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An affirmative answer to a conjecture on the Metoki class

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To the memory of Professor Shoshichi Kobayashi at UC Berkeley

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Abstract. In [**6**], Kotschick and Morita showed that the Gel'fand–Kalinin–Fuks class in $\mathrm{H}^7_{\mathrm{GF}}(\mathfrak{ham}_2, \mathfrak{sp}(2,\mathbb{R}))_8$ is decomposed as a product $\eta \wedge \omega$ of some leaf cohomology class η and a transverse symplectic class ω. We show that the same formula holds for the Metoki class, which is a non-trivial element in $\mathrm{H}^9_{\mathrm{GF}}(\mathfrak{ham}_2, \mathfrak{sp}(2,\mathbb{R}))_{14}$. The result was conjectured in [**6**], where they studied characteristic classes of transversely symplectic foliations due to Kontsevich. Our proof depends on Gröbner Basis theory using computer calculations.

1. Introduction.

Let $\mathfrak{X}(M)$ be the Lie algebra of smooth vector fields of a smooth manifold M. $\mathrm{H}^\bullet_c(\mathfrak{X}(M))$ is the Lie algebra cohomology, where the subscript c means that each cochain is required to be *continuous*. The cohomology group $\mathrm{H}^\bullet_c(\mathfrak{X}(M))$ is often written as $\mathrm{H}^\bullet_{\mathrm{GF}}(M)$ and is called the Gel'fand-Fuks cohomology group of M. It is known that if M is of finite-type (i.e., M has a open cover such that each non-empty finite intersection of the member is diffeomorphic to n-dimensional open disk, where $n = \dim M$), then $\mathrm{H}^\bullet_{\mathrm{GF}}(M)$ is finite dimensional.

Let $\mathfrak{a}_n$ denote the Lie algebra of formal vector fields on $\mathbb{R}^n$, expressed as $\mathbb{R}[[x_1, \ldots, x_n]]\langle \partial/\partial x_1, \ldots, \partial/\partial x_n \rangle$ where $x_1, \ldots, x_n$ are the natural coordinates of $\mathbb{R}^n$. Thus an element of $\mathfrak{a}_n$ is a vector field with coefficients which are formal power series in the coordinate functions. Then $\mathrm{H}^\bullet_c(\mathfrak{a}_n) \cong \mathrm{H}^\bullet_{\mathrm{GF}}(\mathbb{R}^n)$ and so $\dim \mathrm{H}^\bullet_c(\mathfrak{a}_n)$ is finite.

Let $\mathfrak{v}_n$ be the subalgebra of $\mathfrak{a}_n$ consisting of the volume preserving formal vector fields on $\mathbb{R}^n$, and $\mathfrak{ham}_{2n}$ the subalgebra of $\mathfrak{a}_{2n}$ consisting of formal Hamiltonian vector fields on $\mathbb{R}^{2n}$. Then, the next question is still open: Is $\dim \mathrm{H}^\bullet_c(\mathfrak{v}_n)$ or $\dim \mathrm{H}^\bullet_c(\mathfrak{ham}_{2n})$ infinite?

There is a notion of *weight* for cochains of $\mathfrak{ham}_{2n}$. Since the weight is preserved by the coboundary operator, there is a cohomology subgroup corresponding to each weight (cf. Section 2.1(I-3)). In [**4**], for the weight $w \leq 0$, the structure of the relative cohomology $\mathrm{H}^\bullet_c(\mathfrak{ham}_{2n}, \mathfrak{sp}(2n,\mathbb{R}))_w$ is completely determined, and when $n = 1$ and $w > 0$, the following holds true:

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$$\mathrm{H}_c^\bullet(\mathfrak{ham}_2, \mathfrak{sp}(2,\mathbb{R}))_w = 0 \quad \text{for} \quad w = 1, 2, \ldots, 7$$

$$\mathrm{H}_c^m(\mathfrak{ham}_2, \mathfrak{sp}(2,\mathbb{R}))_8 = \begin{cases} \mathbb{R} & \text{if } m = 7 \\ 0 & \text{otherwise.} \end{cases}$$

The generator of $\mathrm{H}_c^7(\mathfrak{ham}_2, \mathfrak{sp}(2,\mathbb{R}))_8$ is called the Gel'fand–Kalinin–Fuks class. Hereafter, we use the notation $\mathrm{H}_{\mathrm{GF}}^\bullet(\mathfrak{ham}_{2n}, \mathfrak{sp}(2n,\mathbb{R}))_w$ instead of $\mathrm{H}_c^\bullet(\mathfrak{ham}_{2n}, \mathfrak{sp}(2n,\mathbb{R}))_w$.

There is a homomorphism from $\mathrm{H}_{\mathrm{GF}}^\bullet(\mathfrak{ham}_2, \mathfrak{sp}(2,\mathbb{R}))$ into $\mathrm{H}^\bullet(B\Gamma_2^{symp})$, where Γ_2^{symp} is the groupoid of germs of local diffeomorphisms of $\mathbb{R}^2$ preserving the symplectic structure of $\mathbb{R}^2$. It is not yet known whether the image of the Gel'fand–Kalinin–Fuks class by the homomorphism is trivial in $\mathrm{H}^7(B\Gamma_2^{symp})$ or not (cf. [**2**]).

The next non-trivial result in succession to the Gel'fand–Kalinin–Fuks class is $\mathrm{H}_{\mathrm{GF}}^9(\mathfrak{ham}_2, \mathfrak{sp}(2,\mathbb{R}))_{14} \cong \mathbb{R}$, which was shown by Metoki ([**7**]) in 1999. He was interested in the volume preserving formal vector fields; when $n = 1$ both $\mathfrak{ham}_2$ and $\mathfrak{v}_2$ are the same.

Let $\mathcal{F}$ be a foliation on a manifold M. We have the foliated cohomology defined by $\mathrm{H}_{\mathcal{F}}^\bullet(M,\mathbb{R}) := \mathrm{H}^\bullet(\Omega_{\mathcal{F}})$ where $\Omega_{\mathcal{F}} = \Omega(M)/\mathrm{I}(\mathcal{F})$, $\Omega(M)$ is the exterior algebra of differential forms on M, and $\mathrm{I}(\mathcal{F})$ is the ideal generated by $\{\sigma \in \Omega^1(M) \mid \langle \sigma, \mathrm{T}\mathcal{F}\rangle = 0\}$. Kontsevich ([**5**]) showed that if $\mathcal{F}$ is a codimension $2n$ foliation endowed with a symplectic form ω in the transverse direction, then there is a commutative diagram:

$$\begin{array}{ccc} \mathrm{H}_{\mathcal{F}}(M,\mathbb{R}) & \xrightarrow{\omega^n\wedge} & \mathrm{H}_{\mathrm{DR}}^{\bullet+2n}(M,\mathbb{R}) \\ \uparrow & & \uparrow \\ \mathrm{H}_{\mathrm{GF}}^\bullet(\mathfrak{ham}_{2n}^0, \mathfrak{sp}(2n,\mathbb{R})) & \xrightarrow{\omega^n\wedge} & \mathrm{H}_{\mathrm{GF}}^{\bullet+2n}(\mathfrak{ham}_{2n}, \mathfrak{sp}(2n,\mathbb{R})) \end{array} \tag{1}$$

where $\mathfrak{ham}_{2n}^0$ is the Lie subalgebra of the Hamiltonian vector fields of the formal polynomial vanishing at the origin of $\mathbb{R}^{2n}$.

Kotschick and Morita ([**6**]) determined the space $\mathrm{H}_{\mathrm{GF}}^\bullet(\mathfrak{ham}_2^0, \mathfrak{sp}(2,\mathbb{R}))_w$ for $w \leq 10$, and concerning Kontsevich homomorphism given in the bottom line of (1), they showed the following, as well as the non-triviality of Kontsevich homomorphism in the case $n = 1$:

THEOREM 1.1 ([**6**]). *There is a unique element $\eta \in \mathrm{H}_{\mathrm{GF}}^5(\mathfrak{ham}_2^0, \mathfrak{sp}(2,\mathbb{R}))_{10} \cong \mathbb{R}$ such that*

$$\textit{Gel'fand–Kalinin–Fuks class} = \eta \wedge \omega \in \mathrm{H}_{\mathrm{GF}}^7(\mathfrak{ham}_2, \mathfrak{sp}(2,\mathbb{R}))_8$$

where ω is the cochain associated with the linear symplectic form of $\mathbb{R}^2$.

Further they stated that it is highly likely that the same thing is true also for Metoki class $\in \mathrm{H}_{\mathrm{GF}}^9(\mathfrak{ham}_2, \mathfrak{sp}(2,\mathbb{R}))_{14}$. That is, there should exist an element $\eta' \in \mathrm{H}_{\mathrm{GF}}^7(\mathfrak{ham}_2^0, \mathfrak{sp}(2,\mathbb{R}))_{16}$ such that

$$\text{Metoki class} = \eta' \wedge \omega \in \mathrm{H}_{\mathrm{GF}}^9(\mathfrak{ham}_2, \mathfrak{sp}(2,\mathbb{R}))_{14}.$$

In the same line of Kotschick and Morita ([**6**]), we determined $\mathrm{H}^{\bullet}_{\mathrm{GF}}(\mathfrak{ham}^0_2, \mathfrak{sp}(2,\mathbb{R}))_w$ for $w \leq 20$ in [**13**]. In this paper, making use of information in [**13**], we will show the following theorem.

THEOREM 1.2. *$\mathrm{H}^9_{\mathrm{GF}}(\mathfrak{ham}_2, \mathfrak{sp}(2,\mathbb{R}))_{14}$ and $\mathrm{H}^7_{\mathrm{GF}}(\mathfrak{ham}^0_2, \mathfrak{sp}(2,\mathbb{R}))_{16}$ are both 1-dimensional and the map of wedging symplectic cocycle, i.e., Kontsevich homomorphism for $n = 1$*

$$\omega\wedge : \mathrm{H}^7_{\mathrm{GF}}(\mathfrak{ham}^0_2, \mathfrak{sp}(2,\mathbb{R}))_{16} \longrightarrow \mathrm{H}^9_{\mathrm{GF}}(\mathfrak{ham}_2, \mathfrak{sp}(2,\mathbb{R}))_{14}$$

is an isomorphism. Thus, there is a unique element $\eta' \in \mathrm{H}^7_{\mathrm{GF}}(\mathfrak{ham}^0_2, \mathfrak{sp}(2,\mathbb{R}))_{16} \cong \mathbb{R}$ such that

$$\textit{Metoki class } = \eta' \wedge \omega \in \mathrm{H}^9_{\mathrm{GF}}(\mathfrak{ham}_2, \mathfrak{sp}(2,\mathbb{R}))_{14}$$

where Metoki class is the generator of $\mathrm{H}^9_{\mathrm{GF}}(\mathfrak{ham}_2, \mathfrak{sp}(2,\mathbb{R}))_{14}$.

2. Preliminaries.

Generalities concerning the (relative) Gel'fand–Fuks cohomologies and symplectic formalism are found in Mikami–Nakae–Kodama's preprint ([**13**]). Here we review the concept of weight of cochain complex of our Lie algebras and the symplectic actions on relative complex and also the description of coboundary operators for further calculations. Although the space we are concerned with in this paper is $\mathbb{R}^2$, we review the notions on the general linear symplectic space $\mathbb{R}^{2n}$, and fix notations we use hereafter.

2.1. Symplectic space $\mathbb{R}^{2n}$.

We fix a linear symplectic manifold $(\mathbb{R}^{2n}, \omega)$ with the standard variables $x_1, x_2, \ldots, x_{2n}$. Let $\mathcal{H}_f$ denote the Hamiltonian vector field of f. Recalling the formula $[\mathcal{H}_f, \mathcal{H}_g] = -\mathcal{H}_{\{f,g\}}$ for Hamiltonian vector fields, we identify each formal Hamiltonian vector field with its potential polynomial function up to the constant term and the Lie bracket of vector fields with the Poisson bracket on polynomial functions. We denote by $\mathfrak{S}_p$ the dual space of homogeneous polynomials in $\{x_i\}$ of degree p. Then

$$\mathfrak{ham}_{2n} = \left(\bigoplus_{p=1}^{\infty} \mathfrak{S}^*_p\right)^{\wedge} \quad \text{is a Lie algebra}$$

and

$$\mathfrak{ham}^0_{2n} = \left(\bigoplus_{p=2}^{\infty} \mathfrak{S}^*_p\right)^{\wedge} \quad \text{is a subalgebra of } \mathfrak{ham}_{2n},$$

where $(\quad)^{\wedge}$ means the completion using the Krull topology.

Using the notation above, we have the following:

(I-1) m-th cochain complexes of $\mathfrak{ham}_{2n}$ and $\mathfrak{ham}^0_{2n}$ are given by

$$\mathrm{C}^m_{\mathrm{GF}}(\mathfrak{ham}_{2n}) = \bigoplus_{k_1+k_2+\cdots=m} \Lambda^{k_1}\mathfrak{S}_1 \otimes \Lambda^{k_2}\mathfrak{S}_2 \otimes \Lambda^{k_3}\mathfrak{S}_3 \otimes \cdots$$

and since $\mathfrak{S}_1$ is the dual space of constant vector fields

$$\mathrm{C}^m_{\mathrm{GF}}(\mathfrak{ham}^0_{2n}) = \bigoplus_{k_2+k_3+\cdots=m} \Lambda^{k_2}\mathfrak{S}_2 \otimes \Lambda^{k_3}\mathfrak{S}_3 \otimes \Lambda^{k_4}\mathfrak{S}_4 \otimes \cdots .$$

(I-2) The coboundary operator d on $\mathrm{C}^\bullet_{\mathrm{GF}}(\mathfrak{ham}_{2n})$ is defined by

$$(d\,\sigma)(f_0, f_1, \ldots, f_m) = \sum_{k<\ell} (-1)^{k+\ell} \sigma(\{f_k, f_\ell\}, \ldots, \widehat{f_k}, \ldots, \widehat{f_\ell}, \ldots) \qquad f_i \in \mathfrak{ham}_{2n}$$

for each m-cochain $\sigma \in \mathrm{C}^m_{\mathrm{GF}}(\mathfrak{ham}_{2n})$.
And the coboundary operator d_0 on $\mathrm{C}^\bullet_{\mathrm{GF}}(\mathfrak{ham}^0_{2n})$ is defined by

$$(d_0\,\sigma)(f_0, f_1, \ldots, f_m) = \sum_{k<\ell} (-1)^{k+\ell} \sigma(\{f_k, f_\ell\}, \ldots, \widehat{f_k}, \ldots, \widehat{f_\ell}, \ldots) \qquad f_i \in \mathfrak{ham}^0_{2n}$$

for each m-cochain $\sigma \in \mathrm{C}^m_{\mathrm{GF}}(\mathfrak{ham}^0_{2n})$.
We will study the difference between two coboundary operators d and d_0 in subsection Section 2.3.

(I-3) There is a notion of *weight* for cochains (cf. [**6**]). For each non-zero cochain

$$\sigma \in \Lambda^{k_1}\mathfrak{S}_1 \otimes \Lambda^{k_2}\mathfrak{S}_2 \otimes \Lambda^{k_3}\mathfrak{S}_3 \otimes \cdots \otimes \Lambda^{k_\ell}\mathfrak{S}_\ell$$

its weight is given by

$$(1-2)k_1 + (2-2)k_2 + (3-2)k_3 + \cdots + (\ell-2)k_\ell = \sum_{i=1}^{\ell} (i-2)k_i.$$

The weight of a cochain is preserved by the coboundary operator, and we can decompose each cochain complex by way of weights and get Gel'fand-Fuks cohomologies with a discrete parameter, namely with *weight* w like as

$$\mathrm{C}^m_{\mathrm{GF}}(\mathfrak{ham}^{j-1}_{2n})_w \quad \text{and} \quad \mathrm{H}^m_{\mathrm{GF}}(\mathfrak{ham}^{j-1}_{2n})_w \quad \text{for } j = 0, 1$$

where $\mathfrak{ham}^{-1}_{2n}$ means the space $\mathfrak{ham}_{2n}$.

In both cases, for given degree m and weight w, we consider the sequences $(k_1, k_2, k_3, \ldots)$ of nonnegative integers with

$$\sum_{j=1}^{\infty} k_j = m \quad \text{and} \quad \sum_{j=1}^{\infty} (j-2)k_j = w. \tag{2}$$

Readers may be anxious about the contribution of k_2 or k_1. In fact, there is a dimensional restriction for each k_j with $0 \leq k_j \leq \dim \mathfrak{S}_j = (j+2n-1)!/j!(2n-1)!$.

From those two relations in (2), we have

$$\sum_{j=1}^{\infty} k_j = m \quad \text{and} \quad \sum_{j=1}^{\infty} jk_j = w + 2m. \tag{3}$$

This means our sequences correspond to all partitions of $w+2m$ of length m, or in other words, to the Young diagrams with $w+2m$ cells of length m (cf. [**13**]). Furthermore, we require dimensional restrictions, and $k_1 = 0$ when $\mathfrak{ham}^0_{2n}$.

2.2. Symplectic action and the relative cohomologies.

We denote the natural action of the Lie group $K = Sp(2n, \mathbb{R})$ on $\mathbb{R}^{2n}$ by φ_a for $a \in K$, i.e., $\varphi_a(\boldsymbol{x}) = a\boldsymbol{x}$ as the multiplication of matrices. The action leaves ω invariant by definition, and we see that $(\varphi_a)_*(\mathcal{H}_f) = \mathcal{H}_{f \circ \varphi_{a^{-1}}}$ for each function f on $\mathbb{R}^{2n}$ and $a \in K$. Let $\mathfrak{k} = \mathfrak{sp}(2n, \mathbb{R})$ be the Lie algebra of K. We denote the fundamental vector field on $\mathbb{R}^{2n}$ of K by $\xi_{\mathbb{R}^{2n}}$ for $\xi \in \mathfrak{k}$. The equivariant (co-)momentum mapping of symplectic action of K is given by

$$\hat{J}(\xi)\boldsymbol{x} = -\frac{1}{2}{}^t\boldsymbol{x} \begin{bmatrix} \{x_1, x_1\} & \cdots & \{x_1, x_{2n}\} \\ \vdots & \dots & \vdots \\ \{x_{2n}, x_1\} & \cdots & \{x_{2n}, x_{2n}\} \end{bmatrix} \xi\boldsymbol{x}$$

where $\boldsymbol{x}$ is the column vector of the natural coordinates of $\mathbb{R}^{2n}$, ${}^t\boldsymbol{x}$ means the transposed row vector of $\boldsymbol{x}$, $\{x_i, x_j\}$ is the Poisson bracket with respect to ω, of i-th and j-th components of $\boldsymbol{x}$ and $\xi \in \mathfrak{k}$. $\hat{J}$ is a Lie algebra monomorphism from the Lie algebra $\mathfrak{sp}(2n, \mathbb{R})$ into the Lie algebra $C^\infty(\mathbb{R}^{2n})$ with the Poisson bracket. We stress that $\hat{J}(\xi)$ is a degree 2 homogeneous polynomial function on $\mathbb{R}^{2n}$ for $\xi \neq 0$. The Hamilton potential of the bracket $[\xi_{\mathbb{R}^{2n}}, \mathcal{H}_f]$ is given by $-\{\hat{J}(\xi), f\}$, because of $[\xi_{\mathbb{R}^{2n}}, \mathcal{H}_f] = [\mathcal{H}_{\hat{J}(\xi)}, \mathcal{H}_f] = -\mathcal{H}_{\{\hat{J}(\xi), f\}}$. This means that $\mathfrak{k}$ is regarded as a subalgebra of $\mathfrak{g} = \mathfrak{ham}_{2n}$ or $\mathfrak{ham}^0_{2n}$ through the equivariant momentum mapping J.

Define the relative cochain group $\mathrm{C}^m(\mathfrak{g}, \mathfrak{k})$ by

$$\mathrm{C}^m(\mathfrak{g}, \mathfrak{k}) = \{\sigma \in \mathrm{C}^m(\mathfrak{g}) \mid i_X\sigma = 0, i_X d\sigma = 0 \quad (\forall X \in \mathfrak{k})\} \quad (m = 0, 1, \ldots).$$

Then $(\mathrm{C}^\bullet(\mathfrak{g}, \mathfrak{k}), d)$ becomes a cochain complex, and we get the relative cohomology group $\mathrm{H}^m(\mathfrak{g}, \mathfrak{k})$. Let K be a Lie group of $\mathfrak{k}$. We also define

$$\mathrm{C}^m(\mathfrak{g}, K) = \{\sigma \in \mathrm{C}^m(\mathfrak{g}) \mid i_X\sigma = 0 \ (\forall X \in \mathfrak{k}), Ad^*_k\sigma = \sigma \ (\forall k \in K)\}$$

and we get the relative cohomology groups $\mathrm{H}^m(\mathfrak{g}, K)$. If K is connected, these two

cochain groups are identical. If K is a closed subgroup of G, then it can be seen $\mathrm{C}^\bullet(\mathfrak{g}, K) = \Lambda^\bullet(G/K)^G$ (the exterior algebra of G-invariant differential forms on G/K).

Since the space $\mathfrak{S}_2^*$ of degree 2 homogeneous polynomials is spanned by the image of momentum mapping $\hat{J}$ of $Sp(2n, \mathbb{R})$, we see that

PROPOSITION 2.1 ([**13**]). *For each cochain σ, $i_\xi\sigma = 0$ $(\forall \xi \in \mathfrak{sp}(2n, \mathbb{R}))$ implies $k_2 = 0$, and the other condition $i_\xi d\sigma = 0$ is equivalent to $L_\xi\sigma = 0$ $(\forall \xi \in \mathfrak{sp}(2n, \mathbb{R}))$. Thus we see for $j = 0, 1$*

$$\mathrm{C}^\bullet_{\mathrm{GF}}(\mathfrak{ham}_{2n}^{j-1}, \mathfrak{sp}(2n, \mathbb{R}))_w = \sum_{\mathrm{Cond}_j} \left(\Lambda^{k_1}\mathfrak{S}_1 \otimes \Lambda^{k_2}\mathfrak{S}_2 \otimes \Lambda^{k_3}\mathfrak{S}_3 \otimes \cdots\right)^{triv}$$

where $(\quad)^{triv}$ means the direct sum of the (underlying) subspaces of the trivial representations. Cond_0 *consists of the conditions* (3) *in the preceding subsection, $k_2 = 0$, and the dimensional restrictions.* Cond_1 *consists of* Cond_0 *and $k_1 = 0$.*

REMARK 2.1. As explained in [**6**], if the weight w is odd, then $\mathrm{C}^\bullet_{\mathrm{GF}}(\mathfrak{ham}_{2n}^{j-1}, \mathfrak{sp}(2n, \mathbb{R}))_w$ and $\mathrm{H}^\bullet_{\mathrm{GF}}(\mathfrak{ham}_{2n}^{j-1}, \mathfrak{sp}(2n, \mathbb{R}))_w$ vanish for $j = 0, 1$. Thus, we have only to deal with even weights.

There is a notion of *type N* for cochains of $\mathfrak{ham}_2$ in [**7**]. The weight w and type N are related by $w = 2N$ when $n = 1$.

There is a general method to decompose $\Lambda^p\mathfrak{S}_q$ into the irreducible subspaces for a general $Sp(2n, \mathbb{R})$-representation space, namely, getting the maximal vectors which are invariant by the maximal unipotent subgroup of $Sp(2n, \mathbb{R})$.

Concerning the decomposition of the tensor product, we have the Clebsch-Gordan rule when $n = 1$. (For $n = 2$, Littlewood-Richardson rule is used in [**12**], and the crystal base theory is used in [**11**] when $n = 3$.)

2.3. Coboundary operators.

By d, we will mean the coboundary operator which acts on $\mathrm{C}^\bullet_{\mathrm{GF}}(\mathfrak{ham}_{2n}, \mathfrak{sp}(2n, \mathbb{R}))_w$ and by d_0, the one acts on $\mathrm{C}^\bullet_{\mathrm{GF}}(\mathfrak{ham}_{2n}^0, \mathfrak{sp}(2n, \mathbb{R}))_w$.

Let ω be the 2-cochain defined by the linear symplectic form of $\mathbb{R}^{2n}$. We see that

$$\omega \in \mathrm{C}^2_{\mathrm{GF}}(\mathfrak{ham}_{2n}, \mathfrak{sp}(2n, \mathbb{R}))_{(-2)} \backslash \mathrm{C}^2_{\mathrm{GF}}(\mathfrak{ham}_{2n}^0, \mathfrak{sp}(2n, \mathbb{R}))_{(-2)}$$

and $\omega^n \in \mathrm{C}^{2n}_{\mathrm{GF}}(\mathfrak{ham}_{2n}, \mathfrak{sp}(2n, \mathbb{R}))_{(-2n)}$.

PROPOSITION 2.2. *The linear map*

$$\omega^n\wedge : \mathrm{C}^\bullet_{\mathrm{GF}}(\mathfrak{ham}_{2n}^0, \mathfrak{sp}(2n, \mathbb{R}))_w \longrightarrow \mathrm{C}^{\bullet+2n}_{\mathrm{GF}}(\mathfrak{ham}_{2n}, \mathfrak{sp}(2n, \mathbb{R}))_{w-2n}$$

satisfies

$$d\,(\omega^n \wedge \sigma) = \omega^n \wedge d_0\,\sigma$$

and the next diagram is commutative

$$\begin{array}{ccc} \mathrm{C}^{\bullet-1+2n}_{\mathrm{GF}}(\mathfrak{ham}_{2n}, \mathfrak{sp}(2n,\mathbb{R}))_{w-2n} & \xrightarrow{d} & \mathrm{C}^{\bullet+2n}_{\mathrm{GF}}(\mathfrak{ham}_{2n}, \mathfrak{sp}(2n,\mathbb{R}))_{w-2n} \\ \omega^n\wedge \uparrow & & \uparrow \omega^n\wedge \\ \mathrm{C}^{\bullet-1}_{\mathrm{GF}}(\mathfrak{ham}^0_{2n}, \mathfrak{sp}(2n,\mathbb{R}))_{w} & \xrightarrow{d_0} & \mathrm{C}^{\bullet}_{\mathrm{GF}}(\mathfrak{ham}^0_{2n}, \mathfrak{sp}(2n,\mathbb{R}))_{w}. \end{array} \tag{4}$$

Thus we have a linear map

$$\omega^n\wedge : \mathrm{H}^{\bullet}_{\mathrm{GF}}(\mathfrak{ham}^0_{2n}, \mathfrak{sp}(2n,\mathbb{R}))_w \longrightarrow \mathrm{H}^{\bullet+2n}_{\mathrm{GF}}(\mathfrak{ham}_{2n}, \mathfrak{sp}(2n,\mathbb{R}))_{w-2n}$$

naturally. This induced map is trivial if and only if

$$\begin{aligned} \omega^n \wedge \ker(d_0 : \mathrm{C}^{\bullet}_{\mathrm{GF}}(\mathfrak{ham}^0_{2n}, \mathfrak{sp}(2n,\mathbb{R}))_w \to \mathrm{C}^{\bullet+1}_{\mathrm{GF}}(\mathfrak{ham}^0_{2n}, \mathfrak{sp}(2n,\mathbb{R}))_w) \\ \subset d\left(\mathrm{C}^{\bullet-1+2n}_{\mathrm{GF}}(\mathfrak{ham}_{2n}, \mathfrak{sp}(2n,\mathbb{R}))_{w-2n}\right). \end{aligned} \tag{5}$$

Proof. We have $d\,\omega = 0$. This is a requirement for a symplectic form. For each $\sigma \in \mathrm{C}^{\bullet}_{\mathrm{GF}}(\mathfrak{ham}^0_2)$, we already know that $\omega^n \wedge (d\,\sigma - d_0\,\sigma) = 0$ and we see that $\omega^n \wedge d\,\sigma = d\,(\omega^n \wedge \sigma)$ because of $d\,\omega = 0$. This states that the diagram (4) is commutative. Then we have

(i) If $d_0\,\sigma = 0$, then $d\,(\omega^n \wedge \sigma) = 0$, namely, $\omega^n \wedge \ker(d_0) \subset \ker(d)$.

(ii) If $\sigma, \tau \in \mathrm{C}^{\bullet}_{\mathrm{GF}}(\mathfrak{ham}^0_{2n}, \mathfrak{sp}(2n,\mathbb{R}))$ satisfy $d_0\,\sigma = 0 = d_0\,\tau$ and $\sigma - \tau = d_0\,\rho$, then $\omega^n \wedge \sigma - \omega^n \wedge \tau = \omega^n \wedge d_0\,\rho = d\,(\omega^n \wedge \rho)$. This means that the wedge product by ω^n induces a well-defined linear map

$$\omega^n : \mathrm{H}^{\bullet}_{\mathrm{GF}}(\mathfrak{ham}^0_{2n}, \mathfrak{sp}(2n,\mathbb{R})) \longrightarrow \mathrm{H}^{\bullet+2n}_{\mathrm{GF}}(\mathfrak{ham}_{2n}, \mathfrak{sp}(2n,\mathbb{R})) \quad \text{by} \quad \sigma \mapsto \omega^n \wedge \sigma.$$

(iii) From (i), we see that the map is trivial if and only if $\omega^n \wedge \ker(d_0) \subset d\,(\mathrm{C}^{\bullet+2n-1}_{\mathrm{GF}}(\mathfrak{ham}_{2n}, \mathfrak{sp}(2n,\mathbb{R}))_{w-2n})$. □

3. Symplectic 2-plane.

In this section, we deal with the symplectic 2-plane $\mathbb{R}^2$. We study the difference between the two coboundary operators d and d_0. Since $\dim \mathfrak{S}_1 = 2$, the domain of definition of d is given by

$$\mathrm{C}^{\bullet}_{\mathrm{GF}}(\mathfrak{ham}_2) = \mathrm{C}^{\bullet}_{\mathrm{GF}}(\mathfrak{ham}^0_2) \oplus \left(\mathfrak{S}_1 \otimes \mathrm{C}^{\bullet-1}_{\mathrm{GF}}(\mathfrak{ham}^0_2)\right) \oplus \left(\Lambda^2\mathfrak{S}_1 \otimes \mathrm{C}^{\bullet-2}_{\mathrm{GF}}(\mathfrak{ham}^0_2)\right).$$

Let x, y be global Darboux coordinates, i.e., $\{x, y\} = 1$. For each positive integer R, $\{\widehat{z}^r_R = (x^r/r!)(y^{R-r}/(R-r)!) \mid r = 0, 1, \ldots, R\}$ is a basis of the space of R-homogeneous polynomials of x, y. Let $\{z^r_R \mid R > 0, r = 0, 1, \ldots, R\}$ be the dual basis of $\{\widehat{z}^r_R \mid R > 0, r = 0, 1, \ldots, R\}$.

The two coboundary operators d, d_0 in those bases, are

$$d\,z_R^r = -\frac{1}{2}\sum_{A+B=2+R}\langle z_R^r, \{\widehat{z}_A^a, \widehat{z}_B^b\}\rangle z_A^a \wedge z_B^b$$

where $A > 0$, $B > 0$, $a \in \{0, \dots, A\}$ and $b \in \{0, \dots, B\}$, and

$$d_0\, z_R^r = -\frac{1}{2}\sum_{A+B=2+R}\langle z_R^r, \{\widehat{z}_A^a, \widehat{z}_B^b\}\rangle z_A^a \wedge z_B^b$$

where $A > 1$, $B > 1$, $a \in \{0, 1, \dots, A\}$ and $b \in \{0, 1, \dots, B\}$. Thus, the difference between d and d_0 for a 1-cochain can be written as follows.

$$\begin{aligned} d\,z_R^r &= d_0\, z_R^r - \sum_{a\in\{0,1\}, b\in\{0,\dots,1+R\}}\langle z_R^r, \{\widehat{z}_1^a, \widehat{z}_{1+R}^b\}\rangle z_1^a \wedge z_{1+R}^b \\ &= d_0\, z_R^r + z_1^0 \wedge z_{1+R}^{1+r} - z_1^1 \wedge z_{1+R}^r. \end{aligned}$$

We may assume that $d_0\, z_1^r = 0$ $(r = 0, 1)$. The 2-cochain ω which comes from the symplectic structure, is written as $\omega = z_1^0 \wedge z_1^1$ in our notation and we see directly that

$$d\,\omega = d\,(z_1^0 \wedge z_1^1) = (z_1^0 \wedge z_2^1 - z_1^1 \wedge z_2^0) \wedge z_1^1 - z_1^0 \wedge (z_1^0 \wedge z_2^2 - z_1^1 \wedge z_2^1) = 0.$$

But, ω is not d-exact because $\{\widehat{z}_1^a, \widehat{z}_1^b\} =$ constant.

4. Proof of Theorem 1.2.

In this section, we give a proof for Theorem 1.2 which asserts that

$$\omega\wedge : \mathrm{H}^7_{\mathrm{GF}}(\mathfrak{ham}_2^0, \mathfrak{sp}(2,\mathbb{R}))_{16} \to \mathrm{H}^9_{\mathrm{GF}}(\mathfrak{ham}_2, \mathfrak{sp}(2,\mathbb{R}))_{14}$$

is an isomorphism. Since we know that the source and the target spaces are both 1-dimensional, it is enough to show the map $\omega\wedge$ is non-trivial. For that purpose, we make use of (5) of Proposition 2.2.

We have information about $\mathrm{C}^\bullet_{\mathrm{GF}}(\mathfrak{ham}_2^0, \mathfrak{sp}(2,\mathbb{R}))_w$ $(w = 12, 14, 16, 18, 20)$ (cf. [**13**]). We show the result of weight =16 in the table below. In the table, C^k is $\mathrm{C}^k_{\mathrm{GF}}(\mathfrak{ham}_2^0, \mathfrak{sp}(2,\mathbb{R}))_{16}$ and rank is the rank of $d_0 : C^k \longrightarrow C^{1+k}$.

$\mathfrak{ham}_2^0$, w=16	**0**	→	C^3	→	C^4	→	C^5	→	C^6	→	C^7	→	C^8	→	**0**
$\dim C^k$			12		61		126		147		95		24		
rank	0		12		49		77		70		24		0		
Betti num			0		0		0		0		1		0		

The table above says that $\dim \mathrm{H}^7_{\mathrm{GF}}(\mathfrak{ham}_2^0, \mathfrak{sp}(2,\mathbb{R}))_{16} = 1$.

Concerning $\mathrm{H}^9_{\mathrm{GF}}(\mathfrak{ham}_2, \mathfrak{sp}(2,\mathbb{R}))_{14}$, we refer to [**7**], where we see the complete data. But, the notation there is different from ours, and it seems hard to find an applicable translation rule. So we need to get suitable bases for our notation and begin searching

bases without the $k_1 = 0$ condition at the beginning in order to get the complete bases. In the following discussion, we only need information about the bases of $\mathfrak{C}^8$, $\mathfrak{C}^9$ and the matrix representation $\overline{M}$ of $d : \mathfrak{C}^8 \to \mathfrak{C}^9$, where $\mathfrak{C}^k = \mathrm{C}^k_{\mathrm{GF}}(\mathfrak{ham}_2, \mathfrak{sp}(2,\mathbb{R}))_{14}$.

A similar table is obtained in the case of $\mathfrak{ham}_2$ and weight 14, rank is the rank of $d : \mathfrak{C}^k \longrightarrow \mathfrak{C}^{1+k}$.

$\mathfrak{ham}_2$, wt=14	$\to$	$\mathfrak{C}^8$	$\to$	$\mathfrak{C}^9$	$\to$	$\mathfrak{C}^{10}$	$\to$	$\mathbf{0}$
$\dim \mathfrak{C}^k$		232		113		25		
rank	145		87		25		0	
Betti num		0		1		0		

Our proof of Theorem 1.2 consists of the 3 steps as follows:

1. To find a vector $\boldsymbol{h} \in \ker(d_0 : \mathrm{C}^7_{\mathrm{GF}}(\mathfrak{ham}^0_2, \mathfrak{sp}(2,\mathbb{R}))_{16} \to \mathrm{C}^8_{\mathrm{GF}}(\mathfrak{ham}^0_2, \mathfrak{sp}(2,\mathbb{R}))_{16})$ but $\boldsymbol{h} \notin d_0\,(\mathrm{C}^6_{\mathrm{GF}}(\mathfrak{ham}^0_2, \mathfrak{sp}(2,\mathbb{R}))_{16})$.
2. To calculate $\omega \wedge \boldsymbol{h}$.
3. To check whether $\omega \wedge \boldsymbol{h} \in d\,(\mathrm{C}^8_{\mathrm{GF}}(\mathfrak{ham}_2, \mathfrak{sp}(2,\mathbb{R}))_{14})$ or not, by counting the dimension of the space generated by $\omega \wedge \boldsymbol{h}$ and $d\,(\mathrm{C}^8_{\mathrm{GF}}(\mathfrak{ham}_2, \mathfrak{sp}(2,\mathbb{R}))_{14})$.

4.1. Gröbner Basis theory for cohomology groups.

To complete the proof, we make use of the Gröbner Basis theory (cf. [**3**]) for linear homogeneous polynomials. Suppose we have μ indeterminate variables $\{y_j\}_{j=1}^{\mu}$ and fix a monomial order Ord_y of $\{y_j\}$ by $y_1 \succ \cdots \succ y_\mu$. If $\{g_i\}_{i=1}^{\lambda}$ are linear homogeneous polynomials of $\{y_j\}$, then we may write $[g_1, \ldots, g_\lambda] = [y_1, \ldots, y_\mu]M$ for some matrix M. It is known that we can deform M into the unique column echelon matrix $\widehat{M}$ by a sequence of the three kinds elementary column operations. Getting the row echelon matrix ${}^t\widehat{M}$ from ${}^tM$ by elementary row operations is well-known as the Gaussian elimination method. The monic Gröbner basis of $\{g_i\}$ with the monomial order, we denote as $\mathrm{mBasis}([g_1, \ldots, g_\lambda], \mathrm{Ord}_y)$, satisfies

$$[\mathrm{mBasis}([g_1, \ldots, g_\lambda], \mathrm{Ord}_y), 0, \ldots, 0] = [y_1, \ldots, y_\mu]\widehat{M}.$$

Thus, $\mathrm{rank}\, M = \mathrm{rank}\, \widehat{M}$ is equal to the cardinality of $\mathrm{mBasis}([g_1, \ldots, g_\lambda], \mathrm{Ord}_y)$ and $\mathrm{mBasis}([g_1, \ldots, g_\lambda], \mathrm{Ord}_y)$ gives a basis for the $\mathbb{R}$-vector space generated by $\{g_i\}$. Hereafter, we use a reduced Gröbner basis, we denote it by $\mathrm{Basis}([g_1, \ldots, g_\lambda], \mathrm{Ord}_y)$, for which we allow that each leading coefficient should not be 1. So, each j-th element of $\mathrm{Basis}([g_1, \ldots, g_\lambda], \mathrm{Ord}_y)$ is a non-zero scalar multiple of j-th element of $\mathrm{mBasis}([g_1, \ldots, g_\lambda], \mathrm{Ord}_y)$.

The normal form of a given polynomial h with respect to a Gröbner basis GB together with a fixed monomial order, for example $\mathrm{NF}(h, \mathrm{GB}, \mathrm{Ord}_y)$, is the "smallest" remainder of h modulo by the Gröbner basis GB. Again, if we restrict our discussion to the linear homogeneous polynomials, then $\mathrm{NF}(h, \mathrm{GB}, \mathrm{Ord}_y) = 0$ is equivalent to $h \in$ the linear space spanned by GB.

We recall key techniques in cohomology theory involving the Gröbner Basis theory. Let X, Y and Z be finite dimensional vector spaces with bases $\{\boldsymbol{q}_i\}_{i=1}^{\lambda}$, $\{\boldsymbol{w}_j\}_{j=1}^{\mu}$ and

$\{\boldsymbol{r}_k\}_{k=1}^{\nu}$ respectively. Assume that there are linear maps $g : X \to Y$ and $f : Y \to Z$ whose matrix representations are M and N respectively: i.e.,

$$[g(\boldsymbol{q}_1), g(\boldsymbol{q}_2), \dots, g(\boldsymbol{q}_\lambda)] = [\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_\mu]M \tag{6}$$

and

$$[f(\boldsymbol{w}_1), f(\boldsymbol{w}_2), \dots, f(\boldsymbol{w}_\mu)] = [\boldsymbol{r}_1, \boldsymbol{r}_2, \dots, \boldsymbol{r}_\nu]N. \tag{7}$$

In the right-hand side of (6), we replace $\boldsymbol{w}_j$ by indeterminate variable y_j ($j = 1, 2, \dots, \mu$), and get a set of linear homogeneous polynomials $[y_1, y_2, \dots, y_\mu]M$. Denote them by $[g_1(y), g_2(y), \dots, g_\lambda(y)]$, i.e., $[g_1(y), g_2(y), \dots, g_\lambda(y)] = [y_1, y_2, \dots, y_\mu]M$.

PROPOSITION 4.1 ([**1**]). *$GB_e = \mathrm{Basis}([g_1, g_2, \dots, g_\lambda], [y_1, y_2, \dots, y_\mu], \mathrm{Ord}_y)$ gives a basis of $g(X)$ in the sense that $\{\varphi(\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_\mu) \mid \varphi \in GB_e\}$ forms a basis of $g(X)$ and $\mathrm{rank}(g) = \#(GB_e)$.*

We study $f^{-1}(0) = \ker(f : Y \to Z)$. Since $\langle f(\boldsymbol{u}), \sigma\rangle = \langle \boldsymbol{u}, f^*(\sigma)\rangle$ for $\boldsymbol{u} \in Y, \sigma \in Z^*$(= the dual space of Z), $f^{-1}(0) = \mathrm{Im}(f^*)^0$, the annihilator subspace of $\mathrm{Im}(f^*)$. By Proposition 4.1, we know well about $\mathrm{Im}(f^*)$ by the Gröbner Basis theory as follows: Since N is the matrix representation of f, ${}^tN$ is a matrix representation of f^*. We put $[c_1, c_2, \dots, c_\mu]({}^tN)$ by $[f_1, f_2, \dots, f_\nu]$. Fix the monomial order Ord_c of $\{c_j\}_{j=1}^{\mu}$ by $c_1 \succ \cdots \succ c_\mu$. We get the Gröbner basis $GB_{tr(f)} = \mathrm{Basis}([f_1, f_2, \dots, f_\nu],\ \mathrm{Ord}_c)$, which gives a basis of $\mathrm{Im}(f^*)$.

Consider the polynomial $h = \sum_{j=1}^{\mu} c_j y_j$, where $\{y_j\}_{j=1}^{\mu}$ are the other auxiliary variables (which appear for the linear map g).

PROPOSITION 4.2 ([**1**]). *The normal form $\mathrm{NF}(h, GB_{tr(f)}, \mathrm{Ord}_c)$ of h is written as $\sum_{\ell=1}^{\mu} c_\ell \tilde{f}_\ell(y)$ where $\tilde{f}_\ell(y)$ is linear in $\{y_j\}$.*

Let $GB_k = \mathrm{Basis}([\tilde{f}_1(y), \tilde{f}_2(y), \dots, \tilde{f}_\mu(y)], \mathrm{Ord}_y)$. Then GB_k gives a basis of the kernel space $f^{-1}(0) = \ker(f)$, and the cardinality of GB_k is $\dim\ker(f)$.

Now assume that $f \circ g = 0$. We use the Gröbner bases GB_e of g, and GB_k of $\ker(f)$ above, then we have the following.

PROPOSITION 4.3 ([**1**]). *The quotient space $\ker(f : Y \to Z)/\mathrm{Im}(g : X \to Y)$ is equipped with the basis*

$$GB_{k/e} = \mathrm{Basis}([\mathrm{NF}(\varphi, GB_e, \mathrm{Ord}_y) \mid \varphi \in GB_k], \mathrm{Ord}_y).$$

In particular, $\dim\left(\ker(f : Y \to Z)/\mathrm{Im}(g : X \to Y)\right) = \#(GB_{k/e})$.

REMARK 4.1. If we follow the way consisting of the 3 steps described just before this subsection, there is some ambiguity in choosing an element $\boldsymbol{h}$, in general. But, if we use the Gröbner Basis theory, we can avoid this ambiguity. This is a main reason why we use the Gröbner Basis theory here. It is hard to handle big matrices, but it is easy to deal with polynomials. This is the second small reason. The last reason why we use the

Gröbner Basis theory is that it is pre-packaged in the symbolic calculus softwares such as Maple, Mathematica, Risa/Asir (this is freeware) and so on. And such softwares are becoming more and more reliable and faster.

Our calculation of Gröbner Bases or normal forms is assisted by symbolic calculus software Maple. There is a proof by the Gröbner Basis theory of Theorem 1.1 with the assistance of Maple in [**10**]. Also, the draft of it is available on URL [**8**] www.math.akita-u.ac.jp/˜mikami/Conj4MetokiClass/ with the title "A proof to Kotschick–Morita theorem for G-K-F class".

Risa/Asir is popular among Japanese mathematicians because it is bundled in the Math Libre Disk which is distributed at annual meetings of the Mathematical Society of Japan. We put the source code and output of our computer argument for Risa/Asir on [**8**] and in appendices in [**10**]. Here, you can find a proof of the Kotschick–Morita theorem by Risa/Asir. You can also compare the two kinds of results calculated by Maple and Risa/Asir, and see that the final normal forms are the same, up to non-zero scalar multiples.

Even in the classical linear algebra argument or the Gröbner Basis argument, our discussion is based on matrix representations of the two coboundary operators. We stress that everything starts from the concrete bases of cochain complexes.

4.2. Selecting a generator h of $\mathbf{H}^{\mathbf{7}}_{\mathbf{GF}}(\mathfrak{ham}^{\mathbf{0}}_{\mathbf{2}}, \mathfrak{sp}(\mathbf{2}, \mathbb{R}))_{\mathbf{16}}$.

As mentioned in Remark 4.1, the existence of concrete bases of our cochain complexes is important. Actually, we got them and can handle them, but as shown in the first table above, the dimensions are large; for example $\dim C^6 = 147$, $\dim C^7 = 95$ and $\dim C^8 = 24$, where $C^k = \mathrm{C}^k_{\mathrm{GF}}(\mathfrak{ham}^0_2, \mathfrak{sp}(2,\mathbb{R}))_{16}$, It is difficult to show them all in this paper. The entire data of our concrete bases of C^k (k = 6,7,8) are found either on [**8**] or in Appendix 1, 2 and 3 in [**9**].

Here we only show several elements, whose number of terms of summation is smaller. The smallest element of our basis of C^6 is next, and consists of 28 terms:

$$
\begin{aligned}
\boldsymbol{q}_{142} = &-\frac{8}{3}z_4^0z_4^1z_4^2z_4^3z_5^2z_7^6 - z_4^0z_4^1z_4^3z_4^4z_5^1z_7^5 + \frac{1}{6}z_4^0z_4^1z_4^2z_4^4z_5^1z_7^6 - \frac{1}{6}z_4^0z_4^1z_4^2z_4^4z_5^0z_7^7\\
&+ \frac{2}{3}z_4^1z_4^2z_4^3z_4^4z_5^4z_7^0 - \frac{1}{2}z_4^0z_4^2z_4^3z_4^4z_5^0z_7^5 - \frac{8}{3}z_4^1z_4^2z_4^3z_4^4z_5^1z_7^3 + z_4^0z_4^2z_4^3z_4^4z_5^3z_7^2\\
&- \frac{7}{3}z_4^0z_4^1z_4^2z_4^4z_5^3z_7^4 + \frac{1}{6}z_4^0z_4^2z_4^3z_4^4z_5^4z_7^1 + \frac{1}{3}z_4^0z_4^1z_4^3z_4^4z_5^0z_7^6 + \frac{2}{3}z_4^0z_4^1z_4^2z_4^3z_5^1z_7^7\\
&+ \frac{2}{3}z_4^0z_4^1z_4^3z_4^4z_5^3z_7^3 - \frac{8}{3}z_4^0z_4^1z_4^2z_4^3z_4^4z_7^4 - \frac{8}{3}z_4^1z_4^2z_4^3z_4^4z_5^3z_7^1 - \frac{7}{3}z_4^0z_4^2z_4^3z_4^4z_5^2z_7^3\\
&+ \frac{11}{6}z_4^0z_4^1z_4^2z_4^4z_5^4z_7^3 + \frac{2}{3}z_4^0z_4^1z_4^2z_4^3z_5^5z_7^3 + z_4^0z_4^1z_4^2z_4^4z_5^2z_7^5 + \frac{1}{3}z_4^0z_4^1z_4^3z_4^4z_5^5z_7^1\\
&+ \frac{11}{6}z_4^0z_4^2z_4^3z_4^4z_5^1z_7^4 - \frac{1}{2}z_4^0z_4^1z_4^2z_4^4z_5^5z_7^2 + \frac{2}{3}z_4^1z_4^2z_4^3z_4^4z_5^0z_7^4 + 4z_4^1z_4^2z_4^3z_4^4z_5^2z_7^2\\
&- \frac{1}{6}z_4^0z_4^2z_4^3z_4^4z_5^5z_7^0 + 4z_4^0z_4^1z_4^2z_4^3z_5^3z_7^5 - z_4^0z_4^1z_4^3z_4^4z_5^4z_7^2 + \frac{2}{3}z_4^0z_4^1z_4^3z_4^4z_5^2z_7^4
\end{aligned}
$$

where we omit the symbol $\wedge$ of wedge product. The two small-size elements of our basis of C^7 are the following:

$$\begin{aligned}\boldsymbol{w}_6 = {} & z_3^0z_3^1z_3^2z_3^3z_6^0z_6^3z_6^6 - 3z_3^0z_3^1z_3^2z_3^3z_6^0z_6^4z_6^5 - 3z_3^0z_3^1z_3^2z_3^3z_6^1z_6^2z_6^6 \\ & + 6z_3^0z_3^1z_3^2z_3^3z_6^1z_6^3z_6^5 - 15z_3^0z_3^1z_3^2z_3^3z_6^2z_6^3z_6^4\end{aligned}$$

and

$$\boldsymbol{w}_{95} = z_4^0z_4^1z_4^2z_4^3z_4^4z_5^0z_5^5 - 5z_4^0z_4^1z_4^2z_4^3z_4^4z_5^1z_5^4 + 10z_4^0z_4^1z_4^2z_4^3z_4^4z_5^2z_5^3.$$

We pick up the smallest element of our basis of C^8:

$$\boldsymbol{r}_7 = z_3^0z_3^1z_3^2z_3^3z_5^0z_5^1z_5^4z_5^5 - 2z_3^0z_3^1z_3^2z_3^3z_5^0z_5^2z_5^3z_5^5 + 10z_3^0z_3^1z_3^2z_3^3z_5^1z_5^2z_5^3z_5^4.$$

We only need a generator of $\mathrm{H}^7_{\mathrm{GF}}(\mathfrak{ham}_2^0, \mathfrak{sp}(2,\mathbb{R}))_{16}$ by the Gröbner Basis theory and we write down our linear functions $\{g_i\}_{i=1}^{147}$ corresponding to $d_0 : C^6 \to C^7$ and linear functions $\{f_j\}_{j=1}^{24}$, giving the kernel condition for $d_0 : C^7 \to C^8$ as follows.

$$[g_1, \dots, g_{147}] = [y_1, \dots, y_{95}]M, \quad [f_1, \dots, f_{24}] = [c_1, \dots, c_{95}]^t N$$

where M and N are matrices of $d_0 : C^6 \to C^7$ and $d_0 : C^7 \to C^8$ with respect to the bases above. Since the size of matrix M is (95, 147) and that of N is (24,95), we will not show them here, but the precise complete data are found either on [**8**] or in Appendix 4 and 5 in [**9**]. Here, we show a few terms as examples:

$$\begin{aligned}g_1 = {} & 176y_1 - \frac{1036}{3}y_8 + \frac{632}{3}y_9 + \frac{544}{3}y_{10} - 60y_{11} - 22y_{12} + 152y_{13} - 802y_{21} \\ & + 531y_{22} + 590y_{23} - \frac{1625}{3}y_{24} + \frac{292}{3}y_{25} + 60y_{26} - \frac{1595}{3}y_{27} + 805y_{28} \\ & - 90y_{29} + 48y_{30} - 108y_{31} - 144y_{32} - 306y_{33} + 144y_{34} + 450y_{35} \\ & + 36y_{36} + 168y_{37} \\ & \vdots \\ g_{147} = {} & \frac{5}{2}y_{60} - \frac{7}{2}y_{61} + \frac{11}{10}y_{62} + \frac{11}{6}y_{63} - \frac{21}{2}y_{65} + \frac{33}{10}y_{66} + \frac{15}{2}y_{67} - \frac{1}{10}y_{68} \\ & + \frac{11}{2}y_{69} - 3y_{79} - \frac{1}{2}y_{80} + \frac{3}{2}y_{86} + \frac{95}{12}y_{87} - \frac{17}{6}y_{88} + 2y_{89} - \frac{209}{30}y_{90} \\ & + \frac{23}{10}y_{91} + \frac{6}{25}y_{92} + \frac{5}{2}y_{93} - \frac{35}{2}y_{94} - \frac{133}{30}y_{95}\end{aligned}$$

and

$$
\begin{aligned}
f_1 = &-\frac{55}{4}c_1 + 25c_3 + 8c_5 - \frac{475}{54}c_8 + \frac{145}{9}c_9 - \frac{995}{54}c_{10} + \frac{70}{3}c_{11} + \frac{1700}{81}c_{12} \\
&- \frac{10}{81}c_{13} - \frac{41}{9}c_{14} - \frac{215}{18}c_{15} - \frac{425}{18}c_{16} + \frac{425}{36}c_{17} + \frac{35}{9}c_{18} - \frac{59}{9}c_{19} + \frac{92}{9}c_{20} \\
&+ \frac{75}{32}c_{21} + \frac{85}{48}c_{22} + \frac{33}{16}c_{23} + \frac{139}{64}c_{24} + \frac{65}{64}c_{25} + \frac{75}{32}c_{26} - \frac{221}{64}c_{27} + \frac{1}{24}c_{28} \\
&- \frac{65}{12}c_{42} + \frac{35}{4}c_{43} + \frac{95}{6}c_{44} + \frac{53}{4}c_{45} + \frac{10}{3}c_{46} + \frac{13}{4}c_{47} + 2c_{48} - \frac{9}{2}c_{49} - \frac{3}{2}c_{50} \\
&\vdots \\
f_{24} = &-15c_{41} - 10c_{42} + 30c_{43} + 35c_{44} + 3c_{45} + 40c_{46} + 18c_{47} - 3c_{48} \\
&- \frac{301839}{740}c_{59} + \frac{256839}{740}c_{60} + \frac{1094769}{740}c_{61} + \frac{73128}{37}c_{62} + \frac{174258}{185}c_{63} \\
&+ \frac{848394}{185}c_{64} - \frac{435089}{740}c_{65} - \frac{105235}{111}c_{66} - \frac{9623}{148}c_{67} - \frac{28657}{37}c_{68} \\
&- \frac{70569}{185}c_{69} - \frac{150326}{111}c_{70} + \frac{35965}{111}c_{71} + \frac{4484}{185}c_{72} - \frac{3827}{37}c_{73} - \frac{52105}{74}c_{74} \\
&+ \frac{68225}{148}c_{75} - \frac{2556}{185}c_{76} + \frac{31601}{370}c_{77} + \frac{37535}{148}c_{78} - \frac{7439}{185}c_{79} + \frac{15139}{37}c_{80} \\
&+ \frac{10657}{148}c_{81} + \frac{56}{3}c_{86} + \frac{53}{3}c_{87} + \frac{193}{3}c_{88} + \frac{76}{3}c_{89} + 7c_{90} + 4c_{91} + \frac{85}{6}c_{92} \\
&- \frac{71}{6}c_{93} - 3c_{94} + 15c_{95}.
\end{aligned}
$$

The Gröbner basis GB_e of $\{g_i\}_{i=1}^{147}$ consists of 70 elements as expected. The whole Gröbner basis GB_e is stored on [**8**] and in Appendix 9 in [**9**]. The first element of sorted GB_e is

$$
\begin{aligned}
&446227638468y_{75} - 258371100400y_{76} + 2677414594200y_{77} \\
&- 2808720072600y_{78} + 483892450500y_{79} + 838357655220y_{80} \\
&- 871685530860y_{81} + 1892343009627y_{82} - 2525687071848y_{83} \\
&- 861370434243y_{84} - 625187443152y_{85} - 6093198421500y_{86} \\
&- 4546246681400y_{87} + 2813196475270y_{88} - 2152132471560y_{89} \\
&+ 15133158761840y_{90} - 9561265966665y_{91} - 2198954966322y_{92} \\
&+ 9680559087150y_{93} + 3770983597200y_{94} + 11367701561860y_{95},
\end{aligned}
$$

and the last element of sorted GB_e is

$$
\begin{aligned}
&7228887743181600y_1 + 26505921724999200y_{47} - 8835307241666400y_{50} \\
&- 40863295992707100y_{51} - 23594575360435200y_{76} + 141145959892004100y_{77} \\
&- 152234378969531760y_{78} + 126419237509059 00y_{79} + 103786265245653540y_{80}
\end{aligned}
$$

$$
\begin{aligned}
&- 23040628976396880y_{81} + 55341457461003915y_{82} - 18213992229930804 0y_{83}\\
&- 33164405973011299 5y_{84} - 37012865309023290y_{85} - 46733030259800940 0y_{86}\\
&- 32710734169626150 0y_{87} + 18200224688328441 0y_{88} - 27682638636383280y_{89}\\
&+ 81151325454211116 0y_{90} - 53069225576874574 5y_{91} - 21650055791402069 4y_{92}\\
&+ 75267796352469015 0y_{93} - 11777478047827755 0y_{94} + 79613644656769006 0y_{95}.
\end{aligned}
$$

The Gröbner basis GB_k corresponding to the kernel space defined by $\{f_j\}_{j=1}^{24}$ consists of 71 elements. The whole Gröbner basis GB_k is stored on [**8**] and in Appendix 10 in [**9**]. The first element of sorted GB_k is:

$$
\begin{aligned}
&2027141067600y_{76} + 6871115344500y_{77} - 8293793595120y_{78} + 1593871052400y_{79}\\
&+ 3342315930030y_{80} + 2188718191440y_{81} + 6047944018587y_{82}\\
&- 7911486513648y_{83} + 1366183084077y_{84} - 1206881491512y_{85}\\
&- 10895090886900y_{86} - 9572836551300y_{87} + 1269138903120y_{88}\\
&+ 3867959161440y_{89} + 28054435525860y_{90} - 23511502274085y_{91}\\
&- 7468180349703y_{92} + 2799062316375y_{93} + 17517045194250y_{94}\\
&+ 24368226519980y_{95},
\end{aligned}
$$

and the last element of sorted GB_k is:

$$
\begin{aligned}
&368571103200y_1 + 1351427378400y_{47} - 450475792800y_{50} - 2083450541700y_{51}\\
&+ 11274054788700y_{77} - 12683683914000y_{78} + 1590428838900y_{79}\\
&+ 7275101984700y_{80} - 10448587809000y_{81} + 6410733790653y_{82}\\
&- 13981567014072y_{83} - 16098409761957y_{84} - 2603346695478y_{85}\\
&- 30292840571400y_{86} - 22358770705700y_{87} + 10032702042550y_{88}\\
&+ 883984609200y_{89} + 58024375642760y_{90} - 41010508144455y_{91}\\
&- 15470397459450y_{92} + 40037017987050y_{93} + 4390494333150y_{94}\\
&+ 55052825955540y_{95}.
\end{aligned}
$$

The Gröbner basis corresponding to $\mathrm{H}^7_{\mathrm{GF}}(\mathfrak{ham}_2^0, \mathfrak{sp}(2,\mathbb{R}))_{16}$ is

$$
\begin{aligned}
h = {}& 2027141067600y_{76} + 6871115344500y_{77} - 8293793595120y_{78}\\
&+ 1593871052400y_{79} + 3342315930030y_{80} + 2188718191440y_{81}\\
&+ 6047944018587y_{82} - 7911486513648y_{83} + 1366183084077y_{84}\\
&- 1206881491512y_{85} - 10895090886900y_{86} - 9572836551300y_{87}\\
&+ 1269138903120y_{88} + 3867959161440y_{89} + 28054435525860y_{90}
\end{aligned}
$$

$$- 23511502274085y_{91} - 7468180349703y_{92} + 2799062316375y_{93}$$
$$+ 17517045194250y_{94} + 24368226519980y_{95}. \tag{8}$$

REMARK 4.2. So far, all the results above, using Maple software, are also calculated by Risa/Asir; the outputs are found either on [**8**] by the title "Results by Risa/Asir for wt=16 type1 C^{6}->C^{7}->C^{8}" or in Appendix 12 and 13 in [**9**].

We have the generator of $\mathrm{H}^7_{\mathrm{GF}}(\mathfrak{ham}^0_2, \mathfrak{sp}(2,\mathbb{R}))_{16}$ by two methods. One is h above by Maple. The generator derived by Risa/Asir is $-h$; namely, the negative sign is the only difference.

4.3. Gröbner basis of $d\,(\mathbf{C}^{\mathbf{8}}_{\mathbf{GF}}(\mathfrak{ham}_2, \mathfrak{sp}(2,\mathbb{R}))_{14})$.

Next, we only need information about the bases of $\mathfrak{C}^8$, $\mathfrak{C}^9$ and the matrix representation $\overline{M}$ of $d : \mathfrak{C}^8 \to \mathfrak{C}^9$, where $\mathfrak{C}^k = \mathrm{C}^k_{\mathrm{GF}}(\mathfrak{ham}_2, \mathfrak{sp}(2,\mathbb{R}))_{14}$. These are found either on [**8**] or in Appendix 6, 7 and 8 in [**9**]. Below we only show one of them: One of the 232 elements of our basis of $\mathfrak{C}^8$ is:

$$\begin{aligned}
\overline{\boldsymbol{q}}_{231} = &-\frac{1}{2}z_3^0z_3^1z_3^2z_3^3z_4^0z_4^1z_4^2z_6^6 + z_3^0z_3^1z_3^2z_3^3z_4^0z_4^1z_4^3z_6^5 - \frac{1}{2}z_3^0z_3^1z_3^2z_3^3z_4^0z_4^1z_4^4z_6^4\\
&- \frac{3}{2}z_3^0z_3^1z_3^2z_3^3z_4^0z_4^2z_4^3z_6^4 + z_3^0z_3^1z_3^2z_3^3z_4^0z_4^2z_4^4z_6^3 - \frac{1}{2}z_3^0z_3^1z_3^2z_3^3z_4^0z_4^3z_4^4z_6^2\\
&+ 2z_3^0z_3^1z_3^2z_3^3z_4^1z_4^2z_4^3z_6^3 - \frac{3}{2}z_3^0z_3^1z_3^2z_3^3z_4^1z_4^2z_4^4z_6^2 + z_3^0z_3^1z_3^2z_3^3z_4^1z_4^3z_4^4z_6^1\\
&- \frac{1}{2}z_3^0z_3^1z_3^2z_3^3z_4^2z_4^3z_4^4z_6^0
\end{aligned}$$

and, one of the 113 elements of our basis of $\mathfrak{C}^9$ is:

$$\overline{\boldsymbol{w}}_{95} = -\frac{1}{5}z_1^0z_1^1z_4^0z_4^1z_4^2z_4^3z_4^4z_5^0z_5^5 + z_1^0z_1^1z_4^0z_4^1z_4^2z_4^3z_4^4z_5^1z_5^4 - 2z_1^0z_1^1z_4^0z_4^1z_4^2z_4^3z_4^4z_5^2z_5^3.$$

The matrix $\overline{M}$ of $d : \mathfrak{C}^8 \to \mathfrak{C}^9$ is of size (113,232) and the linear functions $\{\overline{g}_i\}$ corresponding to $d : \mathfrak{C}^8 \to \mathfrak{C}^9$ are given by

$$[\overline{g}_1, \ldots \overline{g}_{232}] = [y_1, \ldots, y_{113}]\overline{M}.$$

We will continue the same discussion as in subsection Section 4.2. We see that $\operatorname{rank}\overline{M} = 87$ and the Gröbner basis $\overline{GB}_e$ of $\{\overline{g}_i\}$, which corresponds to $d\,(\mathfrak{C}^8)$, consists of 87 elements as expected. The complete data of $\{\overline{g}_i\}$, in other words, that of $\overline{M}$, and the detail of $\overline{GB}_e$ are found either on [**8**] or in Appendix 8 and 11 in [**9**].

4.4. $\omega \wedge h$ is not in $d\,(\mathbf{C}^{\mathbf{8}}_{\mathbf{GF}}(\mathfrak{ham}_2, \mathfrak{sp}(2,\mathbb{R}))_{14})$.

We have the linear function h of $\{y_j\}_{j=1}^{95}$ in (8); we know that the cochain $\boldsymbol{h}(\boldsymbol{w})$ is a non-exact kernel element in $\mathrm{C}^7_{\mathrm{GF}}(\mathfrak{ham}^0_2, \mathfrak{sp}(2,\mathbb{R}))_{16}$. We analyze the next element

$$\omega \wedge h(\boldsymbol{w}) = z_1^0 \wedge z_1^1 \wedge h(\boldsymbol{w})$$

by the basis of $\mathfrak{C}^9$, and we have a linear function $\overline{h}$ of $\{y_i\}_{i=1}^{113}$ satisfying

$$\overline{h}(\overline{\boldsymbol{w}}) = \omega \wedge h(\boldsymbol{w}) = z_1^0 \wedge z_1^1 \wedge h(\boldsymbol{w})$$

which is given by the following:

$$\begin{aligned}\overline{h} = &-6996191251500y_{74} - 1557312364575y_{76} + 2027141067600y_{77}\\ &+ 6871115344500y_{78} - 8293793595120y_{79} + 1593871052400y_{80}\\ &+ 3342315930030y_{81} + 3576568317699y_{82} - 1206881491512y_{83}\\ &- 3952406350359y_{84} - 21353158325775y_{85} - 21096249215580y_{86}\\ &- 9572836551300y_{87} + 3867959161440y_{88} + 10699190322480y_{89}\\ &- 23511502274085y_{90} + 2799062316375y_{91} + 17460883387175y_{92}\\ &+ 17517045194250y_{93} - 43245161055925y_{94} - 121841132599900y_{95}.\end{aligned}$$

The normal form of $\overline{h}$ with respect to $\overline{GB}_e$ is

$$\begin{aligned}-\frac{1}{1191}(&7443523237284708y_{82} + 10932577142466y_{83} - 2773751000717088y_{84}\\ &- 8746061713972800y_{85} - 9309703351771180y_{90}\\ &+ 40450535427124200y_{91} - 30933987324063320y_{92}\\ &+ 24871612999110150y_{93} + 54636855766752700y_{94}\\ &+ 201445748822724700y_{95} + 1180249792365936600y_{112}\\ &+ 3540749377097809800y_{113})\end{aligned}$$

and is not zero, namely $\overline{h}(\overline{\boldsymbol{w}}) = \omega \wedge h(\boldsymbol{w}) = z_1^0 \wedge z_1^1 \wedge h(\boldsymbol{w})$ is not exact, and our proof is complete. □

REMARK 4.3. Throughout this paper, the Gröbner basis and the normal form are computed by Maple. On the other hand, the results by Risa/Asir are found either on [**8**] or in Appendix 12 and 13 in [**9**].

We denote by B_{maple} the normal form of $\overline{h}$ with respect to $\overline{GB}_e$ and by A_{asir}, the normal form calculated by Risa/Asir. The two are related as

$$B_{maple} = -\frac{1}{1191} \cdot \frac{7443523237284708}{5337006161133135636} A_{asir}.$$

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