear operators from X to Y; it is also a Banach space with norm $\|\cdot\|_{\mathcal{L}(X;Y)}$. We simply write $\mathcal{L}(X) = \mathcal{L}(X;X)$.

Let us introduce the Oseen operator in the L^q space, $1 < q < \infty$, of solenoidal vector fields. Set

$$\begin{aligned} L^q_\sigma(\Omega) &= \text{completion of } C^\infty_{0,\sigma}(\Omega) \text{ in } L^q(\Omega) \\ &= \{u \in L^q(\Omega);\ \mathrm{div}\ u = 0,\ \nu \cdot u|_{\partial\Omega} = 0\} \end{aligned}$$

where $C^\infty_{0,\sigma}(\Omega)$ consists of all smooth and divergence free vector fields with compact support, ν is the outer unit normal to the boundary $\partial\Omega$ and $\nu \cdot u|_{\partial\Omega}$ denotes the normal trace of u. Here and hereafter, we use the same symbol for denoting spaces of vector and scalar functions if there is no confusion. It is well known that the space $L^q(\Omega)$ of vector fields admits the Helmholtz decomposition

$$L^q(\Omega) = L^q_\sigma(\Omega) \oplus \{\nabla p \in L^q(\Omega); p \in L^q_{loc}(\overline{\Omega})\}$$

which was proved by Miyakawa [**29**] and by Simader and Sohr [**33**]. By using the projection $P : L^q(\Omega) \to L^q_\sigma(\Omega)$, the Oseen operator L_α is defined by

$$\begin{cases} D(L_\alpha) = \{u \in W^{2,q}(\Omega) \cap L^q_\sigma(\Omega);\ u|_{\partial\Omega} = 0\}, \\ L_\alpha u = -P[\Delta u + 2\alpha\partial_1 u]. \end{cases}$$

As for the Stokes operator $A = L_0 = -P\Delta$, we know the generation of analytic semigroup (the Stokes semigroup) $\{e^{-tA}\}_{t\geq 0}$ in $L^q_\sigma(\Omega)$ due to Giga [**19**], Solonnikov [**35**], Farwig and Sohr [**13**]. Furthermore, it is uniformly bounded $\|e^{-tA}f\|_q \leq C\|f\|_q$ by the result of Borchers and Varnhorn [**4**]. Those results follow from the resolvent estimate: For any $\varepsilon \in (0, \pi/2]$ there is a constant $C = C(\varepsilon, \Omega, q) > 0$ such that

$$\|(\lambda + A)^{-1} f\|_q \leq C|\lambda|^{-1}\|f\|_q \tag{2.2}$$

for $|\arg\lambda| \leq \pi - \varepsilon$ and $f \in L^q_\sigma(\Omega)$. We also have

$$\|\nabla(\lambda + A)^{-1} f\|_q \leq C|\lambda|^{-1/2}\|f\|_q \tag{2.3}$$

for $|\arg\lambda| \leq \pi - \varepsilon$ with $|\lambda| \geq 1$ and $f \in L^q_\sigma(\Omega)$, which is a consequence of $\|\nabla u\|_q \leq C(\|Au\|_q + \|u\|_q)^{1/2}\|u\|_q^{1/2}$ with $u = (\lambda + A)^{-1}f$ and (2.2). The condition $|\lambda| \geq 1$ cannot be removed if $q > 2 = n$; in fact, if it were possible, then $\|\nabla e^{-tA}f\|_q \leq Ct^{-1/2}\|f\|_q$ would hold for large t, which yields a contradiction ([**28**], [**23**]).

It follows from $\|u\|_{W^{2,q}(\Omega)} \leq C(\|Au\|_q + \|u\|_q)$ for $u \in D(A)$ that

$$\|u\|_{W^{2,q}(\Omega)} \leq C\|L_\alpha u\|_q + C(1 + \alpha^2)\|u\|_q \tag{2.4}$$

for $u \in D(L_\alpha) = D(A)$. As in Miyakawa [**29**], by a perturbation argument from the Stokes operator, it is easily verified that the Oseen operator generates an analytic semigroup (the Oseen semigroup) $\{e^{-tL_\alpha}\}_{t\geq 0}$ in $L^q_\sigma(\Omega)$, so that

$$\|L_\alpha e^{-tL_\alpha} f\|_q \leq Ct^{-1}\|f\|_q \qquad (0 < t < 2) \tag{2.5}$$

for $f \in L^q_\sigma(\Omega)$. Thus the solution of (1.4) is given by $u(\cdot, t) = e^{-tL_\alpha}f$. In fact, by

$$\lambda + L_\alpha = \left[1 - 2\alpha P\partial_1(\lambda + A)^{-1}\right](\lambda + A)$$

together with (2.3), there is a constant $c_0 = c_0(\varepsilon, \Omega, q)$ such that if $|\lambda| \geq \max\{c_0\alpha^2, 1\}$ as well as $|\arg\lambda| \leq \pi - \varepsilon$, then $\lambda \in \rho(-L_\alpha)$ and

$$(\lambda + L_\alpha)^{-1} = (\lambda + A)^{-1}\sum_{k=0}^\infty \left[2\alpha P\partial_1(\lambda + A)^{-1}\right]^k$$

that enjoys

$$|\lambda|\,\|(\lambda+L_\alpha)^{-1}f\|_q+|\lambda|^{1/2}\|\nabla(\lambda+L_\alpha)^{-1}f\|_q\le C\|f\|_q \tag{2.6}$$

for λ as above and $f\in L^q_\sigma(\Omega)$. Concerning the resolvent set $\rho(-L_\alpha)$, we will show that it contains $\mathbb{C}\setminus S_\alpha$, see Proposition 6.1, where

$$S_\alpha=\{\lambda\in\mathbb{C};\ 4\alpha^2\mathrm{Re}\ \lambda+(\mathrm{Im}\ \lambda)^2\le 0\}. \tag{2.7}$$

Actually, one can construct the resolvent concretely for every $\lambda\in\mathbb{C}\setminus S_\alpha$ by using a cut-off argument developed by Koboyashi and Shibata [**26**, Theorem 4.4] ($n=3$), Enomoto and Shibata [**10**, Theorem 4.4] ($n\ge 3$). Among ingredients of their proof, the only point which depends on the space dimension n is the uniqueness for the exterior problem (6.2), and it will be provided in Lemma 6.1 for $n=2$. There is further information due to Farwig and Neustupa [**12**, Theorem 1.2], which proves that S_α is exactly the essential spectrum of $-L_\alpha$ for all $\alpha>0$ and $q\in(1,\infty)$. Although they gave a proof in the case of exterior of 3D rotating obstacle around the e_1-axis, it seems to work in 2D without rotation as well. In this paper we will show that the semigroup is uniformly bounded, see (2.15) with $r=q$.

Set $B_R=\{x\in\mathbb{R}^2;|x|<R\}$. We fix $R_0>0$ such that $\mathbb{R}^2\setminus\Omega\subset B_{R_0}$ throughout this paper. For $1<q<\infty$ and $R\ge R_0+1$ we set

$$L^q_{[R]}(\Omega)=\{f\in L^q(\Omega);\ f(x)=0\ a.e.\ |x|\ge R\} \tag{2.8}$$

from which the initial data are taken in the following theorem on local energy decay properties of the Oseen semigroup in $\Omega_R=\Omega\cap B_R$.

Theorem 2.1. *Let $2\le q<\infty$, $R\ge R_0+1$, $M>0$ and $0\le\theta\le 1$. Then there is a constant $C=C(M,\theta;\Omega,q,R)>0$ such that*

$$\|e^{-tL_\alpha}Pf\|_{W^{1,q}(\Omega_R)}\le\frac{C}{\alpha^{1+2\theta}}\,t^{-(1+\theta)}\,(\log t)^\theta\|f\|_q \tag{2.9}$$

for all $t\ge 2$, $f\in L^q_{[R]}(\Omega)$ and $\alpha\in(0,M]$.

For the Stokes semigroup, the rate of local energy decay shown by Dan and Shibata [**8**] is $t^{-1}(\log t)^{-2}$. Therefore, one can expect no singular behavior with respect to α in (2.9) at least for the case $\theta=0$, however, we could not remove α^{-1}. When we fix $\alpha>0$ and take $\theta=1$ in Theorem 2.1, we find that the rate of local energy decay of the Oseen semigroup is $t^{-2}\log t$, which is better than that of the Stokes semigroup.

Remark 2.1. The reason why we have the restriction $q\in[2,\infty)$ is that we are forced to employ L^2-theory in a part of the proof. But, even for the case $1<q<2$, it is obvious that Theorem 2.1 yields

$$\|e^{-tL_\alpha}Pf\|_{W^{1,q}(\Omega_R)}\le\frac{C}{\alpha^{1+2\theta}}\,t^{-(1+\theta)}\,(\log t)^\theta\|f\|_2 \tag{2.10}$$

for $f \in L^2_{[R]}(\Omega)$, which is enough to proceed to the next stage (Theorem 2.2) on account of smoothing effect of analytic semigroups. But the use of L^2-theory causes unpleasant behavior α^{-1}, see Lemma 7.2, while the essential part from spectral analysis yields less singular behavior like $\log \alpha$, see Lemma 7.1.

Let $1 < q < \infty$. By (2.5) together with (2.4) it is easily seen that

$$\begin{aligned} \|e^{-tL_\alpha} Pf\|_{W^{1,q}(\Omega)} &\le C\big\{\|L_\alpha e^{-tL_\alpha} Pf\|_q + (1+\alpha^2)\|e^{-tL_\alpha} Pf\|_q\big\}^{1/2} \|e^{-tL_\alpha} Pf\|_q^{1/2} \\ &\le C(1+\alpha)\, t^{-1/2}\|f\|_q \end{aligned} \tag{2.11}$$

for $0 < t < 2$ and $f \in L^q(\Omega)$. It is convenient to write

$$\|e^{-tL_\alpha} Pf\|_{W^{1,q}(\Omega_R)} \le \frac{C}{\alpha^{1+2\theta}}\, t^{-1/2}(e+t)^{-(1/2+\theta)} \big(\log{(e+t)}\big)^\theta \|f\|_{\max\{q,2\}} \tag{2.12}$$

for all $t > 0$, $f \in L^{\max\{q,2\}}_{[R]}(\Omega)$ and $\alpha \in (0, M]$, by combining (2.11) with (2.9) and (2.10).

In the next step, we still consider the local energy decay, however, for general data from $L^q_\sigma(\Omega)$. By using (2.12) with arbitrary small $\theta > 0$ together with L^q-L^r estimate (3.3) of the Oseen semigroup in the whole plane $\mathbb{R}^2$, see (3.1), we will show the following theorem.

THEOREM 2.2. *Let $1 < q < \infty$, $R \ge R_0 + 1$ and $M > 0$. Suppose $\varepsilon > 0$ is arbitrarily small. Then there is a constant $C = C(M, \varepsilon; \Omega, q, R) > 0$ such that*

$$\|e^{-tL_\alpha} f\|_{W^{1,q}(\Omega_R)} \le \frac{C}{\alpha^{1+\varepsilon}}\, t^{-1/q}\|f\|_q \tag{2.13}$$

for all $t \ge 2$, $f \in L^q_\sigma(\Omega)$ and $\alpha \in (0, M]$.

In view of (1.5) with $n = 2$ and $r = \infty$, we formally observe that the decay rate $t^{-1/q}$ is reasonable. Note that this rate cannot be improved even though we use (2.12) with $\theta = 1$. On the other hand, if we used (2.12) with $\theta = 0$, the decay rate in (2.13) would be $t^{-1/q} \log t$, however, this is not enough to show Theorem 2.3 below. For the Stokes semigroup as well, the rate $t^{-1}(\log t)^{-2}$ of local energy decay for $f \in L^q_{[R]}(\Omega)$ is important in [**8**] since it is summable for large t. As an immediate consequence of (2.13), we obtain

$$\|e^{-tL_\alpha} f\|_{W^{1,q}(\Omega_R)} + \|\partial_t e^{-tL_\alpha} f\|_{L^q(\Omega_R)} \le \frac{C}{\alpha^{1+\varepsilon}}\, (1+t)^{-1/q}\big(\|L_\alpha f\|_q + \|f\|_q\big) \tag{2.14}$$

for all $t \ge 0$, $f \in D(L_\alpha)$ and $\alpha \in (0, M]$. In fact, (2.13) yields

$$\|\partial_t e^{-tL_\alpha} f\|_{L^q(\Omega_R)} = \|e^{-tL_\alpha} L_\alpha f\|_{L^q(\Omega_R)} \le \frac{C}{\alpha^{1+\varepsilon}}\, t^{-1/q}\|L_\alpha f\|_q$$

for $t \ge 2$. On the other hand, by (2.4) we have

$$\|e^{-tL_\alpha}f\|_{W^{1,q}(\Omega_R)}+\|\partial_t e^{-tL_\alpha}f\|_{L^q(\Omega_R)}\le \|e^{-tL_\alpha}f\|_{W^{2,q}(\Omega)}+\|e^{-tL_\alpha}L_\alpha f\|_{L^q(\Omega)}$$
$$\le C\big(\|L_\alpha f\|_q+\|f\|_q\big)$$

for $0\le t<2$. These estimates imply (2.14).

The main result on L^q-L^r estimate of the Oseen semigroup reads as follows.

Theorem 2.3. *Let $\alpha>0$. Then*

$$\|e^{-tL_\alpha}f\|_r\le C\,t^{-1/q+1/r}\|f\|_q \qquad (1<q\le r<\infty), \tag{2.15}$$
$$\|e^{-tL_\alpha}f\|_\infty\le C\,t^{-1/q}\big(\log(e+t)\big)\|f\|_q \qquad (1<q<r=\infty), \tag{2.16}$$
$$\|\nabla e^{-tL_\alpha}f\|_r\le C\,t^{-1/q+1/r-1/2}\|f\|_q \qquad (1<q\le r<2=n), \tag{2.17}$$
$$\|\nabla e^{-tL_\alpha}f\|_2\le C\,t^{-1/q}\big(\log(e+t)\big)\|f\|_q \qquad (1<q\le r=2=n) \tag{2.18}$$

for all $t>0$ and $f\in L^q_\sigma(\Omega)$. Concerning the constant $C>0$, given arbitrary large $M>0$ and small $\varepsilon>0$, there is a constant $\widetilde{C}=\widetilde{C}(M,\varepsilon;\Omega,q,r)>0$ such that $C\le\widetilde{C}/\alpha^\rho$ provided $\alpha\in(0,M]$, where

$$\begin{aligned}
&\rho=\begin{cases}1+\varepsilon, & 1/q-1/r\le 1/2\\ 2+\varepsilon, & 1/q-1/r>1/2\end{cases} \qquad \textit{for } (2.15),\\
&\rho=\begin{cases}1+\varepsilon, & q>2\\ 2+\varepsilon, & q\le 2\end{cases} \qquad \textit{for } (2.16),\\
&\rho=\begin{cases}1+\varepsilon, & q=r\\ 2+\varepsilon, & q<r\end{cases} \qquad \textit{for } (2.17) \textit{ and } (2.18).
\end{aligned} \tag{2.19}$$

Remark 2.2. For the marginal cases, the rate of decay given in (2.16) and (2.18) does not seem to be sharp. In fact, the Stokes semigroup e^{-tA} satisfies

$$\|e^{-tA}f\|_\infty\le C\,t^{-1/q}\|f\|_q \qquad (1<q<\infty), \tag{2.20}$$
$$\|\nabla e^{-tA}f\|_2\le C\,t^{-1/q}\|f\|_q \qquad (1<q\le 2) \tag{2.21}$$

for $t>0$ and $f\in L^q_\sigma(\Omega)$, see [**8**], [**9**] and [**28**]. It is worth while noting that (2.21) with $q=2$ can be deduced by a simple weighted energy method. One can also apply the energy method to the Oseen system (1.4) to obtain

$$\|\nabla e^{-tL_\alpha}f\|_2\le C\big(\sqrt{\alpha}\,t^{-1/4}+t^{-1/2}\big)\|f\|_2 \tag{2.22}$$

with some $C>0$ independent of α.

3. The Oseen semigroup in the whole plane.

In this section we give estimates of the Oseen semigroup in the whole plane $\mathbb{R}^2$, which solves

$$\partial_t u - \Delta u - 2\alpha\partial_1 u = 0, \qquad u(\cdot,0) = f \qquad \text{in } \mathbb{R}^2 \times (0,\infty).$$

It is of the explicit form

$$u(x,t) = \big(U_\alpha(t)f\big)(x) = \int_{\mathbb{R}^2} G(x + 2\alpha t e_1 - y, t) f(y)\, dy \tag{3.1}$$

where $G(x,t)$ denotes the heat kernel

$$G(x,t) = \frac{1}{4\pi t}\, e^{-|x|^2/4t}. \tag{3.2}$$

If in particular the vector field f satisfies $\operatorname{div} f = 0$, then $\operatorname{div} u = 0$ as well as $\partial_t u - \Delta u - 2\alpha\partial_1 u + \nabla p = 0$ with arbitrary function p that depends only on time t.

Let $1 \le q \le r \le \infty$. Obviously there is a constant $C = C(j,q,r)$ independent of α such that

$$\|\nabla^j U_\alpha(t) f\|_{L^r(\mathbb{R}^2)} \le C t^{-j/2-1/q+1/r} \|f\|_{L^q(\mathbb{R}^2)} \tag{3.3}$$

for all $t > 0$, $f \in L^q(\mathbb{R}^2)$ and integer $j \ge 0$. One can also deduce a kind of local energy decay easily.

LEMMA 3.1. *Let $1 < q \le r < \infty$ and $R > 0$. Then there is a constant $C = C(q,r,R) > 0$ such that*

$$\begin{aligned} &\|U_\alpha(t)f\|_{W^{2,r}(B_R)} + \|\partial_t U_\alpha(t) f\|_{L^r(B_R)} \\ &\qquad \le C(1+\alpha^2)\, t^{-1/q+1/r}(1+t)^{-1/r} \|f\|_{W^{2,q}(\mathbb{R}^2)} \end{aligned} \tag{3.4}$$

for all $t > 0$ and $f \in W^{2,q}(\mathbb{R}^2)$.

PROOF. By the equation we have

$$\|\partial_t U_\alpha(t) f\|_{L^r(B_R)} \le \|\Delta U_\alpha(t) f\|_{L^r(B_R)} + 2\alpha \|\partial_1 U_\alpha(t) f\|_{L^r(B_R)}.$$

From (3.3) we obtain

$$\|\nabla^j U_\alpha(t) f\|_{L^r(B_R)} \le C\|\nabla^j U_\alpha(t) f\|_{L^\infty(\mathbb{R}^2)} \le C t^{-j/2-1/q} \|f\|_{L^q(\mathbb{R}^2)}$$

for all $t > 0$ and integer $j \ge 0$. By $L_{\mathbb{R}^2,\alpha} = -\Delta - 2\alpha\partial_1$ we denote the generator of the semigroup $U_\alpha(t)$; then, it has the same estimate as in (2.4). Since $f \in W^{2,q}(\mathbb{R}^2) = D(L_{\mathbb{R}^2,\alpha})$, we have

$$
\begin{aligned}
\|U_\alpha(t)f\|_{W^{2,r}(B_R)} &\le \|U_\alpha(t)f\|_{W^{2,r}(\mathbb{R}^2)} \\
&\le C\|L_{\mathbb{R}^2,\alpha}U_\alpha(t)f\|_{L^r(\mathbb{R}^2)} + C(1+\alpha^2)\|U_\alpha(t)f\|_{L^r(\mathbb{R}^2)} \\
&\le Ct^{-1/q+1/r}\big\{\|L_{\mathbb{R}^2,\alpha}f\|_{L^q(\mathbb{R}^2)} + (1+\alpha^2)\|f\|_{L^q(\mathbb{R}^2)}\big\}
\end{aligned}
$$

for all $t > 0$. Summing up the estimates above, we conclude (3.4). □

Let $P_{\mathbb{R}^2} = (\delta_{jk} + R_jR_k)_{1\le j,k\le 2}$ be the Helmholtz projection in the whole plane, where R_j denotes the Riesz transform. Then the operator $P_{\mathbb{R}^2}$ is bounded on $L^r(\mathbb{R}^2)$ for every $r \in (1,\infty)$. Set $L^r_\sigma(\mathbb{R}^2) = P_{\mathbb{R}^2}L^r(\mathbb{R}^2)$ for $1 < r < \infty$, which consists of all $u \in L^r(\mathbb{R}^2)$ satisfying $\operatorname{div} u = 0$. We note the duality relation $L^{r/(r-1)}_\sigma(\mathbb{R}^2) = L^r_\sigma(\mathbb{R}^2)^*$. Although $P_{\mathbb{R}^2}$ is not bounded on $L^1(\mathbb{R}^2)$, we have the following lemma, which will be needed in the last section.

Lemma 3.2. *Let $r \in (1,\infty)$. Then, for each $t > 0$, the operator $U_\alpha(t)P_{\mathbb{R}^2}$ on $C^\infty_0(\mathbb{R}^2)$ extends uniquely to a bounded operator from $L^1(\mathbb{R}^2)$ to $L^r_\sigma(\mathbb{R}^2)$. Furthermore, for $r \in (1,\infty]$ there is a constant $C = C(r)$ independent of α such that*

$$
\|\nabla^j U_\alpha(t)P_{\mathbb{R}^2}f\|_{L^r(\mathbb{R}^2)} \le Ct^{-j/2-1+1/r}\|f\|_{L^1(\mathbb{R}^2)} \tag{3.5}
$$

for all $t > 0$, $f \in L^1(\mathbb{R}^2)$ and integer $j \ge 0$.

Proof. Let $r \in (1,\infty)$ and $f \in C^\infty_0(\mathbb{R}^2)$. Since $U_\alpha(t)^* = U_{-\alpha}(t)$ also satisfies (3.3), we have

$$
\begin{aligned}
|\langle U_\alpha(t)Pf, \varphi\rangle| = |\langle f, U_{-\alpha}(t)\varphi\rangle| &\le \|f\|_{L^1(\mathbb{R}^2)}\|U_{-\alpha}(t)\varphi\|_{L^\infty(\mathbb{R}^2)} \\
&\le Ct^{-1+1/r}\|f\|_{L^1(\mathbb{R}^2)}\|\varphi\|_{L^{r/(r-1)}(\mathbb{R}^2)}
\end{aligned}
$$

for all $\varphi \in L^{r/(r-1)}_\sigma(\mathbb{R}^2)$. By duality we find that $U_\alpha(t)Pf \in L^r_\sigma(\mathbb{R}^2)$ with

$$
\|U_\alpha(t)Pf\|_{L^r(\mathbb{R}^2)} \le Ct^{-1+1/r}\|f\|_{L^1(\mathbb{R}^2)}
$$

for some $C = C(r) > 0$ independent of t, α and f. By density $U_\alpha(t)P_{\mathbb{R}^2}$ extends uniquely to a bounded operator from $L^1(\mathbb{R}^2)$ to $L^r_\sigma(\mathbb{R}^2)$ together with the same estimate as above. We have also (3.5) even for $r = \infty$ and/or $j \ge 1$ on account of semigroup property and (3.3). This completes the proof. □

4. Fundamental solution of the Oseen resolvent.

In this section we consider the solution

$$
u = A^\alpha_\lambda f = E^\alpha_\lambda * f, \qquad p = \Pi f = \frac{x}{2\pi|x|^2} * f \tag{4.1}
$$

of the resolvent equation in the whole plane

$$\lambda u - \Delta u - 2\alpha\partial_1 u + \nabla p = f, \qquad \text{div } u = 0 \quad \text{in } \mathbb{R}^2 \tag{4.2}$$

for suitable external force f, where $E_\lambda^\alpha(x)$ is the fundamental solution of the Oseen resolvent. Indeed we can derive the standard estimate $|\lambda|\|u\|_{L^q(\mathbb{R}^2)} \le C\|f\|_{L^q(\mathbb{R}^2)}$ for large $|\lambda|$ with $|\arg\lambda| \le 3\pi/4$ (say) and all $f \in L^q(\mathbb{R}^2)$ $(1 < q < \infty)$, yielding the generation of analytic semigroup, but the point here is to analyze the resolvent for small (λ, α) under the assumption that the support of f is compact. This section is devoted to such analysis especially for $\lambda \in \overline{\mathbb{C}_+} = \{\lambda \in \mathbb{C}; \mathrm{Re}\,\lambda \ge 0\}$.

Everything relies upon the fundamental solution, which is given by

$$\begin{aligned} E_\lambda^\alpha(x) &= \mathcal{F}^{-1}\left[\frac{|\xi|^2\mathbb{I} - \xi\otimes\xi}{(\lambda + |\xi|^2 - 2\alpha i\xi_1)|\xi|^2}\right](x) \\ &= \int_0^\infty e^{-\lambda t}(G\,\mathbb{I} + H)(x + 2\alpha t e_1, t)\,dt \end{aligned} \tag{4.3}$$

where

$$\mathbb{I} = (\delta_{jk})_{1\le j,k\le 2}, \qquad \xi\otimes\xi = (\xi_j\xi_k)_{1\le j,k\le 2}$$

and $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. In $(4.3)_2$, $G(x,t)$ is the heat kernel given by (3.2) and

$$\begin{aligned} H(x,t) &= \int_t^\infty \nabla^2 G(x,s)\,ds = \int_t^\infty \frac{e^{-|x|^2/4s}}{4\pi s}\left(\frac{x\otimes x}{4s^2} - \frac{\mathbb{I}}{2s}\right)ds \\ &= \frac{-1}{2\pi|x|^2}\left(1 - e^{-|x|^2/4t}\right)\mathbb{I} + \left[\frac{1}{\pi|x|^2}\left(1 - e^{-|x|^2/4t}\right) - \frac{e^{-|x|^2/4t}}{4\pi t}\right]\frac{x\otimes x}{|x|^2}. \end{aligned} \tag{4.4}$$

Note that $(G\,\mathbb{I} + H)(x,t)$ and $(G\,\mathbb{I} + H)(x + 2\alpha t e_1, t)$ are, respectively, fundamental solutions of unsteady Stokes and Oseen systems. In $(4.3)_1$ we have

$$\lambda + |\xi|^2 - 2\alpha i\xi_1 \ne 0 \quad \text{for all } \xi \in \mathbb{R}^2 \text{ and } \lambda \in \mathbb{C}\setminus S_\alpha, \tag{4.5}$$

where S_α is given by (2.7). As shown in [**26**, Lemma 3.1] and [**10**, Lemma 3.1], the Fourier multiplier theorem leads us to the first assertion of the following lemma. The remaining assertions are easily implied by the Hardy-Littlewood-Sobolev inequality and embedding relations.

LEMMA 4.1. *Let $\alpha > 0$ and $1 < q < \infty$.*

1. *Let $\lambda \in \mathbb{C}\setminus S_\alpha$. Then A_λ^α is a bounded operator from $L^q(\mathbb{R}^2)$ to $W^{2,q}(\mathbb{R}^2)$. Furthermore, the function $\mathbb{C}\setminus S_\alpha \ni \lambda \mapsto A_\lambda^\alpha \in \mathcal{L}(L^q(\mathbb{R}^2); W^{2,q}(\mathbb{R}^2))$ is analytic.*
2. *The operator $\nabla\Pi$ is bounded on $L^q(\mathbb{R}^2)$. If $1 < q < 2$, then Π is bounded from $L^q(\mathbb{R}^2)$ to $L^{q_*}(\mathbb{R}^2)$, where $1/q_* = 1/q - 1/2$.*
3. *Let $\lambda \in \mathbb{C}\setminus S_\alpha$. If $f \in L^q(\mathbb{R}^2)$ satisfies $f(x) = 0$ a.e. $|x| \ge R$ for some $R > 0$, then*

$$\nabla^2 A_\lambda^\alpha f,\ \nabla \Pi f \in L^r(\mathbb{R}^2), \quad \forall r \in (1,q]; \qquad A_\lambda^\alpha f \in L^r(\mathbb{R}^2), \quad \forall r \in (1,\infty),$$

$$\nabla A_\lambda^\alpha f \in L^r(\mathbb{R}^2), \quad \begin{cases} \forall r \in (1,q_*] & \text{if } q \in (1,2), \\ \forall r \in (1,\infty) & \text{if } q \in [2,\infty), \end{cases}$$

$$\Pi f \in L^r(\mathbb{R}^2), \quad \begin{cases} \forall r \in (2,q_*] & \text{if } q \in (1,2), \\ \forall r \in (2,\infty) & \text{if } q \in [2,\infty), \end{cases}$$

where $1/q_* = 1/q - 1/2$ for $q \in (1,2)$. All of them in $L^r(\mathbb{R}^2)$ are estimated from above by $C_R\|f\|_{L^q(\mathbb{R}^2)}$.

To derive a representation of $E_\lambda^\alpha(x)$, following [**20**, Appendix] by Guenther and Thomann, we use the Laplace transform $(4.3)_2$ rather than $(4.3)_1$. Then, by a lengthy but elementary calculation, we obtain

$$\begin{aligned}
E_\lambda^\alpha(x) &= \sum_{j=1}^5 E_{j,\lambda}^\alpha(x) \\
&= \frac{\mathbb{I}}{2\pi} e^{-\alpha x_1}\, K_0\big(\sqrt{\lambda+\alpha^2}\,|x|\big) \\
&\quad - \frac{\mathbb{I}}{4\pi}\int_0^1 e^{-\alpha x_1 s}\, K_0\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big)\, ds \\
&\quad + \frac{x\otimes x}{4\pi|x|}\int_0^1 e^{-\alpha x_1 s}\sqrt{s(\lambda+\alpha^2 s)}\, K_1\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big)\, ds \\
&\quad + \frac{\alpha(x\otimes e_1 + e_1\otimes x)}{4\pi}\int_0^1 s e^{-\alpha x_1 s}\, K_0\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big)\, ds \\
&\quad + \frac{\alpha^2|x| e_1\otimes e_1}{4\pi}\int_0^1 \frac{s^2 e^{-\alpha x_1 s}}{\sqrt{s(\lambda+\alpha^2 s)}}\, K_1\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big)\, ds
\end{aligned} \tag{4.6}$$

for $\operatorname{Re}\lambda \geq 0$ and $\alpha > 0$, where the branch of $\sqrt{\cdot}$ is chosen so that $\operatorname{Re}\sqrt{\cdot} > 0$ (thus $\arg\sqrt{\lambda+\alpha^2} < \pi/4$ and so on) and

$$\begin{aligned}
K_0(z) &= \frac{1}{2}\int_0^\infty \exp\left[\frac{-z}{2}\left(t+\frac{1}{t}\right)\right]\frac{dt}{t}, \\
K_1(z) &= -K_0'(z) = \frac{1}{2}\int_0^\infty \exp\left[\frac{-z}{2}\left(t+\frac{1}{t}\right)\right]\frac{dt}{t^2}
\end{aligned} \tag{4.7}$$

are modified Bessel functions of the second kind (order 0/order 1, respectively); these integral representations are valid for $z \in \mathbb{C}_+$. For the representation (4.7), see for instance Watson [**36**, 6.22].

In fact, by using the simple relation $(1-e^{-\sigma})/\sigma = \int_0^1 e^{-\sigma s} ds$, the Laplace transform of the coefficient of $\mathbb{I}$ in $H(x + 2\alpha t e_1, t)$, see (4.4), is

$$
\begin{aligned}
&\frac{-1}{2\pi}\int_0^\infty \frac{e^{-\lambda t}}{|x+2\alpha te_1|^2}\big(1-e^{-|x+2\alpha te_1|^2/4t}\big)\,dt \\
&\quad= \frac{-1}{8\pi}\int_0^\infty e^{-\lambda t}\int_0^1 e^{-|x+2\alpha te_1|^2\,s/4t}\,ds\,\frac{dt}{t} \\
&\quad= \frac{-1}{8\pi}\int_0^1 e^{-\alpha x_1 s}\int_0^\infty e^{-s(\lambda+\alpha^2 s)|x|^2 t/4}\,e^{-1/t}\,\frac{dt}{t}\,ds
\end{aligned}
$$

which implies the second term $E^\alpha_{2,\lambda}(x)$ since $(1/2)\int_0^\infty \exp[-(z^2t/4+1/t)](dt/t)$ coincides with the RHS of $(4.7)_1$ for $z>0$ and, therefore, for $|\arg z|<\pi/4$ by unicity theorem for holomorphic functions. The derivation of the first term $E^\alpha_{1,\lambda}(x)$ is even easier and classical. Using the relation

$$
\frac{1}{\sigma}\left(\frac{1-e^{-\sigma}}{\sigma}-e^{-\sigma}\right)=\int_0^1 se^{-\sigma s}\,ds
$$

we can rewrite the Laplace transform of the remaining part of $H(x+2\alpha te_1,t)$ as

$$
\begin{aligned}
&\int_0^\infty e^{-\lambda t}\left[\frac{1-e^{-|x+2\alpha te_1|^2/4t}}{\pi|x+2\alpha te_1|^2}-\frac{e^{-|x+2\alpha te_1|^2/4t}}{4\pi t}\right]\frac{(x+2\alpha te_1)\otimes(x+2\alpha te_1)}{|x+2\alpha te_1|^2}\,dt \\
&= \int_0^\infty e^{-\lambda t}\int_0^1 se^{-|x+2\alpha te_1|^2\,s/4t}\,ds\,\frac{(x+2\alpha te_1)\otimes(x+2\alpha te_1)}{16\pi t^2}\,dt \\
&=: I_3+I_4+I_5
\end{aligned}
$$

with

$$
\begin{aligned}
I_3 &= \frac{x\otimes x}{16\pi}\int_0^\infty e^{-\lambda t}\int_0^1 se^{-|x+2\alpha te_1|^2\,s/4t}\,ds\,\frac{dt}{t^2} \\
&= \frac{x\otimes x}{4\pi|x|^2}\int_0^1 e^{-\alpha x_1 s}\int_0^\infty e^{-s(\lambda+\alpha^2 s)|x|^2 t/4}\,e^{-1/t}\,\frac{dt}{t^2}\,ds, \\
I_4 &= \frac{\alpha(x\otimes e_1+e_1\otimes x)}{8\pi}\int_0^\infty e^{-\lambda t}\int_0^1 se^{-|x+2\alpha te_1|^2\,s/4t}\,ds\,\frac{dt}{t} \\
&= \frac{\alpha(x\otimes e_1+e_1\otimes x)}{8\pi}\int_0^1 se^{-\alpha x_1 s}\int_0^\infty e^{-s(\lambda+\alpha^2 s)|x|^2 t/4}\,e^{-1/t}\,\frac{dt}{t}\,ds, \\
I_5 &= \frac{\alpha^2 e_1\otimes e_1}{4\pi}\int_0^\infty e^{-\lambda t}\int_0^1 se^{-|x+2\alpha te_1|^2\,s/4t}\,ds\,dt \\
&= \frac{\alpha^2|x|^2 e_1\otimes e_1}{16\pi}\int_0^1 s^2 e^{-\alpha x_1 s}\int_0^\infty e^{-s(\lambda+\alpha^2 s)|x|^2 t/4}\,e^{-1/t}\,dt\,ds.
\end{aligned}
$$

By the same reasoning as in the second term, we obtain $E^\alpha_{4,\lambda}(x)$ from I_4. Since we have

$$\frac{1}{z}\int_0^\infty \exp\left[-\left(\frac{z^2t}{4}+\frac{1}{t}\right)\right]\frac{dt}{t^2}=\frac{z}{4}\int_0^\infty \exp\left[-\left(\frac{z^2t}{4}+\frac{1}{t}\right)\right]dt=K_1(z)$$

for $z>0$, see $(4.7)_2$, it follows from unicity theorem for holomorphic functions that the same relation holds for $|\arg z|<\pi/4$, and thus I_3 and I_5 respectively yield $E^\alpha_{3,\lambda}(x)$ and $E^\alpha_{5,\lambda}(x)$.

The representation (4.6) also covers the case $\lambda=0$ (the Oseen fundamental solution), which was derived by Guenther and Thomann [**20**]; in this case, another representation of $E^\alpha_0(x)$ without s-integral is also available (even standard), see [**17**], [**27**] and [**30**]. Indeed (4.6) can be regarded as a generalization of the result in [**20**], but to the best of my knowledge it is not found in any other literature.

Remark 4.1. The representation (4.6) itself covers the case $\alpha=0$ (fundamental solution of the Stokes resolvent) as well. Actually, $E^0_\lambda(x)$ of our form coincides with the fundamental solution given by Borchers and Varnhorn [**4**]. This is verified by the relations

$$\int_0^1 K_0(\sqrt{s}\,z)\,ds=\frac{2}{z^2}-\frac{2}{z}\,K_1(z),\qquad \int_0^1 \sqrt{s}\,K_1(\sqrt{s}\,z)\,ds=\frac{4}{z^3}-\frac{4}{z^2}\,K_1(z)-\frac{2}{z}\,K_0(z).$$

But (4.6) never covers the case $(\lambda,\alpha)=(0,0)$, in which the Stokes fundamental solution is given by

$$E^0_0(x)=\frac{1}{4\pi}\left[\left(\log\frac{1}{|x|}\right)\mathbb{I}+\frac{x\otimes x}{|x|^2}\right]. \tag{4.8}$$

As clarified in [**20**], one needs the centering technique to recover $E^0_0(x)$ from the fundamental solution $(G\mathbb{I}+H)(x,t)$ of unsteady Stokes system.

Remark 4.2. If $\lambda\in\mathbb{C}\setminus S_\alpha$, see (2.7), that is equivalent to $\mathrm{Re}\,\sqrt{\lambda+\alpha^2}>\alpha$, then the fundamental solution $E^\alpha_\lambda(x)$ decays exponentially as $|x|\to\infty$ by the asymptotic behavior of the modified Bessel functions (4.7) for $|z|\to\infty$, see [**36**, 7.23]. This gives another interpretation of boundedness of the operator A^α_λ on $L^q(\mathbb{R}^2)$, $1<q<\infty$, for such λ, see Lemma 4.1.

We also provide the representation of $\partial^k_\lambda E^\alpha_\lambda(x)$ $(k=1,2)$ for later use:

$$\begin{aligned}\partial_\lambda E^\alpha_\lambda(x)&=\sum_{j=1}^5\partial_\lambda E^\alpha_{j,\lambda}(x)\\ &=\frac{-|x|\,\mathbb{I}}{4\pi}\,e^{-\alpha x_1}\,\frac{K_1\big(\sqrt{\lambda+\alpha^2}\,|x|\big)}{\sqrt{\lambda+\alpha^2}}\\ &\quad+\frac{|x|\,\mathbb{I}}{8\pi}\int_0^1 e^{-\alpha x_1 s}\sqrt{\frac{s}{\lambda+\alpha^2 s}}\,K_1\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big)\,ds\end{aligned}$$

$$
\begin{aligned}
&+ \frac{x \otimes x}{8\pi|x|} \int_0^1 e^{-\alpha x_1 s} \Big\{ \sqrt{\frac{s}{\lambda+\alpha^2 s}}\, K_1\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big) \\
&\qquad\qquad + |x| s\, K_1'\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big) \Big\} ds \\
&- \frac{\alpha|x|(x\otimes e_1 + e_1 \otimes x)}{8\pi} \int_0^1 \frac{s^{3/2} e^{-\alpha x_1 s}}{\sqrt{\lambda+\alpha^2 s}}\, K_1\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big)\, ds \\
&+ \frac{\alpha^2|x| e_1\otimes e_1}{8\pi} \int_0^1 e^{-\alpha x_1 s} \Big\{ -\Big(\frac{s}{\lambda+\alpha^2 s}\Big)^{3/2} K_1\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big) \\
&\qquad\qquad + \frac{|x| s^2}{\lambda+\alpha^2 s}\, K_1'\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big) \Big\} ds, \qquad (4.9)
\end{aligned}
$$

$$
\begin{aligned}
\partial_\lambda^2 E_\lambda^\alpha(x) &= \sum_{j=1}^5 \partial_\lambda^2 E_{j,\lambda}^\alpha(x) \\
&= \frac{|x|\,\mathbb{I}}{8\pi}\, e^{-\alpha x_1} \Big\{ \frac{K_1\big(\sqrt{\lambda+\alpha^2}\,|x|\big)}{(\lambda+\alpha^2)^{3/2}} - \frac{|x|\, K_1'\big(\sqrt{\lambda+\alpha^2}\,|x|\big)}{\lambda+\alpha^2} \Big\} \\
&+ \frac{|x|\,\mathbb{I}}{16\pi} \int_0^1 e^{-\alpha x_1 s} \Big\{ \frac{-\sqrt{s}}{(\lambda+\alpha^2 s)^{3/2}} K_1\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big) \\
&\qquad\qquad + \frac{|x| s}{\lambda+\alpha^2 s}\, K_1'\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big) \Big\} ds \\
&+ \frac{x\otimes x}{16\pi|x|} \int_0^1 e^{-\alpha x_1 s} \Big\{ \frac{-\sqrt{s}}{(\lambda+\alpha^2 s)^{3/2}}\, K_1\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big) \\
&\qquad\qquad + \frac{|x| s}{\lambda+\alpha^2 s}\, K_1'\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big) \\
&\qquad\qquad + \frac{|x|^2 s^{3/2}}{\sqrt{\lambda+\alpha^2 s}}\, K_1''\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big) \Big\} ds \\
&- \frac{\alpha|x|(x\otimes e_1 + e_1\otimes x)}{16\pi} \int_0^1 e^{-\alpha x_1 s} \Big\{ -\Big(\frac{s}{\lambda+\alpha^2 s}\Big)^{3/2} K_1\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big) \\
&\qquad\qquad + \frac{|x| s^2}{\lambda+\alpha^2 s}\, K_1'\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big) \Big\} ds \\
&+ \frac{\alpha^2|x| e_1\otimes e_1}{16\pi} \int_0^1 e^{-\alpha x_1 s} \Big\{ \frac{3 s^{3/2}}{(\lambda+\alpha^2 s)^{5/2}}\, K_1\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big) \\
&\qquad\qquad - \frac{3|x| s^2}{(\lambda+\alpha^2 s)^2}\, K_1'\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big) \\
&\qquad\qquad + \frac{|x|^2 s^{5/2}}{(\lambda+\alpha^2 s)^{3/2}}\, K_1''\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big) \Big\} ds. \qquad (4.10)
\end{aligned}
$$

We recall the asymptotic expansion of the modified Bessel functions ([**30**, Lemma 2.7])

$$\begin{aligned} K_0(z) &= -\log z + \log 2 - \gamma - \frac{z^2}{4}(\log z - \log 2 + \gamma - 1) + (\log z)O(z^4), \\ K_1(z) &= \frac{1}{z} + \frac{z}{2}\left(\log z - \log 2 + \gamma - \frac{1}{2}\right) + (\log z)O(z^3) \end{aligned} \tag{4.11}$$

as $\mathbb{C}_+ \ni z \to 0$, where $\gamma = \lim_{m\to\infty}\left(\sum_{k=1}^m (1/k) - \log m\right)$ is the Euler constant. Throughout this paper, $\log z$ is understood as the principal branch so that $\mathrm{Im}\,\log z \in (-\pi/2, \pi/2)$ for $z \in \mathbb{C}_+$.

By making full use of (4.11), one can find the following pointwise estimates of the fundamental solution. We observe that the local energy decay like t^{-2} is hopeless in Theorem 2.1 because of the behavior $|\lambda|^{-1}$ near $\lambda = 0$ in (4.14) below. This is not the case for the model operator $\Delta + 2\alpha\partial_1$ without pressure, for which faster local energy decay of the associated semigroup is obtained. In fact, the worst term comes from $E^\alpha_{2,\lambda}(x)$ and $E^\alpha_{5,\lambda}(x)$ in (4.6) (see also Remark 4.3 below).

Lemma 4.2. *Let $R > 0$, $\Lambda > 0$ and $M > 0$. Then there is a constant $C = C(R,\Lambda,M) > 0$ such that*

$$|E^\alpha_\lambda(x)| \le C\left(\left|\log\frac{1}{|\lambda| + \alpha^2}\right| + \left|\log\frac{1}{|x|}\right| + 1\right), \qquad x \in B_R \setminus \{0\}, \tag{4.12}$$

$$\sup_{|x|\le R} |\partial_\lambda E^\alpha_\lambda(x)| \le \frac{C}{\alpha^2}\log\left(\frac{\alpha^2}{|\lambda|} + \sqrt{1 + \frac{\alpha^4}{|\lambda|^2}}\right), \tag{4.13}$$

$$\sup_{|x|\le R} |\partial^2_\lambda E^\alpha_\lambda(x)| \le \frac{C}{|\lambda|(|\lambda| + \alpha^2)} \tag{4.14}$$

for $\lambda \in \Sigma_\Lambda$ and $\alpha \in (0, M]$, where Σ_Λ is given by (2.1)*. The case $\lambda = 0$ is also covered only for* (4.12)*.*

Proof. We first look at (4.9) to show (4.13). Set

$$L^\alpha_{1,\lambda}(x) = \frac{-e^{-\alpha x_1}}{4\pi(\lambda + \alpha^2)}\,\mathbb{I},$$

$$L^\alpha_{2,\lambda}(x) = \frac{\mathbb{I}}{8\pi}\int_0^1 \frac{e^{-\alpha x_1 s}}{\lambda + \alpha^2 s}\,ds,$$

$$L^\alpha_{3,\lambda}(x) = \frac{x \otimes x}{8\pi}\int_0^1 s e^{-\alpha x_1 s}\log\sqrt{\lambda + \alpha^2 s}\,ds,$$

$$L^\alpha_{4,\lambda}(x) = \frac{-\alpha(x \otimes e_1 + e_1 \otimes x)}{8\pi}\int_0^1 \frac{s e^{-\alpha x_1 s}}{\lambda + \alpha^2 s}\,ds,$$

$$L^\alpha_{5,\lambda}(x) = \frac{-\alpha^2 e_1 \otimes e_1}{4\pi}\int_0^1 \frac{s e^{-\alpha x_1 s}}{(\lambda + \alpha^2 s)^2}\,ds,$$

which are respectively dominant parts (that is, the most singular parts) of $\partial_\lambda E^\alpha_{j,\lambda}(x)$ $(j=1,\dots,5)$ for small (λ,α) on account of (4.11). Concerning $\partial_\lambda E^\alpha_{3,\lambda}(x)$, the leading term of modified Bessel function does not contribute to $L^\alpha_{3,\lambda}(x)$ by some cancellation and thus it is less singular for small (λ,α). Since

$$|\lambda+\alpha^2 s|^2=|\lambda|^2+2\alpha^2 s\,\mathrm{Re}\,\lambda+\alpha^4 s^2\geq|\lambda|^2+\alpha^4 s^2,$$

we get

$$\int_0^1\frac{ds}{|\lambda+\alpha^2 s|}\leq\int_0^1\frac{ds}{\sqrt{|\lambda|^2+\alpha^4 s^2}}=\frac{1}{\alpha^2}\log\left(\frac{\alpha^2}{|\lambda|}+\sqrt{1+\frac{\alpha^4}{|\lambda|^2}}\right),$$

$$\int_0^1\frac{s}{|\lambda+\alpha^2 s|}\,ds\leq\int_0^1\frac{s}{\sqrt{|\lambda|^2+\alpha^4 s^2}}\,ds=\frac{1}{\sqrt{|\lambda|^2+\alpha^4}+|\lambda|}\leq\frac{1}{\sqrt{|\lambda|^2+\alpha^4}},$$

$$\int_0^1\frac{s}{|\lambda+\alpha^2 s|^2}\,ds\leq\int_0^1\frac{s}{|\lambda|^2+\alpha^4 s^2}\,ds=\frac{1}{2\alpha^4}\log\left(1+\frac{\alpha^4}{|\lambda|^2}\right).$$

When $|\lambda|$ and α are so small that $|\lambda|+\alpha^2\leq e^{-\pi/4}$, we have

$$\big|\log\sqrt{\lambda+\alpha^2 s}\big|^2=\big(\log|\sqrt{\lambda+\alpha^2 s}|\big)^2+\big(\arg\sqrt{\lambda+\alpha^2 s}\big)^2\leq 2\big\{\log(|\lambda|^2+\alpha^4 s^2)^{1/4}\big\}^2$$

where $\arg(\cdot)$ is understood as the principal branch so that $|\arg\sqrt{\lambda+\alpha^2 s}|\leq\pi/4$. Hence

$$\begin{aligned}\int_0^1 s\big|\log\sqrt{\lambda+\alpha^2 s}\big|\,ds&\leq\frac{-1}{2\sqrt{2}}\int_0^1 s\,\log\,(|\lambda|^2+\alpha^4 s^2)\,ds\\&=\frac{1}{4\sqrt{2}}\left(1+\log\,\frac{1}{|\lambda|^2+\alpha^4}\right)-\frac{|\lambda|^2}{4\sqrt{2}\,\alpha^4}\log\left(1+\frac{\alpha^4}{|\lambda|^2}\right)\\&\leq\frac{1}{4\sqrt{2}}\left(1+\log\,\frac{1}{|\lambda|^2+\alpha^4}\right)\end{aligned}$$

for small (λ,α) as mentioned above. Let $\lambda\in\Sigma_\Lambda,\alpha\in(0,M]$ and $|x|\leq R$, and the constants C below depend on Λ, M and R. We then obtain

$$|L^\alpha_{1,\lambda}(x)|\leq\frac{C}{\sqrt{|\lambda|^2+\alpha^4}},\qquad |L^\alpha_{2,\lambda}(x)|\leq\frac{C}{\alpha^2}\log\left(\frac{\alpha^2}{|\lambda|}+\sqrt{1+\frac{\alpha^4}{|\lambda|^2}}\right),$$

$$|L^\alpha_{3,\lambda}(x)|\leq C\left(1+\left|\log\,\frac{1}{|\lambda|^2+\alpha^4}\right|\right),\qquad |L^\alpha_{4,\lambda}(x)|\leq\frac{C\alpha}{\sqrt{|\lambda|^2+\alpha^4}},$$

$$|L^\alpha_{5,\lambda}(x)|\leq\frac{C}{\alpha^2}\log\left(1+\frac{\alpha^4}{|\lambda|^2}\right)\leq\frac{2C}{\alpha^2}\log\left(\frac{\alpha^2}{|\lambda|}+\sqrt{1+\frac{\alpha^4}{|\lambda|^2}}\right).$$

Since $1/\sqrt{t^2+1}\leq c_0\,\log\,(1/t+\sqrt{1+(1/t^2)})$ for all $t>0$, we put $t=|\lambda|/\alpha^2$ to find

$$\frac{1}{\sqrt{|\lambda|^2+\alpha^4}} \le \frac{c_0}{\alpha^2} \log\left(\frac{\alpha^2}{|\lambda|}+\sqrt{1+\frac{\alpha^4}{|\lambda|^2}}\right). \tag{4.15}$$

Thus, collecting the estimates above yields (4.13). Similarly (4.12) is verified by using (4.11) and so omitted. Finally, the worst term of $\partial_\lambda^2 E_\lambda^\alpha(x)$ given by (4.10) comes from $\partial_\lambda L_{2,\lambda}^\alpha(x)$ and $\partial_\lambda L_{5,\lambda}^\alpha(x)$ (no other parts cause the behavior $|\lambda|^{-1}$ below near $\lambda=0$). The essential parts are respectively given by

$$\int_0^1 \frac{ds}{|\lambda+\alpha^2 s|^2} \le \int_0^1 \frac{ds}{|\lambda|^2+\alpha^4 s^2} = \frac{1}{\alpha^2|\lambda|}\tan^{-1}\frac{\alpha^2}{|\lambda|} \le \frac{\pi}{2|\lambda|\sqrt{|\lambda|^2+\alpha^4}},$$

$$\begin{aligned}\alpha^2\int_0^1 \frac{s}{|\lambda+\alpha^2 s|^3}\,ds &\le \alpha^2\int_0^1 \frac{s}{(|\lambda|^2+\alpha^4 s^2)^{3/2}}\,ds = \frac{\alpha^2}{|\lambda|\sqrt{|\lambda|^2+\alpha^4}\left(|\lambda|+\sqrt{|\lambda|^2+\alpha^4}\right)} \\ &\le \frac{1}{|\lambda|\sqrt{|\lambda|^2+\alpha^4}}.\end{aligned}$$

We thus conclude (4.14). □

Remark 4.3. By (4.15) the RHS of (4.13) is bounded from below (away from zero) like

$$\frac{1}{c_0\sqrt{\Lambda^2+M^4}} \le \frac{1}{\alpha^2}\log\left(\frac{\alpha^2}{|\lambda|}+\sqrt{1+\frac{\alpha^4}{|\lambda|^2}}\right)$$

as long as $\alpha \in (0,M]$ and $|\lambda| \le \Lambda$. Note also that the RHS of (4.13) goes to $C/|\lambda|$ as $\alpha \to 0$. The reason why the RHS of (4.13) is not related to that of (4.12) can be interpreted as follows. The leading term of $E_{2,\lambda}^\alpha(x)$ for small (λ,α) is essentially given by

$$E_{2,\lambda}^\alpha(x) \sim \int_0^1 \log(\lambda+\alpha^2 s)\,ds = \log(\lambda+\alpha^2) - \int_0^1 \frac{\alpha^2 s}{\lambda+\alpha^2 s}\,ds \sim E_{1,\lambda}^\alpha(x)+E_{5,\lambda}^\alpha(x).$$

This behaves like $E_{1,\lambda}^\alpha(x)$ because the leading term of $E_{5,\lambda}^\alpha(x)$ is bounded for small (λ,α). However, look at the leading term of $\partial_\lambda E_{2,\lambda}^\alpha(x)$:

$$\partial_\lambda E_{2,\lambda}^\alpha(x) \sim \int_0^1 \frac{1}{\lambda+\alpha^2 s}\,ds = \frac{1}{\lambda+\alpha^2} + \int_0^1 \frac{\alpha^2 s}{(\lambda+\alpha^2 s)^2}\,ds \sim \partial_\lambda E_{1,\lambda}^\alpha(x)+\partial_\lambda E_{5,\lambda}^\alpha(x)$$

which behaves like $\partial_\lambda E_{5,\lambda}^\alpha(x)$; actually, it is singular for $\lambda \to 0$ even for fixed $\alpha>0$, unlike $\partial_\lambda E_{1,\lambda}^\alpha(x)$.

As a consequence of Lemma 4.2, we have the estimate of the solution operator (4.1).

Proposition 4.1. *Let $1<q<\infty$, $R>0$, $\Lambda>0$ and $M>0$. Then there is a constant $C=C(q,R,\Lambda,M)>0$ such that*

$$\|A_\lambda^\alpha f\|_{W^{1,q}(B_R)} \le C\left(\left|\log\frac{1}{|\lambda|+\alpha^2}\right|+1\right)\|f\|_{L^q(\mathbb{R}^2)}, \tag{4.16}$$

$$\|\partial_\lambda A_\lambda^\alpha f\|_{W^{1,q}(B_R)} \le \frac{C}{\alpha^2}\log\left(\frac{\alpha^2}{|\lambda|}+\sqrt{1+\frac{\alpha^4}{|\lambda|^2}}\right)\|f\|_{L^q(\mathbb{R}^2)}, \tag{4.17}$$

$$\|\partial_\lambda^2 A_\lambda^\alpha f\|_{W^{1,q}(B_R)} \le \frac{C}{|\lambda|(|\lambda|+\alpha^2)}\|f\|_{L^q(\mathbb{R}^2)} \tag{4.18}$$

for $\lambda\in\Sigma_\Lambda$, $\alpha\in(0,M]$ *and* $f\in L^q(\mathbb{R}^2)$ *with* $f(x)=0$ *a.e.* $|x|\ge R$, *where* Σ_Λ *is given by* (2.1). *The case* $\lambda=0$ *is also covered only for* (4.16).

Proof. By the assumption on f and (4.12), we have

$$\begin{aligned}|A_\lambda^\alpha f(x)| &\le \int_{|y|<R}|E_\lambda^\alpha(x-y)||f(y)|\,dy\\ &\le C\left(\left|\log\frac{1}{|\lambda|+\alpha^2}\right|+1\right)\|f\|_{L^1(B_R)}+C\left\|\log\frac{1}{|x-\cdot|}\right\|_{L^{q/(q-1)}(B_{2R}(x))}\|f\|_{L^q(B_R)}\end{aligned}$$

for $x\in B_R$ with $C=C(2R,\Lambda,M)$, where $B_{2R}(x)=\{y\in\mathbb{R}^2; |y-x|<2R\}$. We thus find

$$\|A_\lambda^\alpha f\|_{L^q(B_R)}\le C\|A_\lambda^\alpha f\|_{L^\infty(B_R)}\le C\left(\left|\log\frac{1}{|\lambda|+\alpha^2}\right|+1\right)\|f\|_{L^q(\mathbb{R}^2)}.$$

To consider $\nabla A_\lambda^\alpha f$, we give the representation of $\partial_m E_\lambda^\alpha(x)$ $(m=1,2)$, where $\partial_m=\partial_{x_m}$:

$$\partial_m E_{1,\lambda}^\alpha(x) = -\frac{\mathbb{I}}{2\pi}e^{-\alpha x_1}\left\{\frac{x_m}{|x|}\sqrt{\lambda+\alpha^2}\,K_1\big(\sqrt{\lambda+\alpha^2}\,|x|\big)-\alpha\delta_{1m}\,K_0\big(\sqrt{\lambda+\alpha^2}\,|x|\big)\right\},$$

$$\begin{aligned}\partial_m E_{2,\lambda}^\alpha(x) = \frac{\mathbb{I}}{4\pi}\int_0^1 e^{-\alpha x_1 s}&\left\{\frac{x_m}{|x|}\sqrt{s(\lambda+\alpha^2 s)}\,K_1\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big)\right.\\ &\left.+\alpha\delta_{1m}\,s\,K_0\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big)\right\}ds,\end{aligned}$$

$$\begin{aligned}\partial_m E_{3,\lambda}^\alpha(x) &= \left(\frac{e_m\otimes x+x\otimes e_m}{4\pi|x|}-\frac{x_m(x\otimes x)}{4\pi|x|^3}\right)\\ &\quad\times\int_0^1 e^{-\alpha x_1 s}\sqrt{s(\lambda+\alpha^2 s)}\,K_1\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big)\,ds\\ &\quad+\frac{x\otimes x}{4\pi|x|}\int_0^1 e^{-\alpha x_1 s}\left\{\frac{x_m}{|x|}s(\lambda+\alpha^2 s)\,K_1'\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big)\right.\\ &\qquad\left.-\alpha\delta_{1m}\,s\sqrt{s(\lambda+\alpha^2 s)}\,K_1\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big)\right\}ds,\end{aligned}$$

$$\partial_m E^\alpha_{4,\lambda}(x) = \frac{\alpha(e_m\otimes e_1+e_1\otimes e_m)}{4\pi}\int_0^1 se^{-\alpha x_1 s}\, K_0\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big)\,ds$$
$$+\frac{\alpha(x\otimes e_1+e_1\otimes x)}{4\pi}\int_0^1 e^{-\alpha x_1 s}\bigg\{\frac{-x_m}{|x|}s\sqrt{s(\lambda+\alpha^2 s)}\,K_1\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big)$$
$$-\alpha\delta_{1m}\,s^2\,K_0\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big)\bigg\}\,ds,$$

$$\partial_m E^\alpha_{5,\lambda}(x) = \frac{\alpha^2 x_m\, e_1\otimes e_1}{4\pi|x|}\int_0^1 \frac{s^2 e^{-\alpha x_1 s}}{\sqrt{s(\lambda+\alpha^2 s)}}\,K_1\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big)\,ds$$
$$+\frac{\alpha^2|x|e_1\otimes e_1}{4\pi}\int_0^1 e^{-\alpha x_1 s}\bigg\{\frac{x_m}{|x|}s^2\,K_1'\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big)$$
$$-\frac{\alpha\delta_{1m}\,s^3}{\sqrt{s(\lambda+\alpha^2 s)}}\,K_1\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big)\bigg\}\,ds.$$

Note that $\nabla A^\alpha_\lambda f$ involves the term

$$(Jf)(x)=\int_{\mathbb{R}^2}\frac{(x-y)\otimes f(y)}{|x-y|^2}\,dy,$$

however, it is harmless by the Hardy-Littlewood-Sobolev inequality. In fact, when $q\in(2,\infty)$, we take $r\in(1,2)$ such that $1/r=1/q+1/2$ to obtain $\|Jf\|_{L^q(\mathbb{R}^2)}\le C\|f\|_{L^r(B_R)}\le C\|f\|_{L^q(B_R)}$. When $q\in(1,2)$, we take $r\in(2,\infty)$ such that $1/r=1/q-1/2$ to get $\|Jf\|_{L^q(B_R)}\le C\|Jf\|_{L^r(B_R)}\le C\|f\|_{L^q(\mathbb{R}^2)}$. The case $q=2$ is treated similarly. Since the asymptotic behavior of $\nabla\partial_\lambda^k E^\alpha_\lambda(x)$ $(k=0,1,2)$ for small (λ,α) is respectively better than (4.12)–(4.14), we obtain (4.16)–(4.18). □

The following structure of the fundamental solution (4.6) plays an important role to construct a parametrix of the resolvent in exterior domains, see section 6.

Lemma 4.3. *We have the decomposition*

$$E^\alpha_\lambda(x)=\underbrace{E^0_0(x)+\frac{1}{4\pi}\bigg[\bigg(\log\frac{1}{\alpha}\bigg)\,\mathbb{I}+\mathbb{J}\bigg]+F^\alpha(x)}_{=E^\alpha_0(x)}+S^\alpha_\lambda(x), \tag{4.19}$$

where $E^0_0(x)$ is the Stokes fundamental solution (4.8), *$\mathbb{J}$ is the constant matrix given by*

$$\mathbb{J}=(\log 2-\gamma-1)\,\mathbb{I}+(e_1\otimes e_1),$$

and

$$\sup_{|x|\le R}|F^\alpha(x)|+\|\nabla F^\alpha\|_{L^1(B_R)}=O\bigg(\alpha\log\frac{1}{\alpha}\bigg)\qquad \text{as } \alpha\to 0, \tag{4.20}$$

$$\sup_{|x|\le R} |\nabla^k S_\lambda^\alpha(x)| \le \rho\left(\frac{|\lambda|}{\alpha^2}\right), \qquad \alpha\in(0,M],\ \lambda\in\Sigma_\Lambda,\ k=0,1, \tag{4.21}$$

with a function $\rho=\rho_{R,\Lambda,M}$ *satisfying* $\rho(\varepsilon)=O(\varepsilon\log\frac{1}{\varepsilon})$ *as* $\varepsilon\to+0$ *for given* R, Λ, $M>0$ *arbitralily, where* Σ_Λ *is given by* (2.1).

PROOF. We first derive the structure of the Oseen fundamental solution $E_0^\alpha(x)$ given by (4.6) with $\lambda=0$, which was already shown by [**30**, (2.19)–(2.21)]. By $D_{j,0}^\alpha(x)$ we denote the part of $E_{j,0}^\alpha(x)$ ($j=1,\dots,5$) which is never small even if α is small, while the remaining term (together with its spatial gradient) goes to zero as $\alpha\to 0$ as in (4.20) (note that $\nabla F^\alpha(x)$ possesses the logarithmic singularity at $x=0$). By (4.11) we find

$$D_{1,0}^\alpha(x)=\frac{1}{2\pi}\left(\log\frac{1}{\alpha|x|}\right)\mathbb{I}+\frac{\log 2-\gamma}{2\pi}\,\mathbb{I},$$

$$D_{2,0}^\alpha(x)=\frac{-1}{4\pi}\left(\log\frac{1}{\alpha|x|}\right)\mathbb{I}+\frac{-(\log 2-\gamma+1)}{4\pi}\,\mathbb{I},$$

$$D_{3,0}^\alpha(x)=\frac{x\otimes x}{4\pi|x|^2}, \qquad D_{4,0}^\alpha(x)=0, \qquad D_{5,0}^\alpha(x)=\frac{e_1\otimes e_1}{4\pi}.$$

We collect these terms to obtain the decomposition of $E_0^\alpha(x)$ in (4.19).

We next estimate the difference

$$S_{j,\lambda}^\alpha(x):=E_{j,\lambda}^\alpha(x)-E_{j,0}^\alpha(x) \qquad (j=1,\dots,5).$$

Given R, Λ, $M>0$, let $|x|\le R$, $\lambda\in\Sigma_\Lambda$ and $\alpha\in(0,M]$. Since

$$K_0\big(\sqrt{\lambda+\alpha^2}\,|x|\big)-K_0\big(\alpha\,|x|\big)=\int_0^1\partial_t K_0\big(\sqrt{\lambda t+\alpha^2}\,|x|\big)\,dt$$

it follows from $(4.11)_2$ that

$$\big|K_0\big(\sqrt{\lambda+\alpha^2}\,|x|\big)-K_0\big(\alpha\,|x|\big)\big|\le\frac{|\lambda||x|}{2}\int_0^1\frac{\big|K_1\big(\sqrt{\lambda t+\alpha^2}\,|x|\big)\big|}{|\lambda t+\alpha^2|^{1/2}}\,dt\le C|\lambda|\int_0^1\frac{dt}{|\lambda t+\alpha^2|}$$

which yields

$$|S_{1,\lambda}^\alpha(x)|\le\frac{C|\lambda|}{\alpha^2}. \tag{4.22}$$

Similarly, we have

$$\big|K_0\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big)-K_0\big(\alpha s\,|x|\big)\big|\le C|\lambda|\int_0^1\frac{dt}{|\lambda t+\alpha^2 s|}$$

and

$$\left|\sqrt{s(\lambda+\alpha^2 s)}\,K_1\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big)-\alpha s\,K_1\big(\alpha s\,|x|\big)\right|\le \frac{C|\lambda|}{|x|}\int_0^1\frac{dt}{|\lambda t+\alpha^2 s|}$$

which together with $|\lambda t+\alpha^2 s|^2\ge|\lambda|^2t^2+\alpha^4 s^2$ imply

$$\begin{aligned}
&|S^\alpha_{2,\lambda}(x)|+|S^\alpha_{3,\lambda}(x)|\\
&\quad\le C|\lambda|\int_0^1\int_0^1\frac{dt}{\sqrt{|\lambda|^2t^2+\alpha^4s^2}}\,ds\\
&\quad=\frac{C|\lambda|}{\alpha^2}\int_0^{\alpha^2/|\lambda|}\log\left(\frac{1}{s}+\sqrt{1+\frac{1}{s^2}}\right)ds\\
&\quad=C\log\left(\frac{|\lambda|}{\alpha^2}+\sqrt{1+\frac{|\lambda|^2}{\alpha^4}}\right)+\frac{C|\lambda|}{\alpha^2}\log\left(\frac{|\lambda|/\alpha^2+\sqrt{1+|\lambda|^2/\alpha^4}+1}{|\lambda|/\alpha^2+\sqrt{1+|\lambda|^2/\alpha^4}-1}\right).
\end{aligned}\tag{4.23}$$

We also find

$$|S^\alpha_{4,\lambda}(x)|\le C\alpha|\lambda|\int_0^1 s\int_0^1\frac{dt}{|\lambda t+\alpha^2 s|}\,ds\le\frac{C|\lambda|}{\alpha}.\tag{4.24}$$

Finally, using

$$\left|\frac{K_1\big(\sqrt{s(\lambda+\alpha^2 s)}\,|x|\big)}{\sqrt{s(\lambda+\alpha^2 s)}}-\frac{K_1\big(\alpha s\,|x|\big)}{\alpha s}\right|\le\frac{C|\lambda|}{s|x|}\int_0^1\frac{dt}{|\lambda t+\alpha^2 s|^2}$$

we obtain

$$\begin{aligned}
|S^\alpha_{5,\lambda}(x)|&\le C\alpha^2|\lambda|\int_0^1 s\int_0^1\frac{dt}{|\lambda|^2t^2+\alpha^4s^2}\,ds=\frac{C|\lambda|}{\alpha^2}\int_0^{\alpha^2/|\lambda|}\tan^{-1}\frac{1}{s}\,ds\\
&=C\tan^{-1}\frac{|\lambda|}{\alpha^2}+\frac{C|\lambda|}{\alpha^2}\log\left(1+\frac{\alpha^4}{|\lambda|^2}\right).
\end{aligned}\tag{4.25}$$

Collecting (4.22), (4.23), (4.24) and (4.25), we are led to (4.21) for $k=0$.

Concerning $\nabla S^\alpha_{j,\lambda}(x)$, we should use some cancellation (with respect to x) in the leading term. For instance, for $j=5$ (a delicate case), it follows from (4.11) that

$$\begin{aligned}
S^\alpha_{5,\lambda}(x)&=\frac{\alpha^2|x|e_1\otimes e_1}{4\pi}\int_0^1 s^2e^{-\alpha x_1 s}\\
&\qquad\times\left\{\frac{1}{\sqrt{s(\lambda+\alpha^2 s)}}\left(\frac{1}{\sqrt{s(\lambda+\alpha^2 s)}|x|}+\cdots\right)-\frac{1}{\alpha s}\left(\frac{1}{\alpha s|x|}+\cdots\right)\right\}ds\\
&\quad=:I^\alpha_\lambda(x)+(\text{remainder})
\end{aligned}$$

with

$$I_\lambda^\alpha(x) = \frac{-\lambda e_1 \otimes e_1}{4\pi} \int_0^1 \frac{e^{-\alpha x_1 s}}{\lambda + \alpha^2 s}\, ds.$$

Then we have

$$|\partial_m I_\lambda^\alpha(x)| \le C\alpha|\lambda|\delta_{1m} \int_0^1 \frac{s}{|\lambda + \alpha^2 s|}\, ds \le \frac{C|\lambda|\delta_{1m}}{\alpha}$$

for $m = 1, 2$, where $\partial_m = \partial_{x_m}$. Since ∇(remainder) can be easily treated (as in the argument above for $S_\lambda^\alpha(x)$) to obtain better estimate $|\nabla(\text{remainder})| \le C|\lambda|$, we find

$$|\nabla S_{5,\lambda}^\alpha(x)| \le \frac{C|\lambda|}{\alpha}.$$

The other terms $\nabla S_{j,\lambda}^\alpha(x)$ for $1 \le j \le 4$ are also discussed in the similar way. This completes the proof. $\square$

REMARK 4.4. We note that the structure of $E_0^\alpha(x)$ with respect to α in the first three terms of (4.19) is essentially the same as that of $E_\lambda^0(x)$ (fundamental solution of the Stokes resolvent) with respect to λ, see [**8**, (2.7)].

By (4.19) one can write $A_\lambda^\alpha f = A_0^\alpha f + S_\lambda^\alpha * f$ with

$$A_0^\alpha f = A_0^0 f + \frac{1}{4\pi}\left[\left(\log\frac{1}{\alpha}\right)\mathbb{I} + \mathbb{J}\right]\Gamma f + F^\alpha * f \tag{4.26}$$

where

$$A_0^0 f = E_0^0 * f, \qquad \Gamma f = \int_{\mathbb{R}^2} f(y)\, dy.$$

Let $f \in L^q(\mathbb{R}^2)$ fulfill $f(x) = 0$ a.e. $|x| \ge R$. Then (4.20) together with

$$\begin{aligned} \|F^\alpha * f\|_{L^\infty(B_R)} &\le \|F^\alpha\|_{L^\infty(B_{2R})}\|f\|_{L^1(B_R)}, \\ \|(\nabla F^\alpha) * f\|_{L^q(B_R)} &\le \|\nabla F^\alpha\|_{L^1(B_{2R})}\|f\|_{L^q(B_R)} \end{aligned}$$

implies (4.27) in the following proposition. Similarly, (4.28) follows from (4.21).

PROPOSITION 4.2. *Let $1 < q < \infty$, $R > 0$, $\Lambda > 0$ and $M > 0$. Then*

$$\|F^\alpha * f\|_{W^{1,q}(B_R)} = O\left(\alpha \log\frac{1}{\alpha}\right) \qquad \text{as } \alpha \to 0, \tag{4.27}$$

$$\|A_\lambda^\alpha f - A_0^\alpha f\|_{W^{1,q}(B_R)} \le \pi R^2 \rho\left(\frac{|\lambda|}{\alpha^2}\right)\|f\|_{L^q(\mathbb{R}^2)}, \qquad \alpha \in (0, M],\ \lambda \in \Sigma_\Lambda, \tag{4.28}$$

for $f \in L^q(\mathbb{R}^2)$ with $f(x) = 0$ a.e. $|x| \ge R$, where Σ_Λ is given by (2.1) and $\rho = \rho_{2R,\Lambda,M}$

is the function as in Lemma 4.3.

In the remaining part of this section, let $\lambda = 0$. The following lemma on the continuity of $A_0^\alpha f$ with respect to $\alpha > 0$ is needed in section 6.

Lemma 4.4. *Let $1 < q < \infty$ and $R > 0$. Suppose that $f \in L^q(\mathbb{R}^2)$ satisfies $f(x) = 0$ a.e. $|x| \geq R$. Then the function $\alpha \mapsto A_0^\alpha f$ with values in $W^{1,q}(B_R)$ is continuous on $(0, \infty)$.*

Proof. For $\alpha > \beta > 0$, it follows from (4.19) that

$$E_0^\alpha(x) - E_0^\beta(x) = \frac{1}{4\pi}\left(\log\frac{\beta}{\alpha}\right)\mathbb{I} + F^\alpha(x) - F^\beta(x)$$

has no singularity at $x = 0$. As in the proof of Proposition 4.1, we find

$$\sup_{|x|\leq R} |A_0^\alpha f(x) - A_0^\beta f(x)| \leq \sup_{|x|\leq 2R} |E_0^\alpha(x) - E_0^\beta(x)| \|f\|_{L^1(B_R)}.$$

Since $(\alpha, x) \mapsto E_0^\alpha(x)$ is uniformly continuous in any compact set of $(0, \infty) \times \mathbb{R}^2$, we conclude that the RHS of the estimate above goes to zero when $(\alpha - \beta) \to 0$. In view of the representation of $\nabla E_0^\alpha(x)$ given in the proof of Proposition 4.1, concerning $\nabla\big(E_0^\alpha(x) - E_0^\beta(x)\big) = \nabla\big(F^\alpha(x) - F^\beta(x)\big)$, one can divide $F^\alpha(x)$ into $F^{\alpha,1}(x) + F^{\alpha,2}(x)$ such that $\nabla\big(F^{\alpha,1}(x) - F^{\beta,1}(x)\big)$ has no singularity at $x = 0$, while $\nabla\big(F^{\alpha,2}(x) - F^{\beta,2}(x)\big)$ possesses the logarithmic singularity at $x = 0$ whose coefficient is proportional to $(\alpha - \beta)$. Therefore, as in the proof of Proposition 4.1 again, we have

$$\begin{aligned}\sup_{|x|\leq R} \big|\nabla\big(A_0^\alpha f(x) - A_0^\beta f(x)\big)\big| \leq{}& \sup_{|x|\leq 2R} \big|\nabla\big(F^{\alpha,1}(x) - F^{\beta,1}(x)\big)\big| \|f\|_{L^1(B_R)} \\ &+ C\,(\alpha - \beta)\left\|\log\frac{1}{|x - \cdot|}\right\|_{L^{q/(q-1)}(B_{2R}(x))} \|f\|_{L^q(B_R)}.\end{aligned}$$

By the same reasoning as above, we obtain the assertion. □

For later use, we finally summarize some other knowledge of the case $\lambda = 0$, in particular, asymptotic behavior of the Oseen fundamental solution

$$E_0^\alpha(x) = \begin{pmatrix} E_{0,11}^\alpha(x) & E_{0,12}^\alpha(x) \\ E_{0,21}^\alpha(x) & E_{0,22}^\alpha(x) \end{pmatrix}$$

for large $|x|$ and L^q-estimate of the solution operator A_0^α. The details are found in the book by Galdi [**17**, Chapter VII]. Let $\alpha > 0$ and $R > 0$. Then

$$\begin{aligned} &|E_0^\alpha(x)| \leq C|x|^{-1/2} = C|x|^{-(n-1)/2} \quad \text{for } |x| \geq R, \\ &E_0^\alpha \in L^r(\mathbb{R}^2 \setminus B_R) \quad \text{for } \forall\, r > 3 = (n+1)/(n-1), \end{aligned} \tag{4.29}$$

where n is the space dimension. To be precise, the only component having anisotropic decay structure (slow decay in the wake region) is $E^\alpha_{0,11}(x)$, while the other components have better decay and summability properties. We have also

$$\begin{aligned} &|\nabla E^\alpha_0(x)| \le C|x|^{-1} = C|x|^{-n/2} \quad \text{for } |x| \ge R, \\ &\nabla E^\alpha_0(x) \in L^r(\mathbb{R}^2 \setminus B_R) \quad \text{for } \forall\, r > 3/2 = (n+1)/n. \end{aligned} \tag{4.30}$$

The worst one is actually $\partial_2 E^\alpha_{0,11}(x)$ and the other derivatives have better decay and summability properties. By the Fourier multiplier theorem we find

$$\|\nabla^2 A^\alpha_0 f\|_{L^q(\mathbb{R}^2)} + \alpha\|\partial_1 A^\alpha_0 f\|_{L^q(\mathbb{R}^2)} + \|\nabla \Pi f\|_{L^q(\mathbb{R}^2)} \le C\|f\|_{L^q(\mathbb{R}^2)} \tag{4.31}$$

for $\alpha \ge 0$, $1 < q < \infty$ and $f \in L^q(\mathbb{R}^2)$; moreover, we have $\nabla(A^\alpha_0 f)_2 \in L^q(\mathbb{R}^2)$ as well for the second component $(A^\alpha_0 f)_2$ and this is a particular feature of the 2-dimensional case (although we don't need it in this paper). By using (4.29), (4.30) and (4.31), we conclude this section with the following lemma, which corresponds to the third assertion of Lemma 4.1 for $\lambda \in \mathbb{C} \setminus S_\alpha$. Since the pressure is the same as in Lemma 4.1, it is omitted.

LEMMA 4.5. *Let $\alpha > 0$ and $1 < q < \infty$. If $f \in L^q(\mathbb{R}^2)$ satisfies $f(x) = 0$ a.e. $|x| \ge R$ for some $R > 0$, then*

$$\nabla^2 A^\alpha_0 f \in L^r(\mathbb{R}^2), \quad \forall r \in (1, q],$$

$$\nabla A^\alpha_0 f \in L^r(\mathbb{R}^2), \quad \begin{cases} \forall r \in (3/2, q_*] & \text{if } q \in (1,2), \\ \forall r \in (3/2, \infty) & \text{if } q \in [2,\infty), \end{cases}$$

$$A^\alpha_0 f \in L^r(\mathbb{R}^2), \quad \forall r \in (3, \infty),$$

where $q_* \in (2,\infty)$ is determined by $1/q_* = 1/q - 1/2$ for $q \in (1,2)$. All of them in $L^r(\mathbb{R}^2)$ are estimated from above by $C_R\|f\|_{L^q(\mathbb{R}^2)}$.

PROOF. The first assertion is obvious. We will show the assertion only for $\nabla A^\alpha_0 f$ since the last one can be discussed similarly by using (4.29). We fix $r \in (3/2, \infty)$. Suppose $f \in L^q(\mathbb{R}^2)$ satisfies $f(x) = 0$ a.e.$|x| \ge R$, and consider

$$\begin{aligned} |\nabla A^\alpha_0 f(x)|^r &\le \left(\int_{B_R} |(\nabla E^\alpha_0)(x-y)||f(y)|\,dy\right)^r \\ &\le \|f\|_1^{r-1} \int_{B_R} |(\nabla E^\alpha_0)(x-y)|^r |f(y)|\,dy. \end{aligned}$$

For $|x| \ge 2R$ and $|y| \le R$, we have $|x-y| \ge R$ and, therefore,

$$\|\nabla A^\alpha_0 f\|_{L^r(\mathbb{R}^2 \setminus B_{2R})} \le \|\nabla E^\alpha_0\|_{L^r(\mathbb{R}^2 \setminus B_R)}\|f\|_1$$

follows from (4.30). On the other hand, by embedding relation we have

$$\nabla A_0^\alpha f \in L^r_{loc}(\mathbb{R}^2), \qquad \begin{cases} \forall r \in (1, q_*] & \text{if } q \in (1,2), \\ \forall r \in (1,\infty) & \text{if } q \in [2,\infty). \end{cases}$$

The proof is complete. □

5. The interior problem.

Let D be a bounded domain in $\mathbb{R}^2$ with smooth boundary ∂D (but the space dimension $n = 2$ does not play any role in this section). We will construct a solution of the interior problem for the Oseen resolvent system

$$\begin{cases} \lambda u - \Delta u - 2\alpha\partial_1 u + \nabla p = f, \quad \operatorname{div} u = 0 \quad \text{in } D, \\ u|_{\partial D} = 0 \end{cases} \tag{5.1}$$

and investigate its asymptotic behavior with respect to λ and α. Of course, one can obtain the resolvent estimate like (2.6) for large $|\lambda|$, however, this is not our aim. What we need is not the behavior for $|\lambda| \to \infty$ but the estimate near $(\lambda, \alpha) = (0,0)$. Indeed our result for the interior problem is covered by [**26**] and [**10**], but we give it in our convenient form for completeness.

We fix a subdomain $D_0 \subset D$ with $|D_0| > 0$ (positive Lebesgue measure) and, we find a solution of (5.1) subject to

$$\int_{D_0} p(x)\, dx = 0. \tag{5.2}$$

In the next section we will take the pressure with $\int_{D_0} p(x)\, dx = c_0$ for a specified constant c_0, however, this general case can be easily reduced to the case (5.2) by subtracting the constant $c_0|D_0|$, see (6.7). The only thing by this reduction is lack of the Poincaré inequality $\|p\|_{L^q(D)} \le C\|\nabla p\|_{L^q(D)}$.

Let $1 < q < \infty$. The solenoidal space $L^q_\sigma(D)$ and the Stokes operator $A_D = -P_D\Delta$ on that space are defined in the same way as in section 2, where P_D denotes the Helmholtz projection (Fujiwara and Morimoto [**16**], Simader and Sohr [**33**]). Then $u = M_0^0 f := A_D^{-1}P_D f$ together with the associated pressure $p = N_0^0 f$ solves the Stokes problem

$$-\Delta u + \nabla p = f, \qquad \operatorname{div} u = 0 \qquad \text{in } D, \qquad u|_{\partial D} = 0 \tag{5.3}$$

subject to (5.2) and satisfies (Cattabriga [**5**], Galdi [**17**, Chapter IV])

$$\|M_0^0 f\|_{W^{2,q}(D)} + \|N_0^0 f\|_{W^{1,q}(D)} \le C\|f\|_{L^q(D)} \tag{5.4}$$

from which we see that M_0^0 and ∇M_0^0 are compact operators in $L^q(D)$ by the Rellich theorem.

We will show the following proposition.

PROPOSITION 5.1. *Let $\alpha \geq 0$, $\lambda \in (\mathbb{C} \setminus S_\alpha) \cup \{0\}$ and $1 < q < \infty$, where S_α is given by* (2.7) *for $\alpha > 0$ and $S_0 = (-\infty, 0]$. Then there exist operators M_λ^α and N_λ^α from $L^q(D)$ into $W^{2,q}(D) \times W^{1,q}(D)$ such that the pair $(M_\lambda^\alpha f, N_\lambda^\alpha f)$ provides a unique solution of* (5.1) *subject to* (5.2) *for all $f \in L^q(D)$ and that it is analytic in $(\mathbb{C} \setminus S_\alpha) \cup \{|\lambda| < \rho\}$ for some $\rho > 0$. It also enjoys the following properties.*

1. *Let $M > 0$ and let K be a compact subset of $(\mathbb{C} \setminus S_\alpha) \cup \{0\}$. Then, for every integer $j \geq 0$, there is a constant $C = C(j, M, K; D, q) > 0$ such that*
$$\|\partial_\lambda^j M_\lambda^\alpha f\|_{W^{2,q}(D)} + \|\partial_\lambda^j N_\lambda^\alpha f\|_{W^{1,q}(D)} \leq C\|f\|_q \tag{5.5}$$
for $\alpha \in [0, M]$, $\lambda \in K$ and $f \in L^q(D)$.
2. *Let $\alpha_0 \geq 0$ and let K be a compact subset of $(\mathbb{C} \setminus S_\alpha) \cup \{0\}$. Then*
$$\sup_{\lambda \in K} \left(\|M_\lambda^\alpha f - M_\lambda^{\alpha_0} f\|_{W^{2,q}(D)} + \|N_\lambda^\alpha f - N_\lambda^{\alpha_0} f\|_{W^{1,q}(D)}\right) = O(|\alpha - \alpha_0|),$$
$$\text{as } (0,\infty) \ni \alpha \to \alpha_0 \tag{5.6}$$
for $f \in L^q(D)$.
3. *Let $\lambda_0 \in (\mathbb{C} \setminus S_\alpha) \cup \{0\}$ and $M > 0$. Then*
$$\sup_{0 \leq \alpha \leq M} \left(\|M_\lambda^\alpha f - M_{\lambda_0}^\alpha f\|_{W^{2,q}(D)} + \|N_\lambda^\alpha f - N_{\lambda_0}^\alpha f\|_{W^{1,q}(D)}\right) = O(|\lambda - \lambda_0|),$$
$$\text{as } \mathbb{C} \setminus S_\alpha \ni \lambda \to \lambda_0 \tag{5.7}$$
for $f \in L^q(D)$.

PROOF. We intend to find the solution of the form $(u, p) = (M_0^0 g, N_0^0 g)$ with a suitable $g \in L^q(D)$. Since

$$\lambda u - \Delta u - 2\alpha\partial_1 u + \nabla p = g + \lambda M_0^0 g - 2\alpha\partial_1 M_0^0 g, \quad \text{div } u = 0 \quad \text{in } D, \quad u|_{\partial D} = 0$$

and since M_0^0 and $\partial_1 M_0^0$ are compact, in order to prove that $1 + \lambda M_0^0 - 2\alpha\partial_1 M_0^0$ is invertible in $L^q(D)$, it suffices to show the injectivity by the Fredholm alternative. Let $g \in L^q(D)$ satisfy $(1 + \lambda M_0^0 - 2\alpha\partial_1 M_0^0)g = 0$. Then the pair $(u, p) = (M_0^0 g, N_0^0 g)$ obeys (5.1)–(5.2) with $f = 0$. Since we have $u \in H^2(D)$ and $p \in H^1(D)$ (even though q is close to 1) by bootstrap argument with the aid of regularity theory for the Stokes system, we multiply the equation by $\overline{u}$, integrate and take the real and imaginary parts to find

$$(\text{Re } \lambda)\|u\|_{L^2(D)}^2 + \|\nabla u\|_{L^2(D)}^2 = 0, \qquad (\text{Im } \lambda)\|u\|_{L^2(D)}^2 - 2\alpha \,\text{Im} \int_D (\partial_1 u) \cdot \overline{u}\, dx = 0.$$

When $\text{Re } \lambda \geq 0$, we get $u = \nabla p = 0$ at once (by (5.2), $p = 0$). Even if $\lambda \in \mathbb{C} \setminus S_\alpha$ with $\text{Re } \lambda < 0$, the equalities above imply that

$$(\mathrm{Im}\ \lambda)^2\|u\|_{L^2(D)}^4 \le 4\alpha^2\|\nabla u\|_{L^2(D)}^2\|u\|_{L^2(D)}^2 = -4\alpha^2(\mathrm{Re}\ \lambda)\|u\|_{L^2(D)}^4.$$

Thus the condition $4\alpha^2\mathrm{Re}\ \lambda + (\mathrm{Im}\ \lambda)^2 > 0$ yields $u = \nabla p = 0$; in any case, we obtain $g = 0$. We thus find that the pair

$$\begin{aligned} u &= M_\lambda^\alpha f := M_0^0(1+\lambda M_0^0 - 2\alpha\partial_1 M_0^0)^{-1} f \in W^{2,q}(D), \\ p &= N_\lambda^\alpha f := N_0^0(1+\lambda M_0^0 - 2\alpha\partial_1 M_0^0)^{-1} f \in W^{1,q}(D) \end{aligned} \tag{5.8}$$

provides a unique solution of (5.1) subject to (5.2) for all $\alpha \ge 0$, $\lambda \in (\mathbb{C}\setminus S_\alpha)\cup\{0\}$ and $f \in L^q(D)$. Since $(\lambda,\alpha) \mapsto 1+\lambda M_0^0 - 2\alpha\partial_1 M_0^0$ is continuous from $(\mathbb{C}\setminus S_\alpha)\cup\{0\}\times[0,\infty)$ to $\mathcal{L}(L^q(D))$, so is $(\lambda,\alpha) \mapsto (1+\lambda M_0^0 - 2\alpha\partial_1 M_0^0)^{-1}$. In fact, for any $(\lambda_0,\alpha_0) \in (\mathbb{C}\setminus S_\alpha)\cup\{0\}\times[0,\infty)$, if

$$\begin{aligned} &|\lambda-\lambda_0|\|M_0^0\|_{\mathcal{L}(L^q(D))} + 2|\alpha-\alpha_0|\|\partial_1 M_0^0\|_{\mathcal{L}(L^q(D))} \\ &\quad\le \frac{1}{2\,\|(1+\lambda_0 M_0^0 - 2\alpha_0\partial_1 M_0^0)^{-1}\|_{\mathcal{L}(L^q(D))}} \end{aligned} \tag{5.9}$$

then we obtain the Neumann series representation

$$\begin{aligned} &(1+\lambda M_0^0 - 2\alpha\partial_1 M_0^0)^{-1} \\ &\quad= (1+\lambda_0 M_0^0 - 2\alpha_0\partial_1 M_0^0)^{-1} \\ &\qquad\times\sum_{k=0}^\infty \big[-\big\{(\lambda-\lambda_0)M_0^0 - 2(\alpha-\alpha_0)\partial_1 M_0^0\big\}(1+\lambda_0 M_0^0 - 2\alpha_0\partial_1 M_0^0)^{-1}\big]^k \end{aligned} \tag{5.10}$$

which implies

$$\begin{aligned} &\big\|(1+\lambda M_0^0 - 2\alpha\partial_1 M_0^0)^{-1} - (1+\lambda_0 M_0^0 - 2\alpha_0\partial_1 M_0^0)^{-1}\big\|_{\mathcal{L}(L^q(D))} \\ &\quad\le 2\big\|(1+\lambda_0 M_0^0 - 2\alpha_0\partial_1 M_0^0)^{-1}\big\|_{\mathcal{L}(L^q(D))}^2 \\ &\qquad\times\big(|\lambda-\lambda_0|\|M_0^0\|_{\mathcal{L}(L^q(D))} + 2|\alpha-\alpha_0|\|\partial_1 M_0^0\|_{\mathcal{L}(L^q(D))}\big). \end{aligned} \tag{5.11}$$

Therefore, for any $M > 0$ and compact set $K \subset (\mathbb{C}\setminus S_\alpha)\cup\{0\}$ we have

$$\big\|(1+\lambda M_0^0 - 2\alpha\partial_1 M_0^0)^{-1}\big\|_{\mathcal{L}(L^q(D))} \le C_{M,K} \tag{5.12}$$

provided $\alpha \in [0,M]$ and $\lambda \in K$. As a consequence, for such α and λ, (5.4) yields

$$\|M_\lambda^\alpha f\|_{W^{2,q}(D)} + \|N_\lambda^\alpha f\|_{W^{1,q}(D)} \le C_{M,K}\|f\|_q \tag{5.13}$$

for $f \in L^q(D)$. Furthermore, (5.11) and (5.12) together with (5.4) imply both (5.6) and (5.7) immediately. We next fix $\alpha_0 \ge 0$ and $\lambda_0 \in (\mathbb{C}\setminus S_{\alpha_0})\cup\{0\}$, and set $\alpha = \alpha_0$ in (5.10). Then we find the analyticity of $(1+\lambda M_0^0 - 2\alpha_0\partial_1 M_0^0)^{-1}$ with respect to λ and, hence, so is $(M_\lambda^{\alpha_0}, N_\lambda^{\alpha_0})$.

Finally, for any $M > 0$ and compact set $K \subset (\mathbb{C} \setminus S_\alpha) \cup \{0\}$, let us show (5.5) for $\alpha \in [0, M]$, $\lambda \in K$ and $f \in L^q(D)$. Since we already know (5.13), we will consider the case $j \geq 1$. By taking the differentiation of (5.1), we find that the pair $(\partial_\lambda M_\lambda^\alpha f, \partial_\lambda N_\lambda^\alpha f) \in W^{2,q}(D) \times W^{1,q}(D)$ is a solution of (5.1)–(5.2) with f replaced by $-M_\lambda^\alpha f \in L^q(D)$. By uniqueness of solutions, we see that

$$\partial_\lambda M_\lambda^\alpha f = -(M_\lambda^\alpha)^2 f, \qquad \partial_\lambda N_\lambda^\alpha f = -N_\lambda^\alpha M_\lambda^\alpha f.$$

By induction we find

$$\partial_\lambda^j M_\lambda^\alpha f = (-1)^j j!\,(M_\lambda^\alpha)^{j+1} f, \qquad \partial_\lambda^j N_\lambda^\alpha f = (-1)^j j!\, N_\lambda^\alpha (M_\lambda^\alpha)^j f.$$

Thus, (5.13) implies (5.5) for every $j \geq 1$ as well. We have completed the proof. □

6. Construction of the resolvent in exterior domains.

For the proof of Theorem 2.1 we need spectral analysis. We have the Dunford integral representation formula of the semigroup in terms of the resolvent $(\lambda + L)^{-1}$. Here and in what follows we simply write $L = L_\alpha$. In Proposition 6.1 below it is proved that the spectrum of $-L$ is contained in S_α given by (2.7) along the same argument as in Kobayashi and Shibata [**26**, Theorem 4.4]. It thus seems to be impossible to take the same path of integration as in Dan and Shibata [**8**] for the Stokes semigroup ($\alpha = 0$). Let $f \in L^q_{[R]}(\Omega)$, see (2.8). What we need is then to study the asymptotic behavior of $\partial_\lambda(\lambda + L)^{-1} P f$ for $\lambda \to 0$ (and $\alpha \to 0$ as well) which ensures its summability near the origin, see Lemma 7.1 in the next section. This enables us to justify the representation formula (which was also used in [**26**], [**22**] and [**24**])

$$e^{-tL} P f = \frac{-1}{2\pi i t} \int_{-\infty}^{\infty} e^{i\tau t}\, \partial_\tau (i\tau + L)^{-1} P f \, d\tau \tag{6.1}$$

in $W^{1,q}(\Omega_R)$ provided $f \in L^q_{[R]}(\Omega)$, when we perform integration by parts and then move the path of integration to the imaginary axis. When we derive faster decay than t^{-1}, we have to study further regularity of $\partial_\lambda(\lambda + L)^{-1} P f$ near $\lambda = 0$, see Lemma 7.4.

In order to carry out this strategy, given $f \in L^q_{[R]}(\Omega)$, we construct a parametrix of solutions to the problem

$$\begin{cases} \lambda u - \Delta u - 2\alpha \partial_1 u + \nabla p = f, \quad \operatorname{div} u = 0 \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases} \tag{6.2}$$

where $\lambda \in (\mathbb{C} \setminus S_\alpha) \cup \{0\}$. The behavior along the imaginary axis is particularly important for us. Let $\{A_\lambda^\alpha, \Pi\}$ be the solution operator (4.1) for (4.2) in the whole plane $\mathbb{R}^2$, and $\{M_\lambda^\alpha, N_\lambda^\alpha\}$ that for (5.1)–(5.2) given by Theorem 5.1 with the bounded domain $D = \Omega_{R_0+2} = \Omega \cap B_{R_0+2}$ and

$$D_0 = \{x \in \mathbb{R}^2; R_0 < |x| < R_0 + 1\}. \tag{6.3}$$

We take a cut-off function $\psi \in C^\infty(\mathbb{R}^2; [0,1])$ such that

$$\psi(x) = \begin{cases} 1, & |x| \le R_0, \\ 0, & |x| \ge R_0 + 1 \end{cases} \tag{6.4}$$

and employ the Bogovskii operator B in the annulus D_0, that is, the function $w = Bg \in C_0^\infty(D_0)^2$ is a solution specified by Bogovskii [**1**] among many solutions to

$$\operatorname{div} w = g \quad \text{in } D_0, \qquad w|_{\partial D_0} = 0$$

for given $g \in C_0^\infty(D_0)$ with $\int_{D_0} g(x)dx = 0$. Note that B extends uniquely to a bounded operator from $W_0^{k,q}(D_0)$ to $W_0^{k+1,q}(D_0)^2$ on account of estimate

$$\|\nabla^{k+1} Bg\|_{L^q(D_0)} \le C\|\nabla^k g\|_{L^q(D_0)}, \qquad (1 < q < \infty,\ k = 0, 1, \ldots), \tag{6.5}$$

see [**3**] and [**17**].

Given $f \in L^q_{[R]}(\Omega)$, we set

$$\begin{aligned} v = R_\lambda^\alpha f &:= (1-\psi)A_\lambda^\alpha f + \psi M_\lambda^\alpha f + B\big[(A_\lambda^\alpha f - M_\lambda^\alpha f)\cdot \nabla\psi\big], \\ \sigma = Q_\lambda^\alpha f &:= (1-\psi)\Pi f + \psi \widetilde{N}_\lambda^\alpha f \end{aligned} \tag{6.6}$$

where f is understood as its zero extension (resp. restriction) to $\mathbb{R}^2$ (resp. Ω_{R_0+2}) and the pressure $\widetilde{N}_\lambda^\alpha f$ in Ω_{R_0+2} is chosen in such a way that

$$\widetilde{N}_\lambda^\alpha f := N_\lambda^\alpha f + |D_0| \int_{D_0} (\Pi f)(x)\, dx. \tag{6.7}$$

Because of this choice, we have the Poincaré inequality

$$\big\|\Pi f - \widetilde{N}_\lambda^\alpha f\big\|_{L^q(D_0)} \le C\big\|\nabla(\Pi f - \widetilde{N}_\lambda^\alpha f)\big\|_{L^q(D_0)} = C\|\nabla(\Pi f - N_\lambda^\alpha f)\|_{L^q(D_0)}. \tag{6.8}$$

When $\alpha > 0$, it follows from Lemma 4.1, Lemma 4.5, Proposition 5.1 and embedding relations that

$$\begin{aligned} &\nabla^2 R_\lambda^\alpha f,\ \nabla Q_\lambda^\alpha f \in L^r(\Omega) \quad \forall r \in (1, q], \\ &Q_\lambda^\alpha f \in L^r(\Omega) \quad \begin{cases} \forall r \in (2, q_*] & \text{if } q \in (1,2), \\ \forall r \in (2, \infty) & \text{if } q \in [2,\infty), \end{cases} \end{aligned} \tag{6.9}$$

$$\nabla R_\lambda^\alpha f \in L^r(\Omega) \quad \begin{cases} \forall r \in (1, q_*] & \text{if } q \in (1,2) \\ \forall r \in (1, \infty) & \text{if } q \in [2,\infty) \end{cases} \quad \text{if } \lambda \in \mathbb{C} \setminus S_\alpha, \tag{6.10}$$

$$\nabla R_0^\alpha f \in L^r(\Omega) \quad \begin{cases} \forall r \in (3/2, q_*] & \text{if } q \in (1,2), \\ \forall r \in (3/2, \infty) & \text{if } q \in [2,\infty) \end{cases} \tag{6.11}$$

where $1/q_* = 1/q - 1/2$ for $q \in (1,2)$, and

$$R_\lambda^\alpha f \in L^r(\Omega) \quad \forall r \in (1,\infty) \quad \text{if } \lambda \in \mathbb{C} \setminus S_\alpha; \qquad R_0^\alpha f \in L^r(\Omega) \quad \forall r \in (3,\infty). \tag{6.12}$$

The pair (v, σ) should obey

$$\begin{cases} \lambda v - \Delta v - 2\alpha\partial_1 v + \nabla\sigma = f + T_\lambda^\alpha f, \quad \operatorname{div} v = 0 \quad \text{in } \Omega, \\ v|_{\partial\Omega} = 0 \end{cases} \tag{6.13}$$

where

$$\begin{aligned} T_\lambda^\alpha f &= 2\nabla\psi\cdot\nabla(A_\lambda^\alpha f - M_\lambda^\alpha f) + (\Delta\psi + 2\alpha\partial_1\psi)(A_\lambda^\alpha f - M_\lambda^\alpha f) \\ &\quad - \Delta B\big[(A_\lambda^\alpha f - M_\lambda^\alpha f)\cdot\nabla\psi\big] + \lambda B\big[(A_\lambda^\alpha f - M_\lambda^\alpha f)\cdot\nabla\psi\big] \\ &\quad - 2\alpha\partial_1 B\big[(A_\lambda^\alpha f - M_\lambda^\alpha f)\cdot\nabla\psi\big] - (\nabla\psi)(\Pi f - \widetilde{N}_\lambda^\alpha f). \end{aligned} \tag{6.14}$$

Note that the operator T_λ^α is bounded (even compact) from $L^q_{[R]}(\Omega)$ into itself provided $R \geq R_0 + 1$.

In order to show that $1 + T_\lambda^\alpha$ is bijective in $L^q_{[R]}(\Omega)$, we begin with uniqueness of solutions to (6.2) under weak assumptions at space infinity.

LEMMA 6.1. *Let $\alpha > 0$, $\lambda \in (\mathbb{C} \setminus S_\alpha) \cup \{0\}$ and $1 < q < \infty$. Suppose that $(u, p) \in W^{2,q}_{loc}(\overline{\Omega}) \times W^{1,q}_{loc}(\overline{\Omega})$ solves* (6.2) *with $f = 0$ in the sense of distributions. If $u \in L^r(\Omega)$ and $\nabla p \in L^s(\Omega)$ for some $r, s \in (1,\infty)$, then $u = 0$ and p is a constant.*

PROOF. By regularity theory for the Stokes system we may assume $(u,p) \in H^2_{loc}(\overline{\Omega}) \times H^1_{loc}(\overline{\Omega})$ (even though q is close to 1). We first show that $(u, \nabla p)$ enjoys better decay/summability properties (like the fundamental solution) for $|x| \to \infty$ than those we have assumed. We use a cut-off technique. Consider

$$\widetilde{u} = (1-\psi)u + B[u\cdot\nabla\psi], \qquad \widetilde{p} = (1-\psi)p$$

which obey (4.2) in the whole plane $\mathbb{R}^2$ with

$$\begin{aligned} f = 2\nabla\psi\cdot\nabla u + (\Delta\psi + 2\alpha\partial_1\psi)u - \Delta B[u\cdot\nabla\psi] \\ + \lambda B[u\cdot\nabla\psi] - 2\alpha\partial_1 B[u\cdot\nabla\psi] - (\nabla\psi)p, \end{aligned}$$

where the cut-off function ψ is as in (6.4) and B is the Bogovskii operator in the annulus D_0, see (6.3). Set $v = A_\lambda^\alpha f$ and $\sigma = \Pi f$, see (4.1). Since $f \in L^2(\mathbb{R}^2)$ satisfies $f(x) = 0$ for $|x| \geq R_0 + 1$, the pair (v, σ) belongs to the class specified in Lemma 4.1 or Lemma 4.5 (with $q = 2$). We note that $w = \widetilde{u} - v$ and $\tau = \widetilde{p} - \sigma$ are tempered distributions (see Chemin [**7**, Proposition 1.2.1], from which $\nabla\tau \in \mathcal{S}'(\mathbb{R}^2)$ implies $\tau \in \mathcal{S}'(\mathbb{R}^2)$) which satisfy (4.2) without external force. For $\lambda \in \mathbb{C} \setminus S_\alpha$, we see from (4.5) that w and τ are polynomials. This is also the case for $\lambda = 0$ since $|\xi|^2 - 2\alpha i\xi_1 \neq 0$ for $\xi \in \mathbb{R}^2 \setminus \{0\}$. The summability assumptions on u and ∇p imply that $w = 0$ and $\tau = p_0$ with some constant p_0. Since $u(x) = \widetilde{u}(x) = v(x)$ and $p(x) = \widetilde{p}(x) = \sigma(x) + p_0$, for $|x| \geq R_0 + 1$, we find that $(u, p - p_0)$ has the same summability as in (6.9)–(6.12) (with $q = 2$). We now take $\varphi \in C_0^\infty(\mathbb{R}^2; [0,1])$ satisfying $\varphi(x) = 1$ for $|x| \leq 1$ and $\varphi(x) = 0$ for $|x| \geq 2$, and set $\varphi_R(x) = \varphi(x/R)$. We multiply $\lambda u - \Delta u - 2\alpha\partial_1 u + \nabla(p - p_0) = 0$ by $\varphi_R\overline{u}$ and integrate to obtain

$$\begin{aligned}\lambda \int_{\Omega_{2R}} \varphi_R|u|^2\,dx + \int_{\Omega_{2R}} \varphi_R|\nabla u|^2\,dx + \int_{R<|x|<2R} (\nabla\varphi_R \cdot \nabla u)\cdot \overline{u}\,dx \\ - 2\alpha \int_{\Omega_{2R}} \varphi_R(\partial_1 u)\cdot\overline{u}\,dx - \int_{R<|x|<2R} (p - p_0)(\overline{u}\cdot\nabla\varphi_R)\,dx = 0\end{aligned}$$

where $(\nabla\varphi_R \cdot \nabla u)\cdot\overline{u} = \sum_{j,k}(\partial_j\varphi_R)(\partial_j u_k)\overline{u_k}$. Note that

$$\mathrm{Re} \int_{\Omega_{2R}} \varphi_R(\partial_1 u)\cdot\overline{u}\,dx = \frac{-1}{2}\int_{R<|x|<2R} (\partial_1\varphi_R)|u|^2\,dx.$$

When Re $\lambda \geq 0$, we take the real part and let $R \to \infty$ to obtain $\|\nabla u\|_2^2 = 0$ because

$$\lim_{R\to\infty} \frac{1}{R}\int_{R<|x|<2R} |u(x)|^2\,dx = 0, \tag{6.15}$$

$$\lim_{R\to\infty} \frac{1}{R}\int_{R<|x|<2R} \big(|\nabla u(x)| + |p(x) - p_0|\big)|u(x)|\,dx = 0 \tag{6.16}$$

which follow from the summability of $(u, p - p_0)$ deduced above. This implies $(u, p) = (0, p_0)$. When $\lambda \in \mathbb{C} \setminus S_\alpha$ with Re $\lambda < 0$, we have $u \in L^2(\Omega)$ as well. Thus, taking the real and imaginary parts, letting $R \to \infty$ and using (6.15), (6.16), we get

$$(\mathrm{Re}\ \lambda)\|u\|_2^2 + \|\nabla u\|_2^2 = 0, \qquad (\mathrm{Im}\ \lambda)\|u\|_2^2 - 2\alpha\,\mathrm{Im}\int_\Omega (\partial_1 u)\cdot\overline{u}\,dx = 0.$$

As in the proof of Proposition 5.1, we obtain $(u, p) = (0, p_0)$. This completes the proof. □

Using Lemma 6.1 together with compactness argument, we construct a solution of (6.2) for $f \in L^q_{[R]}(\Omega)$.

Lemma 6.2. *Let $\alpha > 0$, $\lambda \in (\mathbb{C} \setminus S_\alpha) \cup \{0\}$, $1 < q < \infty$ and $R \geq R_0 + 1$. Then the operator $1 + T_\lambda^\alpha$ has a bounded inverse $(1 + T_\lambda^\alpha)^{-1}$ on $L^q_{[R]}(\Omega)$ and thereby*

$$u = R_\lambda^\alpha(1 + T_\lambda^\alpha)^{-1}f, \qquad p = Q_\lambda^\alpha(1 + T_\lambda^\alpha)^{-1}f \tag{6.17}$$

provides a unique solution of (6.2) *for every* $f \in L^q_{[R]}(\Omega)$.

PROOF. When $\lambda \in \mathbb{C} \setminus S_\alpha$, it follows from Lemma 4.1, (5.5) (with (6.7)), (6.5) and (6.8) together with the Rellich theorem that the operator T^α_λ is compact from $L^q_{[R]}(\Omega)$ into itself. For $\lambda = 0$, we have only to replace Lemma 4.1 by (4.16) and (4.31). Let $f \in L^q_{[R]}(\Omega)$ fulfill $(1 + T^\alpha_\lambda)f = 0$, so that the support of f is contained in $\overline{D_0}$, see (6.3). By the class (6.9)–(6.12) of $(v, \sigma) = (R^\alpha_\lambda f, Q^\alpha_\lambda f)$, one can apply Lemma 6.1 to get $(v, \sigma) = (0, 0)$ (by (6.9) we have $\sigma = 0$ as well). In view of (6.6) we observe that $(A^\alpha_\lambda f, \Pi f) = (0, 0)$ for $|x| \geq R_0 + 1$ and that $(M^\alpha_\lambda f, \widetilde{N}^\alpha_\lambda f) = (0, 0)$ for $|x| \leq R_0$. Therefore, both pairs are solutions of the Oseen resolvent system in B_{R_0+2} with homogeneous boundary condition on ∂B_{R_0+2}, and so they should coincide on account of uniqueness for this interior problem under the assumption $\lambda \in (\mathbb{C} \setminus S_\alpha) \cup \{0\}$. Consequently, $(A^\alpha_\lambda f, \Pi f) = (0, 0)$ in the whole plane $\mathbb{R}^2$ and, therefore, $f = 0$. By the Fredholm alternative, $1 + T^\alpha_\lambda$ is bijective and thereby (6.17) provides a solution, that is unique within the class specified in Lemma 6.1. $\square$

By means of a cut-off procedure due to [**26**] with the aid of Lemma 6.2, we will construct the resolvent for $\lambda \in \mathbb{C} \setminus S_\alpha$.

PROPOSITION 6.1. *Let* $\alpha > 0$ *and* $1 < q < \infty$. *Then* $\mathbb{C} \setminus S_\alpha \subset \rho(-L)$, *where* $\rho(-L)$ *is the resolvent set in* $L^q_\sigma(\Omega)$.

PROOF. Let $\lambda \in \mathbb{C} \setminus S_\alpha$ and let $u \in D(L)$ satisfy $(\lambda + L)u = 0$ in $L^q_\sigma(\Omega)$. Then, for some associated pressure p satisfying $\nabla p \in L^q(\Omega)$, the pair (u, p) satisfies (6.2), so that Lemma 6.1 yields $u = 0$. As a consequence, $\lambda + L$ is injective. We next construct a solution of (6.2) for given $f \in L^q(\Omega)$. Set

$$v = (1 - \psi)A^\alpha_\lambda f + B[(A^\alpha_\lambda f) \cdot \nabla\psi], \qquad \sigma = (1 - \psi)(\Pi f - \sigma_0)$$

where f is understood as its zero extension to $\mathbb{R}^2$, ψ is the same cut-off function as in (6.4), B is the Bogovskii operator in the annulus (6.3) and $\sigma_0 = |D_0|^{-1} \int_{D_0} \Pi f \, dx$. We intend to find a solution of the form $u = v + w$ and $p = \sigma + \tau$; then (w, τ) should obey (6.2) with f replaced by

$$\begin{aligned} g = \psi f - 2\nabla\psi \cdot \nabla A^\alpha_\lambda f - (\Delta\psi + 2\alpha\partial_1\psi)A^\alpha_\lambda f + \Delta B[(A^\alpha_\lambda f) \cdot \nabla\psi] - \lambda B[(A^\alpha_\lambda f) \cdot \nabla\psi] \\ + 2\alpha\partial_1 B[(A^\alpha_\lambda f) \cdot \nabla\psi] + (\nabla\psi)(\Pi f - \sigma_0) \end{aligned}$$

that belongs to $L^q_{[R]}(\Omega)$, where $R \geq R_0 + 1$. Thus Lemma 6.2 provides a solution $w = R^\alpha_\lambda(1 + T^\alpha_\lambda)^{-1}g$ and $\tau = Q^\alpha_\lambda(1 + T^\alpha_\lambda)^{-1}g$. From Lemma 4.1, (6.9), (6.10) and (6.12) we find $u = v + w \in D(L)$ together with $(\lambda + L)u = Pf$. Hence, Range $(\lambda + L) = L^q_\sigma(\Omega)$ and thereby $(\lambda + L)^{-1} \in \mathcal{L}(L^q_\sigma(\Omega))$ since L is closed. This completes the proof. $\square$

Combining Proposition 6.1 with Lemma 6.2, we conclude a representation of the resolvent in terms of (6.6) and (6.14) (by uniqueness of the resolvent) when $f \in L^q_{[R]}(\Omega)$.

COROLLARY 6.1. *Let* $\alpha > 0$, $\lambda \in \mathbb{C} \setminus S_\alpha$, $1 < q < \infty$ *and* $R \geq R_0 + 1$. *Then we* *have*

$$(\lambda + L)^{-1} P f = R_\lambda^\alpha \, (1 + T_\lambda^\alpha)^{-1} f \tag{6.18}$$

provided that $f \in L^q_{[R]}(\Omega)$.

Our aim is to derive the asymptotic behavior of the resolvent (6.18) for small (λ, α). The main theorem of this section reads as follows.

Theorem 6.1. *Let* $1 < q < \infty$, $R \geq R_0 + 1$ *and* $M > 0$. *There are constants* $\delta = \delta(M; \Omega, q, R) > 0$ *and* $C = C(M; \Omega, q, R) > 0$ *such that if* $\lambda \in \overline{\mathbb{C}_+}$ *satisfies* $|\lambda| \leq \delta\alpha^2$ *then the bounded inverse* $(1 + T_\lambda^\alpha)^{-1}$ *obtained in Lemma* 6.2 *satisfies*

$$\|(1 + T_\lambda^\alpha)^{-1}\|_{\mathcal{L}(L^q_{[R]}(\Omega))} \leq C \tag{6.19}$$

for all $\alpha \in (0, M]$; *as a consequence, for such* α *and* λ (*except* $\lambda = 0$) *we have*

$$\|(\lambda + L)^{-1} P f\|_{W^{1,q}(\Omega_R)} \leq C\left(\left|\log \frac{1}{|\lambda| + \alpha^2}\right| + 1\right)\|f\|_q \tag{6.20}$$

for $f \in L^q_{[R]}(\Omega)$.

Remark 6.1. Even if $\lambda = 0$, the estimate (6.20) for $R_0^\alpha \, (1 + T_0^\alpha)^{-1} f$ as well as (6.19) holds true.

The compactness argument in Lemma 6.2 provides us little information about the dependence of $(1 + T_\lambda^\alpha)^{-1}$ on (λ, α). We thus need another construction of $(1 + T_\lambda^\alpha)^{-1}$ especially near $\lambda = 0$. The strategy is first to consider $(1 + T_0^\alpha)^{-1}$ and then to estimate $T_\lambda^\alpha - T_0^\alpha$. Let us consider the case $\lambda = 0$. In view of (4.26) we have the logarithmic singularity only in the degenerate part of A_0^α, that enables us to show the following proposition.

Proposition 6.2. *Let* $1 < q < \infty$ *and* $R \geq R_0 + 1$. *Then, for any* $M > 0$, *there is a positive constant* $C = C(M; \Omega, q, R)$ *such that*

$$\|(1 + T_0^\alpha)^{-1}\|_{\mathcal{L}(L^q_{[R]}(\Omega))} \leq C \tag{6.21}$$

for $\alpha \in (0, M]$.

For the proof, the following lemma on nonexistence of solutions of the Stokes problem under a certain condition (what is called the Stokes paradox) plays a crucial role.

Lemma 6.3. *Let* $u_* \in \mathbb{R}^2 \setminus \{0\}$ *be a constant vector. Then the Stokes problem*

$$-\Delta u + \nabla \sigma = 0, \quad \operatorname{div} u = 0 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = u_*, \quad u \to 0 \ \text{ as } |x| \to \infty \tag{6.22}$$

has no solution.

Proof. This was essentially proved by Chang and Finn [**6**], but we give the proof for completeness. In view of the asymptotic expansion of solutions at infinity of the Stokes system with no external force (without assuming any boundary condition on $\partial\Omega$) due to [**6**], the necessity for the decay is that the total net force exerted by the fluid to the obstacle vanishes:

$$\int_{\partial\Omega} \nu\cdot T(u,\sigma)\,ds = 0,$$

where $T(u,\sigma) = Du - \sigma\mathbb{I}$ denotes the Cauchy stress tensor and $Du = \nabla u + (\nabla u)^T$. This is because the net force is the coefficient of the fundamental solution (4.8) in that asymptotic expansion. We have also better decay properties

$$u(x) = O(|x|^{-1}), \qquad \nabla u(x) = O(|x|^{-2}), \qquad \sigma(x) - \sigma_\infty = O(|x|^{-2}) \quad \text{as } |x|\to\infty$$

for some constant $\sigma_\infty\in\mathbb{R}$. Then we multiply the equation $-\mathrm{div}\,T(u,\sigma-\sigma_\infty) = -\Delta u + \nabla(\sigma-\sigma_\infty) = 0$ by u and integrate over Ω_R for any $R>R_0$ to obtain

$$\frac{1}{2}\int_{\Omega_R}|Du|^2\,dx = \int_{\partial\Omega_R} u\cdot(\nu\cdot T(u,\sigma-\sigma_\infty))\,ds_x = \frac{1}{R}\int_{\partial B_R} u\cdot(x\cdot T(u,\sigma-\sigma_\infty))\,ds_x$$

which goes to zero as $R\to\infty$ due to $|u\cdot(x\cdot T(u,\sigma-\sigma_\infty))|\le C/|x|^2$. We thus conclude $Du=0$ in Ω, so that u is a rigid motion and, hence, $u=0$ since it decays at infinity. Thus the inhomogeneous boundary condition $u|_{\partial\Omega} = u_*\neq 0$ is impossible in (6.22). This completes the proof. □

Proof of Proposition 6.2. In the proof of Lemma 6.2, we see that $(0,\infty)\ni\alpha\mapsto(1+T_0^\alpha)^{-1}\in\mathcal{L}(L^q_{[R]}(\Omega))$ is continuous (by means of the Neumann series argument around each $\alpha_0>0$) because so is $\alpha\mapsto T_0^\alpha$, see Lemma 4.4, (5.6), (6.5) and (6.7). Therefore, it is essential to show the boundedness (6.21) near $\alpha=0$. To this end, we reconstruct $(1+T_0^\alpha)^{-1}$ for small $\alpha>0$ by following the argument developed by Dan and Shibata [**8**]. But their argument is not enough to conclude (6.21), and what is particularly new here is (6.30) below, which is proved by use of Lemma 6.3. It is convenient to introduce notation

$$\langle e_j, f\rangle = \int_{\mathbb{R}^2} e_j\cdot f(y)\,dy \qquad (j=1,2), \qquad e_1 = \begin{pmatrix}1\\0\end{pmatrix}, \qquad e_2 = \begin{pmatrix}0\\1\end{pmatrix}$$

so that $\Gamma f = \langle e_1,f\rangle e_1 + \langle e_2,f\rangle e_2$ and

$$\frac{1}{4\pi}\left[\left(\log\frac{1}{\alpha}\right)\mathbb{I}+\mathbb{J}\right]\Gamma f = \frac{\log\frac{2}{\alpha}-\gamma}{4\pi}\langle e_1,f\rangle e_1 + \frac{\log\frac{2}{\alpha}-\gamma-1}{4\pi}\langle e_2,f\rangle e_2.$$

By (4.26), (4.27), (5.6) and (6.7) the remaining term $T_0^\alpha f$ can be written as

$$T_0^\alpha f = T_0^0 f + Y(\alpha)f + Z(\alpha)f \tag{6.23}$$

with

$$Y(\alpha)f = (\log\alpha - \log 2 + \gamma)\langle e_1, f\rangle w_1 + (\log\alpha - \log 2 + \gamma + 1)\langle e_2, f\rangle w_2, \tag{6.24}$$

$$\|Z(\alpha)\|_{\mathcal{L}(L^q_{[R]}(\Omega))} = O\left(\alpha\log\frac{1}{\alpha}\right) \qquad \text{as } \alpha \to 0, \tag{6.25}$$

where

$$w_j = \frac{1}{4\pi}\{(-\Delta\psi)e_j + \Delta B[\partial_j\psi]\} \qquad (j = 1, 2), \tag{6.26}$$

and f is understood as its zero extension to $\mathbb{R}^2$ in (6.24). Concerning the operator T_0^0, see (6.14) with $(\lambda, \alpha) = (0, 0)$ which consists of solution operators for the Stokes problem in $\mathbb{R}^2$ and in Ω_{R_0+2}, we recall the following result due to [**8**, Lemma 3.2–3.5]. It is a compact operator from $L^q_{[R]}(\Omega)$ into itself and $1 + T_0^0$ is injective on the subspace $\{f \in L^q_{[R]}(\Omega); \Gamma f = 0\}$ together with $\dim\ker(1 + T_0^0) \le 2$. By the Fredholm theory one can take $m_1, m_2 \in L^q_{[R]}(\Omega)$ such that $m_1, m_2 \notin$ Range $(1 + T_0^0)$ and $L^q_{[R]}(\Omega) =$ Range $(1 + T_0^0) \oplus$ Span $\{m_1, m_2\}$; furthermore, the operator

$$\Lambda(0)f = (1 + T_0^0)f + \langle e_1, f\rangle m_1 + \langle e_2, f\rangle m_2 \tag{6.27}$$

is bijective with bounded inverse $\Lambda(0)^{-1}$ on $L^q_{[R]}(\Omega)$. When $\dim\ker(1 + T_0^0) \le 1$, m_1 and/or m_2 should be understood as zero. By (6.23) and (6.27) we have

$$(1 + T_0^\alpha)f = \Lambda(\alpha)f + \widetilde{Y}(\alpha)f$$

with

$$\begin{aligned} \Lambda(\alpha)f &= \Lambda(0)f + Z(\alpha)f = \left[1 + Z(\alpha)\Lambda(0)^{-1}\right]\Lambda(0)f, \\ \widetilde{Y}(\alpha)f &= Y(\alpha)f - \langle e_1, f\rangle m_1 - \langle e_2, f\rangle m_2. \end{aligned} \tag{6.28}$$

By (6.25) there is a constant $\alpha_0 > 0$ such that if $\alpha \in (0, \alpha_0]$, then $\Lambda(\alpha)$ is invertible as the Neumann series $\Lambda(\alpha)^{-1} = \Lambda(0)^{-1}\sum_{k=0}^\infty\{-Z(\alpha)\Lambda(0)^{-1}\}^k$ with

$$\frac{C_*}{2} \le \|\Lambda(\alpha)^{-1}\|_{\mathcal{L}(L^q_{[R]}(\Omega))} \le 2C_* \tag{6.29}$$

where $C_* = \|\Lambda(0)^{-1}\|_{\mathcal{L}(L^q_{[R]}(\Omega))} > 0$. Let us consider the degenerate (that is, finite rank) operator $\widetilde{Y}(\alpha)$. A key observation is that w_1 and w_2 given by (6.26) are linearly independent, which follows from

$$\det\begin{pmatrix}\langle e_1,\Lambda(0)^{-1}w_1\rangle & \langle e_1,\Lambda(0)^{-1}w_2\rangle\\ \langle e_2,\Lambda(0)^{-1}w_1\rangle & \langle e_2,\Lambda(0)^{-1}w_2\rangle\end{pmatrix}\neq 0. \tag{6.30}$$

We postpone the proof of (6.30). Set $N=\dim\{\widetilde{Y}(\alpha)f;\ f\in L^q_{[R]}(\Omega)\}$. Then we know $2\le N\le 4$ and take a basis $\{w_j\}_{j=1}^N$ by adding w_3 or $\{w_3,w_4\}$ to $\{w_1,w_2\}$ if necessary. We set $-m_k=\sum_{j=1}^N a_j^k w_j$ $(k=1,2)$, then (6.24) and $(6.28)_2$ lead us to

$$\widetilde{Y}(\alpha)f=\sum_{j=1}^N\langle\phi_j,f\rangle w_j$$

where $\phi_j=\phi_j(\alpha)$ is given by

$$\begin{aligned}
&\phi_1=(\log\alpha-\log 2+\gamma+a_1^1)e_1+a_1^2e_2,\\
&\phi_2=a_2^1e_1+(\log\alpha-\log 2+\gamma+1+a_2^2)e_2,\\
&\phi_3=a_3^1e_1+a_3^2e_2,\qquad \phi_4=a_4^1e_1+a_4^2e_2.
\end{aligned}\tag{6.31}$$

One can regard $1+T_0^\alpha$ as

$$(1+T_0^\alpha)f=\Lambda(\alpha)[1+\Lambda(\alpha)^{-1}\widetilde{Y}(\alpha)]f \tag{6.32}$$

with

$$\Lambda(\alpha)^{-1}\widetilde{Y}(\alpha)f=\sum_{j=1}^N\langle\phi_j,f\rangle\Lambda(\alpha)^{-1}w_j.$$

Set $N\times N$ matrix $\mathbb{M}$ by

$$\mathbb{M}=\mathbb{M}(\alpha)=\mathbb{I}+\big(\langle\phi_j,\Lambda(\alpha)^{-1}w_k\rangle\big)_{1\le j,k\le N} \tag{6.33}$$

which satisfies $\det\mathbb{M}\neq 0$ for all $\alpha>0$. In fact, suppose the contrary, then one can take $c=(c_j)_{j=1}^N\neq 0$ such that

$$c_j+\sum_{k=1}^N\langle\phi_j,\Lambda(\alpha)^{-1}w_k\rangle c_k=0$$

for $1\le j\le N$. Since $\Lambda(\alpha)^{-1}w_k$ $(1\le k\le N)$ are linearly independent, we have

$$g:=\sum_{k=1}^N c_k\Lambda(\alpha)^{-1}w_k\neq 0.$$

On the other hand, we find

$$\Lambda(\alpha)^{-1}\widetilde{Y}(\alpha)g=\sum_{j=1}^N\Big\langle\phi_j,\sum_{k=1}^N c_k\Lambda(\alpha)^{-1}w_k\Big\rangle\Lambda(\alpha)^{-1}w_j=-\sum_{j=1}^N c_j\Lambda(\alpha)^{-1}w_j=-g,$$

so that $(1+T_0^\alpha)g=0$ by (6.32). We already know, however, that $1+T_0^\alpha$ is invertible on $L^q_{[R]}(\Omega)$ and, therefore, $g=0$, which leads us to a contradiction. By using the inverse matrix $\mathbb{M}^{-1}=(b_{jk})$ with $b_{jk}=b_{jk}(\alpha)$, we define the degenerate operator $V(\alpha)$ by

$$V(\alpha)f=\sum_{j,k=1}^N\langle\phi_k,f\rangle\, b_{jk}\,\Lambda(\alpha)^{-1}w_j$$

for $f\in L^q_{[R]}(\Omega)$. Then we find that $1+\Lambda(\alpha)^{-1}\widetilde{Y}(\alpha)$ is invertible and

$$\big[1+\Lambda(\alpha)^{-1}\widetilde{Y}(\alpha)\big]^{-1}=1-V(\alpha).$$

Hence it follows from (6.32) that

$$(1+T_0^\alpha)^{-1}f=(1-V(\alpha))\Lambda(\alpha)^{-1}f=\Lambda(\alpha)^{-1}f-\sum_{j,k=1}^N\langle\phi_k,\Lambda(\alpha)^{-1}f\rangle\, b_{jk}\,\Lambda(\alpha)^{-1}w_j.$$

For each j, look at

$$\sum_{k=1}^N\langle\phi_k,\cdot\,\rangle\, b_{jk}=\sum_{k=1}^N\langle\phi_k,\cdot\,\rangle\frac{\widetilde{M}_{kj}}{\det\mathbb{M}}$$

where $\widetilde{M}_{kj}$ denotes (k,j)-cofactor of $\mathbb{M}$. By (6.31) and (6.33) the order of singularities of both $\det\mathbb{M}$ and $\sum_{k=1}^N\langle\phi_k,\cdot\,\rangle\widetilde{M}_{kj}$ are at most $(\log\alpha)^2$. One can write $\det\mathbb{M}=\sum_{j=0}^2 C_j(\alpha)(\log\alpha)^j$, where $C_j(\alpha)$ has no singularity for $\alpha\to 0$; then, it is easy to verify

$$C_2(\alpha)=\det\begin{pmatrix}\langle e_1,\Lambda(\alpha)^{-1}w_1\rangle & \langle e_1,\Lambda(\alpha)^{-1}w_2\rangle\\ \langle e_2,\Lambda(\alpha)^{-1}w_1\rangle & \langle e_2,\Lambda(\alpha)^{-1}w_2\rangle\end{pmatrix}$$

no matter what $N\in\{2,3,4\}$ is. Since $C_2(\alpha)\to C_2(0)$ as $\alpha\to 0$, (6.30) together with (6.29) implies (6.21).

It remains to prove (6.30). Suppose the contrary. Then there is a constant $c\neq 0$ such that $\Gamma\Lambda(0)^{-1}w_1+c\Gamma\Lambda(0)^{-1}w_2=0$. We consider (R_0^0,Q_0^0) defined by (6.6) with $(\lambda,\alpha)=(0,0)$, and set $v_j=R_0^0\Lambda(0)^{-1}w_j$ and $\sigma_j=Q_0^0\Lambda(0)^{-1}w_j$ $(j=1,2)$. Then (6.27) implies

$$-\Delta v_j+\nabla\sigma_j=(1+T_0^0)\Lambda(0)^{-1}w_j=w_j-\langle e_1,\Lambda(0)^{-1}w_j\rangle m_1-\langle e_2,\Lambda(0)^{-1}w_j\rangle m_2$$

subject to $\operatorname{div} v_j=0$ and $v_j|_{\partial\Omega}=0$. Therefore, $v=v_1+cv_2$ and $\sigma=\sigma_1+c\sigma_2$ obey

$$-\Delta v + \nabla \sigma = w_1 + cw_2$$

subject to div $v = 0$ and $v|_{\partial\Omega} = 0$. For $|x| \geq R_0 + 1$, we find

$$v = A_0^0 \Lambda(0)^{-1}(w_1 + cw_2) = E_0^0 * [\Lambda(0)^{-1}(w_1 + cw_2)] = O(|x|^{-1}),$$

see (4.8), because $\Gamma\Lambda(0)^{-1}(w_1 + cw_2) = \int_{\mathbb{R}^2} \Lambda(0)^{-1}(w_1 + cw_2) = 0$. Set

$$g(x) = \frac{1}{4\pi}\left\{ -\psi \begin{pmatrix} 1 \\ c \end{pmatrix} + B[\partial_1 \psi + c\partial_2 \psi] \right\}.$$

Since div $g = 0$, the pair of $u = v + g$ and σ solves (6.22) with the constant vector $u_* = (-1/4\pi)\binom{1}{c}$, however, solutions cannot exist by Lemma 6.3. We have completed the proof. □

Remark 6.2. It is reasonable to expect that $\liminf_{\alpha\to 0} \|(1+T_0^\alpha)^{-1}\|_{\mathcal{L}(L^q_{[R]}(\Omega))} = 0$ because of nonexistence of $(1 + T_0^0)^{-1}$. The results of [**8**] and [**30**] suggest that $\|(1 + T_0^\alpha)^{-1}\|_{\mathcal{L}(L^q_{[R]}(\Omega))} = O(|\log \alpha|^{-1})$ (however, I could not find the proof).

Proof of Theorem 6.1. We now regard $1 + T_\lambda^\alpha$ as

$$1 + T_\lambda^\alpha = \left[1 + (T_\lambda^\alpha - T_0^\alpha)(1 + T_0^\alpha)^{-1}\right](1 + T_0^\alpha).$$

Since

$$\begin{aligned} \|(T_\lambda^\alpha - T_0^\alpha)f\|_q &\leq C\|(A_\lambda^\alpha - A_0^\alpha)f\|_{W^{1,q}(D_0)} + C\|(M_\lambda^\alpha - M_0^\alpha)f\|_{W^{1,q}(D_0)} \\ &\quad + C|\lambda|\|A_\lambda^\alpha f\|_{L^q(D_0)} + C|\lambda|\|M_\lambda^\alpha f\|_{L^q(D_0)} + C\|(\widetilde{N}_\lambda^\alpha - \widetilde{N}_0^\alpha)f\|_{L^q(D_0)} \end{aligned}$$

for $f \in L^q_{[R]}(\Omega)$ and $\alpha \in (0, M]$, we collect (4.28), (4.16) (with $R = R_0 + 1$), (5.7) with $\lambda_0 = 0$, (6.7) and (5.5) together with Proposition 6.2 to find a constant $\delta > 0$ such that if $\lambda \in \overline{\mathbb{C}_+}$ satisfies $|\lambda| \leq \delta\alpha^2$ as well as $\alpha \in (0, M]$, then

$$\|(T_\lambda^\alpha - T_0^\alpha)(1 + T_0^\alpha)^{-1}\|_{\mathcal{L}(L^q_{[R]}(\Omega))} \leq \frac{1}{2}$$

which yields the representation

$$\begin{aligned} (1 + T_\lambda^\alpha)^{-1} &= (1 + T_0^\alpha)^{-1}\left[1 + (T_\lambda^\alpha - T_0^\alpha)(1 + T_0^\alpha)^{-1}\right]^{-1} \\ &= (1 + T_0^\alpha)^{-1}\sum_{k=0}^{\infty}\left\{ -(T_\lambda^\alpha - T_0^\alpha)(1 + T_0^\alpha)^{-1}\right\}^k. \end{aligned}$$

This combined with (6.21) implies (6.19). Finally, the desired estimate follows from (6.19), (4.16) and (5.5). This completes the proof. □

7. Local energy decay.

In this section we will prove Theorem 2.1 and Theorem 2.2. The proof of Theorem 2.1 is based on the representation

$$\partial_\tau (i\tau+L)^{-1}Pf = (\partial_\tau R^\alpha_{i\tau})(1+T^\alpha_{i\tau})^{-1}Pf - R^\alpha_{i\tau}\,(1+T^\alpha_{i\tau})^{-1}(\partial_\tau T^\alpha_{i\tau})(1+T^\alpha_{i\tau})^{-1}Pf \tag{7.1}$$

and (7.18) below, which follow from (6.18). By Proposition 4.1, (5.5) and (6.19) we find

$$\|\partial_\tau (i\tau+L)^{-1}Pf\|_{W^{1,q}(\Omega_R)} \le C\left(\left|\log\frac{1}{|\tau|+\alpha^2}\right|+1\right)\frac{1}{\alpha^2}\log\left(\frac{\alpha^2}{|\tau|}+\sqrt{1+\frac{\alpha^4}{\tau^2}}\right)\|f\|_q \tag{7.2}$$

for $\alpha\in(0,M]$, $0<|\tau|\le\delta\alpha^2$ and $f\in L^q_{[R]}(\Omega)$. In fact, the worst part arises from the product of (4.16) and (4.17). As a consequence, we have the following lemma.

Lemma 7.1. *Let $1<q<\infty$, $R\ge R_0+1$ and $M>0$. Suppose δ is the constant as in Theorem 6.1. Then there is a constant $C=C(M;\Omega,q,R)>0$ such that*

$$\int_{|\tau|\le\delta\alpha^2}\|\partial_\tau (i\tau+L)^{-1}Pf\|_{W^{1,q}(\Omega_R)}\,d\tau \le C(|\log\alpha|+1)\|f\|_q \tag{7.3}$$

for $f\in L^q_{[R]}(\Omega)$ and $\alpha\in(0,M]$.

Due to Lemma 7.1 the representation (6.1) of the semigroup makes sense in $W^{1,q}(\Omega_R)$ for $f\in L^q_{[R]}(\Omega)$.

On the other hand, (2.6) implies that there is a constant $c_1=c_1(\Omega,q)>0$ independent of α such that

$$\|\partial_\tau (i\tau+L)^{-1}Pf\|_{W^{1,q}(\Omega)} = \|(i\tau+L)^{-2}Pf\|_{W^{1,q}(\Omega)} \le c_1|\tau|^{-3/2}\|f\|_q \tag{7.4}$$

for $|\tau|\ge\max\{c_0\alpha^2,\,1\}$ and $f\in L^q(\Omega)$, where $c_0=c_0(\pi/2,\Omega,q)>0$ is the constant for which (2.6) holds as long as $|\lambda|\ge\max\{c_0\alpha^2,\,1\}$ and $\lambda\in\overline{\mathbb{C}_+}$. We thus have (7.4) for $|\tau|\ge 2$ when $\alpha\le\sqrt{2/c_0}$. In the subsequent argument it suffices to consider the case when $\alpha>0$ is small.

It remains to estimate the integral for $\delta\alpha^2\le|\tau|\le 2$, which is worse than (7.3). In fact, we cannot use any knowledge from spectral analysis and are forced to employ L^2-theory.

Lemma 7.2. *Let $1<q<\infty$ and $R\ge R_0+1$. Given $\delta>0$, there is a constant $C=C(\Omega,q,R,\delta)>0$ such that*

$$\int_{\delta\alpha^2\le|\tau|\le 2}\|\partial_\tau (i\tau+L)^{-1}Pf\|_{W^{1,q}(\Omega_R)}\,d\tau \le \frac{C}{\alpha}\,\|f\|_2 \tag{7.5}$$

for $f \in L^2(\Omega)$ *and* $\alpha \in (0, \sqrt{2/\delta})$.

Proof. We employ the standard energy method. Let $\lambda = i\tau \,(\in i\mathbb{R})$ and $f \in L^2(\Omega)$. We write (6.2) as $(i\tau + L)u = (i\tau + A - 2\alpha P\partial_1)u = Pf$ in $L^2_\sigma(\Omega)$, where $A = -P\Delta$ denotes the Stokes operator. We take the scalar product in $L^2(\Omega)$ with u to obtain

$$\|\nabla u\|_2^2 \le \|f\|_2\|u\|_2, \qquad |\tau|\|u\|_2 \le 2\alpha\|\partial_1 u\|_2 + \|f\|_2$$

from the real and imaginary parts. They imply

$$\|(i\tau + L)^{-1}Pf\|_2 \le 2\left(\frac{1}{|\tau|} + \frac{2\alpha^2}{\tau^2}\right)\|f\|_2 \tag{7.6}$$

which was already observed in [**2**]. We also have

$$\|\nabla(i\tau + L)^{-1}Pf\|_2 \le \left\{2\left(\frac{1}{|\tau|} + \frac{2\alpha^2}{\tau^2}\right)\right\}^{1/2}\|f\|_2. \tag{7.7}$$

Let $g \in L^2_\sigma(\Omega)$. For $(i\tau + L)u = g$, we take the scalar product in $L^2(\Omega)$ with Au to get

$$\|Au\|_2^2 \le 8\alpha^2\|\partial_1 u\|_2^2 + 2\|g\|_2^2$$

which together with $\|\nabla^2 u\|_2 \le C(\|Au\|_2 + \|\nabla u\|_2)$, see [**21**], as well as (7.7) implies

$$\|\nabla^2(i\tau + L)^{-1}g\|_2 + \|\nabla(i\tau + L)^{-1}g\|_2 \le C\left(\frac{1}{|\tau|} + \frac{\alpha^2}{\tau^2}\right)^{1/2}\|g\|_2 \tag{7.8}$$

as long as $|\tau| \le 2$. In Ω_R we have the Poincaré inequality $\|u\|_{L^2(\Omega_R)} \le CR\|\nabla u\|_{L^2(\Omega_R)}$ since $u|_{\partial\Omega} = 0$, by which combined with the Sobolev inequality in 2D we find

$$\begin{aligned}\|\partial_\tau(i\tau + L)^{-1}Pf\|_{W^{1,q}(\Omega_R)} &\le C\|(i\tau + L)^{-2}Pf\|_{H^2(\Omega_R)} \\ &\le C\big(\|\nabla^2(i\tau + L)^{-2}Pf\|_2 + \|\nabla(i\tau + L)^{-2}Pf\|_2\big)\end{aligned} \tag{7.9}$$

for $f \in L^2(\Omega)$. We collect (7.6), (7.8) with $g = (i\tau + L)^{-1}Pf$ and (7.9) to conclude

$$\|\partial_\tau(i\tau + L)^{-1}Pf\|_{W^{1,q}(\Omega_R)} \le C\left(\frac{1}{|\tau|} + \frac{\alpha^2}{\tau^2}\right)^{3/2}\|f\|_2 \tag{7.10}$$

for $|\tau| \le 2$, which leads us to (7.5). □

It follows from Lemma 7.1, (7.4) and Lemma 7.2 that

$$\int_{-\infty}^{\infty}\|\partial_\tau(i\tau + L)^{-1}Pf\|_{W^{1,q}(\Omega_R)}\,d\tau \le \frac{C}{\alpha}\,\|f\|_q \tag{7.11}$$

for $q \in [2,\infty)$ and $f \in L^q_{[R]}(\Omega)$, and that

$$\int_{-\infty}^{\infty} \|\partial_\tau (i\tau + L)^{-1} Pf\|_{W^{1,q}(\Omega_R)}\, d\tau \le \frac{C}{\alpha}\, \|f\|_2$$

for $q \in (1,2)$ and $f \in L^2_{[R]}(\Omega)$. As a consequence, we obtain (2.9) ($q \ge 2$) and (2.10) ($q < 2$), respectively, with $\theta = 0$.

We take a cut-off function $\eta \in C^\infty(\mathbb{R};[0,1])$ such that $\eta(\tau) = 1$ for $|\tau| \le 1$ and $\eta(\tau) = 0$ for $|\tau| \ge 2$, and divide the integral (6.1) into

$$\frac{-1}{2\pi i t} \int_{-\infty}^{\infty} e^{i\tau t}\, \eta(\tau)\partial_\tau (i\tau + L)^{-1} Pf\, d\tau \tag{7.12}$$

and the other part which decays like t^{-2} by integration by parts once more since (2.6) yields

$$\|\partial_\tau^2 (i\tau + L)^{-1} Pf\|_{W^{1,q}(\Omega)} = 2\|(i\tau + L)^{-3} Pf\|_{W^{1,q}(\Omega)} \le c|\tau|^{-5/2}\|f\|_q$$

for $|\tau| \ge \max\{c_0\alpha^2, 1\}$ (thus, for $|\tau| \ge 1$ when $\alpha \le \sqrt{1/c_0}$) and $f \in L^q(\Omega)$, where c_0 is as in (7.4). For notational simplicity, we set

$$v(\tau) = \eta(\tau) w(\tau), \qquad w(\tau) = \partial_\tau (i\tau + L)^{-1} Pf. \tag{7.13}$$

For further decay of (7.12), we use the following lemma with $E = W^{1,q}(\Omega_R)$, which tells us a relation between the modulus of continuity of v and the rate of decay of the (inverse) Fourier transform of v.

Lemma 7.3. *Let E be a Banach space with norm $\|\cdot\|$ and $v \in L^1(\mathbb{R}; E)$. Then*

$$V(t) = \int_{-\infty}^{\infty} e^{i\tau t}\, v(\tau)\, d\tau$$

enjoys

$$\|V(t)\| \le C \int_{-\infty}^{\infty} \left\| v\left(\tau + \frac{1}{t}\right) - v(\tau) \right\| d\tau \tag{7.14}$$

for $t \in \mathbb{R} \setminus \{0\}$.

Proof. The proof is very simple by using the representation

$$V(t) = \frac{e^{iht}}{1 - e^{iht}} \int_{-\infty}^{\infty} e^{i\tau t} \big(v(\tau + h) - v(\tau) \big)\, d\tau$$

for $h \in \mathbb{R}$ satisfying $ht \ne 0$ (mod. 2π). Given $t \ne 0$, we take $h = 1/t$ to obtain (7.14). $\square$

REMARK 7.1. The idea of the proof above is found first in [**22**], however, the origin of this lemma goes back to Shibata [**31**]. But estimate of the form (7.14) is more straightforward for applications when $h \mapsto \int_{-\infty}^{\infty} \|v(\tau+h)-v(\tau)\| d\tau$ is not simple (this is just the case in Lemma 7.4 below).

In order to apply (7.14) to (7.12), we show the following lemma on further regularity of the resolvent near $\tau = 0$ along the imaginary axis.

LEMMA 7.4. *Suppose $v(\tau)$ is the function given by* (7.13). *Let $2 \le q < \infty$, $R \ge R_0 + 1$ and $M > 0$. Then there is a constant $C = C(M;\Omega,q,R) > 0$ such that*

$$\int_{-\infty}^{\infty} \|v(\tau+h)-v(\tau)\|_{W^{1,q}(\Omega_R)}\, d\tau \le C\left(\frac{|\log\alpha|+1}{\alpha^2}\left|\log\frac{\alpha^2}{h}\right| + \frac{1}{\alpha^3}\right) h\, \|f\|_q \tag{7.15}$$

for $f \in L^q_{[R]}(\Omega)$, $\alpha \in (0,M]$ and $h \in (0,1/2]$.

PROOF. Let δ be the constant as in Theorem 6.1. Suppose $h \in (0, \delta\alpha^2/3)$ and consider

$$\int_{-\infty}^{\infty} \eta(\tau)\|w(\tau+h)-w(\tau)\|_{W^{1,q}(\Omega_R)}\, d\tau = \int_{|\tau|\le 2h} + \int_{|\tau|>2h}.$$

It follows from (7.2) that

$$\begin{aligned}
&\int_{|\tau|\le 2h} \eta(\tau)\|w(\tau+h)-w(\tau)\|_{W^{1,q}(\Omega_R)}\, d\tau \\
&\quad \le 2\int_{|\tau|\le 3h} \|w(\tau)\|_{W^{1,q}(\Omega_R)}\, d\tau \\
&\quad \le C\bigg[(|\log\alpha|+1)\int_0^{3h/\alpha^2} \log\left(\frac{1}{\sigma}+\sqrt{1+\frac{1}{\sigma^2}}\right) d\sigma \\
&\qquad\quad + \int_0^{3h/\alpha^2} \log(1+\sigma)\,\log\left(\frac{1}{\sigma}+\sqrt{1+\frac{1}{\sigma^2}}\right) d\sigma\bigg]\|f\|_q.
\end{aligned}$$

Since $3h/\alpha^2 < \delta$, the dominant integral is the first one. Although it is exactly calculated as in (4.23), it is easier to find

$$\begin{aligned}
\int_0^{3h/\alpha^2} \log\left(\frac{1}{\sigma}+\sqrt{1+\frac{1}{\sigma^2}}\right) d\sigma &\le \int_0^{3h/\alpha^2} \log\left(1+\frac{2}{\sigma}\right) d\sigma \\
&= \frac{3h}{\alpha^2}\log\left(1+\frac{2\alpha^2}{3h}\right) + 2\log\left(1+\frac{3h}{2\alpha^2}\right).
\end{aligned}$$

Since $2\log(1+3h/2\alpha^2) \le 3h/\alpha^2$, we obtain

$$\int_{|\tau|\le 2h} \eta(\tau)\|w(\tau+h)-w(\tau)\|_{W^{1,q}(\Omega_R)}\,d\tau$$
$$\le \frac{C\big(|\log\alpha|+1\big)\,h}{\alpha^2}\left\{\log\left(1+\frac{2\alpha^2}{3h}\right)+1\right\}\|f\|_q \tag{7.16}$$

for $f\in L^q_{[R]}(\Omega)$. To estimate the other integral we use

$$\|w(\tau+h)-w(\tau)\|_{W^{1,q}(\Omega_R)} \le \int_\tau^{\tau+h} \|\partial_s^2(is+L)^{-1}Pf\|_{W^{1,q}(\Omega_R)}\,ds. \tag{7.17}$$

Let $2h<|\tau|\le 2\delta\alpha^2/3$, so that $|\tau|/2<|\tau+h|<\delta\alpha^2$. Then, for $s\in(\tau,\tau+h)$, we have the relation

$$\begin{aligned}
&\partial_s^2(is+L)^{-1}Pf\\
&\quad=(\partial_s^2R^\alpha_{is})(1+T^\alpha_{is})^{-1}Pf-2\,(\partial_sR^\alpha_{is})(1+T^\alpha_{is})^{-1}(\partial_sT^\alpha_{is})(1+T^\alpha_{is})^{-1}Pf\\
&\qquad+2\,R^\alpha_{is}\,(1+T^\alpha_{is})^{-1}(\partial_sT^\alpha_{is})(1+T^\alpha_{is})^{-1}(\partial_sT^\alpha_{is})(1+T^\alpha_{is})^{-1}Pf\\
&\qquad-R^\alpha_{is}\,(1+T^\alpha_{is})^{-1}(\partial_s^2T^\alpha_{is})(1+T^\alpha_{is})^{-1}Pf
\end{aligned} \tag{7.18}$$

as well as (7.1); thus, Proposition 4.1, (5.5) combined with (6.19) implies that

$$\|\partial_s^2(is+L)^{-1}Pf\|_{W^{1,q}(\Omega_R)}\le C\left(\left|\log\frac{1}{|s|+\alpha^2}\right|+1\right)\frac{1}{|s|\,(|s|+\alpha^2)}\|f\|_q$$

for $f\in L^q_{[R]}(\Omega)$. Note that

$$\frac{1}{\alpha^4}\left\{\log\left(\frac{\alpha^2}{|s|}+\sqrt{1+\frac{\alpha^4}{s^2}}\right)\right\}^2\le\frac{c_1}{|s|(|s|+\alpha^2)}$$

since $\big\{\log\big(1/t+\sqrt{1+1/t^2}\big)\big\}^2\le c_1/t(t+1)$ for all $t>0$. Let $\tau>2h$. Then we have by (7.17)

$$\begin{aligned}
&\int_{2h}^{2\delta\alpha^2/3}\eta(\tau)\|w(\tau+h)-w(\tau)\|_{W^{1,q}(\Omega_R)}\,d\tau\\
&\quad\le\left[\frac{C\big(|\log\alpha|+1\big)}{\alpha^2}\int_{2h}^{2\delta\alpha^2/3}\int_{\tau/\alpha^2}^{(\tau+h)/\alpha^2}\frac{1}{\sigma(\sigma+1)}\,d\sigma\,d\tau\right.\\
&\qquad\left.+\frac{C}{\alpha^2}\int_{2h}^{2\delta\alpha^2/3}\int_{\tau/\alpha^2}^{(\tau+h)/\alpha^2}\frac{\log(1+\sigma)}{\sigma(\sigma+1)}\,d\sigma\,d\tau\right]\|f\|_q.
\end{aligned}$$

The dominant integral is the first one and it is estimated as

$$\int_{2h}^{2\delta\alpha^2/3}\int_{\tau/\alpha^2}^{(\tau+h)/\alpha^2}\frac{1}{\sigma(\sigma+1)}\,d\sigma\,d\tau\le h\int_{2h}^{2\delta\alpha^2/3}\frac{d\tau}{\tau}=h\log\frac{\delta\alpha^2}{3h}.$$

The case $\tau<-2h$ is also discussed similarly. We thus obtain

$$\int_{2h<|\tau|\le 2\delta\alpha^2/3}\eta(\tau)\|w(\tau+h)-w(\tau)\|_{W^{1,q}(\Omega_R)}\,d\tau\le\frac{C\big(|\log\alpha|+1\big)\,h}{\alpha^2}\,\log\frac{\delta\alpha^2}{3h}\,\|f\|_q \tag{7.19}$$

for $f\in L^q_{[R]}(\Omega)$. It remains to estimate of the integral over $2\delta\alpha^2/3<|\tau|\le 2$, in which we have no longer (7.18). We are thus forced to rely on L^2-theory an in Lemma 7.2; indeed, by (7.6), (7.8) with $g=(i\tau+L)^{-2}Pf$ and (7.9) in which $(i\tau+L)^{-2}$ should be replaced by $(i\tau+L)^{-3}$ we have

$$\|\partial_\tau^2(i\tau+L)^{-1}Pf\|_{W^{1,q}(\Omega_R)}\le C\left(\frac{1}{|\tau|}+\frac{\alpha^2}{\tau^2}\right)^{5/2}\|f\|_2$$

for $|\tau|\le 2$ and $f\in L^2(\Omega)$. Consequently,

$$\int_{2\delta\alpha^2/3<|\tau|\le 2}\eta(\tau)\|w(\tau+h)-w(\tau)\|_{W^{1,q}(\Omega_R)}\,d\tau\le\frac{Ch}{\alpha^3}\,\|f\|_2 \tag{7.20}$$

for $f\in L^2(\Omega)$. Collecting (7.16), (7.19) and (7.20) leads us to

$$\int_{-\infty}^{\infty}\eta(\tau)\|w(\tau+h)-w(\tau)\|_{W^{1,q}(\Omega_R)}\,d\tau\le C\left\{\frac{|\log\alpha|+1}{\alpha^2}\left|\log\frac{\alpha^2}{h}\right|+\frac{1}{\alpha^3}\right\}h\|f\|_q \tag{7.21}$$

for $q\in[2,\infty)$, $f\in L^q_{[R]}(\Omega)$ and $h\in(0,\delta\alpha^2/3)$. When $\delta\alpha^2/3\le h\le 1/2$ (that is not the important case), one needs a small modification. In fact, by (7.3) and (7.10) we replace (7.16) by

$$\int_{|\tau|\le 3h}\|w(\tau)\|_{W^{1,q}(\Omega_R)}\,d\tau=\int_{|\tau|\le\delta\alpha^2}+\int_{\delta\alpha^2\le|\tau|\le 3h}\le C\big(|\log\alpha|+1\big)\|f\|_q+\frac{C}{\alpha}\,\|f\|_2$$

for $q\in[2,\infty)$ and $f\in L^q_{[R]}(\Omega)$, but the RHS can be written as $(Ch/\alpha^3)\|f\|_q$ because of $h\ge\delta\alpha^2/3$. And also we don't have (7.19), but instead we see

$$\int_{|\tau|>2h}\eta(\tau)\|w(\tau+h)-w(\tau)\|_{W^{1,q}(\Omega_R)}\,d\tau\le\int_{2\delta\alpha^2/3<|\tau|\le 2}\le\frac{Ch}{\alpha^3}\,\|f\|_2$$

as in (7.20). After all, we have (7.21) even for $h\in(0,1/2]$. On the other hand, from (7.11) we obtain

$$\int_{-\infty}^{\infty}|\eta(\tau+h)-\eta(\tau)|\,\|w(\tau+h)\|_{W^{1,q}(\Omega_R)}\,d\tau\le Ch\int_{-\infty}^{\infty}\|w(\tau)\|_{W^{1,q}(\Omega_R)}\,d\tau\le\frac{Ch}{\alpha}\,\|f\|_q$$

which combined with (7.21) implies (7.15). This completes the proof. □

We are now in a position to show Theorem 2.1 and Theorem 2.2.

Proof of Theorem 2.1. In view of (6.1) and (7.11), estimate (2.9) with $\theta = 0$ has been already proved. Given $t \geq 2$, we take $h = 1/t$ in Lemma 7.4. We then apply Lemma 7.3 to (7.12) to accomplish the proof of (2.9) with $\theta = 1$. □

Proof of Theorem 2.2. Given $f \in L^q_\sigma(\Omega)$, we set $g = e^{-L}f \in D(L) \subset W^{2,q}(\Omega)$ and consider the decay property of $v(t) := e^{-tL}g = e^{-(t+1)L}f$ for $t \geq 1$. Let ψ be the same cut-off function as in (6.4) and B the Bogovskii operator in the annulus (6.3). Set

$$\widetilde{g} = (1-\psi)g + B[g \cdot \nabla\psi],$$

then it follows from (6.5) and (2.4) that $\widetilde{g} \in W^{2,q}(\mathbb{R}^2)$ and $\operatorname{div} \widetilde{g} = 0$ with

$$\|\widetilde{g}\|_{W^{2,q}(\mathbb{R}^2)} \leq C\|g\|_{W^{2,q}(\Omega)} \leq C\big(\|Lg\|_q + \|g\|_q\big) \leq C\|f\|_q. \tag{7.22}$$

Let $U(t) = U_\alpha(t)$ be the Oseen semigroup in the whole plane $\mathbb{R}^2$ given by (3.1). We need another cut-off function $\phi \in C^\infty(\mathbb{R}^2; [0,1])$ such that $\phi(x) = 0$ for $|x| \geq R_0$ and $\phi(x) = 1$ in a neighborhood of the obstacle $\mathbb{R}^2 \setminus \Omega$. By B' we denote the Bogovskii operator in the bounded domain Ω_{R_0}. We set

$$w(t) = (1-\phi)U(t)\widetilde{g} + B'[(U(t)\widetilde{g}) \cdot \nabla\phi].$$

Since $\operatorname{div} \widetilde{g} = 0$, we have $\operatorname{div} (U(t)\widetilde{g}) = 0$, so that $\operatorname{div} w(t) = 0$. We fix $R \geq R_0 + 1$. By (3.4) (with $r = q$) and (6.5) for B' (in which D_0 is replaced by Ω_{R_0}) together with (7.22), we have

$$\|w(t)\|_{W^{1,q}(\Omega_R)} \leq C\|U(t)\widetilde{g}\|_{W^{1,q}(B_R)} \leq C(1+t)^{-1/q}\|\widetilde{g}\|_{W^{2,q}(\mathbb{R}^2)} \leq C(1+t)^{-1/q}\|f\|_q$$

for $t > 0$. Our main task is thus to derive the decay property of the difference $u(t) := v(t) - w(t)$, which together with the pressure p associated with $v = e^{-tL}g$ satisfies

$$\begin{cases} \partial_t u - \Delta u - 2\alpha\partial_1 u + \nabla p = F, \quad \operatorname{div} u = 0 \quad \text{in } \Omega \times (0,\infty), \\ u|_{\partial\Omega} = 0, \\ u(\cdot, 0) = \psi g - B[g \cdot \nabla\psi] =: h \end{cases}$$

where

$$\begin{aligned} F &= -\partial_t w + \Delta w + 2\alpha\partial_1 w \\ &= -2\nabla\phi \cdot \nabla U(t)\widetilde{g} - (\Delta\phi + 2\alpha\partial_1\phi)U(t)\widetilde{g} - B'[(\partial_t U(t)\widetilde{g}) \cdot \nabla\phi] \\ &\quad + \Delta B'[(U(t)\widetilde{g}) \cdot \nabla\phi] + 2\alpha\partial_1 B'[(U(t)\widetilde{g}) \cdot \nabla\phi]. \end{aligned}$$

Set $\widetilde{q}=\max\{q,2\}$. Then $h\in L^{\widetilde{q}}_{[R]}(\Omega)$ ($h\in L^2_{[R]}(\Omega)$ by embedding even if $q<2$) and

$$\|h\|_{\widetilde{q}}\le C\|g\|_{L^{\widetilde{q}}(\Omega_R)}\le C\|g\|_{W^{1,q}(\Omega_R)}\le C\|e^{-L}f\|_{W^{1,q}(\Omega)}\le C\|f\|_q. \tag{7.23}$$

In view of Lemma 3.1 with $r=\widetilde{q}=\max\{q,2\}$ and (7.22) we observe $F(t)\in L^{\widetilde{q}}_{[R]}(\Omega)$ subject to

$$\begin{aligned}\|F(t)\|_{\widetilde{q}}&\le C\big(\|U(t)\widetilde{g}\|_{W^{1,\widetilde{q}}(B_{R_0})}+\|\partial_t U(t)\widetilde{g}\|_{L^{\widetilde{q}}(B_{R_0})}\big)\\&\le Ct^{-1/q+1/\widetilde{q}}(1+t)^{-1/\widetilde{q}}\|\widetilde{g}\|_{W^{2,q}(\mathbb{R}^2)}\\&\le Ct^{-1/q+1/\widetilde{q}}(1+t)^{-1/\widetilde{q}}\|f\|_q\end{aligned} \tag{7.24}$$

for $t>0$. Since div $F(t)=0$ and $F(t)|_{\partial\Omega}=0$, we have $F(t)\in L^{\widetilde{q}}_\sigma(\Omega)$ as well (so that $PF(t)=F(t)$). In order to estimate

$$u(t)=e^{-tL}h+\int_0^t e^{-(t-\tau)L}F(\tau)\,d\tau$$

we employ (2.12) together with (7.23) and (7.24) to find

$$\|e^{-tL}h\|_{W^{1,q}(\Omega_R)}\le\frac{C}{\alpha^{1+2\theta}}\,t^{-(1+\theta)}\big(\log(e+t)\big)^\theta\|f\|_q$$

and

$$\begin{aligned}&\int_0^t\|e^{-(t-\tau)L}F(\tau)\|_{W^{1,q}(\Omega_R)}d\tau\\&\quad\le\frac{C\|f\|_q}{\alpha^{1+2\theta}}\int_0^t(t-\tau)^{-1/2}(e+t-\tau)^{-(1/2+\theta)}\big(\log(e+t-\tau)\big)^\theta\tau^{-1/q+1/\widetilde{q}}(1+\tau)^{-1/\widetilde{q}}\,d\tau\\&\quad=\frac{C\|f\|_q}{\alpha^{1+2\theta}}\,(I_1+I_2)\end{aligned}$$

where $I_1=\int_0^{t/2}$ and $I_2=\int_{t/2}^t$. We easily see that

$$\begin{aligned}I_1&\le(t/2)^{-1/2}(e+t/2)^{-(1/2+\theta)}\big(\log(e+t)\big)^\theta\int_0^{t/2}\tau^{-1/q+1/\widetilde{q}}(1+\tau)^{-1/\widetilde{q}}\,d\tau\\&\le Ct^{-(1/q+\theta)}\big(\log(e+t)\big)^\theta\end{aligned}$$

and that

$$
\begin{aligned}
I_2 &\leq (t/2)^{-1/q} \int_{t/2}^{t} (t-\tau)^{-1/2}(e+t-\tau)^{-(1/2+\theta)}\big(\log(e+t-\tau)\big)^{\theta}\,d\tau \\
&\leq (t/2)^{-1/q} \int_{0}^{\infty} \sigma^{-1/2}(e+\sigma)^{-(1/2+\theta)}\big(\log(e+\sigma)\big)^{\theta}\,d\sigma = Ct^{-1/q}
\end{aligned}
$$

as long as $\theta > 0$. The proof is complete. □

8. L^q-L^r estimate.

In this section we conclude the paper with the proof of L^q-L^r estimate of the Oseen semigroup by using local energy decay properties.

Proof of Theorem 2.3. We fix $R \geq R_0 + 1$. By virtue of Theorem 2.2, our main task is to deduce the decay property in the other region $\Omega \setminus \Omega_R$. Given $f \in L^q_\sigma(\Omega)$, $1 < q < \infty$, as in the proof of Theorem 2.2, we set $g = e^{-L}f$ and $v(t) = e^{-tL}g = e^{-(t+1)L}f$. We denote by p the pressure associated with v satisfying $\int_{D_0} p = 0$, where D_0 is the annulus given by (6.3). Then we have the inequality ([**17**, Chapter III], [**22**, Remark 4.1])

$$
\|p(t)\|_{L^q(D_0)} \leq C\|\nabla p(t)\|_{W^{-1,q}(D_0)} \tag{8.1}
$$

where $W^{-1,q}(D_0)$ denotes the dual space of $W_0^{1,q/(q-1)}(D_0)$, that is the completion of $C_0^\infty(D_0)$ in $W^{1,q/(q-1)}(D_0)$. Let us consider

$$
w(t) = (1-\psi)v(t) + B[v(t)\cdot\nabla\psi], \qquad \sigma(t) = (1-\psi)p(t)
$$

where ψ is the cut-off function as in (6.4) and B denotes the Bogovskii operator in D_0. Since $w = v$ for $|x| \geq R$, we have only to estimate w for $t > 0$. Since (w, σ) obeys

$$
\begin{cases}
\partial_t w - \Delta w - 2\alpha\partial_1 w + \nabla\sigma = K, & \operatorname{div} w = 0 \quad \text{in } \mathbb{R}^2 \times (0,\infty), \\
w(\cdot,0) = (1-\psi)g + B[g\cdot\nabla\psi] =: \widetilde{g}
\end{cases}
$$

where

$$
\begin{aligned}
K = {} & 2\nabla\psi\cdot\nabla v + (\Delta\psi + 2\alpha\partial_1\psi)v + B[(\partial_t v)\cdot\nabla\psi] \\
& - \Delta B[v\cdot\nabla\psi] - 2\alpha\partial_1 B[v\cdot\nabla\psi] - (\nabla\psi)p,
\end{aligned}
$$

we are going to estimate

$$
w(t) = U(t)\widetilde{g} + \int_0^t U(t-\tau)P_{\mathbb{R}^2}K(\tau)\,d\tau
$$

where $U(t) = U_\alpha(t)$ is the semigroup in the whole plane $\mathbb{R}^2$ given by (3.1). By (7.22) and (3.3) we obtain

$$\|\nabla^j U(t)\widetilde{g}\|_{L^r(\mathbb{R}^2)} \le Ct^{-j/2-1/q+1/r}\|\widetilde{g}\|_{L^q(\mathbb{R}^2)} \le Ct^{-j/2-1/q+1/r}\|f\|_q \tag{8.2}$$

for $t > 0$, $r \in [q, \infty]$ and $j = 0, 1$. We fix both $M > 0$ and $\varepsilon > 0$, and put $c_\alpha = C/\alpha^{1+\varepsilon}$ for simplicity of notation, where $C = C(M, \varepsilon)$ is the constant in (2.14). By (8.1) we observe

$$\begin{aligned}\|p(t)\|_{L^q(D_0)} &\le C\|\partial_t v(t) - \Delta v(t) - 2\alpha\partial_1 v(t)\|_{W^{-1,q}(D_0)} \\ &\le C\|\partial_t v(t)\|_{L^q(D_0)} + C\|v(t)\|_{W^{1,q}(D_0)}\end{aligned}$$

which combined with (2.14) implies

$$\begin{aligned}\|K(t)\|_{L^1(\mathbb{R}^2)} + \|K(t)\|_{L^q(\mathbb{R}^2)} &\le C\|K(t)\|_{L^q(D_0)} \le C\|v(t)\|_{W^{1,q}(\Omega_R)} + C\|\partial_t v(t)\|_{L^q(\Omega_R)} \\ &\le Cc_\alpha(1+t)^{-1/q}\big(\|Lg\|_q + \|g\|_q\big) \\ &\le Cc_\alpha(1+t)^{-1/q}\|f\|_q.\end{aligned} \tag{8.3}$$

It follows from (3.3), (3.5) and (8.3) that

$$\begin{aligned}&\int_0^t \|\nabla^j U(t-\tau)P_{\mathbb{R}^2}K(\tau)\|_{L^r(\mathbb{R}^2)}\, d\tau \\ &\le Cc_\alpha\|f\|_q \int_0^t (t-\tau)^{-j/2-1/q+1/r}(1+t-\tau)^{-1+1/q}(1+\tau)^{-1/q}\, d\tau \\ &=: Cc_\alpha\|f\|_q\big(I_1^{(j)} + I_2^{(j)}\big)\end{aligned} \tag{8.4}$$

for all $t > 0$ and $j = 0, 1$, where $I_1^{(j)} = \int_0^{t/2}$ and $I_2^{(j)} = \int_{t/2}^t$. Then

$$I_1^{(j)} \le Ct^{-j/2-1+1/r}\int_0^{t/2}(1+\tau)^{-1/q}\, d\tau \le Ct^{-j/2-1/q+1/r} \tag{8.5}$$

for $t > 0$ and $r \in [q, \infty]$, while

$$\begin{aligned}I_2^{(j)} &\le Ct^{-1/q}\int_{t/2}^t (t-\tau)^{-j/2-1/q+1/r}(1+t-\tau)^{-1+1/q}\, d\tau \\ &= Ct^{-1/q}\int_0^{t/2}\sigma^{-j/2-1/q+1/r}(1+\sigma)^{-1+1/q}\, d\sigma\end{aligned}$$

which yields

$$I_2^{(0)} \le \begin{cases} Ct^{-1/q+1/r}, & 1 < q \le r < \infty, \\ Ct^{-1/q}\log(e+t), & 1 < q < r = \infty \end{cases} \tag{8.6}$$

and

$$I_2^{(1)} \le \begin{cases} Ct^{-1/2-1/q+1/r}, & 1 < q \le r < 2, \\ Ct^{-1/q} \log(e+t), & 1 < q \le r = 2 \end{cases} \tag{8.7}$$

for $t > 0$. We collect (8.2), (8.4), (8.5), (8.6) and (8.7) to obtain

$$\begin{aligned} \|e^{-(t+1)L} f\|_{L^r(\Omega \setminus \Omega_R)} &\le \|w(t)\|_{L^r(\mathbb{R}^2)} \\ &\le \begin{cases} Cc_\alpha t^{-1/q+1/r} \|f\|_q, & 1 < q \le r < \infty, \\ Cc_\alpha t^{-1/q} \big(\log(e+t)\big) \|f\|_q, & 1 < q < r = \infty \end{cases} \end{aligned} \tag{8.8}$$

and

$$\begin{aligned} \|\nabla e^{-(t+1)L} f\|_{L^r(\Omega \setminus \Omega_R)} &\le \|\nabla w(t)\|_{L^r(\mathbb{R}^2)} \\ &\le \begin{cases} Cc_\alpha t^{-1/2-1/q+1/r} \|f\|_q, & 1 < q \le r < 2, \\ Cc_\alpha t^{-1/q} \big(\log(e+t)\big) \|f\|_q, & 1 < q \le r = 2 \end{cases} \end{aligned} \tag{8.9}$$

for $t > 0$. We first consider the following cases:

$$\begin{aligned} &\text{case (i)} && 1 < q < 2,\ q \le r \le q_*, \\ &\text{case (ii)} && 2 \le q \le r < \infty, \\ &\text{case (iii)} && 2 < q < r = \infty \end{aligned}$$

where $1/q_* = 1/q - 1/2$. For those cases, it follows from Theorem 2.2 that

$$\|e^{-tL} f\|_{L^r(\Omega_R)} + \|\nabla e^{-tL} f\|_{L^q(\Omega_R)} \le Cc_\alpha t^{-1/q} \|f\|_q$$

for $t \ge 2$, which combined with (8.8) and (8.9) furnishes

$$\|e^{-tL} f\|_r \le \begin{cases} Cc_\alpha t^{-1/q+1/r} \|f\|_q, & \text{for case (i), case (ii),} \\ Cc_\alpha t^{-1/q} \big(\log(e+t)\big) \|f\|_q, & \text{for case (iii)} \end{cases} \tag{8.10}$$

and

$$\|\nabla e^{-tL} f\|_r \le \begin{cases} Cc_\alpha t^{-1/2} \|f\|_r, & 1 < r < 2, \\ Cc_\alpha t^{-1/2} \big(\log(e+t)\big) \|f\|_r, & r = 2 \end{cases} \tag{8.11}$$

for $t \ge 2$. Concerning the estimate for $0 < t < 2$, we employ the interpolation inequality with use of (2.4), (2.5) and (2.11) to obtain the desired estimate of $\|\nabla^j e^{-tL} f\|_r$ from above by $t^{-j/2-1/q+1/r} \|f\|_q$. Hence, we conclude (2.15) for case (i) and case (ii), (2.16) for case (iii), and (2.17) and (2.18) for $q = r$. All the other cases then follow from the semigroup property. The proof is complete. $\square$

REMARK 8.1. From the proof of (8.11), we observe that if $r > 2$, then $\|\nabla e^{-tL} f\|_r \leq C c_\alpha t^{-1/r} \|f\|_r$ for $t \geq 2$, which does not seem to be sharp. According to consideration by [**23**], there is no contradiction even if we have the optimal decay estimate $\|\nabla e^{-tL} f\|_r \leq C t^{-1/2} \|f\|_r$ for $t \geq 2$ and $r < 6$.

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