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Infinite dimensional oscillatory integrals as projective systems of functionals

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Dedicated to the memory of Professor Kiyosi Itô

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Abstract. The theory of infinite dimensional oscillatory integrals and some of its applications are discussed, with special attention to the relations with the original work of K. Itô in this area. A recent general approach to infinite dimensional integration which unifies the case of oscillatory integrals and the case of probabilistic type integrals is presented, together with some new developments.

1. Introduction.

It is a special honor and pleasure for us to be able to dedicate the present pages to the memory of K. Itô, on the occasion of the centenary of his birth. His early work on infinite dimensional oscillatory integrals stimulated very much subsequent research on this topic.

We shall first give a short exposition of the theory of a particular class of oscillatory integrals and relate them to corresponding probabilistic integrals. Oscillatory integrals are objects of the form:

$$I^{\Phi/\epsilon}(f) = \text{“}\, C^{-1}\int_\Gamma e^{i(\Phi/\epsilon)(\gamma)} f(\gamma) d\gamma \,\text{”} \tag{1}$$

where Γ denotes either a finite dimensional space (e.g. $\mathbb{R}^n$, or an n-dimensional differential manifold M^n), or an infinite dimensional space (e.g. a “path space”). $\Phi : \Gamma \to \mathbb{R}$ is called phase function, while $f : \Gamma \to \mathbb{C}$ is the function to be integrated and $\epsilon \in \mathbb{R}\setminus\{0\}$ is a parameter. The symbol $d\gamma$ denotes a “flat” Lebesgue (or Haar-) type volume measure, while C plays the role of a “normalization” constant. In the case where Γ is a finite dimensional vector space, i.e. $\Gamma = \mathbb{R}^n$, $n \in \mathbb{N}$, then $d\gamma$ is the Riemann–Lebesgue volume measure and expression (1) is a basic object of study in classical analysis, e.g. [**20**], [**31**].

In the case where Γ is infinite dimensional no analogue of Riemann–Lebesgue volume measure is mathematically well defined and $d\gamma$ is, to start with, just an heuristic expression. We shall focus ourselves basically on this latter case, showing how to generalize the concept of integration giving a well defined mathematical meaning to (1) when Γ can be

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realized as a real separable infinite dimensional Hilbert space (or other spaces, closely related to such Hilbert spaces).

The main motivation is the study of the "Feynman path integrals", a class of (heuristic) functional integrals introduced by R. P. Feynman in 1942 in order to propose an alternative, Lagrangian, formulation of quantum mechanics. In the traditional "von Neumann" formulation of quantum theory the states of a non relativistic particle moving in a d-dimensional Euclidean space are described by normalized vectors ψ belonging to the complex Hilbert space $L^2(\mathbb{R}^d)$. The time evolution is given by a (strongly continuous) one parameter group of unitary operators $\{U(t)\}_{t\in\mathbb{R}}$ on $L^2(\mathbb{R}^d)$, generated, by Stone's theorem, by a self-adjoint operator $H : D(H) \subset L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, with domain $D(H)$, the *quantum Hamiltonian*, and we can write $U(t) = e^{-(i/\hbar)Ht}$. In the case where the particle moves in a force field associated to a classical (real-valued) potential $V \in C(\mathbb{R}^d)$, the action of the Hamiltonian operator on functions $\varphi \in C_0^\infty(\mathbb{R}^d)$ with compact support (looked upon as vectors in $L^2(\mathbb{R}^d)$) is given by

$$H\varphi(x) = -\frac{\hbar^2}{2m}\Delta\varphi(x) + V(x)\varphi(x), \qquad x \in \mathbb{R}^d,$$

where Δ is the Laplacian, $m > 0$ is the mass of the particle and $\hbar$ is the reduced Planck constant. The time evolution of the state vector, or *wave function*, $\psi(t) = U(t)\psi_0$ is described by the Schrödinger equation:

$$\begin{cases} i\hbar\dfrac{\partial}{\partial t}\psi = -\dfrac{\hbar^2}{2m}\Delta\psi + V\psi \\ \psi(0,x) = \psi_0(x), \qquad\qquad \psi_0 \in C_0^\infty(\mathbb{R}^d). \end{cases} \tag{2}$$

In his PhD thesis [**22**] (see also [**23**]) R. Feynman proposed an alternative suggestive description of time evolution in quantum mechanics, in terms of an heuristic formula where the state vector $\psi(t)$ at time t should be given by an integral over the space of paths $\gamma : [0,t] \to \mathbb{R}^d$ with fixed end point

$$\psi(t,x) = \text{“ } C^{-1}\int_{\{\gamma|\gamma(t)=x\}} e^{(i/\hbar)S_t(\gamma)}\psi_0(\gamma(0))d\gamma \text{ ”} \tag{3}$$

where $S_t(\gamma) = S^0(\gamma) - \int_0^t V(\gamma(s))ds$, $S^0(\gamma) = (m/2)\int_0^t |\dot\gamma(s)|^2 ds$, is the classical action of the system evaluated along the path γ, while $d\gamma$ stands for a heuristic "flat" measure on the space of paths and $C =$ "$\int_{\{\gamma|\gamma(t)=x\}} e^{(i/\hbar)S^0(\gamma)}d\gamma$" plays the role of a normalization constant. It is well known and first observed in [**47**] (see references in [**3**]) that a derivation of Feynman's heuristic formula (3) can be obtained by means of Lie–Trotter's product formula, which gives the unitary group $U(t)$ generated by the operator sum of $-\Delta/2$ (regarded as a positive self-adjoint operator with domain the Sobolev space $H^2(\mathbb{R}^d)$) and the bounded multiplication operator associated to the continuous bounded potential $V \in C_b(\mathbb{R}^d)$ (i.e. defined as $(V\psi)(x) = V(x)\psi(x)$, $\psi \in L^2(\mathbb{R}^d)$) as the strong limit in $L^2(\mathbb{R}^d)$:

$$U(t)\psi_0 = \lim_{n\to\infty} \left(e^{it\Delta/2n} e^{-(it/n)V}\right)^n \psi_0$$

(we have set, for simplicity, $\hbar = m = 1$).

By passing to a subsequence and introducing the fundamental solution G of the Schrödinger equation, namely $e^{it\Delta/2}\psi_0(x) = \int_{\mathbb{R}^d} G_t(x,y)\psi_0(y)dy$, one obtains that for Lebesgue a.e. $x \in \mathbb{R}^d$ the action of the group $U(t)$ can be described in terms of the limit of a sequence of (finite dimensional) integrals of the form:

$$U(t)\psi_0(x) = \lim_{n\to\infty} \int_{\mathbb{R}^{dn}} \psi_0(x_0) e^{-i\sum_{j=1}^n V(x_{j-1})(t/n)} \prod_{j=1}^n G_{t/n}(x_j, x_{j-1}) dx_{j-1}, \qquad (4)$$

with $x_n \equiv x$. By introducing the explicit form of the Green function of the Schrödinger equation (i.e. of (2) with $\hbar = m = 1$ and $V = 0$), namely $G_t(x,y) = e^{i(x-y)^2/2t}/(2i\pi t)^{d/2}$, the integrals appearing on the right hand side of (4) assume the following form

$$U(t)\psi_0(x) = \lim_{n\to\infty} \int_{\mathbb{R}^{dn}} \frac{e^{i\sum_{j=1}^n ((x_j - x_{j-1})^2/2(t/n)^2 - V(x_{j-1}))(t/n)}}{(2\pi i t/n)^{nd/2}} \psi_0(x_0) dx_0 \cdots dx_{n-1}.$$

The term $\sum_{j=1}^n ((x_j - x_{j-1})^2/2(t/n)^2 - V(x_{j-1}))(t/n)$ appearing in the exponent can be regarded as a Cauchy–Riemann sums approximation of the classical action integral $S_t(\gamma)$. By taking heuristically the limit as $n \to \infty$, one obtains formula (3), that, at this stage, is just a symbolic expression which suggests a limiting procedure. Indeed formula (3), as it stands, lacks of mathematical rigor, in particular the "flat" Lebesgue-type measure $d\gamma$ appearing in (1), (3) and (5) below has no mathematical meaning. Going beyond this "minimal interpretation" and trying to realize the heuristic formula (3) in terms of a well defined integral on a space Γ of paths is not trivial. This problem is connected with the implementation of infinite dimensional integration techniques of oscillatory type, as the Feynman path integrals (3) can be regarded as oscillatory integrals of the form (1), where

$$\Gamma = \{\text{paths } \gamma : [0,t] \to \mathbb{R}^s,\ \gamma(t) = x \in \mathbb{R}^d\},$$

the phase function Φ being then the classical action functional S_t and the "integrand f" being given by $f(\gamma) = \psi_0(\gamma(0))$. The parameter ϵ is interpreted as the reduced Planck constant $\hbar$ and $d\gamma$ denotes heuristically

$$d\gamma = \text{“} \prod_{s\in[0,t]} d\gamma(s) \text{”}. \qquad (5)$$

In 1949 Kac [**36**], [**37**] observed that, by considering the heat equation with potential (again with $m = \hbar = 1$ for simplicity)

$$\begin{cases} \dfrac{\partial}{\partial t} u = \dfrac{1}{2}\Delta u - Vu \\ u(0,x) = u_0(x) \end{cases} \tag{6}$$

instead of Schrödinger equation and by replacing the oscillatory factor $e^{iS_t(\gamma)}d\gamma$ by the non oscillatory $e^{-S_t^{-V}(\gamma)}d\gamma$ (with S_t^{-V} defined as S_t but with V replaced by $-V$), one can give (for "good" V) a mathematical meaning to Feynman's formula in terms of a well defined Gaussian integral on the space of continuous paths: an integral with respect to the well known Wiener measure

$$u(t,x) = \mathbb{E}\big[e^{-\int_0^t V(\omega(s)+x)ds} u_0(\omega(t)+x)\big], \tag{7}$$

with $\mathbb{E}$ standing for expectation with respect to the standard Wiener process (mathematical Brownian motion) ω started at time 0 at the origin in $\mathbb{R}^d$. Equation (7) is called Feynman–Kac formula.

In 1956 I. M. Gelfand and A. M. Yaglom [**26**] tried to realize Feynman's heuristic complex measure $e^{(i/\hbar)\Phi(\gamma)}d\gamma$ by means of a limiting procedure:

$$e^{(i/\hbar)\Phi(\gamma)}d\gamma := \lim_{\sigma\downarrow 0} e^{(i/(\hbar - i\sigma))\Phi(\gamma)}d\gamma.$$

In 1960 Cameron [**18**] proved however that the resulting measure cannot be $\sigma-$ additive and of bounded variation, even on very "nice" subsets of paths' space, and it is not possible to implement an integration in the Lebesgue traditional sense. As a consequence since then mathematicians tried to realized the integral (3) as a linear continuous functional on a suitable (Banach) algebra of integrable functions.

A particularly interesting approach can be found in the two pioneering papers by K. Itô [**33**], [**34**]. He was aware of the interest of Feynman's formula, as well as of the mathematical problems involved in it. In the first paper in 1961 the author starts to study the problem by assuming that the potential V is linear, postponing the study of a more general case. Very shortly, what Itô did is to define rigorously the "generalized measure" (5), hence the heuristic integral (3), for V of linear type and ψ_0 having a Fourier transform of compact support, as a linear functional, taken to be the limit for $n \to \infty$ of finite dimensional approximations

$$I_n(\psi_0) = C_n^{-1} \int_{L_x} e^{(i/2\hbar)\int_0^t \dot{\gamma}(s)^2 ds} \psi_0(\gamma(t)) P_n^{(x)}(d\gamma),$$

with L_x the "translate by x of Cameron–Martin space", $P_n^{(x)}$ a suitable Gaussian measure associated with a certain compact operator T concentrated on L_x, $C_n \equiv \prod_j (1 + n^2\nu_j/\hbar i)^{-1/2}$, and $\{\nu_j\}$ being the eigenvalues of T. In the second paper [**34**] on this subject K. Itô extended the class of potentials which can be handled and covers the case where the function $V : \mathbb{R}^d \to \mathbb{C}$ is the Fourier transform of a complex bounded variation measure on $\mathbb{R}^d$ (V belongs thus to the class $\mathcal{F}(\mathbb{R}^d)$ discussed below). In this paper Itô's definition of the Feynman integral for the Green function $G(t; x, y)$ of the

Schrödinger equation (2) is of the form:

$$\frac{1}{\sqrt{2\pi\hbar i t}}\lim_{A}\prod_{j=1}^{\infty}\sqrt{1+\frac{\mu_j}{i\hbar}}\,\mathbb{E}_{a,A}\left[e^{(i/\hbar)\int_0^t(\dot\gamma(s)^2/2-V(\gamma(s)))ds}\right],$$

where the integral is computed on the space of all paths $\gamma : [0,t] \to \mathbb{R}$ such that $\gamma(0) = x$, $\gamma(t) = y$ and with weak derivative $\dot\gamma$ belonging to $L^2([0,t])$. $\mathbb{E}_{a,A}$ means expectation with respect to the Gaussian measure in L_x with mean a and a nuclear covariance operator A with eigenvalues μ_j (the limit being taken along the directed system of all such A's and being independent of a). Itô's method for the definition of Feynman's functional applies also to the Wiener integral and to the path integral representation (7) of the solution of the heat equation:

> *Our definition is also applicable to the Wiener integral*; *namely, using it, we shall prove that the solution of the heat equation* (6) *is given by*
>
> $$u(t,x) = \int_\Gamma e^{-\int_0^t(\dot\gamma^2(s)/2+V(\gamma(s)))ds}u_0(\gamma(t))d\gamma$$
>
> *for any bounded continuous function $V(x)$.... This should be called the Feynman's version of Kac's theorem. Now that Kac's theorem is well known to probabilists, no one bothers with its Feynman version. However it is interesting that Kac had the Feynman version ... in mind ...*

Itô's papers [**33**], [**34**] contain important ideas that have been further developed in the 70's, leading to the definition of infinite dimensional Fresnel integrals [**3**], [**4**]. Additional developments of these techniques as well as some interesting applications to quantum dynamical systems can be found in [**2**], [**5**], [**6**], [**8**], [**21**], [**44**] and will be shortly described in Section 2. Furthermore, Itô's constructions of oscillatory integrals (1) is based on the replacement of the concept of *integrals* by the concept of *linear functionals* with a suitable domain of "integrable functions", in the spirit of Riesz–Markov theorem, that states a one to one correspondence between complex measures (on suitable topological spaces X) with finite total variation and linear continuous functionals on $C_\infty(X)$ (the continuous functions on X vanishing at ∞). The systematic implementation of a generalized integration theory on infinite dimensional spaces based on these ideas has been recently developed by us in [**7**] and is presented in Section 3. In particular, Section 4 describes the application of this theory to the construction of infinite dimensional oscillatory integrals.

2. Infinite dimensional Fresnel integrals.

The study of finite dimensional oscillatory integrals of the type (1) is a classical topic, largely developed in connection with several applications in mathematics (such as the theory of Fourier integral operators [**31**]) and physics. Interesting examples of integrals of the form (1) in the case where the phase function is a quadratic form are the Fresnel integrals:

$$\int_{\mathbb{R}^n} \frac{e^{(i/2\hbar)\|x\|^2}}{(2\pi i\hbar)^{n/2}} f(x)dx, \tag{8}$$

(where $\hbar \in \mathbb{R} \setminus \{0\}$ is a real parameter and $f : \mathbb{R}^n \to \mathbb{C}$ a bounded continuous function) that are applied in optics and in the theory of wave diffraction. Particular interest has been devoted to the study of the asymptotic behavior of oscillatory integrals when $\hbar$ is regarded as a small parameter converging to 0. Originally introduced by Stokes and Kelvin and successively developed by several mathematicians, in particular van der Corput, the "stationary phase method" provides a powerful tool to handle the asymptotic behavior of the finite dimensional version of (1) (i.e. (8) where $\|x\|^2$ can be replaced by a more general smooth function) as $\hbar \downarrow 0$ (see, e.g. [**11**], [**5**]). There exist interesting connections with the theory of singularities of algebraic and geometric maps or structures (including catastrophe theory), see e.g. [**20**], [**31**] and also references in [**3**], [**4**].

In the following we are going to present an extension of integrals of the form (8) to the case where $\mathbb{R}^n$ is replaced by a real separable infinite dimensional Hilbert space $\mathcal{H}$.

Given a Schwartz test function $f \in S(\mathbb{R}^n)$, the Fresnel integral (8) can be computed in terms of the following Parseval's identity:

$$\int_{\mathbb{R}^n} \frac{e^{(i/2\hbar)\|x\|^2}}{(2\pi i\hbar)^{n/2}} f(x)dx = \int_{\mathbb{R}^n} e^{-(i\hbar/2)\|x\|^2} \hat{f}(x)dx, \tag{9}$$

$\hat{f}$ being the suitably normalized Fourier transform of f (see, e.g., [**31**], [**5**]).

Let $(\mathcal{H}, \langle\ ,\ \rangle)$ be a real separable Hilbert space and let $\mathcal{M}(\mathcal{H})$ be the Banach space of complex Borel measures on $\mathcal{H}$ with finite total variation, endowed with the total variation norm, denoted by $\|\mu\|_{\mathcal{M}(\mathcal{H})}$. $\mathcal{M}(\mathcal{H})$ is a commutative Banach algebra under convolution, where the unit is the δ point measure concentrated at 0. Let us consider the space $\mathcal{F}(\mathcal{H})$ of complex functions f on $\mathcal{H}$ of the form:

$$f(x) = \hat{\mu}(x) = \int_{\mathcal{H}} e^{i\langle x,y\rangle} d\mu(y), \qquad x \in \mathcal{H} \tag{10}$$

for some $\mu \in \mathcal{M}(\mathcal{H})$, $f \in \mathcal{F}(\mathcal{H})$ being thus the Fourier transform of μ. By introducing on $\mathcal{F}(\mathcal{H})$ the norm $\|f\|_{\mathcal{F}(\mathcal{H})} = \|\mu\|_{\mathcal{M}(\mathcal{H})}$, the map (10) becomes an isometry and $\mathcal{F}(\mathcal{H})$ endowed with the norm $\|\ \|_{\mathcal{F}(\mathcal{H})}$ becomes a commutative Banach algebra of continuous functions (with the pointwise product).

DEFINITION 1. Let $f \in \mathcal{F}(\mathcal{H})$. The *infinite dimensional Fresnel integral* of f, denoted by $\widetilde{\int} e^{(i/2)\|x\|^2} f(x)dx$, is defined as:

$$\widetilde{\int} e^{(i/2\hbar)\|x\|^2} f(x)dx := \int_{\mathcal{H}} e^{-(i\hbar/2)\|x\|^2} d\mu_f(x), \tag{11}$$

where $f(x) = \int_{\mathcal{H}} e^{i\langle x,y\rangle} d\mu_f(y)$, $\mu_f \in \mathcal{M}(\mathcal{H})$.

REMARK 1. The right hand side of (11) is a well defined (absolutely convergent)

Lebesgue integral. Moreover the application $f \mapsto \widetilde{\int} e^{(i/2\hbar)\|x\|^2} f(x)dx$ is a linear continuous functional on $\mathcal{F}(\mathcal{H})$.

In [**3**] the functional defined by (11) has been applied to the construction of a representation for the solution of the Schrödinger equation (2) in the cases where the potential V belongs to $\mathcal{F}(\mathbb{R}^d)$. As we mentioned above an analogous result had been already obtained in Itô's 1967 paper [**34**]. In order to handle more general potentials V, in particular those also interesting from a physical point of view, it is necessarily to enlarge the class of "integrable functions" f, i.e. the domain of the functional. This program has been started in [**21**] and further developed in [**2**], [**5**], [**6**] (see also [**44**] for a review of this topic). In the case of finite dimensional oscillatory integrals, it is convenient to introduce Hörmander's definition [**31**], which allows to define (8) even in the case where the function f has polynomial growth, by exploiting the cancellations due to the oscillatory behavior of the integrand, via a limiting procedure. More precisely, Fresnel integrals can be defined as the limit of a sequence of regularized, hence absolutely convergent, Lebesgue integrals.

Definition 2. A function $f : \mathbb{R}^n \to \mathbb{C}$ is called *Fresnel integrable* if for each Schwartz test function $\phi \in \mathcal{S}(\mathbb{R}^n)$, such that $\phi(0) = 1$, the limit

$$\lim_{\epsilon \to 0} (2\pi i\hbar)^{-n/2} \int e^{(i/2\hbar)\langle x,x\rangle} f(x)\phi(\epsilon x)dx$$

exists and is independent of ϕ. In this case the limit is called *Fresnel integral* of f and is denoted by

$$\widetilde{\int} e^{(i/2\hbar)\langle x,x\rangle} f(x)dx.$$

One shows that for $f \in S(\mathbb{R}^n)$ this Fresnel integral is given by (9).

In [**21**] this definition is generalized to the case where $\mathbb{R}^n$ is replaced by an infinite dimensional real separable Hilbert space $(\mathcal{H}, \langle\, ,\, \rangle)$. More precisely, an *infinite dimensional Fresnel integral* can be defined as the limit of a sequence of finite dimensional approximations.

Definition 3. A function $f : \mathcal{H} \to \mathbb{C}$ is said to be *Fresnel integrable* if for any sequence $\{P_n\}_{n\in\mathbb{N}}$ of projectors onto n-dimensional subspaces of $\mathcal{H}$, such that $P_n \leq P_{n+1}$ and $P_n \to 1$ strongly as $n \to \infty$ (1 being the identity operator in $\mathcal{H}$), the finite dimensional approximations

$$\widetilde{\int}_{P_n\mathcal{H}} e^{(i/2\hbar)\langle P_n x, P_n x\rangle} f(P_n x)d(P_n x)$$

are well defined (in the sense of Definition 2) and the limit

$$\lim_{n\to\infty} \widetilde{\int}_{P_n\mathcal{H}} e^{(i/2\hbar)\langle P_n x, P_n x\rangle} f(P_n x) d(P_n x)$$

exists and is independent of $\{P_n\}$.

In this case the limit is called *Fresnel integral* of f and is denoted by

$$\widetilde{\int} e^{(i/2\hbar)\langle x,x\rangle} f(x)dx.$$

REMARK 2. At a first glance, there are no evident relations between Definition 1 and Definition 3. Actually the latter is a generalization of the former, since, according to Theorem 1, a Fresnel integrable function in the sense of Definition 1 is also integrable in the sense of Definition 3.

A complete "direct description" of the largest class of Fresnel integrable functions is still missing, even in finite dimension. However, it is possible to find some interesting subsets of it. In particular the following theorem shows that the functions belonging to $\mathcal{F}(\mathcal{H})$ are Fresnel integrable in the sense of Definition 3.

THEOREM 1. *Let $L : \mathcal{H} \to \mathcal{H}$ be a self adjoint trace-class operator, such that $(I-L)$ is invertible. Let $y \in \mathcal{H}$ and let $f : \mathcal{H} \to \mathbb{C}$ be the Fourier transform of a complex bounded variation measure μ_f on $\mathcal{H}$. Then the function $g : \mathcal{H} \to \mathbb{C}$ defined by*

$$g(x) = e^{-(i/2\hbar)\langle x,Lx\rangle} e^{i\langle x,y\rangle} f(x), \qquad x \in \mathcal{H}$$

is Fresnel integrable and the corresponding Fresnel integral can be explicitly computed in terms of a well defined absolutely convergent integral with respect to a σ-additive measure on $\mathcal{H}$, by means of the Parseval-type equality:

$$\widetilde{\int} e^{(i/2\hbar)\langle x,x\rangle} g(x)dx = (\det(I-L))^{-1/2} \int_{\mathcal{H}} e^{-(i\hbar/2)\langle x+y,(I-L)^{-1}(x+y)\rangle} d\mu_f(x), \tag{12}$$

where $\det(I-L) = |\det(I-L)|\, e^{-\pi i \operatorname{Ind}(I-L)}$ *is the Fredholm determinant of the operator $I-L$, $|\det(I-L)|$ its absolute value and* $\operatorname{Ind}(I-L)$ *the number of negative eigenvalues of $I-L$, counted with their multiplicity.*

These techniques allow to give a rigorous mathematical meaning to formula (3) for the solution of the Schrödinger equation (2) as an infinite dimensional oscillatory integral on the Hilbert space $\mathcal{H}_t$ of absolutely continuous "paths" $\gamma : [0,t] \to \mathbb{R}^d$ such that $\gamma(t) = 0$ and $\int_0^t |\dot\gamma(s)|^2 ds < \infty$, endowed with the inner product $\langle \gamma_1, \gamma_2\rangle = \int_0^t \dot\gamma_1(s)\cdot\dot\gamma_2(s)ds$.

THEOREM 2. *Let A be a $d \times d$ symmetric positive matrix and let $V_1, \psi_0 \in \mathcal{F}(\mathbb{R}^d)$. Then the functional $f : \mathcal{H}_t \to \mathbb{C}$ defined as*

$$f(\gamma) := e^{-(i/2\hbar)\int_0^t (\gamma(s)+x)A^2(\gamma(s)+x)ds} e^{-(i/\hbar)V_1(\gamma(s)+x)ds} \psi_0(\gamma(0)+x), \quad \gamma \in \mathcal{H}_t$$

is Fresnel integrable and its Fresnel integral is a representation of the solution of the Cauchy problem (2) *with potential* $V(x) = (1/2)xA^2x + V_1(x)$, $x \in \mathbb{R}^d$:

$$\psi(t,x) = \widetilde{\int_{\mathcal{H}_t}} e^{(i/2\hbar)\langle\gamma,\gamma\rangle} e^{-(i/2\hbar)\int_0^t(\gamma(s)+x)A^2(\gamma(s)+x)ds} e^{-(i/\hbar)V_1(\gamma(s)+x)ds} \cdot \psi_0(\gamma(0)+x)d\gamma.$$

For a detailed proof of these results as well as for their applications to the Feynman path integral representation of the solution of the Schrödinger equation, see, e.g., [**21**], [**2**], [**6**], [**44**]. For other applications see, e.g., [**1**], [**3**], [**44**]; for other approaches to the mathematical theory of Feynman path integrals see, e.g., [**28**], [**35**], [**10**], [**25**], [**32**], [**39**], [**40**], [**43**], [**50**], [**51**].

3. Projective systems of functionals.

In this section we show how Definition 1 and Definition 3 of infinite dimensional oscillatory integrals can be regarded as particular cases of a general integration theory which generalizes Kolmogorov's construction of probability measures on infinite dimensional spaces (see [**7**]).

Definition 4. Let us consider a family $\{E_J\}_{J\in A}$ of (non-empty) sets E_J labeled by the elements of a non-empty directed set A, called index set. Let us assume that for any $J, K \in A$, $J \le K$, there exists a surjective map $\pi_J^K : E_K \to E_J$ such that π_K^K is the identity on E_K and for all $J \le K \le R$, $R \in A$, one has $\pi_J^R = \pi_J^K \circ \pi_K^R$ ("consistency property"). Such a family $\{E_J, \pi_J^K\}_{J,K\in A}$ is called a *projective* (*or inverse*) *family of sets.*

The projective family $\{E_J, \pi_J^K\}_{J,K\in A}$ is called *topological* if each E_J, $J \in A$, is a topological space and the maps $\pi_J^K : E_K \to E_J$, $J \le K$, are continuous.

The *projective* (*or inverse*) *limit* $E_A := \varprojlim E_J$ of the projective family $\{E_J, \pi_J^K\}_{J,K\in A}$ is defined as the following subset of the direct (or Cartesian) product of the family $\{E_J\}_{J\in A}$:

$$E_A := \left\{ (x_J) \in \prod_{J\in A} E_J, \middle| x_J = \pi_J^K(x_K) \text{ for all } J \le K,\, J, K \in A \right\}.$$

Let $\tilde{E} := \prod_{J\in A} E_J$. Let $\tilde{\pi}_J : \tilde{E} \to E_J$ be the coordinate projection of $\tilde{E}$ into E_J, so that if $\tilde{\omega} = \{\omega_J, J \in A\} \in \tilde{E}$ then $\tilde{\pi}_J(\tilde{\omega}) = \omega_J$. Let $\pi_J := \tilde{\pi}_J|E_A$ be the restriction of $\tilde{\pi}_J$ to E_A. One has that for any $J, K \in A$, with $J \le K$

$$\pi_J = \pi_J^K \circ \pi_K. \tag{13}$$

If $(E_J, \pi_J^K)_{J,K\in A}$ is a topological projective family, then $E_A = \varprojlim E_J$ (as defined by Definition 4) will be endowed with the coarsest topology making all the projection maps $\pi_J : E_A \to E_J$ continuous. This is also called *initial* or *inductive* topology [**17**].

Given a general projective family $\{E_J, \pi_J^K\}_{J,K\in A}$, two problems may occur. First of all, even if $E_J \neq \emptyset$ for any $J \in A$, it might happen that $E_A = \emptyset$. See, e.g. [**27**]. Secondly, even if all the projections $(\pi_J^K)_{J,K\in A}$ are surjective, the maps $\pi_J : E_A \to E_J$ may fail to be surjective. See, e.g. [**41**]. This motivates the following definition.

DEFINITION 5. A projective family $\{E_J, \pi_J^K\}_{J,K\in A}$ is called a *perfect inverse system* if for all $J \in A$, $x_J \in E_J$, there exist an $x \in E_A$ (with E_A as in Definition 4) such that $x_J = \pi_J x$. In this case all the projections are surjective.[1]

In the following we shall always assume, unless otherwise stated, that the inverse systems we are considering are perfect.

The concepts of projective system and projective (or inverse) limit are connected with those of direct systems and direct limit, which we recall here below.

DEFINITION 6. Let $(A, \leq)$ be a directed set. Let $\{E_J\}_{J\in A}$ be a family of (non empty) sets indexed by the elements of A, endowed with a family of maps $F_{JK} : E_J \to E_K$, for $J \leq K$, such that

- F_{JJ} is the identity of E_J for any $J \in A$,
- $F_{KR} \circ F_{JK} = F_{JR}$ for all $J \leq K \leq R$.

The pair $(E_J, F_{JK})_{J,K\in A}$ is called a *direct system* on A.

The *direct* (*or inductive*) *limit* of the direct system $(E_J, F_{JK})_{J,K\in A}$ is denoted by $\varinjlim E_J$ and defined as the disjoint union $\bigcup_J E_J$ modulo an equivalence relation $\sim$:

$$\varinjlim E_J = \bigcup_J E_J / \sim$$

where, if $\omega_J \in E_J$ and $\omega_K \in E_K$, then $\omega_J \sim \omega_K$ if there is some $R \in A$ with $J \leq R$, $K \leq R$, and $F_{JR}(\omega_J) = F_{KR}(\omega_K)$.

A family of maps $F_J : E_J \to \varinjlim E_J$ naturally arises, where $F_J : E_J \to \varinjlim E_J$ maps each element of E_J into its equivalence class. Further $F_J = F_K \circ F_{JK}$ for all $J, K \in A$, with $J \leq K$.

If the sets $(E_J)_{J\in A}$ are topological spaces and the maps $(F_{JK})_{J,K\in A}$ are continuous, the family $(E_J, F_{JK})_{J,K\in A}$ is called a *topological direct system*. Its direct limit is the space $\varinjlim E_J$ endowed with the finest topology making all the maps $F_J : E_J \to \varinjlim E_J$ continuous.

Direct and projective limits are dual in the sense of category theory, see, e.g., [**42**].

Given a projective family $\{E_J, \pi_J^K\}_{J,K\in A}$, we shall consider complex-valued functions f_J defined on E_J, for any $J \in A$. f_J is thus a map from E_J into $\mathbb{C}$. We shall call $\hat{E}_J$ the space of all such functions on E_J.

Let $f_J \in \hat{E}_J$, $J \in A$. For any $K \in A$ with $J \leq K$ we can define the extension $\mathcal{E}_J^K(f_J)$ of f_J to E_K as the function belonging to $\hat{E}_K$ given by:

[1] In the terminology of [**14**] (see also [**15**]) a perfect inverse system $\{E_J, \pi_J^K\}_{J,K\in A}$ is called *simply maximal*.

$$\mathcal{E}_J^K(f_J)(\omega_K) := f_J\big(\pi_J^K(\omega_K)\big), \qquad \omega_K \in E_K.$$

If $\{E_J, \pi_J^K\}_{J,K\in A}$ is a topological projective family and $f_J \in \hat{E}_J$ is a continuous function, then for any $K \in A$ with $J \le K$, the extension $\mathcal{E}_J^K(f_J)$ is a continuous function on E_K.

Let us now consider linear maps from subsets $\hat{E}_J^0 \subseteq \hat{E}_J$ of $\hat{E}_J$ to $\mathbb{C}$, called functionals. L_J is thus such a functional if L_J associates to a function $f \in \hat{E}_J^0$ a complex number $L_J(f)$ and for any $\alpha, \beta \in \mathbb{C}$, $f, g \in \hat{E}_J^0$ the following holds:

$$L_J(\alpha f + \beta g) = \alpha L_J(f) + \beta L_J(g).$$

$\hat{E}_J^0$ is called domain of L_J, the set $\{L_J(f)\}_{f\in\hat{E}_J^0}$ is called range of L_J. We shall call $\mathrm{Map}(\hat{E}_J)$ the family of all such functionals.

For $J \le K$, let us define the map $\hat{\pi}_J^K : \mathrm{Map}(\hat{E}_K) \to \mathrm{Map}(\hat{E}_J)$ as the transport of any functional $L_K \in \mathrm{Map}(\hat{E}_K)$ induced by the map $\mathcal{E}_J^K$ from $\hat{E}_J$ to $\hat{E}_K$, given by:

$$\hat{\pi}_J^K(L_K)(f_J) := L_K\big((\mathcal{E}_J^K(f_J)\big), \qquad L_K \in \mathrm{Map}(\hat{E}_K), \tag{14}$$

where the domain of $\hat{\pi}_J^K(L_K)$ is given by

$$\mathrm{Dom}\,\big(\hat{\pi}_J^K(L_K)\big) = \big\{f_J \in \hat{E}_J,\, |\mathcal{E}_J^K(f_J) \in \hat{E}_K^0\big\}.$$

Let us consider a family of functionals $\{L_J, \hat{E}_J^0\}_{J\in A}$ labeled by the elements of an index set A.

Definition 7. We call the family $\{L_J, \hat{E}_J^0\}_{J\in A}$ a *projective system of functionals* if for all $J, K \in A$ with $J \le K$ the projective (or coherence or compatibility) conditions hold

$$\begin{aligned} \mathcal{E}_J^K(f_J) &\in \hat{E}_K^0, & \forall f_J &\in \hat{E}_J^0, \\ \hat{\pi}_J^K(L_K)(f_J) &= L_J(f_J), & \forall f_J &\in \hat{E}_J^0. \end{aligned} \tag{15}$$

Given a function $f_J \in \hat{E}_J$, $J \in A$, it can be extended to a function $\mathcal{E}_J^A f_J := \mathcal{E}_J^A(f_J)$ on the projective limit E_A in the following way

$$\mathcal{E}_J^A f_J(\omega) := f_J(\pi_J \omega), \qquad \omega \in E_A.$$

By Equation (13), the extension maps $\mathcal{E}_J^A : \hat{E}_J \to \hat{E}_A$ satisfy the following condition for any $J, K \in A$, with $J \le K$:

$$\mathcal{E}_J^A = \mathcal{E}_K^A \circ \mathcal{E}_J^K. \tag{16}$$

If $(E_J, \pi_J^K)_{J,K\in A}$ is a topological projective family, then all the extensions $\mathcal{E}_J^K$: $\hat{E}^J \to \hat{E}_A$ and $\mathcal{E}_J^A : \hat{E}^J \to \hat{E}_A$ map continuous functions into continuous functions.

We shall write $\mathcal{C}$ for the family

$$\mathcal{C} = \bigcup_{J\in A} \mathcal{E}_J^A(\hat{E}_J).$$

The functions belonging to $\mathcal{C}$ will be called *cylindrical* (*or cylinder*) *functions*.

Given a projective system of functionals $\{L_J, \hat{E}_J^0\}_{J\in A}$, we shall denote by $\mathcal{C}_0 \subset \mathcal{C}$ the subfamily of cylindrical functions consisting of those cylindrical functions which are obtained by extensions $\mathcal{E}_J^A f_J$ of $f_J \in \hat{E}_J^0$ to the projective limit E_A, i.e.:

$$\mathcal{C}_0 := \bigcup_{J\in A} \mathcal{E}_J^A(\hat{E}_J^0) = \{f \in \mathcal{C} \mid f = \mathcal{E}_J^A f_J, \text{ for some } J \in A, f_J \in \hat{E}_J^0\}.$$

We remark that the pair $(E_J^0, \mathcal{E}_J^K)_{J,K\in A}$ forms a direct system in the sense of Definition 6 and $\mathcal{C}_0$ is its direct limit.

DEFINITION 8. A *projective extension* $(L, D(L))$ of a projective system of functionals $\{L_J, \hat{E}_J^0\}_{J\in A}$ is a functional L with domain $D(L) \subseteq \hat{E}_A$ ($\hat{E}_A$ being the complex-valued functions on E_A), such that

- $\mathcal{C}_0 \subseteq D(L)$,
- $L(\mathcal{E}_J^A f_J) = L_J(f_J)$, for all $f_J \in \hat{E}_J^0$.

In fact, if $\{E_J, \pi_J^K\}_{J,K\in A}$ is a perfect inverse system, any projective system of functionals $\{L_J, \hat{E}_J^0\}_{J\in A}$ admits at least a projective extension. It is the functional $(L, D(L))$ defined as

$$D(L) := \{f \in \hat{E}_A, \mid \text{there exists } J \in A, f_J \in \hat{E}_J^0,\ f = \mathcal{E}_J^A f_J\} = \mathcal{C}_0$$
$$L(f) := L_J(f_J), \qquad f = \mathcal{E}_J^A f_J,\ f_J \in \hat{E}_J^0.$$

This functional is "minimal", in the sense that any other extension $(L', D(L'))$ of the projective system of functionals $\{L_J, \hat{E}_J^0\}_{J\in A}$ is such that $D(L) \subseteq D(L')$ and $L'(f) = L(f)$ for all $f \in D(L)$. For this reason in the following the minimal extension will be denoted with $(L_{\min}, D(L_{\min}))$. Its domain can be described in terms of the inductive limit of the direct system $\{\hat{E}_J^0, \mathcal{E}_J^K\}_{J,K\in A}$ (see [**7**] for further details).

A problem which naturally arises is the existence of a "maximal" extension of a projective system of functionals $\{L_J, \hat{E}_J^0\}_{J\in A}$, namely a functional $(L_{\max}, D(L_{\max}))$ such that for any extension $(\tilde{L}, D(\tilde{L}))$ of $\{L_J, \hat{E}_J^0\}_{J\in A}$ one has that

$$D(\tilde{L}) \subseteq D(L_{\max})$$
$$L_{\max}(f) = \tilde{L}(f), \qquad \forall f \in D(\tilde{L}).$$

The problem is strictly connected to the uniqueness property of the extensions of a projective system. Indeed if there are two extensions $(L, D(L))$ and $(L', D(L'))$ such that there exists an element $f \in D(L) \cap D(L')$, with $L(f) \neq L'(f)$, then it is not possible to construct an extension $\tilde{L}$ of both L and L', as $\tilde{L}$ would be ambiguously defined on the element f. The converse is also true, as is stated in the following proposition.

PROPOSITION 1. *Let $\{L_J, \hat{E}^0_J\}_{J\in A}$ be a projective system of functionals and let $\mathcal{F} = \{(L, D(L))\}$ be a non void family of projective extensions of $\{L_J, \hat{E}^0_J\}_{J\in A}$. The family $\mathcal{F}$ has a maximal element if and only if it satisfies the following "uniqueness property":*
whenever two extensions $(L, D(L)), (L', D(L')) \in \mathcal{F}$ have a common element $f \in D(L) \cap D(L')$, one has that $L(f) = L'(f)$.

For a proof of Proposition 1 as well as for an example of a projective system of functionals which does not satisfy the uniqueness property and the study of this problem from a topological point of view see [**7**].

4. Projective systems of measure spaces and Fresnel integrals.

In the following we shall focus on cases where each element $(E_J)_{J\in A}$ of a projective family $\{E_J, \pi^K_J\}$ is endowed with a σ-algebra Σ_J of subsets of E_J. We shall also assume that the maps π^K_J, for $J \leq K$, are measurable. The family $\{E_J, \Sigma_J, \pi^K_J\}$ is called a *projective family of measurable spaces* [**53**], [**54**], [**55**].

Let us assume that there is a measure μ_J, not necessarily real or positive, associated to each (E_J, Σ_J). In particular we shall focus on the case where μ_J is a signed or complex measure with finite total variation [**49**], [**52**]. Let us consider the space $L^1(E_J, \Sigma_J, \mu_J)$, i.e. the subset of $\hat{E}_J$ consisting of (real resp. complex) functions on E_J which are μ_J-integrable, and the family of functionals $\{L_J, \hat{E}^0_J\}_{J\in A}$, given by

$$\hat{E}^0_J := L^1(E_J, \Sigma_J, \mu_J), \qquad L_J(f) := \int_{E_J} f d\mu_J,\; f \in \hat{E}^0_J. \tag{17}$$

By construction, L_J is a linear functional (real resp. complex valued). $\mathcal{E}^K_J$ is defined as before, as a map from $\hat{E}_J$ to $\hat{E}_K$, $J \leq K$, $J, K \in A$. According to Definition 7 the family $\{L_J, \hat{E}^0_J\}_{J\in A}$ is projective on $\hat{E}^0_J$ if for all $J \leq K$, $K, J \in A$, $f_J \in \hat{E}^0_J$ the following compatibility conditions hold:

$$\begin{aligned} &\mathcal{E}^K_J(f_J) \in \hat{E}^0_K, \\ &\hat{\pi}^K_J(L_K)(f_J) = L_J(f_J). \end{aligned} \tag{18}$$

Due to the relation between L_J and μ_J and (14), we have that (18) implies

$$\int_{E_J} f_J d\mu_J = \int_{E_K} f_J \circ \pi^K_J d\mu_K, \qquad f_J \in L^1(E_J, \Sigma_J, \mu_J). \tag{19}$$

For f_J taken to be the characteristic function of $A_J \in \Sigma_J$ this implies

$$\pi^K_J(\mu_K) = \mu_J, \tag{20}$$

which is the analogue of the usual projectivity property for measures on projective spaces (see, e.g., [**12**], [**13**]). We shall say shortly that $\{\mu_J\}_{J\in A}$ is a *projective family of measures*. Conversely, if (20) holds, then by approximating L^1-functions by finite linear combinations of characteristic functions we have that (19) holds, which by the relation (17) between L_J and μ_J implies that (18) holds. Hence the family $(L_J, \hat{E}^0_J)_{J\in A}$ is projective if and only if μ_J is a projective family of measures (in the sense of (20)).

Given a projective family $(E_J, \Sigma_J, \pi^K_J)_{J,K\in A}$ of measure spaces its projective limit (E_A, Σ_A) is the measure space defined as $E_A = \varprojlim E_J$ and $\Sigma_A := \Sigma_\infty \cap E_A$, where $\Sigma_\infty = \bigotimes_{J\in A} \Sigma_J$ is the σ-algebra associated with the product space $\prod_{J\in A} E_J$, which coincides with the smallest σ-algebra making all projection maps $\tilde{\pi}_J$ from $\prod_{J\in A} E_J$ onto E_J measurable. By construction we thus have that $\Sigma_A = \bigcup_{J\in A} \pi_J^{-1}\Sigma_J$ is the σ-algebra generated by the sets of the form $\pi_J^{-1}(B)$, $B \in \Sigma_J$ (where we recall that π_J is the restriction of $\tilde{\pi}_J$ to E_A). We shall write $(E_A, \Sigma_A) = \varprojlim(E_J, \Sigma_J)$.

Let μ be a (complex bounded) measure on (E_A, Σ_A), and let us define the measures μ_J on (E_J, Σ_J) by:

$$\mu_J := \pi_J \circ \mu, \qquad J \in A. \tag{21}$$

It is easy to verify that the family of measures $(\mu_J)_{J\in A}$ satisfies the compatibility condition (20). More generally, we shall say that the members of a family of measures $(\mu_J)_{J\in A}$ on $(E_J, \Sigma_J)_{J\in A}$ are *compatible* (or shortly *the family is compatible*) if they satisfy the compatibility condition (20).

Let us consider now the converse problem to the one solved by (21), namely about when we can find a (signed or complex bounded) measure μ on (E_A, Σ_A) such that (21) holds, starting only from integration on $(E_J, \Sigma_J)_{J\in A}$ and condition (20). If such a μ exists, then the projective system of functionals $\{L_J, \hat{E}^0_J\}_{J\in A}$ given by (17) would admit a projective extension L given by

$$\begin{aligned} D(L) &:= L^1(E_A, \Sigma_A, \mu) \\ L(f) &:= \int_{E_A} f d\mu, \qquad f \in D(L). \end{aligned} \tag{22}$$

If $\{\mu_J\}_{J\in A}$ is a projective system of compatible *probability* measures, then Kolmogorov's existence theorem [**14**], [**12**], [**13**], [**19**], [**48**], [**53**], [**54**], [**55**], [**30**] assures the existence and uniqueness of μ. On the other hand if the measures $\{\mu_J\}_{J\in A}$ of the projective family are not real and positive, i.e. if we consider a projective family of *signed* or *complex* bounded measures, then in general such a μ cannot exists, as stated in the following theorem [**52**], [**7**]

THEOREM 3. *Let $(E_J, \Sigma_J, \pi^K_J)_{J,K\in A}$ be a projective family of measure spaces and let $\{\mu_J\}_{J\in A}$ be a projective family of signed or complex bounded measures, satisfying the compatibility condition* (20). *A necessary condition for the existence of a* (*signed or complex*) *bounded measure μ on (E_A, Σ_A) satisfying the relation* (21) *is the following uniform bound on the total variation of the measures belonging to the family $\{\mu_J\}_{J\in A}$:*

$$\sup_{J \in A} |\mu_J| < +\infty, \tag{23}$$

where $|\mu_J|$ *denotes the total variation of the measure* μ_J*.*

In fact if μ_J are signed or complex bounded measures, in many interesting cases condition (23) cannot be satisfied, as in the case of infinite dimensional oscillatory integrals or in the case of Feynman path integrals.

As an example, let us consider on $\mathbb{R}^n$ the complex measure ν absolutely continuous with respect to the Lebesgue measure dx, $x \in \mathbb{R}^n$, with a density of the form $\rho(x) = e^{(i/2)\|x\|^2}/(2\pi i)^{n/2}$. The total variation of ν on $\mathbb{R}^n$ is infinite, however for any Borel bounded set $B \subset \mathbb{R}^n$, the total variation of ν on B is finite. Hence, given a bounded Borel function $f : \mathbb{R}^n \to \mathbb{C}$ the *Fresnel integral* of f can be defined as the limit

$$\lim_{R \to +\infty} \int_{[-R,R]^n} f(x) \frac{e^{(i/2)\|x\|^2}}{(2\pi i)^{n/2}} dx$$

if this limit exists. In this case it is denoted by $\widetilde{\int}_{\mathbb{R}^n} f(x)\nu(dx)$. According to this definition and the properties of the classical Fresnel integrals one gets that the $\widetilde{\int}$-integral of the function identically equal to 1 on $\mathbb{R}^n$ is 1, i.e. $\widetilde{\int}_{\mathbb{R}^n} d\nu = 1$.

One can see that this family of functionals forms a projective system. Indeed let $A = \mathcal{F}(\mathbb{N})$ be the directed set of finite subsets of $\mathbb{N}$ and let us consider for any $J \in A$ the set $E_J := \mathbb{R}^J$ endowed with the Borel σ-algebra and the complex measure (with finite total variation on bounded sets) $\times_{n \in J} d\mu_n$, μ_n being the complex measure on $\mathbb{R}$ defined as $d\mu_n := (e^{(i/2)\|x\|^2}/\sqrt{2\pi i})dx$, for all $n \in J$. Let us define for any $J \in A$ the functional $L_J : \hat{E}^0_J \to \mathbb{C}$, given by:

$$\hat{E}^0_J := \left\{ f \in \mathcal{B}_b(E_J) \, : \, \exists \lim_{R \to +\infty} \int_{[-R,R]^{|J|}} f(x) \times_{n \in J} \mu_n(dx) \right\}$$

$$L_J(f) := \lim_{R \to +\infty} \int_{[-R,R]^{|J|}} f(x) \times_{n \in J} \mu_n(dx), \quad f \in \hat{E}^0_J.$$

One can easily verify that $(L_J, \hat{E}^0_J)_{J \in A}$ is a projective system of functionals. However it is impossible to construct a projective extension on $E_A \equiv \mathbb{R}^{\mathbb{N}}$ in terms of a (Lebesgue type) improper integral. Indeed, contrary to the case of finite dimension, if we consider the infinite product measure $\times_{n \in \mathbb{N}} d\mu_n$, on $\mathbb{R}^{\mathbb{N}}$ endowed with the product σ algebra, we have that its total variation is infinite even on products of bounded sets.

More generally let $(\mathcal{H}, \langle \ , \ \rangle)$ be a real separable Hilbert space and let $(A, \leq)$ be the directed set of its finite dimensional subspaces, i.e. $A = \{V \subset \mathcal{H} : \dim(V) < \infty\}$ and $V \leq W$ if V is a subspace of W. For $V \leq W$ let $\pi^W_V : W \to V$ be the natural projection from W onto V. For any $V \in A$ let Σ_V be the Borel σ-algebra on V. $(V, \Sigma_V, \pi^W_V)_{V,W \in A}$ is then a projective system of measure spaces. For any $V \in A$ let $L_V : D(L_V) \to \mathbb{C}$ be the linear functional defined by:

$$D(L_V) := \left\{ f \in \mathcal{B}_b(V) : \exists \lim_{R\to+\infty} \int_{[-R,R]^{|V|}} f(x) e^{(i/2)\|x\|^2} dx \right\}$$

$$L_V(f) := \lim_{R\to+\infty} \int_{[-R,R]^{|V|}} f(x) \frac{e^{(i/2)\|x\|^2}}{(2\pi i)^{|V|/2}} dx, \quad f \in \hat{D}(L_V),$$

where dx denotes the Lebesgue measure on V, $\| \;\|$ is the norm in V and $|V|$ denotes the dimension of V.

The family $(L_V, D(L_V))_{V\in A}$ constitutes a projective system of linear functionals. According to Theorem 3 it is not possible to define on the projective limit E_A a complex measure with finite variation on bounded sets obtained as the projective limit of the complex measures $\mu_V(dx) := (e^{(i/2)\|x\|^2}/(2\pi i)^{|V|/2})dx$, $x \in V$. Consequently there cannot be an extension of the projective system of functionals $(L_V, D(L_V))_{V\in A}$ of the form $L(f) = \int_{E_A} f(x)\mu(dx)$, even if f is supported in a product of bounded sets.

A similar phenomenon occurs in the mathematical construction of Feynman path integrals. Let us consider the fundamental solution $G_t \in \mathcal{S}'(\mathbb{R}^d)$ of the Schrödinger equation (2) with $V = 0$ for a non-relativistic quantum particle moving freely in the d-dimensional Euclidean space, i.e. for $t \neq 0$, G_t is the $C^\infty(\mathbb{R}^d)$ function $G_t(x,y) = e^{(i/2t\hbar)|x-y|^2}/(2\pi i t\hbar)^{d/2}$. In fact G_t can be regarded as the density with respect to the Lebesgue measure of a complex measure μ_t on $\mathbb{R}^d$, with finite variation on bounded sets, of the form $d\mu_t(x) = G_t(x)dx$, while G_0 is defined as the Dirac measure at 0. Further, by the group property $\int_{\mathbb{R}^d} G_t(x,z)G_s(z,y)dz = G_{t+s}(x,y)$, $s,t \in \mathbb{R}$, $x,y \in \mathbb{R}^d$, (where the integral is meant as an improper Riemann integral) one has that the family of functionals $(L_J, \hat{E}^0_J)$ defined by:

$$\hat{E}^0_J = L^1(E_J, dx), \quad E_J = (\mathbb{R}^d)^J, \; J = (t_1, \ldots, t_n), \tag{24}$$

$$L_J(f) \equiv \int_{(\mathbb{R}^d)^J} f(x_1,\ldots,x_n) G_{t_1}(x_0,x_1)\cdots G_{t_n - t_{n-1}}(x_{n-1},x_n)dx_1\cdots dx_n$$

is a projective family of functionals, but there cannot exist a complex bounded variation measure μ on $(\mathbb{R}^d)^{[0,+\infty)}$ and a projective extension $(L, D(L))$ of $(L_J, \hat{E}^0_J)$ such that $L(f) = \int_{(\mathbb{R}^d)^{[0,+\infty)}} f d\mu$ (see [**18**]). According to this negative result, it is not possible to realize the limit (4) leading to the heuristic formula (3) as in integral with respect to a complex measure; what is only possible is to define (3) as a particular extension of the projective system of functional $(L_J, \hat{E}^0_J)$. This problem is deeply connected with the rigorous mathematical definition of Feynman path integrals which has been provided in different ways, but always, because of the above obstruction, only in the sense of continuous functionals not directly expressible as integrals with respect to σ-additive measures [**3**], [**28**], [**35**], [**44**]. We present here a construction of the infinite dimensional Fresnel integrals of Section 2 as particular cases of a class of projective systems of functionals.

Let $\mathcal{H}$ be a real separable infinite dimensional Hilbert space and let A be the directed set of its finite dimensional subspaces, ordered by inclusion. For $V, W \in A$, with $V \leq W$, let $\pi^W_V : W \to V$ be the projection from W onto V and $i^W_V : V \to W$ be the inclusion map. One has that $(V, \pi^W_V)_{V,W\in A}$ is a projective family of sets, while $(V, i^W_V)_{V,W\in A}$ forms a direct system on A. Let us consider the projective limit space $E_A := \varprojlim_{V\in A} V$, the direct limit $\tilde{E}_A := \varinjlim_{V\in A} V$, the projection $\pi_V : E_A \to V$ and the inclusion maps $i_V : V \to \tilde{E}_A$. Considered on each $V \in A$ the topology induced by the finite dimensional Hilbert space structure of V, the space $\varprojlim_{V\in A} V$ is endowed with the weakest topology making all the projections $\pi_V : E_A \to V$ continuous, while the space $\varinjlim_{V\in A} V$ is endowed with the *final topology*, i.e. the strongest topology making all the inclusion maps $i_V : V \to \tilde{E}_A$ continuous.

The inverse system $(V, \pi_V^W)_{V,W\in A}$ and the direct system $(V, i_V^W)_{V,W\in A}$ are linked by dualization. Indeed if we identify the dual of a finite dimensional vector space V with itself, we have that the inclusion $i_V^W : V \to W$, $V \le W$, can be identified with the transpose map $(\pi_V^W)^* : V^* \to W^*$ of the projection $\pi_V^W : W \to V$. Further the direct limit space $\tilde{E}_A$ can be naturally identified with a subspace of $(E_A)^*$. Indeed any $\eta \in \tilde{E}_A$ can be associated with the element of $(E_A)^*$ whose action on any $\omega \in E_A$ is given by

$$\eta(\omega) := \langle v, \pi_V \omega \rangle, \tag{25}$$

$v \in V$ being any representative of the equivalence class of vectors associated to η. The definition (25) is well posed, indeed chosen a different representative of the equivalence class η, i.e. a vector $v' \in V'$ such that there exists a $W \in A$, with $V \le W$, $V' \le W$ and $i_V^W v = i_{V'}^W v'$, one has that:

$$\langle v, \pi_V \omega \rangle = \left\langle v, \pi_V^W \circ \pi_W \omega \right\rangle = \left\langle i_V^W v, \pi_W \omega \right\rangle = \left\langle i_{V'}^W v', \pi_W \omega \right\rangle = \langle v', \pi_{V'} \omega \rangle.$$

Further the explicit form (25) of the functional η shows its continuity on E_A.

Analogously the transpose map $\pi_V^* : V^* \to E_A^*$ can be identified with the map $i_V : V \to \tilde{E}_A$, giving:

$$\langle v, \pi_V \omega \rangle = {}_{E_A^*}\langle i_V v, \omega \rangle_{E_A},$$

where the symbol $\langle\ ,\ \rangle$ on the left hand side denotes the inner product in V, while the symbol ${}_{E_A^*}\langle\ ,\ \rangle_{E_A}$ denotes the dual pairing between E_A and E_A^*.

Let us consider on any $V \in A$ the Borel σ-algebra Σ_V and a bounded signed or complex measure $\mu_V : \Sigma_V \to \mathbb{C}$ in such a way that the family $(\mu_V)_{V\in A}$ satisfies the compatibility condition (20). Let us also consider, for any $V \in A$, the Fourier transform $\hat{\mu}_V : V \to \mathbb{C}$ of the measure μ_V, i.e.

$$\hat{\mu}_V(v) = \int_V e^{i\langle v', v\rangle} \mu_V(dv'), \qquad v \in V.$$

By the projectivity condition (20) of the family of measures $(\mu_V)_{V\in A}$, one deduces the following relation (compatibility relation) among the Fourier transforms:

$$\hat{\mu}_V(v) = \hat{\mu}_W(i_V^W v), \qquad V \le W. \tag{26}$$

Let us now define the map $F : \tilde{E}_A \to \mathbb{C}$ by:

$$F(\eta) := \hat{\mu}_V(v), \qquad \eta \in \tilde{E}_A,$$

where $v \in V$ is any representative of the equivalence class $\eta \in \tilde{E}_A$. F is unambiguously defined, indeed given a $v' \sim v$ (in the sense that v, v' are in the same equivalence class), with $v' \in V'$, there exists a $W \in A$, with $V \leq W$ and $V' \leq W$, such that $i_V^W v = i_{V'}^W v'$. By the compatibility condition (26)

$$\hat{\mu}_V(v) = \hat{\mu}_W\left(i_V^W v\right) = \hat{\mu}_W\left(i_{V'}^W v'\right) = \hat{\mu}_{V'}(v').$$

Further, the map F is continuous on $\tilde{E}_A$ in the final topology.

If there exists a measure on μ on E_A such that $\mu_V = \pi_V \circ \mu$ for all $V \in A$, then its Fourier transform $\hat{\mu}$ coincides with F on $\tilde{E}_A$ and

$$\|\hat{\mu}\|_\infty = \sup_{\eta \in (E_A)*} |\hat{\mu}| \leq |\mu|,$$

where $|\mu|$ is the total variation of μ and $\|\hat{\mu}\|_\infty$ stands for the sup-norm of $\hat{\mu}$.

Let us consider the projective system of functionals $(L_V, D(L_V))_{V \in A}$, where $D(L_V) \equiv \mathcal{F}(V)$ is the space of functions $f : V \to \mathbb{C}$ of the form $f(v) = \int_V e^{i\langle v', v\rangle} \nu_f(dv')$ for some complex bounded measure ν_f on V and norm $\|f\| = |\nu_f|$, $|\nu_f|$ being the total variation of ν_f. Let $L_V : D(L_V) \to \mathbb{C}$ be the linear functional defined by

$$\begin{aligned} D(L_V) &:= \mathcal{F}(V) \\ L_V(f) &:= \int_V \hat{\mu}_V(v) \mu_f(dv). \end{aligned} \tag{27}$$

One can easily verify that L_V is continuous in the $\mathcal{F}(V)$-norm and the family $(L_V, D(L_V))_{V \in A}$ forms a projective system of functionals. If $\sup_{V \in A} |\mu_V| = +\infty$, according to Theorem 3, there cannot exist a complex bounded measure μ on E_A which is the projective limit of the measures $(\mu_V)_{V \in A}$. Hence there cannot exists a projective extension $(L, D(L))$ of the projective system $(L_V, D(L_V))_{V \in A}$ of the form

$$\begin{aligned} D(L) &:= L^1(E_A, |\mu|) \\ L(f) &:= \int_{E_A} f(\omega) \mu(d\omega). \end{aligned} \tag{28}$$

However, even if μ does not exist, the map $F : \tilde{E}_A \to \mathbb{C}$ is still well defined, and can be used in the construction of an alternative projective extension of $(L_V, D(L_V))_{V \in A}$, alternative namely to (28).

Consider on $\tilde{E}_A$ the Borel σ-algebra $\mathcal{B}(\tilde{E}_A)$, then one has that the continuous map $F : \tilde{E}_A \to \mathbb{C}$ is measurable. If the condition:

$$\sup_{V \in A} \|\hat{\mu}_V\|_\infty < +\infty \tag{29}$$

is satisfied, then the functional $L : D(L) \to \mathbb{C}$ given by

$$D(L) := \mathcal{F}(E_A)$$

$$L(f) = \int_{\tilde{E}_A} F(\eta)\nu_f(d\eta)$$

is well defined on the Banach algebra $\mathcal{F}(E_A)$ of functions $f : E_A \to \mathbb{C}$ of the form $f(\omega) = \int_{\tilde{E}_A} e^{i\langle \eta, \omega \rangle}\nu_f(d\eta)$ for some complex bounded measure ν_f on $\tilde{E}_A$. L is a projective extension of the projective system of functionals (27).

Depending on the regularity properties of the function $F : \tilde{E}_A \to \mathbb{C}$ one can construct different extensions, in other words construct the functional L on different domains. Let $\mathcal{B}$ be a Banach space where $\tilde{E}_A$ is densely embedded, i.e. $\tilde{E}_A \subset \mathcal{B}$ densely, and let F be continuous with respect to the $\mathcal{B}$-norm. Then F can be extended to a function $\tilde{F} : \mathcal{B} \to \mathbb{C}$, with $\tilde{F}(\eta) = F(\eta)$ for all $\eta \in \tilde{E}_A$. Let $\mathcal{F}(\mathcal{B}^*)$ be the Banach algebra of functions $f : \mathcal{B}^* \to \mathbb{C}$ of the form

$$f(x) = \int_{\mathcal{B}} e^{i\langle y, x \rangle}\nu_f(dy), \quad x \in \mathcal{B}^*,$$

for some complex bounded variation measure ν_f on $\mathcal{B}$. Then the functional $L' : \mathcal{F}(\mathcal{B}^*) \to \mathbb{C}$ defined by

$$D(L') := \mathcal{F}(\mathcal{B}^*)$$

$$L'(f) = \int_{\mathcal{B}} \tilde{F}(x)\nu_f(dx)$$

is an alternative projective extension of the system of functionals (27). In particular, if $F(v) = e^{-(i\hbar/2)\|v\|^2}$, as in the case of Fresnel integrals, the function $F : \tilde{E}_A \to \mathbb{C}$ is continuous in the $\mathcal{H}$-norm and the functional L' becomes the infinite dimensional Fresnel integral of Definition 1, i.e.

$$D(L') := \mathcal{F}(\mathcal{H})$$

$$L'(f) = \int_{\mathcal{H}} e^{-(i\hbar/2)\|x\|^2}\nu_f(dx), \qquad f \in \mathcal{F}(\mathcal{H}),\ f = \hat{\nu}_f.$$

We remark that analogous techniques can be applied in the construction of generalized Feynman–Kac formulae for the representation of the solution of any high-order heat type equation of the form:

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = (-i)^p \alpha \dfrac{\partial^p}{\partial x^p}u(t,x) \\ u(0,x) = u_0(x), \qquad\qquad\qquad x \in \mathbb{R},\ t \in [0, +\infty) \end{cases} \tag{30}$$

where $p \in \mathbb{N}$, $p \geq 2$, and $\alpha \in \mathbb{C}$ is a complex constant. Indeed when $p > 2$, the fundamental solution G_t^p of (30), i.e. $G_t^p(x,y) := (1/2\pi)\int e^{ik(x-y)}e^{\alpha t k^p}dk$, $x, y \in \mathbb{R}$, $t > 0$, is not positive and can be interpreted as the density of a *signed* measure. Hence the projective family of functionals $(L_J, \hat{E}_J^0)$ defined by formula (24), with $G_t^p(x,y)$ replacing the fundamental solution of the Schrödinger equation $G_t(x,y)$, does not admit a projective extension $(L, D(L))$ of the form $L(f) = \int_{(\mathbb{R}^d)^{[0,+\infty)}} f d\mu$, with μ a signed measure with finite total variation (see [**38**]). In fact a functional integral representation of the solution of Equation (30) can be realized only in terms of linear continuous functionals on a suitable domain of "integrable functions" (for such realizations see [**38**], [**29**], [**16**], [**24**], [**45**], [**46**]).

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