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Hypoelliptic Laplacian and probability

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Abstract. The purpose of this paper is to describe the probabilistic aspects underlying the theory of the hypoelliptic Laplacian, as a deformation of the standard elliptic Laplacian. The corresponding diffusion on the total space of the tangent bundle of a Riemannian manifold is a geometric Langevin process, that interpolates between the geometric Brownian motion and the geodesic flow. Connections with the central limit theorem for the occupation measure by the geometric Brownian motion are emphasized. Spectral aspects of the hypoelliptic deformation are also provided on tori. The relevant hypoelliptic deformation of the Laplacian in the case of Riemann surfaces of constant negative curvature is briefly described, in connection with Selberg's trace formula.

Introduction.

The purpose of this paper is to describe the probabilistic aspects that underlie the construction of the hypoelliptic Laplacian, at the crossroads of two unrelated theories: index theory, and the central limit theorem.

1. The index of an elliptic operator is the difference of the dimensions of its kernel and its cokernel. It only depends on the principal symbol of the operator. The Atiyah-Singer index theorem [**AS68a**], [**AS68b**] gives a cohomological formula for the index in terms of characteristic classes.
2. The particular version of the central limit theorem we think of describes the asymptotics of the occupation measure by a geometric Brownian motion as time tends to infinity in terms of a Gaussian random field. This result can also be formulated as the study of the lowest eigenvalue of a family of second order differential operators as a parameter b tends to 0, all the other eigenvalues tending to $+\infty$.

A possibly common feature of these two theories is that they are both concerned with small eigenvalues, the zero eigenvalue of certain operators in the case of index theory, and small eigenvalues in the case of the central limit theorem.

Before we explain the content of the present paper, we will give the proper perspective to the theory of the hypoelliptic Laplacian.

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0.1. The scalar version of the hypoelliptic Laplacian.

Let us describe the hypoelliptic Laplacian in its simplest form. Let X be a compact Riemannian manifold of dimension n, and let $\mathcal{X}$ be the total space of its tangent bundle. Let H^{TX}, A^{TX} be the harmonic oscillators along the fibres TX

$$H^{TX} = \frac{1}{2}(-\Delta^V + |Y|^2 - n), \qquad A^{TX} = \frac{1}{2}(-\Delta^V + 2\nabla^V_Y). \tag{0.1}$$

In (0.1), Δ^V denotes the Laplacian along the fibre TX, Y is the generic element of TX, and ∇^V_Y denote differentiation along the fibre with respect to the radial vector field Y. The two harmonic oscillators are conjugate via the gaussian function $\exp(-|Y|^2/2)$.

Let U be the vector field on $\mathcal{X}$ that generates the geodesic flow. In geodesic coordinates centred at $x \in X$, we have

$$U(x, Y) = \sum_{i=1}^{n} Y^i \frac{\partial}{\partial x^i}. \tag{0.2}$$

For $b > 0$, set

$$L^X_b = \frac{H^{TX}}{b^2} - \frac{U}{b}, \qquad M^X_b = \frac{A^{TX}}{b^2} - \frac{U}{b}. \tag{0.3}$$

The operators in (0.3) are conjugate by the same Gaussian as before. They are not self-adjoint. By a theorem of Hörmander [**H67**] on second operators of the form $\sum_{i=1}^m X_i^2 + X_0$[1], if t is an extra time coordinate in $\mathbf{R}_+$, the operators $\partial/\partial t + L^X_b$, $\partial/\partial t + M^X_b$ are hypoelliptic. This is actually the form of Hörmander's theorem in which X_0 plays an essential role[2]. The scalar operators L^X_b, M^X_b are the simplest version of the hypoelliptic Laplacian.

Such operators have appeared before. The operator of Kolmogorov [**K34**] on $\mathbf{R}^2$ given by $K = -(1/2)(\partial^2/\partial y^2) - y(\partial/\partial x)$ can be thought of as a prototype of the hypoelliptic Laplacian. When $X = \mathbf{R}^3$, the operators L^X_b, M^X_b are known as Fokker-Planck operators and appear in statistical physics. In this case, M^X_b is the infinitesimal generator of a process $(x_\cdot, Y_\cdot)$ that projects on X to a Langevin process [**L08**].

One first result is that the equivalent families of operators L^X_b, M^X_b interpolate in the proper sense between the standard Laplacian $-\Delta^X/2$ on X as $b \to 0$, and the proper version of the generator of the geodesic flow $(1/2)|Y|^2 - U$ as $b \to +\infty$. To be more precise on the $b \to 0$ convergence, the families of operators L^X_b, M^X_b acting on $\mathcal{X}$ collapse[3] to the operator $-\Delta^X/2$ acting on X.

[1]A sufficient condition for the hypoellipticity of such operators is that the distribution spanned by $X_0, \dots, X_m$ and their Lie brackets of arbitrary order is the full tangent bundle.

[2]In this respect, our hypoelliptic operators are different from the Heisenberg Laplacians in subriemannian geometry.

[3]In Riemannian geometry, if the metric on the fibres of a fibration tends to zero, one says that the original manifold collapses to the base manifold. Here, while $\mathcal{X}$ is not viewed as a Riemannian manifold, certain aspects of collapsing remain true.

0.2. Hypoelliptic Laplacian and the central limit theorem.

Let V be a smooth function $X \to \mathbf{R}$ such that $\int_X V(x)dx = 0$. One way to obtain a central limit theorem for the associated Brownian motion $x.$ on X is to study the spectral asymptotics as $b \to 0$ of the family of operators

$$S_b^X = -\frac{\Delta^X}{2b^2} + \frac{V}{b}. \tag{0.4}$$

Indeed if $b^2 = 1/t$, then

$$S_b^X = t\Big(-\frac{\Delta^X}{2} + \frac{V}{\sqrt{t}}\Big), \tag{0.5}$$

and making $b \to 0$ is equivalent to making $t \to +\infty$. By comparing (0.3) and (0.4), the structures of M_b^X and S_b^X are very similar, when replacing $-\Delta^X/2$ by the harmonic oscillator A^{TX}. Since the fibres TX are the fibres of a vector bundle, the analytic and geometric situations are not the same.

0.3. The dynamical version of the interpolation.

The diffusion process corresponding to the operator M_b^X can be described in terms of the second order stochastic differential equation on X

$$b^2\ddot{x} + \dot{x} = \dot{w}. \tag{0.6}$$

In (0.6), $w.$ is a Brownian motion along the fibres TX suitably transported using the Levi-Civita connection on TX, and $\dot{w}$ denotes its differential in the sense of Stratonovitch. The Markov diffusion process associated with M_b^X is just given by $(x., b\dot{x}.)$. The speed $\dot{x}.$ of $x.$ is an Ornstein-Uhlenbeck process $Z.$ with covariance $(1/b^2)\exp(-|t-s|/b^2)$.

If one ignores the probabilistic difficulties, we find that when $b = 0$, equation (0.6) is just $\dot{x} = \dot{w}$, the equation for the Brownian motion on X, and when $b = +\infty$, it becomes $\ddot{x} = 0$, the equation of geodesics on X. At the algebraic level, the interpolation property described at the end of Subsection 0.1 is obvious.

0.4. The preservation of the spectrum.

For a suitable nonscalar version of the hypoelliptic Laplacian $\mathcal{L}_b^X$ described in [**B11**] when X is a locally symmetric space, the spectrum of $-\Delta^X/2$ remains rigidly embedded in the spectrum of $\mathcal{L}_b^X$. In [**B11**], the trace of the heat kernel $\exp(s\Delta^X/2)$ is viewed as a generalized Euler characteristic, and the above rigidity property as a formal consequence of index theory.

The version of the central limit theorem we described in (0.4), (0.5) is concerned with the lower part of the spectrum of the operators S_b^X, while here the whole spectrum of $-\Delta^X/2$ is preserved. There is no contradiction. Indeed the operator $-\Delta^X/2$ acts on $C^\infty(X, \mathbf{R})$, which can be identified with the kernel of the nonnegative operator A^{TX}.

A construction by Witten [**W82**] shows that the harmonic oscillator is just the restriction to smooth functions of the Hodge Laplacian associated with the Witten complex of the considered vector space. Unsurprisingly, the corresponding family of Witten

Laplacians along TX plays an important role in the construction of $\mathcal{L}_b^X$ in [**B11**].

As a consequence of the above, in [**B11**] we show that as $b \to +\infty$, the supertrace of the heat kernel for $\mathcal{L}_b^X$ localizes near the closed geodesics in X. Ultimately we obtain the Selberg's trace formula [**M72**] as a consequence of the interpolation procedure we just described. This version is valid not only for Riemann surfaces of genus $g \geq 2$, but for compact locally symmetric spaces of arbitrary dimension.

0.5. The hypoelliptic Laplacians.

The principal symbol of $-\Delta^X$ is just $|\xi|^2$. The geometric Laplacians, like the Hodge Laplacians in de Rham or Dolbeault theory, or the squares of Dirac operators have the same scalar principal symbol $|\xi|^2$. These operators have canonical hypoelliptic deformations in their category [**B05**], [**B08a**], [**B11**]. This means that there is not only one hypoelliptic Laplacian, but many. In the present paper, we will mostly limit ourselves to the deformation of the scalar operator $-\Delta^X/2$.

The original version of the hypoelliptic Laplacian in de Rham theory developed in [**B05**] exists over any compact Riemannian manifold. It can be viewed as a semiclassical version of the Witten deformation [**W82**] of the standard Hodge Laplacian on the loop space LX[4], that would be associated with the energy functional. The preservation under hypoelliptic deformation of certain spectral invariants like the analytic torsion [**RS71**] was established by Lebeau and ourselves in [**BL08**]. For the relevant probabilistic aspects of the hypoelliptic Laplacian in de Rham theory, we refer to [**B08b**]. The Malliavin calculus also plays an essential role in [**B11**] to obtain the proper uniform control of the hypoelliptic heat kernels. For applications of the hypoelliptic Laplacian to complex geometry, we refer to [**B08a**], [**B13**].

0.6. The organization of the paper.

In the first two sections of the paper, we state a number of results concerning the central limit theorem for the occupation time of a geometric Brownian motion and for the Ornstein-Uhlenbeck process. These two sections give the proper perspective to the construction of the hypoelliptic Laplacian in the three sections that follow.

More precisely, in Section 1, we recall certain aspects of the central limit theorem for the Brownian motion on a compact Riemannian manifold, in commutative and noncommutative form.

In Section 2, we explain the commutative and noncommutative aspects of the central limit theorem associated with the Ornstein-Uhlenbeck process.

In Section 3, we construct the scalar hypoelliptic Laplacian on a vector space and on a torus.

In Section 4, we explain elementary aspects of index theory in connection with the constructions of Subsection 3.

Finally, in Section 5, we construct the scalar hypoelliptic Laplacians L_b^X, M_b^X, and we describe their main properties. Also we show how the index theoretic aspects of the constructions of Section 4, that are valid for vector spaces or for tori, can be suitably

[4]The proper cohomological theory is equivariant cohomology with respect to the obvious action of S^1 on LX. No rigorous construction of the corresponding Hodge Laplacian or of its Witten deformation is available.

extended to the case of Riemann surfaces with constant negative curvature.

The probabilistic aspects of the hypoelliptic Laplacian have played a dominant role in its development. For a survey of other aspects of the hypoelliptic Laplacian, in particular in connection with Morse theory, and with the classical Witten Laplacian, we refer to [**B08b**], [**B08c**].

1. The central limit theorem for the geometric Brownian motion.

In this section, we review some aspects of the central limit theorem for the Brownian motion on a compact Riemannian manifold, in its commutative and noncommutative versions. This will permit us to give the proper perspective to the construction of the hypoelliptic Laplacian, and to some of its properties.

At least in the scalar case, the results contained in this section are already known in a form or another. They are the elementary starting point of the theory of large deviations for occupation times by a Markov process or a diffusion process [**DSt89**].

This section is organized as follows. In Subsection 1.1, we study the behaviour of the spectrum of the operator S_b^X in (0.4) and of its heat kernel as $b \to 0$.

In Subsection 1.2, we interpret the results of Subsection 1.1 in terms of a central limit theorem for the occupation 'density' by the Brownian motion.

Finally, in Subsection 1.3, we give a noncommutative version of the above results for Bochner Laplacians acting on sections of a Hermitian vector bundle.

1.1. The spectrum of S_b^X as $b \to 0$.

Let X be a compact Riemannian manifold, let g^{TX} be the corresponding metric on TX, and let dx be the associated volume form. Let Δ^X be the Laplace-Beltrami operator. For $1 \leq p \leq +\infty$, let L_p^X denote the real L_p space associated with dx. In particular $\langle\,\rangle_{L_2^X}$ denotes the canonical scalar product on L_2^X, and its extension to a symmetric complex valued bilinear form on $L_2^X \otimes_{\mathbf{R}} \mathbf{C}$.

Here $\mathbf{R}$ is the vector space of constants in $C^\infty(X, \mathbf{R})$. Let $P : L_2^X \to \mathbf{R}$ denote the corresponding orthogonal projection operator. If $f \in L_2^X$, then

$$Pf = \frac{1}{\mathrm{Vol}(X)} \int_X f(x)dx. \tag{1.1}$$

Let $\mathbf{R}^\perp$ be the orthogonal space to $\mathbf{R}$ in L_2^X. The corresponding orthogonal projection operator is given by

$$P^\perp = 1 - P. \tag{1.2}$$

For V, $W \in C^\infty(X, \mathbf{C})$, $b > 0$, let S_b^X be the second order elliptic operator on X

$$S_b^X = -\frac{\Delta^X}{2b^2} + \frac{V}{b} + W. \tag{1.3}$$

For $s > 0$, let $\exp(-sS_b^X)(x, x')$ be the smooth kernel associated with $\exp(-sS_b^X)$ with respect to dx'. Let $\mathrm{Tr}[\exp(-sS_b^X)]$ denote the trace of $\exp(-sS_b^X)$ so that

$$\operatorname{Tr}\left[\exp(-sS_b^X)\right] = \int_X \exp\left(-S_b^X\right)(x,x)dx. \tag{1.4}$$

Let $(-\Delta^X/2)^{-1}$ be the inverse of $-\Delta^X/2$ restricted to $\mathbf{R}^\perp$. Then $(-\Delta^X/2)^{-1}$ is a bounded self-adjoint positive operator.

Theorem 1.1. *If $V \in C^\infty(X, \mathbf{C})$ is such that $\int_X V(x)dx = 0$, for $s > 0$, as $b \to 0$, we have the uniform convergence of smooth functions and their derivatives of any order on $X \times X$,*

$$\exp\left(-sS_b^X\right)(x,x') \to \frac{\exp\left(\frac{s}{\operatorname{Vol}(X)}\left(\langle V, (-\Delta^X/2)^{-1}V\rangle_{L_2^X} - \int_X W(x)dx\right)\right)}{\operatorname{Vol}(X)}. \tag{1.5}$$

In particular, for $s > 0$, as $b \to 0$,

$$\operatorname{Tr}\left[\exp(-sS_b^X)\right] \to \exp\left(\frac{s}{\operatorname{Vol}(X)}\left(\left\langle V, (-\Delta^X/2)^{-1}V\right\rangle_{L_2^X} - \int_X W(x)dx\right)\right). \tag{1.6}$$

If V, W are real, if σ_b is the lowest eigenvalue of the self-adjoint operator S_b^X, as $b \to 0$,

$$\sigma_b \to \frac{1}{\operatorname{Vol}(X)}\left(-\left\langle V, (-\Delta^X/2)^{-1}V\right\rangle_{L_2^X} + \int_X W(x)dx\right), \tag{1.7}$$

and the other eigenvalues tend to $+\infty$. When V, W are arbitrary, equation (1.7) *also holds for one simple eigenvalue of S_b^X, the real part of the other eigenvalues tending to $+\infty$.*

Proof. In the sequel, we will write our operators as $(2,2)$ matrices with respect to the orthogonal splitting $L_2^X = \mathbf{R} \oplus \mathbf{R}^\perp$. In particular, we have identities of the form

$$-\frac{1}{2}\Delta^X = \begin{bmatrix} 0 & 0 \\ 0 & \alpha \end{bmatrix}, \qquad V = \begin{bmatrix} 0 & \beta_2 \\ \beta_3 & \beta_4 \end{bmatrix}, \quad W = \begin{bmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_4 \end{bmatrix}. \tag{1.8}$$

Put

$$S_1 = \gamma_1, \quad S_2 = \beta_2 + \gamma_2 b, \quad S_3 = \beta_3 + \gamma_3 b, \quad S_4 = \alpha + \beta_4 b + \gamma_4 b^2. \tag{1.9}$$

Then

$$S_b^X = \begin{bmatrix} S_1 & \dfrac{S_2}{b} \\ \dfrac{S_3}{b} & \dfrac{S_4}{b^2} \end{bmatrix}. \tag{1.10}$$

Now we proceed as in [**BL08**, Section 17.2]. If $\lambda \in \mathbf{C}$, put

$$D_4 = S_4 - b^2\lambda. \tag{1.11}$$

Let $\lambda \in \mathbf{C}$ be such that D_4 is invertible. Set

$$H = \lambda - S_1 + S_2 D_4^{-1} S_3. \tag{1.12}$$

We also assume that H is invertible. By (1.9), at least formally, $\lambda - S_b^X$ is invertible, and we can write $(\lambda - S_b^X)^{-1}$ in the form

$$(\lambda - S_b^X)^{-1} = \begin{bmatrix} H^{-1} & -bH^{-1}S_2D_4^{-1} \\ -bD_4^{-1}S_3H^{-1} & -b^2D_4^{-1} + b^2D_4^{-1}S_3H^{-1}S_2D_4^{-1} \end{bmatrix}. \tag{1.13}$$

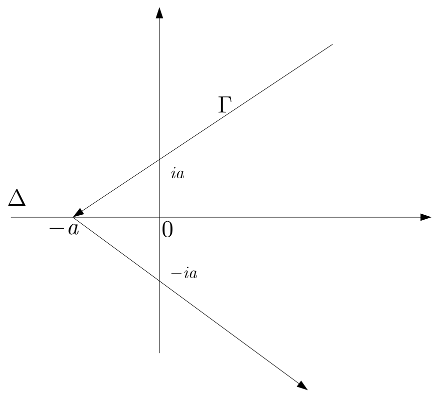

Figure 1.

For $a > 0$, let $\Gamma = \Gamma_a$ be the contour in $\mathbf{C}$ described in Figure 1, and let $\Delta = \Delta_a$ be the closure of the connected component of $\mathbf{C} \setminus \Gamma$ not containing 0. One verifies easily that there exist $a > 0, b_0 > 0$ such that for $0 < b \leq b_0$, if $\lambda \in \Delta$, D_4 and H are invertible, and moreover, if $u \in \mathbf{R}^\perp$,

$$\|D_4^{-1}u\|_0 \leq C\|u\|_0. \tag{1.14}$$

Moreover, given $k \in \mathbf{N}$, there exists $m_k \in \mathbf{N}$ such that for $0 < b \leq b_0$, $\lambda \in \Delta$, $u \in H^k \cap \mathbf{R}^\perp$, then

$$\|D_4^{-1}u\|_{k+1} \leq C_k(1 + |\lambda|)^{m_k}\|u\|_k. \tag{1.15}$$

We have the trivial identity

$$D_4^{-1} - S_4^{-1} = b^2\lambda S_4^{-1}D_4^{-1}. \tag{1.16}$$

By (1.16), we find that if u is taken as before,

$$\left\|(D_4^{-1} - S_4^{-1})u\right\|_0 \leq Cb^2|\lambda|\|u\|_0, \tag{1.17}$$

and moreover, for any $k \in \mathbf{N}$,

$$\left\|(D_4^{-1} - S_4^{-1})u\right\|_{k+1} \leq Cb^2(1+|\lambda|)^{m_k+1}\|u\|_k. \tag{1.18}$$

By (1.13) and by the above, we find that if $u \in C^\infty(X, \mathbf{C})$, if $\lambda \in \Delta$, then

$$\left\|(\lambda - S_b^X)^{-1}u - P\big(\lambda + PVP^\perp(-\Delta^X/2)^{-1}P^\perp VP - PWP\big)^{-1}Pu\right\|_0 \leq Cb\|u\|_0, \tag{1.19}$$

and moreover, for any $k \in \mathbf{N}$, we have

$$\begin{aligned}\Big\|(\lambda - S_b^X)^{-1}u - P\big(\lambda + PVP^\perp(-\Delta^X/2)^{-1}P^\perp VP - PWP\big)^{-1}Pu\Big\|_{k+1} \\ \leq C_k b(1+|\lambda|)^{m_k+1}\|u\|_k.\end{aligned} \tag{1.20}$$

Clearly, for $0 < b \leq b_0, s > 0$, we have the identity

$$\exp(-sS_b^X) = \frac{1}{2i\pi}\int_\Gamma \frac{\exp(-s\lambda)}{\lambda - S_b^X}d\lambda. \tag{1.21}$$

By integration by parts, for any $k \in \mathbf{N}$, we get

$$\exp(-sS_b^X) = (-1)^k\frac{k!}{2i\pi s^k}\int_\Gamma \frac{\exp(-s\lambda)}{(\lambda - S_b^X)^{k+1}}d\lambda. \tag{1.22}$$

By (1.19) and (1.21), we find that for $s > 0$, as $b \to 0$, we have the strong convergence of operators acting on $L_2^X \otimes_{\mathbf{R}} \mathbf{C}$

$$\exp(-sS_b^X) \to \exp\bigg(\frac{s}{\mathrm{Vol}(X)}\bigg(\big\langle V, (-\Delta^X/2)^{-1}V\big\rangle_{L_2^X} - \int_X W(x)dx\bigg)\bigg)P. \tag{1.23}$$

Using (1.20) and (1.22), we deduce easily that the convergence in (1.23) is a uniform convergence of smooth kernels together with their derivatives of arbitrary order. By (1.1), we obtain the first part of our theorem. By integrating the left-hand side of (1.5) on the diagonal, we get (1.6).

If V, W are real, the eigenvalues of S_b^X are real. Also $s > 0 \to \mathrm{Tr}[\exp(-sS_b^X)]$ is the Laplace transform of the spectral measure for S_b^X. From (1.6), we get the last part of our theorem. In the general case, using the convergence of the resolvents in (1.19), (1.20), we obtain the corresponding result on the eigenvalues.

We will now give a second proof of our theorem when $V \in C^\infty(X, \mathbf{R})$. Let $f \in C^\infty(X, \mathbf{R})$, let $\square_T^X$ be the Witten Laplacian [**W82**] associated with f, and let $-\Delta_f^X$ be its restriction in degree 0. Then

$$-\Delta_f^X = -\Delta^X - \Delta^X f + |\nabla f|^2. \tag{1.24}$$

The operator $-\Delta_f^X$ is self-adjoint and nonnegative, and its kernel is 1-dimensional and spanned by e^{-f}.

In the sequel, we take $V \in C^\infty(X, \mathbf{R})$ such that $PV = 0$. Set

$$f = (-\Delta^X/2)^{-1} V. \tag{1.25}$$

Equation (1.3) can be rewritten in the form

$$S_b^X = -\frac{\Delta_{bf}^X}{2b^2} - \frac{1}{2}|\nabla f|^2 + W. \tag{1.26}$$

Let $\mathbf{R}_b$ be the vector space spanned by e^{-bf} and let $\mathbf{R}_b^\perp$ be its orthogonal in L_2^X. We write our operators as $(2,2)$ matrices with respect to the splitting $L_2^X = \mathbf{R}_b \oplus \mathbf{R}_b^\perp$. In particular

$$-\frac{1}{2}\Delta_{bf}^X = \begin{bmatrix} 0 & 0 \\ 0 & \alpha_{bf} \end{bmatrix}, \qquad -\frac{1}{2}|\nabla f|^2 + W = \begin{bmatrix} \delta_1 & \delta_2 \\ \delta_3 & \delta_4 \end{bmatrix}. \tag{1.27}$$

Note that for $b = 0$, $\alpha_{bf} = \alpha$.

By (1.26), (1.27), we have the analogue of (1.10)

$$S_b^X = \begin{bmatrix} \delta_1 & \delta_2 \\ \delta_3 & \dfrac{1}{b^2}(\alpha_{bf} + b^2\delta_4) \end{bmatrix}. \tag{1.28}$$

Using (1.28), we can now proceed exactly as before, except that the matrix structure is simpler. Of course, the splitting of L_2^X now depends on b. However, this dependence on b can be easily dropped by using the fact that $\mathbf{R}_b$ is spanned by e^{bf}. By proceeding as in (1.19), (1.20), we find in particular that as $b \to 0$, if $\lambda \in \Delta$,

$$\left(\lambda - S_b^X\right)^{-1} \to P\left(\lambda - P\left(-\frac{1}{2}|\nabla f|^2 + W\right)P\right)^{-1} P. \tag{1.29}$$

The same arguments as in (1.23) show that for any $s > 0$, as $b \to 0$

$$\exp\left(-sS_b^X\right) \to \exp\left(sP\left(\frac{1}{2}|\nabla f|^2 - W\right)P\right)P, \tag{1.30}$$

the convergence in (1.30) being a convergence of smooth kernels on $X \times X$. By (1.1), we get

$$P\left(\frac{1}{2}|\nabla f|^2 - W\right)P = \frac{1}{\mathrm{Vol}(X)}\int_X \left(\frac{1}{2}|\nabla f|^2 - W\right)(x)dx. \tag{1.31}$$

By (1.25), we obtain

$$\int_X |\nabla f|^2(x)dx = 2\big\langle V, (-\Delta^X/2)^{-1}V\big\rangle_{L_2^X}. \tag{1.32}$$

By (1.30)–(1.32), we get (1.5). The second proof of our theorem is completed. □

REMARK 1.2. The first proof of Theorem 1.1 given above gives some idea of how corresponding results for the hypoelliptic Laplacian are established in [**BL08**].

1.2. The central limit theorem for the geometric Brownian motion. Let $C(\mathbf{R}_+, X)$ denote the space of continuous functions from $\mathbf{R}_+$ into X equipped with the topology of uniform convergence over compact sets, and let $s \in \mathbf{R}_+ \to x_s \in X$ denote its generic element. Given $x \in X$, let P_x denote the probability measure on $C(\mathbf{R}_+, X)$ associated with the Brownian motion starting at x at time 0.

By the ergodic theorem, we know that as $t \to +\infty$,

$$\frac{1}{t}\int_0^t W(x_s)ds \to \frac{\int_X W(x)dx}{\mathrm{Vol}(X)} \; P_x \text{ a.s.}. \tag{1.33}$$

For $a > 0$, let $N(a)$ denote the probability law on $\mathbf{R}$ of the centred Gaussian with variance a.

THEOREM 1.3. *If $V \in C^\infty(X, \mathbf{R})$ is such that $\int_X V(x)dx = 0$, for any $x \in X$, as $t \to +\infty$, the probability law of the process $s \in \mathbf{R}_+ \to (1/\sqrt{t})\int_0^{st} V(x_u)du$ converges to the probability law of the process $s \in \mathbf{R}_+ \to cb_s$, where b_s is a real Brownian motion starting at 0, and*

$$c^2 = 2\frac{\langle(-\Delta^X/2)^{-1}V, V\rangle_{L_2^X}}{\mathrm{Vol}(X)}. \tag{1.34}$$

As $t \to +\infty$, the probability law of the process $s \in \mathbf{R}_+ \to (1/t)\int_0^{st} W(x_u)du$ converges to the deterministic process $s \in \mathbf{R}_+ \to s(\int_X W(x)dx/\mathrm{Vol}(X))$.

For $0 < s_1 < \cdots < s_m$, as $t \to +\infty$, the probability law of $(x_{s_1 t}, x_{s_2 t}, \dots, x_{s_m t})$ converges to the product of the probability laws $dx/\mathrm{Vol}(X)$.

Finally, as $t \to +\infty$, the joint law of the above random processes and random variables converges to the corresponding product law.

PROOF. We use the notation of Subsection 1.1 with V replaced by iV, and $W = 0$. Observe that for $t > 0$,

$$S^X_{1/\sqrt{t}} = t\Big(-\frac{1}{2}\Delta^X + i\frac{V}{\sqrt{t}}\Big). \tag{1.35}$$

By (1.35) and by Feynman-Kac's formula, if $u \in C^\infty(X, \mathbf{R})$, for $s > 0$, we get

$$\exp\big(-sS^X_{1/\sqrt{t}}\big)u(x) = E^{P_x}\Big[\exp\Big(-i\frac{\int_0^{st} V(x_u)du}{\sqrt{t}}\Big)u(x_{st})\Big]. \tag{1.36}$$

By Theorem 1.1, as $t \to +\infty$, we have the uniform convergence of smooth functions and their derivatives of any order

$$\exp\big(-sS^X_{1/\sqrt{t}}\big)u \to \exp\bigg(-s\frac{\langle V, (-\Delta^X/2)^{-1}V\rangle_{L_2^X}}{\mathrm{Vol}(X)}\bigg)\frac{\int_X u(x)dx}{\mathrm{Vol}(X)}. \tag{1.37}$$

By (1.36), (1.37), as $t \to +\infty$

$$E^{P_x}\bigg[\exp\bigg(-i\frac{\int_0^{st} V(x_u)du}{\sqrt{t}}\bigg)\bigg] \to \exp\bigg(-s\frac{\langle V, (-\Delta^X/2)^{-1}V\rangle_{L_2^X}}{\mathrm{Vol}(X)}\bigg). \tag{1.38}$$

For $\alpha \in \mathbf{R}$, we may as well replace V by αV in (1.38). Using Paul Lévy's theorem, as $t \to +\infty$, given $s > 0$, we have the convergence of probability laws,

$$\frac{1}{\sqrt{t}}\int_0^{st} V(x_u)du \to N\bigg(2s\frac{\langle(-\Delta^X/2)^{-1}V, V\rangle_{L_2^X}}{\mathrm{Vol}(X)}\bigg). \tag{1.39}$$

Let us now give another proof of (1.39), that will be the probabilistic counterpart to the second proof of Theorem 1.1. We still define f as in (1.25). By Itô's formula[5], we get

$$f(x_t) = f(x) - \int_0^t V(x_u)du + \int_0^t \langle\nabla f(x_u), \delta x_u\rangle, \tag{1.40}$$

where the last term in (1.40) is a classical Itô stochastic integral. By (1.40), we get

$$\frac{\int_0^{st} V(x_s)ds}{\sqrt{t}} = \frac{f(x) - f(x_{st})}{\sqrt{t}} + \frac{\int_0^{st}\langle\nabla f(x_u), \delta x_u rangle}{\sqrt{t}}. \tag{1.41}$$

Now we proceed as Franchi-Le Jan [**FL12**, Lemma 8.7.4] in their proof of the central limit theorem for martingales. For $\alpha \in \mathbf{R}$, put

$$Z_{\alpha,t,s} = \exp\bigg(\sqrt{-1}\alpha\frac{\int_0^s\langle\nabla f(x_u), \delta x_u\rangle}{\sqrt{t}} + \frac{\alpha^2}{2}\frac{\int_0^s |\nabla f(x_u)|^2du}{t}\bigg). \tag{1.42}$$

Given $t > 0$, $Z_{\alpha,t,s}|_{s\in\mathbf{R}_+}$ is a martingale and

$$E^{P_x}[Z_{\alpha,t,s}] = 1. \tag{1.43}$$

By the ergodic theorem, given $s \geq 0$, as $t \to +\infty$, we get

[5] In the sequel, we denote by δ the differential in the sense of Itô, and by d the differential in the sense of Stratonovitch.

$$\frac{\int_0^{st} |\nabla f(x_u)|^2 du}{t} \to s \frac{\int_X |\nabla f(x)|^2 dx}{\mathrm{Vol}(X)} \, P_x \text{ a.e..} \tag{1.44}$$

By (1.42)–(1.44), we find easily that as $t \to +\infty$,

$$E^{P_x}\left[\exp\left(\sqrt{-1}\alpha \frac{\int_0^{st} \langle \nabla f(x_s), \delta x_s \rangle}{\sqrt{t}}\right)\right] \to \exp\left(-\frac{\alpha^2 s}{2} \frac{\int_X |\nabla f(x)|^2 dx}{\mathrm{Vol}(X)}\right). \tag{1.45}$$

By Paul Lévy's theorem, we deduce from (1.45) that given $s \geq 0$, as $t \to +\infty$, we have the convergence of probability laws

$$\frac{\int_0^{st} \langle \nabla f(x_s), \delta x_s \rangle}{\sqrt{t}} \to N\left(s \frac{\int_X |\nabla f(x)|^2 dx}{\mathrm{Vol}(X)}\right). \tag{1.46}$$

Since f is bounded, from (1.41), (1.46), as $t \to +\infty$, we have the convergence of probability laws

$$\frac{\int_0^{st} V(x_u) du}{\sqrt{t}} \to N\left(s \frac{\int_X |\nabla f(x)|^2 dx}{\mathrm{Vol}(X)}\right). \tag{1.47}$$

By (1.32), (1.47), we get another proof of (1.39).

To complete the proof of our theorem, the essential point is to show that the probability laws of the continuous processes $s \in \mathbf{R}_+ \to (1/\sqrt{t}) \int_0^{st} V(x_u) du$ form a tight set of probability measures. By Burkholder-Davis-Gundy's inequalities, given $p > 1$, for $0 \leq s \leq s'$

$$\left\| \frac{1}{\sqrt{t}} \int_{st}^{s't} \langle \nabla f(x_u), \delta x_u \rangle \right\|_p \leq C_p (s' - s)^{1/2}. \tag{1.48}$$

By (1.40), (1.48), we obtain the required compactness argument. This concludes the proof of the first part of our theorem.

By (1.33), if s varies in compact sets in $\mathbf{R}_+^*$, P_x a.s., $(1/t) \int_0^{st} W(x_u) du$ converges uniformly to $s \to s(\int_X W(x) dx / \mathrm{Vol}(X))$. Since W is bounded, this family of functions is equicontinuous, so that uniform convergence also takes place near 0.

Using Theorem 1.1 with $V = 0$, $W = 0$, we obtain our result on the convergence as $t \to +\infty$ of the joint law of $(x_{s_1 t}, \dots, x_{s_m t})$. By using the full strength of Theorem 1.1, we obtain the required convergence of the joint probability laws. The proof of our theorem is completed. □

Remark 1.4. Theorem 1.3 can be reformulated as a result of convergence in law of a scaled local time centred field for the process $x.$ to the free field on X.

1.3. The case of vector bundles.

We take X as before. Let (F, g^F, ∇^F) be a complex Hermitian vector bundle on X equipped with a Hermitian connection. Let $L_2^{X,F}$ be the vector space of square-integrable sections of F. Let $A^{X,F}$ be an elliptic self-adjoint nonnegative operator of order 2 acting on $C^\infty(X,F)$. Set

$$H=\ker A^{X,F}. \tag{1.49}$$

Then H is a finite dimensional vector subspace of $C^\infty(X,F)$. Let P be the orthogonal projection operator on H. Then P is given by a smooth kernel $P(x,x')$. Set

$$P^\perp=1-P. \tag{1.50}$$

Then $P^\perp$ is the orthogonal projection on the orthogonal space $H^\perp$ to H in $L_2^{X,F}$. Let $(A^{X,F})^{-1}$ denote the inverse of $A^{X,F}$ acting on $H^\perp$.

Let $V,W\in C^\infty(X,\mathrm{End}(F))$. In the sequel, we assume that

$$PVP=0. \tag{1.51}$$

For $b>0$, set

$$S_b^{X,F}=\frac{A^{X,F}}{b^2}+\frac{V}{b}+W. \tag{1.52}$$

Note that $PVP^\perp(A^{X,F})^{-1}P^\perp VP$ is an endomorphism of H. It can also be viewed as an operator acting on $C^\infty(X,F)$ with a smooth kernel.

Theorem 1.5. *For any $s>0$, as $b\to 0$, we have the uniform convergence of smooth kernels together with their derivatives of any order on $X\times X$*

$$\exp\big(-sS_b^{X,F}\big)(x,x')\to\big(P\exp(s(PVP^\perp(A^{X,F})^{-1}P^\perp VP-PWP))P\big)(x,x'). \tag{1.53}$$

In particular, as $b\to+\infty$

$$\mathrm{Tr}\left[\exp(-sS_b^{X,F})\right]\to\mathrm{Tr}\left[P\exp(s(PVP^\perp(A^{X,F})^{-1}P^\perp VP-PWP))P\right]. \tag{1.54}$$

When $b\to 0$, a finite number of eigenvalues of $S_b^{X,F}$ converges to the eigenvalues of $-PVP^\perp(A^{X,F})^{-1}P^\perp VP+PWP$, the real part of the other eigenvalues tending to $+\infty$.

Proof. The proof of our theorem follows the same lines as the proof of Theorem 1.1. In particular, the convergence of the heat kernels is obtained via the convergence of the resolvents. □

Remark 1.6. If $\Delta^{X,F}$ is the Bochner Laplacian acting on $C^\infty(X,F)$ associated with the Riemannian metric g^{TX} and the connection ∇^F, if $A^{X,F}=-\Delta^{X,F}/2$, then $\exp(-sS_b^{X,F})$ can be evaluated using a matrix version of the Feynman-Kac formula.

2. The central limit theorem for the Ornstein-Uhlenbeck process.

In this section, we explain certain aspects of the central limit theorem for the Ornstein-Uhlenbeck process in its commutative and noncommutative form, and we develop corresponding analytic results for its infinitesimal generator, the harmonic oscillator. The results of the present section are closely related to the results of Section 1.

This section is organized as follows. In Subsection 2.1, we state elementary properties of the harmonic oscillator on an Euclidean vector space E.

In Subsection 2.2, we study the behaviour of the spectrum of a family of elliptic operators M_b^E, an analogue of the family of operators S_b^X that was considered in Subsection 1.1.

In Subsection 2.3, we interpret the results of Subsection 2.2 in terms of a central limit theorem for the occupation 'density' of the Ornstein-Uhlenbeck process.

In Subsection 2.4, we obtain a matrix version of the central limit theorem.

Finally, in Subsection 2.5, we study the special case of the perturbation of the harmonic oscillator by a matrix V depending linearly on $Y \in E$. The hypoelliptic Laplacian will turn out to be an infinite dimensional version of this kind of operator.

2.1. The harmonic oscillator.

Let E be a finite dimensional Euclidean vector space of dimension n, and let Y be its generic element. Let Δ^E be the Laplacian of E, and let ∇_Y be the radial vector field on E. Let A^E be the harmonic oscillator

$$A^E = \frac{1}{2}(-\Delta^E + 2\nabla_Y). \tag{2.1}$$

Then A^E is a formally self-adjoint operator on $L_2^E(e^{-|Y|^2}dY)$, whose kernel is spanned by the function 1.

Set

$$H^E = \exp(-|Y|^2/2)A^E\exp(|Y|^2/2), \tag{2.2}$$

so that

$$H^E = \frac{1}{2}(-\Delta^E + |Y|^2 - n). \tag{2.3}$$

Then H^E is a self-adjoint operator acting on L_2^E, and its kernel is spanned by $\exp(-|Y|^2/2)$. It is well-known that

$$\operatorname{Sp} A^E = \operatorname{Sp} H^E = \mathbf{N}. \tag{2.4}$$

Given $m \in \mathbf{N}$, the Hermite polynomials of degree m on E span the eigenspace of A^E associated with the eigenvalue m.

Let p be the orthogonal projection operator from $L_2^E(e^{-|Y|^2}(dY/\pi^{n/2}))$ on $\ker A^E$, and let P be the orthogonal projection operator from L_2^E on $\ker H^E$. We identify p with the corresponding smooth kernel with respect to $\exp(-|Y'|^2)(dY'/\pi^{n/2})$, and P with its smooth kernel with respect to $dY'/\pi^{n/2}$. In particular,

$$p(Y,Y')=1, \qquad P(Y,Y')=\exp\Big(-\frac{1}{2}(|Y|^2+|Y'|^2)\Big). \tag{2.5}$$

2.2. The spectrum of L_b^E as $b\to 0$.

Let $V,W\in C^\infty(E,\mathbf{C})$ such that V,W and their derivatives of arbitrary order grow at most linearly as $|Y|\to+\infty$. Note that

$$\begin{aligned} pVp&=\int_E V(Y)\exp(-|Y|^2)\frac{dY}{\pi^{n/2}}p,\\ PVP&=\int_E V(Y)\exp(-|Y|^2)\frac{dY}{\pi^{n/2}}P. \end{aligned} \tag{2.6}$$

In the sequel, we assume that

$$pVp=0. \tag{2.7}$$

For $b>0$, set

$$L_b^E=\frac{H^E}{b^2}+\frac{V}{b}+W, \qquad M_b^E=\frac{A^E}{b^2}+\frac{V}{b}+W, \tag{2.8}$$

so that

$$L_b^E=\exp(-|Y|^2/2)M_b^E\exp(|Y|^2/2). \tag{2.9}$$

Let $\exp(-sL_b^E)(Y,Y')$ be the smooth kernel associated with $\exp(-sL_b^E)$ with respect to $dY'/\pi^{n/2}$, and let $\exp(-sM_b^E)(Y,Y')$ be the smooth kernel associated with $\exp(-sM_b^E)$ with respect to $\exp(-|Y'|^2)(dY'/\pi^{n/2})$. Then

$$\exp\big(-sL_b^E\big)(Y,Y')=\exp\Big(-\frac{1}{2}(|Y|^2+|Y'|^2)\Big)\exp\big(-sM_b^E\big)(Y,Y'). \tag{2.10}$$

Let $(A^E)^{-1}$ denote the inverse of A^E acting on the orthogonal vector space to the constants.

THEOREM 2.1. *For $s>0$, as $b\to 0$, we have the uniform convergence of smooth kernels and their derivatives on compact subsets of $E\times E$,*

$$\begin{aligned} \exp(-sM_b^E)(Y,Y')\to\exp\Big(s\int_E V(Y)((A^E)^{-1}V)(Y)\exp(-|Y|^2)\frac{dY}{\pi^{n/2}}\\ -s\int_E W(Y)\exp(-|Y|^2)\frac{dY}{\pi^{n/2}}\Big). \end{aligned} \tag{2.11}$$

In particular, as $b \to 0$,

$$\operatorname{Tr}\left[\exp(-sM_b^E)\right] \to \exp\Big(s\int_E V(Y)((A^E)^{-1}V)(Y)\exp(-|Y|^2)\frac{dY}{\pi^{n/2}} - s\int_E W(Y)\exp(-|Y|^2)\frac{dY}{\pi^{n/2}}\Big). \tag{2.12}$$

If V, W are real, if σ_b is the lowest eigenvalue of the self-adjoint operator M_b^E, as $b \to 0$,

$$\sigma_b \to -\int_E V(Y)((A^E)^{-1}V)(Y)\exp(-|Y|^2)\frac{dY}{\pi^{n/2}} + \int_E W(Y)\exp(-|Y|^2)\frac{dY}{\pi^{n/2}}, \tag{2.13}$$

and the other eigenvalues tend to $+\infty$. If V, W are arbitrary, equation (2.13) *still holds for one eigenvalue of M_b^E, the real part of the other eigenvalues tending to $+\infty$.*

PROOF. It is easier to replace M_b^E by L_b^E. Indeed the operator H^E is classically self-adjoint, and the associated Sobolev spaces can be expressed in terms of classical Schwartz spaces. The proof is then exactly the same as the proof of Theorem 1.1. □

2.3. The central limit theorem for the Ornstein-Uhlenbeck process.

To construct the semigroup $\exp(-sA^E)$, one can use the associated stochastic differential equation. Namely let $w_\cdot$ be a Brownian motion with values in E such that $w_0 = 0$. Given $Y_0 \in E$, consider the stochastic differential equation

$$dY = -Y ds + \dot{w}_s, \qquad Y_0 = Y_0, \tag{2.14}$$

so that

$$Y_s = e^{-s}Y_0 + e^{-s}\int_0^s e^u \delta w_u. \tag{2.15}$$

There is another Brownian motion $B_\cdot$ such that

$$Y_s = e^{-s}Y_0 + e^{-s}B_{(e^{2s}-1)/2}. \tag{2.16}$$

Let Q_{Y_0} be the probability law of the process $Y_\cdot$.

We will establish an analogue of Theorem 1.3.

THEOREM 2.2. *If $V \in C^\infty(E, \mathbf{R})$ has at most linear growth and is such that $\int_E V(Y)\exp(-|Y|^2)dY = 0$, as $t \to +\infty$, for any $Y_0 \in E$, the probability law of the process $s \in \mathbf{R}_+ \to (1/\sqrt{t})\int_0^{st} V(Y_u)du$ converges to the probability law of the process $s \in \mathbf{R}_+ \to cw_s$, with*

$$c^2 = 2\int_E V(Y)((A^E)^{-1}V)(Y)\exp(-|Y|^2)\frac{dY}{\pi^{n/2}}. \tag{2.17}$$

As $t \to +\infty$, the probability law of the process $s \in \mathbf{R}_+ \to (1/t)\int_0^{st} W(Y_u)du$ converges to the deterministic process $s \in \mathbf{R}_+ \to s\int_E W(Y)\exp(-|Y|^2)(dY/\pi^{n/2})$.

For $0 < s_1 < \cdots < s_m$, as $t \to +\infty$, the probability law of $(Y_{s_1 t}, Y_{s_2 t}, \ldots, Y_{s_m t})$ converges to the product of the probability laws $\exp(-|Y|^2)dY/\pi^{n/2}$.

Finally, as $t \to +\infty$, the joint law of the above random processes and random variables converges to the corresponding product law.

Proof. By making $t = 1/b^2$, and using Theorem 2.1, the proof is the same as the proof of Theorem 1.3. □

Remark 2.3. The content of Remark 1.4 still applies here. The relevant free field now refers to the Gaussian random field with covariance $2(A^E)^{-1}$ or $2(H^E)^{-1}$.

2.4. The matrix version of the central limit theorem.

Let F be a finite dimensional Hermitian vector space. We assume that $V, W \in C^\infty(E, \mathrm{End}(F))$ are such that V, W and their derivatives of arbitrary order grow at most linearly as $|Y| \to +\infty$. We still assume that (2.6), (2.7) hold. Set

$$L_b^{E,F} = \frac{H^E}{b^2} + \frac{V}{b} + W, \qquad M_b^{E,F} = \frac{A^E}{b^2} + \frac{V}{b} + W. \tag{2.18}$$

Also we use the same conventions on the kernels of $\exp(-sL_b^{E,F}), \exp(-sM_b^{E,F})$ as in Subsection 2.2.

Theorem 2.4. *For $s > 0$, as $b \to 0$, we have the uniform convergence of smooth kernels and their derivatives on $E \times E$*

$$\exp\big(-sM_b^{E,F}\big)(Y, Y') \to \exp\Big(s\int_E V(Y)((A^E)^{-1}V)(Y)\exp(-|Y|^2)\frac{dY}{\pi^{n/2}} - s\int_E W(Y)\exp(-|Y|^2)\frac{dY}{\pi^{n/2}}\Big). \tag{2.19}$$

As $b \to 0$,

$$\mathrm{Tr}\big[\exp(-sM_b^{E,F})\big] \to \mathrm{Tr}\Big[\exp\Big(s\int_E V(Y)((A^E)^{-1}V)(Y)\exp(-|Y|^2)\frac{dY}{\pi^{n/2}} - s\int_E W(Y)\exp(-|Y|^2)\frac{dY}{\pi^{n/2}}\Big)\Big]. \tag{2.20}$$

As $b \to 0$, the eigenvalues of $M_b^{E,F}$ converge either to the eigenvalues of

$$-\int_E V(Y)((A^E)^{-1}V)(Y)\exp(-|Y|^2)\frac{dY}{\pi^{n/2}} + \int_E W(Y)\exp(-|Y|^2)\frac{dY}{\pi^{n/2}},$$

or their real part tends to $+\infty$.

PROOF. The proof of our theorem is the same as the proof of Theorems 1.5 and 2.1. □

2.5. The case where V is linear.

We will now consider a special case of Theorem 2.4. Indeed let $v : E \to \mathrm{End}(F)$ be a linear map. Let $R \in \mathrm{End}(F)$ be such that if $e_1, \dots, e_n$ is an orthonormal basis of E, then

$$R = \frac{1}{2} \sum_{i=1}^{n} v(e_i)^2. \tag{2.21}$$

Set

$$L_b^{E,F} = \frac{H^E}{b^2} + \frac{v(Y)}{b}, \qquad M_b^{E,F} = \frac{A^E}{b^2} + \frac{v(Y)}{b}. \tag{2.22}$$

THEOREM 2.5. *For $s > 0$, as $b \to 0$, we have the uniform convergence of smooth kernels and their derivatives on compact subsets of $E \times E$*

$$\exp\big(-sM_b^{E,F}\big)(Y, Y') \to \exp(sR). \tag{2.23}$$

As $b \to +\infty$, the eigenvalues of $M_b^{E,F}$ converge to the eigenvalues of $-R$, or their real parts tend to $+\infty$.

PROOF. The eigenspace of A^E associated with the eigenvalue 1 is spanned by the linear functions of Y. It follows that if $V(Y) = v(Y)$, we have the identity

$$\int_E V(Y)((A^E)^{-1}V)(Y) \exp(-|Y|^2) \frac{dY}{\pi^{n/2}} = R. \tag{2.24}$$

Our theorem now follows from Theorem 2.4. □

Let us give the probabilistic counterpart to Theorem 2.5. We still take $Y_\cdot$ as in (2.14). Set

$$Z_s = \frac{Y_{s/b^2}}{b}. \tag{2.25}$$

There is another Brownian motion $w_\cdot$ such that

$$dZ_s = \frac{1}{b^2}(-Z_s + dw_s), \qquad Z_0 = Y_0/b. \tag{2.26}$$

Let $U_{b,s} \in \mathrm{End}(F)$ be the solution of

$$\frac{dU_{b,s}}{ds} = -U_{b,s} v(Z_s), \qquad U_0 = 1. \tag{2.27}$$

If $f(Y)$ is a bounded element of $C^{\infty}(E,F)$, a version of Feynman-Kac formula shows that

$$\exp\big(-sM_b^{E,F}\big)f(Y_0)=E^{Q_{Y_0}}\big[U_{b,s}f(Y_{s/b^2})\big]. \tag{2.28}$$

Let $U_{0,s}$ be the solution of the stochastic differential equation in the sense of Stratonovitch

$$dU_{0,s}=-U_{0,s}v(dw_s),\qquad U_{0,0}=1. \tag{2.29}$$

Equation (2.29) can be rewritten as the stochastic differential equation in the sense of Itô

$$dU_{0,s}=U_{0,s}(Rds-v(\delta w)),\qquad U_{0,0}=1. \tag{2.30}$$

By (2.30), we deduce that

$$E[U_{0,s}]=\exp(sR). \tag{2.31}$$

THEOREM 2.6. *As $b\to 0$, the distribution valued process $s\in\mathbf{R}_+\to Z_s$ converges in probability law to $s\in\mathbf{R}_+\to\dot w_s$. As $b\to 0$, the probability law of $s\in\mathbf{R}_+\to U_{b,s}$ converges to the probability law of $s\in\mathbf{R}_+\to U_{0,s}$. Also the joint law of $(Z_\cdot,U_\cdot)$ converges to the joint law of $(\dot w,U_{0,\cdot})$.*

For $0<s_1<\cdots s_m$, as $b\to 0$, the probability law of $(Y_{s_1/b^2},Y_{s_2/b^2},\dots Y_{s_m/b^2})$ converges to the product of probability laws $\exp(-|Y|^2)dY/\pi^{n/2}$.

Finally, the joint law of $(Z_\cdot,U_\cdot)$ and $(Y_{s_1/b^2},Y_{s_2/b^2},\dots Y_{s_m/b^2})$ converges to the corresponding product law.

PROOF. For the proof of a more difficult result, we refer to [**B11**, Theorem 12.8.1]. □

3. The hypoelliptic Laplacian on a vector space and on a torus.

The purpose of this section is to study the hypoelliptic Laplacian in its simplest version. Namely, if E is an Euclidean vector space, the companion hypoelliptic Laplacians $\mathcal{L}_b^E,\mathcal{M}_b^E$ are scalar operator over $E\times E$. They are infinite dimensional versions of the operators $L_b^{E,F},M_b^{E,F}$ that were considered in Subsection 2.5, and their properties are intimately related to what we did in Sections 1 and 2. Also if E/Λ is a torus modelled on E, we also consider the hypoelliptic Laplacians $\mathcal{L}_b^{E/\Lambda},\mathcal{M}_b^{E/\Lambda}$ that act over on $E/\Lambda\times E$. We study the analytic and spectral properties of the above operators, and the behaviour of their heat kernel as $b\to 0$. Also we describe the associated diffusion process, which is a Langevin process.

This section is organized as follows. In Subsection 3.1, we construct the hypoelliptic Laplacians $\mathcal{L}_b^E,\mathcal{M}_b^E$ on $E\times E$.

In Subsection 3.2, we show that the diffusion process on $E\times E$ associated with $\mathcal{M}_b^E$ projects to a Langevin process on E.

In Subsection 3.3, by taking the Fourier transform of $\mathcal{L}_b^E$ on the first copy of E, we obtain an operator $\widehat{\mathcal{L}}_{b,\xi}^E$ which can be diagonalized explicitly.

In Subsection 3.4, we show that the hypoelliptic non-self adjoint operator $\mathcal{L}_b^E$ is conjugate to an elliptic self-adjoint operator, by an unbounded conjugation.

In Subsection 3.5, if $\Lambda \subset E$ is a lattice, using the previous conjugation, we compute the spectrum of $\mathcal{L}_b^{E/\Lambda}$ explicitly, and we show that as $b \to 0$, from a spectral point of view, $\mathcal{L}_b^{E/\Lambda}$ converges to $-(1/2)\Delta^{E/\Lambda}$.

In Subsection 3.6, we recall a formula of [**B11**] for the heat kernel for $\mathcal{L}_b^E$ on $E \times E$.

In Subsection 3.7, we give a nontrivial identity that expresses an integral along the second copy of E of the heat kernel for $\mathcal{L}_b^E$ in terms of the heat kernel of $\Delta^E/2$ on E.

Finally, in Subsection 3.8, we study the limit as $b \to +\infty$ of the hypoelliptic heat kernel.

3.1. The hypoelliptic Laplacian associated with a vector space *E*.

We use the notation of Subsection 2.5. The generic element of $E \times E$ will be denoted (x, Y). Differentiation along the first copy of E will be denoted ∇^H, while differentiation on the second copy will be denoted ∇^V. Here, the harmonic oscillators A^E, H^E will act on the second copy of E in $E \times E$.

Put

$$F = C^\infty(E, \mathbf{R}). \tag{3.1}$$

In (3.1), E is identified with the corresponding first copy in $E \times E$.

Let $v : E \to \mathrm{End}(F)$ be the linear map

$$v(Y) = -\nabla_Y^H. \tag{3.2}$$

Then $v(Y) \in \mathrm{End}(F)$. Let $\Delta^{E,H}$ denote the Laplacian along the first copy of E. With the conventions in (2.21), we get

$$R = \frac{1}{2}\Delta^{E,H}. \tag{3.3}$$

We denote by $\mathcal{L}_b^E, M_b^E$ the operators $L_b^{E,F}, M_b^{E,F}$ in (2.22), so that

$$\mathcal{L}_b^E = \frac{H^E}{b^2} - \frac{\nabla_Y^H}{b}, \qquad \mathcal{M}_b^E = \frac{A^E}{b^2} - \frac{\nabla_Y^H}{b}. \tag{3.4}$$

The operators $\mathcal{L}_b^E, \mathcal{M}_b^E$ are known as Fokker-Planck operators. They are the sum of a self-adjoint piece, A^E/b^2 or H^E/b^2, and of an antisymmetric piece $-\nabla_Y^H/b$.

In [**K34**], Kolmogorov introduced the operator

$$K = -\frac{1}{2}\Delta^V - \nabla_Y^H, \tag{3.5}$$

and computed the smooth heat kernel for K. The operator K is the model of the second order hypoelliptic differential operators studied by Hörmander [**H67**]. Let just mention that if $t \in \mathbf{R}_+$, the operator $\partial/\partial t + K$ is also hypoelliptic, which, in retrospect explains the smoothness of the heat operator for K.

The same argument also shows that $\partial/\partial t + \mathcal{L}_b^E$, $\partial/\partial t + \mathcal{M}_b^E$ are hypoelliptic. In particular $\mathcal{L}_b^E, \mathcal{M}_b^E$ are hypoelliptic. An operator like $\mathcal{L}_b^E, \mathcal{M}_b^E$ is called a hypoelliptic Laplacian.

3.2. The Langevin process associated with $\mathcal{M}_b^E$.

The stochastic differential equation corresponding to the semigroup $\exp(-s\mathcal{M}_b^E)$ is given by

$$\begin{aligned} \dot{x} &= \frac{Y}{b}, \qquad \dot{Y} = -\frac{Y}{b^2} + \frac{\dot{w}}{b}, \\ x_0 &= x, \qquad Y_0 = Y. \end{aligned} \tag{3.6}$$

If $Z = Y/b$, we get instead

$$\begin{aligned} \dot{x} &= Z, \qquad \dot{Z} = \frac{1}{b^2}(-Z + \dot{w}), \\ x_0 &= x, \qquad Z_0 = \frac{Y}{b}. \end{aligned} \tag{3.7}$$

If $f \in C^\infty(E \times E, \mathbf{R})$ is bounded, then

$$\exp\big(-s\mathcal{M}_b^E\big) f(x, Y) = E[f(x_s, Y_s)]. \tag{3.8}$$

By (3.6), (3.7), we obtain

$$b^2 \ddot{x} + \dot{x} = \dot{w}. \tag{3.9}$$

Equation (3.9) is a Langevin equation [**L08**]. For $b = 0$, (3.9) reduces to the equation of Brownian motion $\dot{x} = \dot{w}$ in the first copy of E, for $b = \infty$, it reduces to the equation of geodesics $\ddot{x} = 0$.

An obvious application of Theorem 2.6 gives the following result.

Theorem 3.1. *As $b \to 0$, the probability law of $(x_\cdot, Z_\cdot)$ converges to the probability law of $(x + w_\cdot, \dot{w}_\cdot)$. For $0 < s_1 < \cdots < s_m$, the probability law of $(Y_{s_1}, \ldots, Y_{s_m})$ converges to the product of the probability laws* $\exp(-|Y|^2) dY/\pi^{n/2}$.

Finally, the joint law of $(x_\cdot, Z_\cdot)$ and $(Y_{s_1}, \ldots, Y_{s_m})$ converges to the corresponding product law.

Remark 3.2. Theorem 3.1 legitimates the naive idea that for $b = 0$, equation (3.9) reduces to $\dot{x} = \dot{w}$. Ultimately, the infinite dimensional version of Theorem 2.6 with $F = C^\infty(E, \mathbf{R})$ and $v(Y) = -\nabla_Y^H$ is correct.

The diffusion process corresponding to the operator $\mathcal{L}_b^E$ is given by the stochastic differential equation

$$\dot{x} = \frac{Y}{b}, \qquad \dot{Y} = \frac{\dot{w}}{b}, \tag{3.10}$$
$$x_0 = x, \qquad Y_0 = Y.$$

By (3.10), we get

$$b^2 \ddot{x} = \dot{w}. \tag{3.11}$$

If $f \in C^\infty(E \times E, \mathbf{R})$ is bounded, then

$$\exp\big(-s\mathcal{L}_b^E\big) f(x,Y) = E\left[\exp\left(-\frac{1}{2b^2}\left(\int_0^s |Y_u|^2 du - ns\right)\right) f(x_s, Y_s)\right]. \tag{3.12}$$

Equation (3.12) can be rewritten in the form

$$\exp\big(-s\mathcal{L}_b^E\big) f(x,Y) = E\left[\exp\left(-\frac{1}{2}\left(\int_0^s |\dot{x}|^2 du - \frac{ns}{b^2}\right)\right) f(x_s, Y_s)\right]. \tag{3.13}$$

Equation (3.13) is of special interest, since the energy of the path $x_\cdot$ appears explicitly in the right-hand side, while for the usual Brownian motion, this energy is infinite, and remains conceptually in the shadow.

3.3. The Fourier transform $\widehat{\mathcal{L}}_{b,\xi}$.

Let $\widehat{\mathcal{L}}_{b,\xi}^E$ denote the Fourier transform of the operator $\mathcal{L}_b^E$ in the variable x. If $\xi \in E^*$, then

$$\widehat{\mathcal{L}}_{b,\xi}^E = \frac{1}{2b^2}\big(-\Delta^{E,V} + |Y|^2 - n\big) - \frac{i}{b}\langle Y, \xi\rangle. \tag{3.14}$$

Given $\xi \in \mathbf{R}$, the proper theory of the operator $\widehat{\mathcal{L}}_{b,\xi}$ can be set up so that the operator $\widehat{\mathcal{L}}_{b,\xi}$ has compact resolvent acting on L_2^E [**BL08**].

We identify E and E^* by the scalar product. Then we can rewrite (3.14) in the form

$$\widehat{\mathcal{L}}_{b,\xi}^E = \frac{1}{2b^2}\big(-\Delta^{E,H} + |Y - ib\xi|^2 - n\big) + \frac{1}{2}|\xi|^2. \tag{3.15}$$

Let $\mathcal{A}(E)$ be the vector space of complex valued analytic functions on E. Then $\widehat{\mathcal{L}}_{b,\xi}^E$ acts on $\mathcal{A}(E)$. If $a \in E \otimes_{\mathbf{R}} \mathbf{C}$, let T_a be the map acting on $\mathcal{A}(E)$,

$$T_a f(Y) = f(Y + a). \tag{3.16}$$

Equivalently

$$T_a = \exp(\nabla_a^V). \tag{3.17}$$

By (3.14), we have the identity

$$T_{ib\xi}\widehat{\mathcal{L}}^E_{b,\xi}T^{-1}_{ib\xi} = \frac{H^E}{b^2} + \frac{1}{2}|\xi|^2. \tag{3.18}$$

This identity should be understood as purely algebraic. Note that $T_{ib\xi}$ does not act on any natural Sobolev space on E. Therefore, the operators in (3.16) cannot safely be considered as isospectral[6].

For $k \in \mathbf{N}^n$, let $P_k(Y)$ be the Hermite polynomial of multiindex k. As we saw before, the $\exp(-|Y|^2/2)P_k(Y)$ are the eigenfunctions of H^E. Note that

$$T_{-ib\xi}\exp(-|Y|^2/2)P_k(Y) = \exp\left(-\frac{1}{2}|Y - ib\xi|^2\right)P_k(Y - ib\xi). \tag{3.19}$$

Given $\xi \in E^*$, the function in (3.19) lies in the Schwartz space $\mathcal{S}(E, \mathbf{C})$. Ultimately, one finds easily that the operator $\widehat{\mathcal{L}}_{b,\xi}$ is explicitly diagonalizable with eigenfunctions in (3.19), and the vector space spanned by the linear combinations of these eigenfunctions is dense in L^E_2. In particular, the operators in (3.18) are indeed isospectral, in spite of the fact that the intertwining map $T_{ib\xi}$ does not act in a standard way.

It follows from the above that

$$\operatorname{Sp}\widehat{\mathcal{L}}^E_{b,\xi} = \frac{\mathbf{N}}{b^2} + \frac{1}{2}|\xi|^2. \tag{3.20}$$

By (3.20), as $b \to 0$, the spectrum of $\widehat{\mathcal{L}}_{b,\xi}$ converges to $(1/2)|\xi|^2$, which fits with Theorem 2.5.

3.4. A nontrivial conjugation.

Let us now go back to the original operators $\mathcal{L}^E_b$. Let $\Delta^{E,H}$ be the Laplacian on the first copy of E in $E \times E$. We have the analogue of (3.15)

$$\mathcal{L}^E_b = \frac{1}{2b^2}\big(-\Delta^{E,V} + |Y - b\nabla^H_{\cdot}|^2 - n\big) - \frac{1}{2}\Delta^{E,H}. \tag{3.21}$$

Again, by [**BL08**], the proper theory of the operator $\mathcal{L}^E_b$ can be set up so that its resolvent acts on $L^{E\times E}_2$. If $\Lambda \subset E$ is a lattice, because of hypoellipticity, the corresponding operator $\mathcal{L}^{E/\Lambda}_b$ on $E/\Lambda \times E$ has compact resolvent.

Let $e_1, \dots, e_n$ be an orthonormal basis of E, and let $(x^1, \dots, x^n)$ and $(Y^1, \dots, Y^n)$ be the corresponding coordinates on $E \oplus E$. Set

$$N = \sum_{i=1}^n \frac{\partial^2}{\partial x^i \partial Y^i}. \tag{3.22}$$

Then

[6]The situation is completely different if ξ is replaced by $i\xi$.

$$\widehat{N}_\xi = i\nabla^V_\xi . \tag{3.23}$$

Note that N is a hyperbolic operator. Let $C(E)$ be the complex vector space of linear combinations of functions of the type $x \to \exp(i\langle x, \xi\rangle)$. Then $\exp(bN)$ acts on $C(E) \otimes \mathcal{A}(E)$. However, $\exp(bN)$ does not act on any Sobolev space.

Still we have the formal analogue of (3.18),

$$\exp(bN)\mathcal{L}^E_b \exp(-bN) = \frac{H^E}{b^2} - \frac{1}{2}\Delta^{E,H}. \tag{3.24}$$

Equation (3.24) should be viewed as an algebraic identity. Again, the operators in (3.23) are not conjugate in the classical sense, since the conjugating operator $\exp(bN)$ is not a honest operator.

3.5. The spectrum of $\mathcal{L}^{E/\Lambda}_b$.

To make our argument simpler, we will consider instead the operators $\mathcal{L}^{E/\Lambda}_b, \mathcal{M}^{E/\Lambda}_b$ acting on $E/\Lambda \times E$. Let $\Lambda^* \subset E^*$ be the dual lattice to Λ

$$\Lambda^* = \{\xi \in E^*, \langle \xi, \Lambda\rangle \subset 2\pi\mathbf{Z}\}. \tag{3.25}$$

Then equation (3.24) descends to

$$\exp(bN)\mathcal{L}^{E/\Lambda}_b \exp(-bN) = \frac{H^E}{b^2} - \frac{1}{2}\Delta^{E/\Lambda}. \tag{3.26}$$

The arguments in (3.18)–(3.20) show that $\mathcal{L}^{E/\Lambda}_b$ can be properly diagonalized. If $\lambda \in \Lambda^*$, the eigenfunctions of $\mathcal{L}^E_b$ are given by

$$\exp\left(-\frac{1}{2}|Y - ib\lambda|^2\right)P_k(Y - ib\lambda)\exp(i\langle\lambda, x\rangle), \tag{3.27}$$

the corresponding eigenvalue being given by $|k|/b^2 + |\lambda|^2/2$. Again the span of these eigenfunctions is the full $L_2^{E/\Lambda\times E}$.

If A, B are subsets of $\mathbf{R}$, we denote by $A + B$ the subset of $\mathbf{R}$ that consists of all the possible sums $a + b$, $a \in A$, $b \in B$. It follows from the above that

$$\operatorname{Sp}\mathcal{L}^{E/\Lambda}_b = \frac{\mathbf{N}}{b^2} + \operatorname{Sp}\left(-\frac{1}{2}\Delta^{E/\Lambda}\right). \tag{3.28}$$

Ultimately, even though equation (3.26) is not a proper conjugation, its consequences are correct, i.e., the operators in (3.26) are isospectral.

Observe the remarkable fact that the eigenvalues of $\mathcal{L}^{E/\Lambda}_b$ are real. As $b \to 0$, either they tend to the eigenvalues of $-\Delta^{E/\Lambda}/2$ or they tend to $+\infty$. From a spectral theoretic point of view, as $b \to 0$, the operator $\mathcal{L}^{E/\Lambda}_b$ converges to the operator $-\Delta^{E/\Lambda}/2$. The consequences of Theorem 2.5 are still valid even though the twisting $F = C^\infty(E/\Lambda, \mathbf{R})$

is infinite dimensional. The fundamental fact is that the eigenvalues of $-\Delta^{E/\Lambda}/2$ are rigidly embedded in the eigenvalues of $\mathcal{L}_b^{E/\Lambda}$.

THEOREM 3.3. *The following identity holds*:

$$\operatorname{Tr}\left[\exp(-s\mathcal{L}_b^{E/\Lambda})\right] = \operatorname{Tr}\left[\exp(-sH^E/b^2)\right] \operatorname{Tr}\left[\exp(s\Delta^{E/\Lambda}/2)\right]. \tag{3.29}$$

PROOF. By (3.28), we get

$$\operatorname{Tr}[\exp(-s\mathcal{L}_b^{E/\Lambda})] = \left(1 - e^{-s/b^2}\right)^{-n} \operatorname{Tr}\left[\exp(s\Delta^{E/\Lambda}/2)\right]. \tag{3.30}$$

Moreover, we have the identity

$$\operatorname{Tr}\left[\exp(-sH^E/b^2)\right] = \left(1 - e^{-s/b^2}\right)^{-n}. \tag{3.31}$$

By (3.30), (3.31), we get (3.29). □

REMARK 3.4. One can give a direct proof of (3.29) that uses (3.26).

3.6. The heat kernel for $\mathcal{L}_b^E$.

Set

$$\begin{aligned}\underline{H}_{b,s}\big((x,Y),(x',Y')\big) &= \frac{1}{2}\Big(\tanh(s/2b^2)(|Y|^2 + |Y'|^2) + \frac{|Y'-Y|^2}{\sinh(s/b^2)}\Big) \\ &\quad + \frac{1}{2(s - 2b^2\tanh(s/2b^2))}\left|x' - x - b\tanh(s/2b^2)(Y+Y')\right|^2, \\ \underline{K}_{b,s}\big((x,Y),(x',Y')\big) &= \frac{1}{2\sinh(s/b^2)}\left|e^{-s/2b^2}Y - e^{s/2b^2}Y'\right|^2 \\ &\quad + \frac{1}{2(s - 2b^2\tanh(s/2b^2))}\left|x' - x - b\tanh(s/2b^2)(Y+Y')\right|^2.\end{aligned} \tag{3.32}$$

Then

$$\underline{K}_{b,s}\big((x,Y),(x',Y')\big) = \underline{H}_{b,s}\big((x,Y),(x',Y')\big) + \frac{1}{2}(|Y'|^2 - |Y|^2). \tag{3.33}$$

Let $\exp(-s\mathcal{L}_b^E)((x,Y),(x',Y'))$, $\exp(-s\mathcal{M}_b^E)((x,Y),(x',Y'))$ be the smooth kernels of $\exp(-s\mathcal{L}_b^E)$, $\exp(-s\mathcal{M}_b^E)$ with respect to $dx'(dY'/\pi^{n/2})$, $\exp(-|Y'|^2)dx'(dY'/\pi^{n/2})$. By [**B11**, Proposition 10.5.1], we get

$$\begin{aligned}&\exp\big(-s\mathcal{L}_b^E\big)((x,Y),(x',Y')) \\ &= \left[\frac{e^{s/b^2}}{4\pi\sinh(s/b^2)(s - 2b^2\tanh(s/2b^2))}\right]^{n/2} \exp\big(-\underline{H}_{b,s}((x,Y),(x',Y'))\big),\end{aligned}$$

$$\exp\big(-s\mathcal{M}_b^E\big)((x,Y),(x',Y'))$$
$$= \left[\frac{e^{s/b^2}}{4\pi\sinh(s/b^2)(s-2b^2\tanh(s/2b^2))}\right]^{n/2}\exp\big(|Y'|^2-\underline{K}_{b,s}((x,Y),(x',Y'))\big). \tag{3.34}$$

The computation in [**B11**] uses (3.24) explicitly.

Another explanation for equation (3.34) given in [**B11**, Section 10.5] is as follows. Since the operator $\mathcal{M}_b^E$ has total weight 2 in the variables x, Y and in the differentiation operators ∇^H, ∇^V, its heat kernel $\exp(-s\mathcal{M}_b^E)$ is Gaussian, and can be computed by solving a corresponding variational problem associated with the corresponding large deviation functional [**B11**, Section 10.1]. If $x_\cdot : [0,s] \to E$ is a smooth path, set

$$\underline{H}_{b,s}(x) = \frac{1}{2}\int_0^s \big(|\dot{x}|^2 + b^4|\ddot{x}|^2\big)du. \tag{3.35}$$

Here we fix $(x_0, \dot{x}_0) = (x, Y/b)$ and $(x_s, \dot{x}_s) = (x', Y'/b)$. The problem of minimizing (3.35) has a unique solution. In [**B11**, Proposition 10.3.2], it is shown that $\underline{H}_{b,s}((x,Y),(x',Y'))$ is precisely the minimum value of $\underline{H}_{b,s}(x)$. This ultimately explains equation (3.34). The first factor in the right-hand side can be obtained using the fact that the heat kernel for $\exp(-s\mathcal{M}_b^E)$ consists of probability measures.

By (3.32), we recover the fact established in [**B11**, Equation (10.3.49)] that as $b \to 0$,

$$\underline{H}_{b,s}((x,Y),(x',Y')) \to \underline{H}_{0,s}((x,Y),(x',Y')) = \frac{1}{2s}|x'-x|^2 + \frac{1}{2}(|Y|^2+|Y'|^2). \tag{3.36}$$

By (3.34), (3.36), as $b \to 0$, we have the convergence of smooth kernels and their derivatives on compact sets of $E \times E$

$$\exp\big(-s\mathcal{L}_b^E\big)((x,Y),(x',Y'))$$
$$\to \exp\bigg(-\frac{1}{2}(|Y|^2+|Y'|^2)\bigg)\frac{1}{(2\pi s)^{n/2}}\exp\bigg(-\frac{1}{2s}|x'-x|^2\bigg). \tag{3.37}$$

Let $\exp(s\Delta^E/2)(x)$ be the smooth kernel of $\exp(s\Delta^E/2)$ with respect to dx. Using (2.5), equation (3.37) can be rewritten in the form

$$\exp\big(-s\mathcal{M}_b^E\big)((x,Y),(x',Y')) \to \exp\bigg(\frac{s\Delta^E}{2}\bigg)(x'-x). \tag{3.38}$$

Equation (3.38) is the strict analogue at the level of smooth kernels of equation (2.23) in Theorem 2.5, which is only valid when F is finite dimensional.

3.7. A nonperturbative identity of smooth kernels on E.

First, we state an identity in [**B11**, Equation (10.6.17)].

Proposition 3.5. *For any $a \in E$, the following identity holds*:

$$\int_E \exp\big(-s\mathcal{L}_b^E\big)_{b,s}((0,Y),(a,Y))\frac{dY}{\pi^{n/2}} = \big(1-e^{-s/b^2}\big)^{-n}(2\pi s)^{-n/2}\exp\Big(-\frac{|a|^2}{2s}\Big). \quad (3.39)$$

Proof. This just follows from equation (3.34). □

Now we reinterpret the left-hand side of (3.39) as a partial trace in the variable Y. We denote this partial trace by Tr^a.

Theorem 3.6. *The following identity holds*:

$$\mathrm{Tr}^a\big[\exp(-s\mathcal{L}_b^E)\big] = \mathrm{Tr}\big[\exp(-sH^E/b^2)\big]\exp(s\Delta^E/2)(a). \quad (3.40)$$

Remark 3.7. In [**B11**, Section 10.6], it is shown how to derive (3.40) from (3.24). A version of Poisson's formula shows that Theorem 3.3 can be derived from Theorem 3.6.

3.8. The limit as $b \to +\infty$.

As $b \to +\infty$, by (3.28), an infinite number of eigenvalues of $\mathcal{L}_b^E$ accumulate to 0. Let K_b be the map $f(Y) \to f(bY)$, or $f(x,Y) \to f(x,bY)$. Set

$$\underline{\mathcal{L}}_b^E = K_b\mathcal{L}_b^E K_b^{-1}. \quad (3.41)$$

Then

$$\underline{\mathcal{L}}_b^E = \frac{1}{2}\Big(-\frac{\Delta^V}{b^4} + |Y|^2 - \frac{n}{b^2}\Big) - \nabla_Y^H. \quad (3.42)$$

By (3.42), we get

$$\widehat{\underline{\mathcal{L}}}_{b,\xi}^E = \frac{1}{2}\Big(-\frac{\Delta^V}{b^4} + |Y|^2 - \frac{n}{b^2}\Big) - i\langle Y,\xi\rangle. \quad (3.43)$$

By (3.19), the eigenfunctions of $\widehat{\underline{\mathcal{L}}}_{b,\xi}^E$ are given by

$$\exp\Big(-\frac{b^2}{2}|Y-i\xi|^2\Big)P_k(b(Y-i\xi)). \quad (3.44)$$

By (3.42), (3.43), as $b \to +\infty$, we have the convergence of operators in the naive sense

$$\underline{\mathcal{L}}_b^E \to \underline{\mathcal{L}}_\infty^E = \frac{1}{2}|Y|^2 - \nabla_Y^H, \qquad \widehat{\underline{\mathcal{L}}}_{b,\xi}^E \to \widehat{\underline{\mathcal{L}}}_{\infty,\xi}^E = \frac{1}{2}|Y|^2 - i\langle Y,\xi\rangle. \quad (3.45)$$

Also if $f \in \mathcal{S}(E \times E, \mathbf{C})$, then

$$\exp(-s\underline{\mathcal{L}}_{\infty}^{E})f(x,Y)=\exp\Big(-\frac{s}{2}|Y|^{2}\Big)f(x+sY,Y). \tag{3.46}$$

The operator $-\underline{\mathcal{L}}_{\infty}^{E}$ is essentially the generator of the geodesic flow $(x,Y)\to(x+sY,Y)$ on $E\times E$.

If $\widehat{f}(\xi,Y)$ is the Fourier transform of $f(x,Y)$ in the variable x, we get

$$\exp\big(-s\widehat{\underline{\mathcal{L}}}_{\infty,\xi}^{E}\big)\widehat{f}(\xi,Y)=\exp\Big(-\frac{s}{2}|Y|^{2}+i\langle sY,\xi\rangle\Big)\widehat{f}(\xi,Y). \tag{3.47}$$

Of course, (3.47) is the Fourier transform of (3.46).

The first equation (3.45) indicates that as $b\to+\infty$, the heat equation for $\underline{\mathcal{L}}_{b}^{E}$ propagates more and more along the geodesic flow. This can be seen directly from (3.10), (3.11).

Set

$$H_{b,s}((x,Y),(x',Y'))=\underline{H}_{b,s}((x,bY),(x',bY')). \tag{3.48}$$

Now we proceed as in [**B11**, Section 10.3]. By (3.32), one finds easily that as $b\to+\infty$, $H_{b,s}((x,Y),(x',Y'))$ tends to $+\infty$ unless

$$Y'=Y,\qquad x'=x+sY, \tag{3.49}$$

which is exactly the equation of propagation of the geodesic flow. As shown in [**B11**, Equation (10.3.57)], when such conditions are verified, as $b\to+\infty$,

$$H_{b,s}((x,Y),(x',Y'))\to\frac{1}{2s}|x'-x|^{2}. \tag{3.50}$$

The above considerations indicate that the concentration around the geodesic flow can also be seen at the level of heat kernels.

When considering traces instead, by Theorems 3.3 and 3.6, as $b\to+\infty$,

$$\begin{aligned}\operatorname{Tr}\big[\exp(-s\mathcal{L}_{b}^{E/\Lambda})\big]&\sim(b^{2}/s)^{n}\operatorname{Tr}\big[\exp(s\Delta^{E/\Lambda}/2)\big],\\ \operatorname{Tr}^{a}\big[\exp(-s\mathcal{L}_{b}^{E})\big]&\simeq(b^{2}/s)^{n}\exp(s\Delta^{E}/2)(a).\end{aligned} \tag{3.51}$$

4. Index theory and the hypoelliptic Laplacian on a vector space.

In this section, we combine the results of Section 3 with elementary arguments of index theory to prove that in the proper sense, the supertrace of the heat kernel of a suitable modification of the operator $\mathcal{L}_{b}^{E}$ is just the trace of the original elliptic heat kernel. The price to pay is that the new hypoelliptic operators are no longer scalar.

This section is organized as follows. In Subsection 4.1, we introduce some elementary tools of linear algebra that are relevant in index theory.

In Subsection 4.2, we construct the Witten complex on the Euclidean vector space E. Its corresponding Hodge Laplacian is a simple modification of the harmonic oscillator H^E.

In Subsection 4.3, we construct a first order differential operator D_b^E acting on $C^\infty(E \times E, \Lambda^\cdot(E^*))$, and we obtain the nonscalar hypoelliptic Laplacian $\mathscr{L}_b^E$.

In Subsection 4.4, we obtain the result mentioned before on the invariance of the trace of the elliptic Laplacian by the hypoelliptic deformation.

Finally, in Subsection 4.5, we make $b \to +\infty$ in our fundamental identity, and we recover standard identities on the heat kernel of E or on E/Λ. It turns out that the algebraic machine which produces such identities will extend to a more general geometric context.

4.1. Linear algebra.

Let $H = H_+ \oplus H_-$ be a $\mathbf{Z}_2$-graded real or complex vector space. Let $\tau = \pm 1$ be the involution of H that defines the grading, i.e., $\tau = \pm 1$ on $H_\pm$. The algebra $\mathrm{End}(H)$ is a $\mathbf{Z}_2$-graded algebra, the even part being made of morphisms commuting with τ, the odd part of the morphisms that anticommute with τ.

If $a, b \in \mathrm{End}(H)$, we define the supercommutator $[a, b]$ by the formula

$$[a, b] = ab - (-1)^{\deg a \deg b} ba. \tag{4.1}$$

Note that if a, b are both odd, $[a, b]$ is the anticommutator of a, b.

If $a \in \mathrm{End}(H)$, we define its supertrace $\mathrm{Tr_s}[a]$ by the formula

$$\mathrm{Tr_s}[a] = \mathrm{Tr}[\tau a]. \tag{4.2}$$

A fundamental fact [**Q85**] is that supertraces vanish on supercommutators. Indeed, if $a, b \in \mathrm{End}(H)$, the only nontrivial case is when a, b are both odd. In this case

$$\tau[a, b] = \tau ab - b\tau a. \tag{4.3}$$

The fact that $\mathrm{Tr_s}[[a, b]]$ vanishes reduces to the fact that the trace of a commutator vanishes.

Note that $D \in \mathrm{End}^{\mathrm{odd}}(H)$ can be written in matrix form with respect to the splitting $H = H_+ \oplus H_-$ as

$$D = \begin{bmatrix} 0 & D_- \\ D_+ & 0 \end{bmatrix}. \tag{4.4}$$

Proposition 4.1. *Let $D \in \mathrm{End}^{\mathrm{odd}}(H)$. Then*

$$\mathrm{Tr_s}[\exp(-D^2)] = \dim H_+ - \dim H_-. \tag{4.5}$$

Proof. We only need to show that $\mathrm{Tr_s}[\exp(-sD^2)]$ does not depend on s. Using the fact that

$$D^2 = \frac{1}{2}[D, D], \tag{4.6}$$

we get

$$\frac{\partial}{\partial s}\mathrm{Tr_s}[\exp(-sD^2)] = -\frac{1}{2}\mathrm{Tr_s}[[D, D]\exp(-sD^2)]. \tag{4.7}$$

Since

$$[D, D^2] = 0 \tag{4.8}$$

a form of Jacobi's identity shows that

$$[D, D]\exp(-sD^2) = [D, D\exp(-sD^2)]. \tag{4.9}$$

By (4.7), (4.9), we get

$$\frac{\partial}{\partial s}\mathrm{Tr_s}[\exp(-sD^2)] = -\frac{1}{2}\mathrm{Tr_s}[[D, D\exp(-sD^2)]]. \tag{4.10}$$

Since supertraces vanish on supercommutators, (4.10) vanishes, which concludes the proof. □

In the sequel, we will also use the above formalism in an infinite dimensional setting.

4.2. The Witten complex.

Let E be a finite dimensional Euclidean vector space. Let $(\Omega^{\cdot}(E), d^E)$ denote its de Rham complex. The Witten twist [**W82**] of the operator d^E by the Gaussian function is given by

$$\exp(-|Y|^2/2)d^E\exp(|Y^2/2|) = d^E + Y\wedge. \tag{4.11}$$

We equip $\Omega^{\cdot}_2(E)$, the vector space of L_2 forms on E, with its L_2 scalar product

$$\langle s, s'\rangle = \int_E \langle s, s'\rangle_{\Lambda^{\cdot}(E^*)}dY. \tag{4.12}$$

The formal adjoint of $d^E + Y\wedge$ is just $d^{E*} + i_Y$, where d^{E*} is just the formal adjoint of d^E with respect to (4.12), and i_Y is the contraction by Y. Then $[d + Y\wedge, d^* + i_Y]$ (the supercommutator of the two operators) is the associated Hodge Laplacian.

Let $N^{\Lambda^{\cdot}(E^*)}$ be the number operator of the exterior algebra $\Lambda^{\cdot}(E^*)$, that acts by multiplication by p on $\Lambda^p(E^*)$. Recall that H is the harmonic oscillator on E. Put

$$D^E = d + Y\wedge + d^* + i_Y. \tag{4.13}$$

An easy computation in [**W82**] shows that

$$\frac{1}{2}D^{E,2} = \frac{1}{2}\big[d^E + Y\wedge, d^{E*} + i_Y\big] = H^E + N^{\Lambda^{\cdot}(E^*)}. \tag{4.14}$$

By (4.14), the kernel of the Hodge Laplacian in (4.14) is concentrated in degree 0 in $\Lambda^{\cdot}(E^*)$ and is generated by the Gaussian $\exp(-|Y|^2/2)$, on which $d + Y\wedge, d^* + i_Y$ both vanish.

It is equivalent to equip the vector space of bounded smooth forms in $\Omega^{\cdot}(E)$ with the scalar product

$$\langle s, s' \rangle' = \int_E \langle s, s' \rangle \exp(-|Y|^2) dY, \tag{4.15}$$

and to consider instead the formal adjoint of d^E with respect to (4.15) given by $d^{E*}+2i_Y$. Put

$$D^{E'} = d^E + d^{E*} + 2i_Y. \tag{4.16}$$

Then

$$D^{E\prime} = \exp(|Y|^2/2) D^E \exp(-|Y|^2/2). \tag{4.17}$$

The associated Hodge Laplacian is such that

$$\frac{1}{2}D^{E',2} = \frac{1}{2}\big[d^E, d^{E*} + 2i_Y\big] = A^E + N^{\Lambda^{\cdot}(E^*)}. \tag{4.18}$$

The kernel of the operator in (4.18) is concentrated in degree 0 and is generated by the function 1.

Proposition 4.2. *For any $s > 0$, the following identity holds*:

$$\mathrm{Tr_s}\big[\exp(-sD^{E,2}/2)\big] = 1. \tag{4.19}$$

Proof. By (4.14), we get

$$\mathrm{Tr_s}[\exp(-sD^{E,2}/2)] = \mathrm{Tr}[\exp(-sH^E)]\mathrm{Tr_s}\big[\exp(-sN^{\Lambda^{\cdot}(E^*)})\big]. \tag{4.20}$$

By (3.31), we get

$$\mathrm{Tr}[\exp(-sH^E)] = (1 - e^{-s})^{-n}. \tag{4.21}$$

Also it is elementary to show that

$$\mathrm{Tr_s}\big[\exp(-sN^{\Lambda^{\cdot}(E^*)})\big] = (1 - e^{-s})^n. \tag{4.22}$$

By (4.20)–(4.22), we get (4.19).

Another proof is based on the proof of Proposition 4.1. Indeed, the fact that (4.19) does not depend on s can be shown as in Proposition 4.1. By making instead $s \to +\infty$, we get (4.19). □

Remark 4.3. The method used in the second proof of Proposition 4.2 can be extended to more general index problems.

4.3. The hypoelliptic de Rham Hodge complex.

The generic coordinate in $E \times E$ is still denoted (x, Y). The exterior algebra $\Lambda^{\cdot}(E^*)$ should be thought as the exterior algebra of the dual of the second copy of E in $E \oplus E$. The exterior algebras of the two copies of E will be identified. If $e_1, \ldots, e_n$ is an orthonormal basis of E, using coordinates on $E \times E$ with respect to this base, we have the identity

$$dx^i = dY^i, \quad 1 \leq i \leq n. \tag{4.23}$$

Now we will consider the vector space $C^\infty(E \times E, \Lambda^{\cdot}(E^*))$. Let d_x, d_Y be the de Rham operators along the two copies of E, and let d_x^*, d_Y^* denote their classical formal adjoints. For $b > 0$, set

$$D_b^E = -d_x^E + d_x^{E*} + \frac{1}{b}\big(d_Y^E + Y \wedge + d_Y^{E*} + i_Y\big). \tag{4.24}$$

Equation (4.24) can be written in the form

$$D_b^E = \frac{dy^i}{b}\left(\frac{\partial}{\partial y^i} + y^i - b\frac{\partial}{\partial x^i}\right) + \frac{i_{\partial/\partial y^i}}{b}\left(-\frac{\partial}{\partial y^i} + y^i - b\frac{\partial}{\partial x^i}\right). \tag{4.25}$$

Let $\widehat{D}_{b,\xi}^E$ be the Fourier transform of D_b^E in the variable x. By (4.24), we get

$$\widehat{D}_{b,\xi}^E = \frac{1}{b}\big(d_Y^E + Y \wedge -ib\xi \wedge + d_Y^{E*} + i_{Y-ib\xi}\big). \tag{4.26}$$

The operators in (4.25), (4.26) are not self-adjoint. Also using the notation in (3.17), (3.22), by (4.25), (4.26), we get

$$\exp(bN) D_b^E \exp(-bN) = \frac{D^E}{b}, \qquad T_{ib\xi} \widehat{D}_{b,\xi}^E T_{ib\xi}^{-1} = \frac{D^E}{b}. \tag{4.27}$$

By (4.14), (4.27), we conclude that

$$\begin{aligned} \exp(bN)\frac{1}{2} D_b^{E\times E,2} \exp(-bN) &= \frac{1}{b^2}\big(H^E + N^{\Lambda^{\cdot}(E^*)}\big), \\ T_{ib\xi}\frac{1}{2}\widehat{D}_{b,\xi}^{E,2} T_{ib\xi}^{-1} &= \frac{1}{b^2}\big(H^E + N^{\Lambda^{\cdot}(E^*)}\big). \end{aligned} \tag{4.28}$$

By the results of Subsection 3.3, we conclude easily that $\widehat{D}_{b,\xi}^E$ is isospectral to D^E/b.

If $\Lambda \subset E$ is a lattice, we can replace $E \times E$ by $E/\Lambda \times E$. By the above, we conclude that $D_b^{E/\Lambda}$ is isospectral to an infinite number of copies of D^E/b indexed by the Fourier modes of E/Λ.

Recall that the operator $\mathcal{L}_b^E$ was defined in Subsection 3.1.

Definition 4.4. Set

$$\mathscr{L}_b^E = \mathcal{L}_b^E + \frac{N^{\Lambda^\cdot(E^*)}}{b^2}, \qquad \widehat{\mathscr{L}_{b,\xi}^E} = \widehat{\mathcal{L}}_{b,\xi}^E + \frac{N^{\Lambda^\cdot(E^*)}}{b^2}. \tag{4.29}$$

Then $\mathscr{L}_b^E$ is a hypoelliptic operator acting on $C^\infty(E \times E, \Lambda^\cdot(E^*))$, and $\widehat{\mathscr{L}_{b,\xi}^E}$ is its Fourier transform in the variable x.

Proposition 4.5. *The following identities hold*:

$$\mathscr{L}_b^E = -\frac{1}{2}\Delta^{E,H} + \frac{1}{2}D_b^{E,2}, \qquad \widehat{\mathscr{L}_{b,\xi}^E} = \frac{1}{2}|\xi|^2 + \frac{1}{2}\widehat{D}_{b,\xi}^{E\times E,2}. \tag{4.30}$$

Proof. By (4.14) and (4.28), we get

$$D_b^{E,2} = \frac{1}{2b^2}\big(- \Delta^{E,V} + |Y - b\nabla_\cdot^H|^2 - n\big) + \frac{N^{\Lambda^\cdot(E^*)}}{b^2}. \tag{4.31}$$

By (4.29), (4.31), we get (4.30). □

Remark 4.6. If $\Lambda \subset E$ is a lattice, we define the operator $\mathscr{L}_b^{E/\Lambda}$ on $E/\Lambda \times E$ by a similar formula.

4.4. A nonperturbative identity.

Now we give another version of Theorems 3.3 and 3.6.

Theorem 4.7. *For any $s > 0, a \in E$, the following identities hold*:

$$\begin{aligned} \operatorname{Tr}_s^a\left[\exp(-s\mathscr{L}_b^E)\right] &= \exp(s\Delta^E/2)(a), \\ \operatorname{Tr_s}\left[\exp(-s\mathscr{L}_b^{E/\Lambda})\right] &= \operatorname{Tr}\left[\exp(s\Delta^{E/\Lambda}/2)\right]. \end{aligned} \tag{4.32}$$

Proof. Our theorem follows from Theorems 3.3, 3.6, and from Proposition 4.2. □

Remark 4.8. It follows from the above that the operator $D_b^{E/\Lambda,2}$ is isospectral to the operator $D^{E,2}/b^2$ acting on $C^\infty(E/\Lambda, \mathbf{R}) \otimes C^\infty(E, \Lambda^\cdot(E^*))$. The corresponding eigenvalues have infinite multiplicity, because $C^\infty(E/\Lambda, \mathbf{R})$ is infinite dimensional. Adding $-\Delta^{E,H}/2$ has the effect of making $\mathscr{L}_b^{E/\Lambda}$ hypoelliptic, and the corresponding heat operator to be trace class. A similar interpretation can be given on the operator $\mathscr{L}_b^E$.

The fact that equation (4.32) holds when $b \to 0$ can still be viewed as a consequence of Theorems 2.5 and 3.6. It is similar in spirit to equation (4.5) in Proposition 4.1. It gives a form of the McKean-Singer index formula in index theory [**MS67**].

4.5. The limit $b \to +\infty$.

We can now take the limit as $b \to +\infty$ in equation (4.32). Contrary to what happens in equation (3.51), there is no singularity as $b \to \infty$. By using (3.45) explicitly, from (4.32) we get

$$\begin{aligned} \exp(s\Delta^E/2)(a) &= \frac{\exp(-|a|^2/2)}{(2\pi s)^{n/2}}, \\ \operatorname{Tr}\left[\exp(s\Delta^{E/\Lambda}/2)\right] &= \frac{\operatorname{Vol}(E/\Lambda)}{(2\pi s)^{n/2}} \sum_{\lambda\in\Lambda} \exp(-|\lambda|^2/2s). \end{aligned} \tag{4.33}$$

The identities in (4.33) should be viewed as a consequence of taking the limits as $b \to 0$ and $b \to +\infty$ in (4.32). The second identity is just the Poisson formula.

Of course these identities are well-known! They have acquired an index theoretic flavour, i.e., they express a global (or operator theoretic) quantity in local terms, exactly like the index theorem of Atiyah-Singer [**AS68a**], [**AS68b**]. But most importantly, they will have a nontrivial extension in more general geometric situations.

5. The geometric hypoelliptic Laplacians.

The purpose of this section is to explain the construction of the geometric hypoelliptic Laplacian. More precisely, we describe the scalar hypoelliptic Laplacian acting on the total space of the tangent bundle of a compact Riemannian manifold, and also the associated diffusion, which projects on X as a geometric Langevin process. Also, we briefly explain the extension to symmetric spaces and compact locally symmetric spaces of the index theoretic constructions of Section 4.

This section is organized as follows. In Subsection 5.1, we construct the scalar hypoelliptic Laplacian.

In Subsection 5.2, we describe the corresponding geometric Langevin process, and we state various results on the behaviour as $b \to 0$ of this hypoelliptic diffusion and of its heat kernel.

In Subsection 5.3, we show that as $b \to +\infty$, the geometric Langevin process converges to a suitable version of the geodesic flow.

In Subsection 5.4, we give a few details on the construction of the hypoelliptic Laplacian in de Rham theory.

Finally, in Subsection 5.5, we explain some aspects of the construction of the hypoelliptic Laplacian on symmetric and locally symmetric spaces, by focusing on the case of the Poincaré upper half plane, and on Riemann surfaces of constant negative curvature.

5.1. The hypoelliptic Laplacian on a compact manifold.

Let X be a compact Riemannian manifold. Let ∇^{TX} be the Levi-Civita connection on the tangent bundle TX. Let $\pi : \mathcal{X} \to X$ be the total space of the tangent bundle of X. The fibres of π will be denoted $\widehat{TX}$, to distinguish these fibres from the tangent bundle TX. Let $T^H\mathcal{X} \subset T\mathcal{X}$ denote the horizontal subbundle of associated with ∇^{TX}.

If $U \in TX$, let $U^H \in T^H \mathcal{X}$ denote its horizontal lift.

Let Y denote the tautological section on $\mathcal{X}$ of $\pi^* \widehat{TX}$. Let Z denote the vector field on $\mathcal{X}$ that generates the geodesic flow. If $Y \in \widehat{TX}$, we identify Y with the corresponding element of TX. Then we have the identity

$$Z = Y^H. \tag{5.1}$$

The corresponding differentiation operator will be denoted ∇_Y^H.

Let $H^{\widehat{TX}}, A^{\widehat{TX}}$ denote the harmonic oscillators along the fibres $\widehat{TX}$ with respect to the given Riemannian metric g^{TX}.

DEFINITION 5.1. For $b > 0$, let $\mathcal{L}_b^X, \mathcal{M}_b^X$ be the operators acting on $C^\infty(\mathcal{X}, \mathbf{R})$

$$\mathcal{L}_b^X = \frac{H^{\widehat{TX}}}{b^2} - \frac{\nabla_Y^H}{b}, \qquad \mathcal{M}_b^X = \frac{A^{\widehat{TX}}}{b^2} - \frac{\nabla_Y^H}{b}. \tag{5.2}$$

As in (2.9), we get

$$\mathcal{L}_b^X = \exp(-|Y|^2/2)\mathcal{M}_b^X \exp(|Y|^2/2). \tag{5.3}$$

The operators $\mathcal{L}_b^X, \mathcal{M}_b^X$ are as canonical as the ordinary Laplacian $-\Delta^X/2$. They are still hypoelliptic. Contrary to the hypoelliptic Laplacians on Heisenberg manifolds that require a special geometry, our hypoelliptic operators exist universally. Even though their structure is similar to the structure of the operators $\mathcal{L}_b^E, \mathcal{M}_b^E$ in (3.4), the situation is somewhat different, because the harmonic oscillators $A^{\widehat{TX}}, H^{\widehat{TX}}$ now act along the fibres of a vector bundle.

The operators $\mathcal{L}_b^X, \mathcal{M}_b^X$ are geometric versions of Fokker-Planck operators.

5.2. The geometric Langevin process.

The probabilistic constructions of the heat equation semigroups $\exp(-s\mathcal{L}_b^X)$, $\exp(-s\mathcal{M}_b^X)$ is formally the same as in (3.6)–(3.8) and in (3.10)–(3.12). The only very significant difference is that the Brownian motion $w.$ takes its values in T_xX and is transported along the trajectory $x.$ with respect to the Levi-Civita connection. Also $\dot{Y}$ should be interpreted here as the covariant derivative of Y with respect to the Levi-Civita connection ∇^{TX}. In particular as in (3.9), the stochastic differential equation corresponding to $\mathcal{M}_b^X$ projects on X to the second order stochastic differential equation

$$b^2\ddot{x} + \dot{x} = \dot{w}. \tag{5.4}$$

In (5.4), $\dot{w}$ is the Stratonovitch differential of Brownian motion in TX. Again $\ddot{x}$ is calculated using the Levi-Civita connection.

One remarkable aspect of the above equations is that the stochastic differential equations can be solved pointwise, for every trajectory $w.$. This is in dramatic contrast with the stochastic differential equation

$$\dot{\boldsymbol{x}} = \dot{\boldsymbol{w}}, \qquad \boldsymbol{x}_0 = x \tag{5.5}$$

corresponding to Brownian motion on X.

For $s > 0$, let $\exp(s\Delta^X/2)(x, x')$ be the smooth kernel on X for $\exp(s\Delta^X/2)$ with respect to the volume dx'. Let $\exp(-s\mathcal{M}_b^X)((x, Y), (x', Y'))$ be the smooth kernel for $\exp(-s\mathcal{M}_b^X)$ with respect to $\exp(-|Y'|^2)dx'(dY'/\pi^{n/2})$. Set

$$Z_\cdot = \frac{Y_\cdot}{b}. \tag{5.6}$$

THEOREM 5.2. *As $b \to 0$, the probability law of $(x_\cdot, Z_\cdot)$ converges to the probability law of $(\boldsymbol{x}_\cdot, \dot{\boldsymbol{w}}_\cdot)$. For $0 < s_1 < \cdots < s_m$, the law of $((x_{s_1}, Y_{s_1}), \ldots, (x_{s_m}, Y_{s_m}))$ converges to the law of $((\boldsymbol{x}_{s_1}, \boldsymbol{Y}_{s_1}), \ldots, (\boldsymbol{x}_{s_m}, \boldsymbol{Y}_{s_m}))$, where conditionally on $\boldsymbol{x}_{s_1}, \ldots, \boldsymbol{x}_{s_m}$, the probability law of $(\boldsymbol{Y}_{s_1}, \ldots, \boldsymbol{Y}_{s_m})$ is the product of the probability laws $\exp(-|Y|^2)dY/\pi^{n/2}$ in the fibres $\widehat{TX}_{\boldsymbol{x}_{s_1}}, \ldots, \widehat{TX}_{\boldsymbol{x}_{s_m}}$.*

The joint laws of $(x_\cdot, Z_\cdot)$ and of $Y_{s_1}, \ldots, Y_{s_m}$ converge to the corresponding product law.

As $b \to 0$, for any $s > 0$, we have the uniform convergence of smooth kernels and their derivatives of any order over compact sets of $\mathcal{X}$

$$\exp\big(-s\mathcal{M}_b^X\big)((x, Y), (x', Y')) \to \exp(s\Delta^X/2)(x, x'). \tag{5.7}$$

As $b \to 0$, for any $s > 0$

$$\operatorname{Tr}\big[\exp(-s\mathcal{M}_b^X)\big] \to \operatorname{Tr}[\exp(s\Delta^X/2)]. \tag{5.8}$$

PROOF. The first part of our theorem was established in the proof of [**B11**, Theorem 12.8.1] when X is a noncompact symmetric space. The proof when X is an arbitrary compact manifold is exactly the same.

The convergence of the smooth kernels in (5.7) was proved by Bismut-Lebeau [**BL08**, Equation (3.4.10) and chapter 17] using pseudodifferential operators. Part of the functional analysis developed in [**BL08**] is based on the analogue of the matrix splitting in the proof of Theorem 1.1. Another proof, also valid in the noncompact case, was given in [**B08c**, Theorem 12.8.1], that is based on probabilistic arguments. The Malliavin calculus plays an important role in this last proof. □

REMARK 5.3. By Theorem 5.2, the analogues of Theorem 3.1 and of equation (3.38) still hold in the geometric situation considered above. The behaviour of the lower part of the spectrum of $\mathcal{L}_b^X$ was studied in detail in [**BL08**, Chapter 10]. In [**B11**, Section 14.10], the ergodic theorem for the Ornstein-Uhlenbeck process $Y_\cdot$ plays an important role in the analysis of the above convergence as $b \to 0$. This is not surprising in view of the considerations of Sections 1 and 2. Finally, note that the analogue of equation (2.24) is given by

$$\int_{\widehat{TX}} \nabla_Y^H \big(A^{\widehat{TX}}\big)^{-1} \nabla_Y^H \exp(-|Y|^2) \frac{dY}{\pi^{n/2}} = \frac{1}{2}\Delta^X. \tag{5.9}$$

Equation (5.9) is the simplest formal explanation for some of the above results.

5.3. The dynamical aspects of the limit $b \to +\infty$.

For $b > 0$, let k_b be the morphism of $C^\infty(\mathcal{X}, \mathbf{R})$ given by $f(x, Y) \to f(x, bY)$. As in (3.41), put

$$\underline{\mathcal{L}}_b^X = K_b \mathcal{L}_b^X K_b^{-1}. \tag{5.10}$$

As in (3.42), we get

$$\underline{\mathcal{L}}_b^X = \frac{1}{2}\left(-\frac{\Delta^V}{b^4} + |Y|^2 - \frac{n}{b^2}\right) - \nabla_Y^H. \tag{5.11}$$

As in (3.45), as $b \to +\infty$,

$$\underline{\mathcal{L}}_b^X \to \underline{\mathcal{L}}_\infty^X = \frac{1}{2}|Y|^2 - \nabla_Y^H. \tag{5.12}$$

The above indicates that as in Subsection 3.8, as $b \to +\infty$, the heat equation for $\underline{\mathcal{L}}_b^X$ propagates more and more along the geodesic flow. This is also obvious by equation (5.4).

Intuitively, it should be clear that as $b \to +\infty$, given $s > 0$, $\mathrm{Tr_s}[\exp(-s\mathcal{L}_b^X)]$ localizes around closed geodesics on the time interval $[0, s]$. Since there are an infinite number of those, handling the localization is a priori not so easy.

5.4. The hypoelliptic Laplacian in de Rham theory.

It is not possible to establish a nonperturbative identity for $\mathrm{Tr}[\exp(-s\mathcal{L}_b^X)]$ similar to equation (3.29) in Theorem 3.3, because there is no conjugation identity like (3.26). In general, the spectrum of $\mathcal{L}_b^X$ cannot be computed explicitly.

In [**B05**], we have shown that in the proper sense, in the same way as $-\Delta^X$ can be viewed as the restriction to 0-forms of a classical elliptic Hodge Laplacian $\square^X$, the hypoelliptic operators $\mathcal{L}_b^X, \mathcal{M}_b^X$ can be viewed as the restriction to 0-forms of a hypoelliptic Hodge Laplacian acting on $\mathcal{X}$. In [**BL08**], with Lebeau, we have shown that in full generality, certain spectral invariants like the Ray-Singer analytic torsion are invariant under the deformation from elliptic to hypoelliptic Hodge theory. For explicit connections between the hypoelliptic Laplacian and the Witten Laplacian, we refer to recent work by Shen [**Sh14**].

5.5. The hypoelliptic Laplacian on locally symmetric spaces.

The purpose of our later work [**B11**] is to show that on symmetric or locally symmetric spaces of noncompact type, the nonperturbative index theoretic identities of Theorem 4.7 can be suitably extended. The proper description of locally symmetric spaces is in terms of Lie groups, and that part of the mystery of the construction outlined below lies in group theoretic considerations which go beyond the scope of this paper.

If X is a locally symmetric space of noncompact type, the associated hypoelliptic Laplacian $\mathscr{L}_b^X$ acts on the total space $\widehat{\mathcal{X}}$ of a larger bundle $TX \oplus N$ than TX, N being a natural orthocomplement to TX. More precisely, $\mathscr{L}_b^X$ acts on smooth sections over $\widehat{\mathcal{X}}$ of the exterior algebra $\Lambda^{\cdot}(T^*X \oplus N^*)$.

Let us explain the construction of $\mathscr{L}_b^X$ when X is the Poincaré upper half-plane. We will write X as the symmetric space

$$X = \mathrm{SL}_2(\mathbf{R})/S^1. \tag{5.13}$$

The Poincaré metric on X, which has constant curvature -1, just comes from the Killing form on the Lie algebra $\mathrm{sl}_2(\mathbf{R})$. Also $\mathrm{PSL}_2(\mathbf{R}) = \mathrm{SL}_2(\mathbf{R})/\{\pm 1\}$ acts on X as its group of isometries.

The Lie algebra $\mathrm{sl}_2(\mathbf{R})$ splits as

$$\mathrm{sl}_2(\mathbf{R}) = \mathfrak{p} \oplus \mathbf{R}. \tag{5.14}$$

Indeed $\mathrm{sl}_2(\mathbf{R})$ is the Lie algebra of $(2,2)$ trace free real matrices, $\mathfrak{p}$ is the vector space of trace free symmetric matrices, and $\mathbf{R}$ consists of the antisymmetric matrices.

The tangent bundle TX can easily be obtained from the adjoint action of $S^1 \subset \mathrm{SL}_2(\mathbf{R})$ on $\mathfrak{p}$. Still, the splitting (5.14) indicates that there is a canonical 1-dimensional real line bundle N on X such that $F = TX \oplus N$ is canonically flat. Here, this means that over X, $TX \oplus N$ can be canonically identified with the trivial vector bundle $\mathrm{sl}_2(\mathbf{R})$. The metric on $TX \oplus N$ comes from the adjoint action of S^1 on the right-hand side of (5.14). However, the canonical flat connection does not preserve the splitting on F and does not preserve the metric.

To better explain the construction of $TX \oplus N$, let us consider the sphere S^2 embedded in $\mathbf{R}^3$. The orthogonal bundle N to TS^2 in $\mathbf{R}^3$ is such that $TS^2 \oplus N = \mathbf{R}^3$, and $\mathbf{R}^3$ is canonically flat, but the canonical flat connection does not preserve the splitting of $\mathbf{R}^3$. Exactly the same construction can be used for X, when identifying X with its hyperbolic model, the Euclidean product on $\mathbf{R}^3$ being replaced by the canonical bilinear symmetric form of signature $(2,1)$.

Over X, we will extend the considerations of Subsection 4.3. Namely $E \times E$ is replaced by the total space $\widehat{\mathcal{X}}$ of $TX \oplus N$. The Witten complex over the second copy of E is replaced by the corresponding Witten complex along the fibres of the Euclidean vector bundle $TX \oplus N$.

In the constructions of Subsection 4.3, we have used the following identity

$$\big(- d_x^E + d_x^{E*} \big)^2 = \Delta^E. \tag{5.15}$$

Equivalently, the standard Laplacian Δ^E on E has a natural square root which is a differential operator.

Let Y_1, Y_2, Y_3 be the vector fields on X that correspond to an orthonormal basis of $\mathrm{sl}_2(\mathbf{R}) = \mathfrak{p} \oplus \mathfrak{k}$. Then we have the elementary formula

$$\Delta^X = Y_1^2 + Y_2^2 - Y_3^2. \tag{5.16}$$

The Dirac operator $\widehat{D}^K$ introduced by Kostant [**Ko97**], which is an operator acting on

$C^\infty(X,\Lambda^\cdot(\mathrm{sl}_2(\mathbf{R})^*))$[7], is such that

$$\widehat{D}^{K,2}=\Delta^X+c. \tag{5.17}$$

In (5.17), c is an explicitly computable constant. This identity is a strict analogue of (5.15). Recall that $TX\oplus N$ is the trivial vector bundle $\mathrm{sl}_2(\mathbf{R})$. Therefore $\widehat{D}^K$ can be viewed as an operator acting on $C^\infty(X,\Lambda^\cdot(T^*X\oplus N^*))$.

Remarkably enough, the Witten Laplacian of the fibres $TX\oplus N$ acts on sections of $\Lambda^\cdot(T^*X\oplus N^*)$ along $TX\oplus N$, and $\widehat{D}^K$ acts on sections on X on $\Lambda^\cdot(T^*X\oplus N^*)$. Let $\widehat{\pi}:\widehat{\mathcal{X}}\to X$ be the obvious projection. These considerations at least suggest that these two kinds of operators can be combined into a single operator $\mathscr{L}_b^X$ acting on $C^\infty(\widehat{\mathcal{X}},\widehat{\pi}^*\Lambda^\cdot(T^*X\oplus N^*))$. This is precisely what is done in [**B11**].

If Σ is the Riemann surface obtained by quotienting X by the action of a discrete cocompact torsion free subgroup Γ of $\mathrm{SL}_2(\mathbf{R})$, the flat vector bundle $F=TX\oplus N$ on X descends to the flat vector bundle $F=T\Sigma\oplus N$ on Σ. Ultimately, the whole construction descends to Σ.

The main result in [**B08c**] is that Theorem 4.7 can be suitably extended, X and Σ replacing E and E/Λ. For Σ, the analogue of the second equation in (4.33) is Selberg's trace formula [**M72**].

The results of [**B11**] are valid for compact locally symmetric spaces associated with arbitrary reductive Lie groups.

Remark 5.4. A question which has been repeatedly raised is why, instead of considering the hypoelliptic Laplacian and the corresponding Langevin process, one does not pick instead a process whose speed would be the Brownian motion along the sphere bundle in TX, and the corresponding infinitesimal generator. First, the hypoelliptic Laplacian emerges as a natural answer to questions connected with isospectral deformations of the elliptic Laplacian. More fundamentally, the hypoelliptic Laplacian behaves properly under natural operations like taking products of manifolds. This is not the case with diffusions whose speed lies in a sphere bundle, simply because the product of spheres is not a sphere. Finally, as is made clear by the Poisson formula on the circle, the trace of the heat kernel is expressed as an infinite sum indexed by the lengths of all closed geodesics in a given time t. No such sum would ever emerge from a stochastic process with constant speed. There is certainly a counterpart to the $b\to 0$ properties of the hypoelliptic Laplacian using sphere bundles, but the $b\to+\infty$ aspect would be lost.

The sphere bundle emerges naturally when considering instead the wave equation on X, which propagates at constant speed. Note that the product of solutions of the wave equation on two manifolds is not a solution of the wave equation on the product. As explained in [**B11**, Section 12.3] and [**B12**, Section 5.3], deep connections exist between the hypoelliptic Laplacian and the wave equation. Indeed it can be shown that as $b\to 0$, the integral along the fibre $\widehat{TX}$ of the hypoelliptic heat kernel is an approximate solution of the wave equation on X that propagates at speed $1/b$. This fact does not contradict

[7] The Dirac operator of Kostant acts on $C^\infty(\mathrm{SL}_2(\mathbf{R}),\Lambda^\cdot(\mathrm{sl}_2(\mathbf{R})^*))$. We just give here a simplified version of the Kostant operator.

the nonfunctorial behaviour of solutions of the wave equation, because as $b \to 0$, the speed of propagation becomes infinite.

Important work has been done by Franchi and Le Jan [**FL07**], [**FL11**] on diffusions on the Lorentzian sphere bundle of Lorentzian manifolds. In this case, the functorial obstruction to naturality disappears, because the product of Lorentzian manifolds is not Lorentzian.

Recollections of Professor K. Itô. I first met Professor Kiyosi Itô at a conference funded by the Taniguchi Foundation in Katata in 1982. Accompanying me were my wife and our one year and a half son. Professor Itô struck me as a genuinely kind and attentionate person. He told us of his mathematical life, first under duress in war time, his fundamental work on stochastic differential equations later facing temporary misunderstanding. This was said in a quiet tone, with a genuine sense of humour. His wife and himself were extremely kind to our son, whom they tried to provide with the best milk possible.

Part of the conference was revolving around Paul Malliavin's stochastic calculus of variations. Professor Itô liked to mention he was now a student of Paul Malliavin, which provoked bursts of laughter on Paul Malliavin's side.

Once he asked to talk to me privately. He gave me such an astonishing and personal piece of advice and encouragement that I remember it to this day.

With Professor Kiyosi Itô, we lost a top mathematician, and a gentleman.

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