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Scaling functions generating fractional Hilbert transforms of a wavelet function

By Ryuichi Ashino, Takeshi Mandai and Akira Morimoto

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Abstract. It is well-known that an orthonormal scaling function generates an orthonormal wavelet function in the theory of multiresolution analysis. We consider two families of unitary operators. One is a family of extensions of the Hilbert transform called fractional Hilbert transforms. The other is a new family of operators which are a kind of modified translation operators. A fractional Hilbert transform of a given orthonormal wavelet (resp. scaling) function is also an orthonormal wavelet (resp. scaling) function, although a fractional Hilbert transform of a scaling function has bad localization in many cases. We show that a modified translation of a scaling function is also a scaling function, and it generates a fractional Hilbert transform of the corresponding wavelet function. We also show a good localization property of the modified translation operators. The modified translation operators act on the Meyer scaling functions as the ordinary translation operators. We give a class of scaling functions, on which the modified translation operators act as the ordinary translation operators.

1. Introduction.

This article is concerned with an orthonormal basis of $L^2(\mathbb{R})$, called orthonormal wavelets, where $\mathbb{R}$ denotes the set of real numbers. We denote the inner product of $L^2(\mathbb{R})$ by $\langle f, g\rangle := \int_{\mathbb{R}} f(x)\overline{g(x)}\,dx$ and the norm by $\|f\| := \sqrt{\langle f, f\rangle}$. Let us define two unitary operators in $L^2(\mathbb{R})$:

$$T_b : \text{Translation operator, } b \in \mathbb{R}, \quad (T_b f)(x) := f(x-b),$$

$$D_a : \text{Dilation operator, } a \in \mathbb{R}_+, \quad (D_a f)(x) := a^{-1/2} f(x/a),$$

where $\mathbb{R}_+$ (resp. $\mathbb{R}_-$) denotes the set of positive (resp. negative) real numbers. For $\psi \in L^2(\mathbb{R})$ and $(j,k) \in \mathbb{Z}^2$, where $\mathbb{Z}$ denotes the set of integers, we set

$$\psi_{j,k}(x) = (D_{2^{-j}} T_k \psi)(x) = 2^{j/2}\psi(2^j x - k). \tag{1.1}$$

If $\{\psi_{j,k}\}_{(j,k)\in\mathbb{Z}^2}$ constitutes an orthonormal basis of $L^2(\mathbb{R})$, then ψ is called an *orthonormal wavelet function*, and $\psi_{j,k}$, $j, k \in \mathbb{Z}$ are called orthonormal wavelets. In order to

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construct an orthonormal wavelet function, a system of subspaces called a *multiresolution approximation* or a *multiresolution analysis* (MRA) ([**8**], [**12**]) is used, where an orthonormal *scaling function* ϕ plays an important role. An orthonormal wavelet function ψ is constructed from an orthonormal scaling function ϕ. Then, we say ψ is *associated with* ϕ. Scaling functions are important not only for the construction of wavelet functions, but also for step-wise decomposition and reconstruction of functions, based on the orthonormal basis $\{\psi_{j,k}\}_{(j,k)\in\mathbb{Z}^2}$.

In many applications of wavelets, Hilbert pairs $(\psi, \mathcal{H}\psi)$ of wavelet functions play important roles, where $\mathcal{H}\psi$ is the Hilbert transform ([**10**], [**5**] and so on) of ψ defined as follows. Let $\widehat{f}(\xi)$ be the Fourier transform of f:

$$\widehat{f}(\xi) = (f)^\wedge(\xi) = \mathcal{F}[f](\xi) := \int_{\mathbb{R}} f(x)e^{-i\xi x}\,dx,$$

where the operator $\mathcal{F}$: $f \mapsto \widehat{f}$ can be considered to be a bounded operator from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$. The Hilbert transform $\mathcal{H}f$ of $f \in L^2(\mathbb{R})$ is defined by

$$(\mathcal{H}f)^\wedge(\xi) = -i(\operatorname{sgn}\xi)\widehat{f}(\xi), \tag{1.2}$$

where

$$\operatorname{sgn}\xi = \begin{cases} 1, & \xi > 0, \\ -1, & \xi < 0. \end{cases}$$

Since $\mathcal{H}$ is a unitary operator which commutes with translations and dilations, if ψ is an orthonormal wavelet function, then $\mathcal{H}\psi$ is also an orthonormal wavelet function. The problem is what is the scaling function with which $\mathcal{H}\psi$ is associated. Let ψ be the orthonormal wavelet function associated with a scaling function ϕ. Although $\mathcal{H}\psi$ is the orthonormal wavelet function associated with the scaling function $\mathcal{H}\phi$, the scaling function $\mathcal{H}\phi$ is usually very bad function as for the localization, while the wavelet function $\mathcal{H}\psi$ is not. When ψ is a so-called (Lemarié-)Meyer wavelet, Toda and Zhang [**15**], [**16**] pointed out that $\mathcal{H}\psi$ is the orthonormal wavelet function associated with the scaling function $T_{1/2}\phi$. This seems very unexpected and attractive.

In this article, we consider two families of translation-invariant unitary operators $\mathcal{H}_c$ and $T_c^\dagger$ ($c \in \mathbb{R}$), where $\mathcal{H}_c$ is a *fractional Hilbert transform* ([**11**], [**5**]) with $\mathcal{H}_{1/2} = \mathcal{H}$, and $T_c^\dagger$ is a newly defined operator, a kind of modified translation operator. Let ϕ be an arbitrary orthonormal scaling function, and ψ be the wavelet function associated with ϕ. For every $c \in \mathbb{R}$, we prove that $T_c^\dagger\phi$ is also an orthonormal scaling function, and that $\mathcal{H}_c\psi$ is the wavelet function associated with the scaling function $T_c^\dagger\phi$. Further, we can easily show that $T_c^\dagger f = T_c f$ if $\operatorname{supp}\widehat{f} \subset [-2\pi, 2\pi]$. These clarify the remarkable situation explained above, since $\operatorname{supp}\widehat{\phi} \subset [-2\pi, 2\pi]$ for Meyer scaling functions. We also prove that $T_c^\dagger$ has a good localization property under vanishing moments condition. A part of the results was announced without proofs in [**3**].

In the next two sections, we give a short sketch of a theory of orthonormal wavelets. In Section 4, we explain the Hilbert transform and our problem. In Section 5, we define two families of translation-invariant unitary operators $\mathcal{H}_c$ and $T_c^\dagger$ ($c \in \mathbb{R}$). In Section 6, the main results are given, that is, answers to our problem. In Section 7, good properties of $T_c^\dagger$ are given. Proofs of the results in these two sections are given in Section 8. As an extension of Meyer scaling functions, a family of scaling functions satisfying the condition $\operatorname{supp}\widehat{\phi} \subset [-2\pi, 2\pi]$ is given in the final section.

2. Orthonormal wavelets.

If $\{\psi_{j,k}\}_{(j,k)\in\mathbb{Z}^2}$ is an orthonormal basis of $L^2(\mathbb{R})$, then ψ is called an *orthonormal wavelet function* ([**9**], [**17**] and so on), which is referred to as a wavelet function for short in this article. As important examples, we give the Shannon wavelet and the Meyer wavelets.

EXAMPLE 2.1. (1) The *Shannon wavelet*:

$$\psi(x) = 2\operatorname{sinc}(2x) - \operatorname{sinc} x,$$

where $\operatorname{sinc} x := \sin \pi x/(\pi x)$, is a wavelet function called the Shannon wavelet. In this case, $\psi(x-1/2)$ is also a wavelet function, and it is sometimes called the Shannon wavelet instead of $\psi(x)$. The Fourier transform of ψ has a simple form.

$$\widehat{\psi}(\xi) = \begin{cases} 1, & \pi < |\xi| < 2\pi, \\ 0, & \text{otherwise.} \end{cases}$$

This $\psi(x)$ is an entire function, but has a bad localization. In fact, $\psi \notin L^1(\mathbb{R})$.

(2) The *Meyer wavelets*: These wavelet functions belong to the Schwartz class $\mathcal{S}$ (called the space of testing functions of rapid descent in [**18**]), that is, these are of C^∞ class and all the derivatives are rapidly decreasing. It is known that there is no orthonormal wavelet function ψ with exponential decay such that $\psi \in C^\infty(\mathbb{R})$ and all the derivatives are bounded ([**8**, Corollary 5.5.3]). Hence, the Meyer wavelets have a good balance between the smoothness and the localization as wavelet functions.

We explain the Meyer wavelets more precisely. Take a real-valued function $b(\xi)$ of C^∞ class as

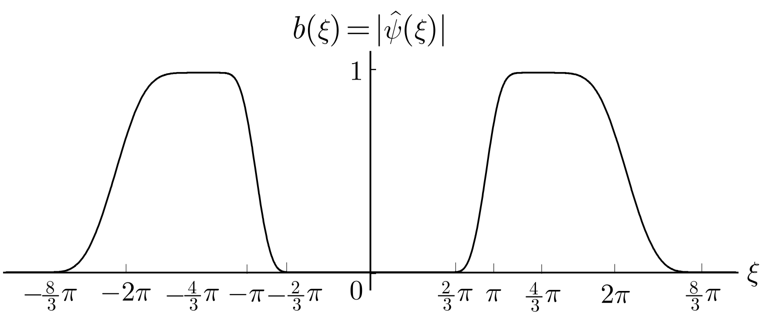

Figure 1. $b(\xi) = |\widehat{\psi}(\xi)|$ for a Meyer wavelet.

$$b(\xi) \geq 0, \quad b(-\xi) = b(\xi),$$
$$\operatorname{supp} b \subset \left[-\frac{8}{3}\pi, -\frac{2}{3}\pi\right] \cup \left[\frac{2}{3}\pi, \frac{8}{3}\pi\right],$$
$$b(\pi + \xi) = b\big(2(\pi - \xi)\big) \quad \text{for } |\xi| \leq \frac{\pi}{3},$$
$$b(\pi + \xi)^2 + b(\pi - \xi)^2 = 1 \quad \text{for } |\xi| \leq \frac{\pi}{3},$$

and define ψ by $\widehat{\psi}(\xi) := b(\xi)e^{-i\xi/2}$ (Figure 1). Sometimes, we take $b(\xi)$ not necessarily of C^∞ class, but only sufficiently smooth (for example [**8**], [**12**]).

3. MRA.

In order to construct orthonormal wavelet functions systematically, a concept called multiresolution analysis (MRA) was developed.

Definition 3.1. If V_j, $j \in \mathbb{Z}$, are closed linear subspaces of $L^2(\mathbb{R})$ satisfying the following conditions (i)–(v), then the sequence $\{V_j\}_{j\in\mathbb{Z}}$ is called a *multiresolution analysis* (*MRA*).

(i) $V_j \subset V_{j+1}$, $j \in \mathbb{Z}$.
(ii) $f \in V_j \iff f(2\cdot) \in V_{j+1}$.
(iii) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$.
(iv) $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$.
(v) there exists a function $\phi \in V_0$ such that $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis of V_0.

The function ϕ is very important and called an orthonormal *scaling function*, which is referred to as a scaling function for short in this article. In this article, we do not assume any further conditions to ϕ, unless otherwise specified. In particular, it can be that $\phi \notin L^1(\mathbb{R})$, and the familiar condition $\int_{\mathbb{R}} \phi(x)\,dx = 1$ or $\widehat{\phi}(0) = 1$ is not assumed.

If ϕ is a scaling function, then there exists a unique 2π-periodic function $m_0(\xi) \in L^1_{loc}(\mathbb{R})$ such that

$$\widehat{\phi}(2\xi) = m_0(\xi)\widehat{\phi}(\xi) \text{ a.e. on } \mathbb{R}.$$

This equation is called the *two scale equation*, and $m_0(\xi)$ is called the *low-pass filter* associated with ϕ. The low-pass filter $m_0(\xi)$ is uniquely determined from ϕ, for example by $m_0(\xi) = \sum_{k\in\mathbb{Z}} \widehat{\phi}(2\xi + 4\pi k)\,\overline{\widehat{\phi}(\xi + 2\pi k)}$.

It is well-known that from ϕ we can construct a wavelet function as follows. (See, for example, [**9**], [**17**].)

Theorem 3.2. *Let ϕ be a scaling function and m_0 be the low-pass filter. Let $\nu \in L^1_{loc}(\mathbb{R})$ be a 2π-periodic function such that $|\nu(\xi)| = 1$ a.e. We set*

$$m_1(\xi) = e^{-i\xi}\,\overline{m_0(\xi+\pi)}\,\nu(2\xi). \tag{3.1}$$

If we define ψ by

$$\widehat{\psi}(\xi) = m_1(\xi/2)\,\widehat{\phi}(\xi/2), \tag{3.2}$$

then ψ is a wavelet function.

The 2π-periodic function m_1 is called the *high-pass filter*. These m_1 and ψ are said to be *associated with* ϕ. There are many choices of ν. In this article, if we take $\nu(\xi) = 1$, then we say that m_1 and ψ are *naturally* associated with ϕ:

$$\widehat{\psi}(\xi) = e^{-i\xi/2}\,\overline{m_0(\xi/2+\pi)}\,\widehat{\phi}(\xi/2). \tag{3.3}$$

EXAMPLE 3.3. (1) (Shannon) Let $\phi(x) = \operatorname{sinc} x$. Note that $\widehat{\phi}(\xi) = \chi_{[-\pi,\pi]}(\xi)$. In this case,

$$V_j := \{f \in L^2(\mathbb{R}) \mid \operatorname{supp}\widehat{f} \subset [-2^j\pi, 2^j\pi]\}.$$

We have $m_0(\xi) = \chi_{[-\pi/2,\pi/2]}(\xi)$ for $|\xi| \le \pi$, that is, $m_0(\xi) = \sum_{k\in\mathbb{Z}} \chi_{[-\pi/2,\pi/2]}(\xi + 2k\pi) = \chi_S(\xi)$, where $S = \bigcup_{k\in\mathbb{Z}}[-\pi/2 + 2k\pi, \pi/2 + 2k\pi]$. In this case, the naturally associated wavelet function is $\psi(x - 1/2)$ in Example 2.1 (1). In Shannon's case, by taking a suitable $\nu(\xi)$, we can omit the factor $e^{-i\xi}$ in the definition of $m_1(\xi)$, and can take $m_1(\xi) = m_0(\xi+\pi)$, which is real-valued. This leads to the Shannon wavelet function $\psi(x)$ in Example 2.1 (1).

(2) (Meyer) Let ϕ be a function satisfying the following conditions (Figure 2).

- $\widehat{\phi} \in C^\infty(\mathbb{R})$, $\widehat{\phi} \ge 0$, $\widehat{\phi}$ is an even function.
- $\operatorname{supp}\widehat{\phi} \subset \left[-\dfrac{4}{3}\pi, \dfrac{4}{3}\pi\right]$.
- $\widehat{\phi}(\xi) = 1$ for $|\xi| \le \dfrac{2}{3}\pi$.
- $|\widehat{\phi}(\xi+\pi)|^2 + |\widehat{\phi}(\xi-\pi)|^2 = 1$ for $|\xi| \le \dfrac{\pi}{3}$.

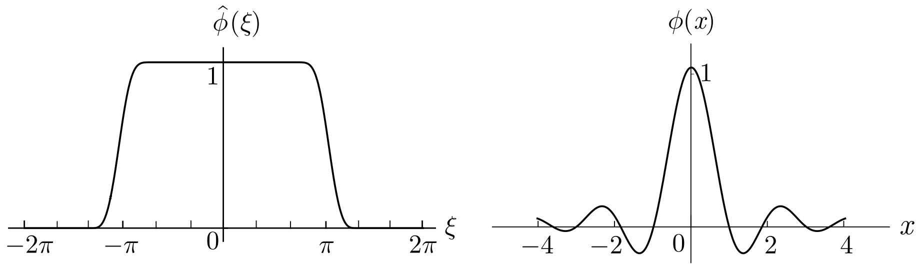

Figure 2. $\widehat{\phi}(\xi)$ and $\phi(x)$ for a Meyer wavelet.

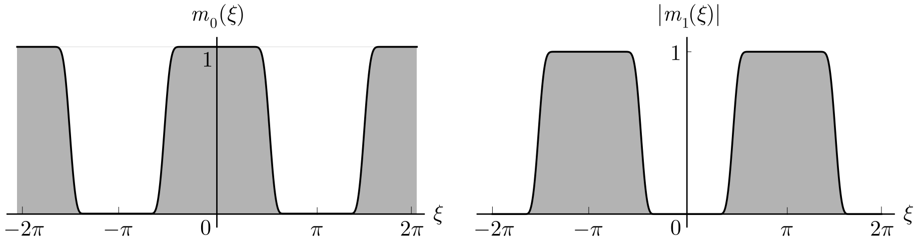

Figure 3. $m_0(\xi)$ and $|m_1(\xi)|$ for a Meyer wavelet.

Then, ϕ is a scaling function and $m_0(\xi) = \widehat{\phi}(2\xi)$ for $|\xi| \leq \pi$, that is, $m_0(\xi) = \sum_{k\in\mathbb{Z}} \widehat{\phi}(2\xi + 4k\pi)$ (Figure 3, Left).

Further,

$$m_1(\xi) = e^{-i\xi}\widehat{\phi}(2\xi + 2\pi) \quad \text{for } -2\pi \leq \xi \leq 0, \qquad \text{(Figure 3, Right)}$$

$$\widehat{\psi}(\xi) = e^{-i\xi/2}\{\widehat{\phi}(\xi + 2\pi) + \widehat{\phi}(\xi - 2\pi)\}\widehat{\phi}(\xi/2)$$

This ψ is a Meyer wavelet in Example 2.1 (2).

4. Hilbert transform.

Although the Hilbert transform $\mathcal{H}$ is defined on many function spaces in several ways, it is simply defined on $L^2(\mathbb{R})$ by (1.2). If f is real-valued, then $\mathcal{H}f$ is also real-valued and $\mathcal{H}f$ is orthogonal to f. Moreover, $\mathcal{H}$ commutes with T_b for every $b \in \mathbb{R}$ and with D_a for every $a \in \mathbb{R}_+$. Hence, $(\mathcal{H}f)_{j,k} = \mathcal{H}(f_{j,k})$ for every $j, k \in \mathbb{Z}$.

The Hilbert transform is important not only theoretically, but also in many applications. A pair of a function (a signal) and its Hilbert transform are often useful ([**7**], [**14**], [**2**] and so on). Chaudhury-Unser [**6**] investigated several properties of $\mathcal{H}\psi$ for a wavelet function ψ.

In the signal processing community, filter design is important. Selesnick [**13**] designed a low-pass filter corresponding to $\mathcal{H}\psi$. This low-pass filter turns out to be the low-pass filter associated with the scaling function $T^{\dagger}_{1/2}\phi$ defined in the next section. Toda-Zhang [**15**], [**16**] pointed out the essential part of the following theorem, which shows that the Hilbert transform of ψ is associated with $T_{1/2}\phi$ in the case of Meyer wavelet.

Theorem 4.1. *Let ϕ be a Meyer scaling function and ψ be the wavelet function naturally associated with ϕ. Fix arbitrary $b \in \mathbb{R}$, and set $\phi_b := T_b\phi$. Then we have the following.*

(1) *ϕ_b is also a scaling function.*
(2) *If ψ_b is the wavelet function naturally associated with ϕ_b, then $\mathcal{H}\psi_b$ is the wavelet function naturally associated with $T_{1/2}\phi_b = \phi_{b+1/2}$.*

The statement (1) is already well-known in the field of wavelets. (2) is very unexpected and attractive. It is very natural to ask the following questions.

Main Questions:

[Q1] What happens for $T_c\phi_b$, $c \neq 1/2$?
[Q2] Which characteristics of the Meyer scaling function, do the properties described in the theorem come from?
[Q3] What happens for other wavelets than the Meyer wavelets?

In order to give our answers, we will define two families of unitary operators $\mathcal{H}_c$ and $T_c^\dagger$, $c \in \mathbb{R}$, in the next section.

5. Unitary operators $\mathcal{H}_c$ and $T_c^\dagger$.

In this section, we define two families of unitary operators $\mathcal{H}_c$ and $T_c^\dagger$, $c \in \mathbb{R}$. The operators $\mathcal{H}_c$ are extensions of the Hilbert transform, called *fractional Hilbert transforms* ([**11**], [**5**] and so on).

DEFINITION 5.1. We define unitary operators $\mathcal{H}_c$ on $L^2(\mathbb{R})$ by

$$\mathcal{H}_c = (\cos c\pi)I + (\sin c\pi)\mathcal{H}, \ c \in \mathbb{R}, \tag{5.1}$$

where I is the identity operator. In other words,

$$(\mathcal{H}_c f)^\wedge(\xi) = \{\cos c\pi - i(\sin c\pi)\operatorname{sgn}\xi\}\widehat{f}(\xi) = e^{-ic\pi(\operatorname{sgn}\xi)}\widehat{f}(\xi). \tag{5.2}$$

We have $\mathcal{H}_{1/2} = \mathcal{H}$, and $\mathcal{H}_c$ is called a *fractional Hilbert transform.* Here, we use a different parametrization from the definition in [**5**] for the compatibility with the other family of operators $T_c^\dagger$.

If f is real-valued, then $\mathcal{H}_c f$ is also real-valued. Further, we have

$$\langle f, \mathcal{H}_c f\rangle = (\cos c\pi)\,\|f\|^2, \tag{5.3}$$

which means that the "angle" between f and $\mathcal{H}_c f$ is $c\pi$.

The family $\{\mathcal{H}_c\}_{c\in\mathbb{R}}$ constitutes a one-parameter group of unitary operators: $\mathcal{H}_c\mathcal{H}_d = \mathcal{H}_{c+d}$, $\mathcal{H}_0 = I$. Further, we have $\mathcal{H}_{c+1} = -\mathcal{H}_c$, $\mathcal{H}_{c+2} = \mathcal{H}_c$, $\mathcal{H}_1 = -I$, $\mathcal{H}_c^* = \mathcal{H}_c^{-1} = \mathcal{H}_{-c}$, where U^* denotes the adjoint operator of U.

We also have the commutativity with translations and dilations:

$$\mathcal{H}_c T_b = T_b\mathcal{H}_c, \ \mathcal{H}_c D_a = D_a\mathcal{H}_c \ \text{ for } \ b, c \in \mathbb{R}, \ a \in \mathbb{R}_+. \tag{5.4}$$

In particular, $\mathcal{H}_c(f_{j,k}) = (\mathcal{H}_c f)_{j,k}$, $j, k \in \mathbb{Z}$.

The unitary operators $\mathcal{H}_c$ are natural operators in the sense of the following proposition. A limited version was given in [**5**, Theorem 3.1], where the domain of the operators consists of only real-valued functions.

PROPOSITION 5.2. *Let U be a unitary operator which is commutative with T_b, D_a for every $b \in \mathbb{R}$, $a \in \mathbb{R}_+$. Then, we have the following.*

(1) *There exist constants* $\theta, c \in \mathbb{R}$ *such that* $U = e^{i\theta}\mathcal{H}_c$.
(2) *If further* U *maps real-valued functions to real-valued functions, then there exists* $c \in \mathbb{R}$ *such that* $U = \mathcal{H}_c$.
(3) *Moreover, if* $\langle Uf, f\rangle = 0$ *for every real-valued* f, *then* $U = \pm\mathcal{H}_{1/2} = \pm\mathcal{H}$.

Proof. (1) By Lemma 2.5 in [**1**], we have that U is a Fourier multiplier operator: $\widehat{Uf}(\xi) = \alpha(\xi)\widehat{f}(\xi)$ for which the multiplier $\alpha(\xi)$ is positively homogeneous of degree zero, which implies that $\alpha(\xi)$ is constant on each of the intervals $\mathbb{R}_\pm$. Thus, there exist $\alpha, \beta \in \mathbb{C}$ such that $\widehat{Uf}(\xi) = \alpha\widehat{f}(\xi)$ if $\xi > 0$ and $\widehat{Uf}(\xi) = \beta\widehat{f}(\xi)$ if $\xi < 0$. Since U is unitary, we have $|\alpha| = |\beta| = 1$, and hence we can write $\alpha = e^{i\theta - ic\pi}$ and $\beta = e^{i\theta + ic\pi}$ with $\theta, c \in \mathbb{R}$. This means that $U = e^{i\theta}\mathcal{H}_c$.

(2) Since $\mathcal{H}_c$ maps real-valued functions to real-valued functions, we have $e^{i\theta} \in \mathbb{R}$, that is $e^{i\theta} = \pm 1$. If $e^{i\theta} = 1$, then we have the result. If $e^{i\theta} = -1$, then by $-\mathcal{H}_c = \mathcal{H}_{c+1}$, we also have the result.

(3) Since (5.3) holds for every real-valued function f, we have $\cos c\pi = 0$, that is, $c = 1/2 + n$ $(n \in \mathbb{Z})$, and hence $\mathcal{H}_c = \pm\mathcal{H}_{1/2}$. □

Next, let us define the unitary operators $T_c^\dagger$, a kind of modified translation operators.

Definition 5.3. We define a function $\tau(\xi)$ (Figure 4) by

$$\begin{aligned} \tau(\xi) &= \xi && \text{for } |\xi| \le 2\pi, \\ \tau(\xi) &= \tau(\xi + 2\pi) && \text{for } \xi < -2\pi, \\ \tau(\xi) &= \tau(\xi - 2\pi) && \text{for } \xi > 2\pi. \end{aligned}$$

We also define unitary operators $T_c^\dagger$, $c \in \mathbb{R}$, by $(T_c^\dagger f)^\wedge(\xi) = e^{-ic\tau(\xi)}\widehat{f}(\xi)$.

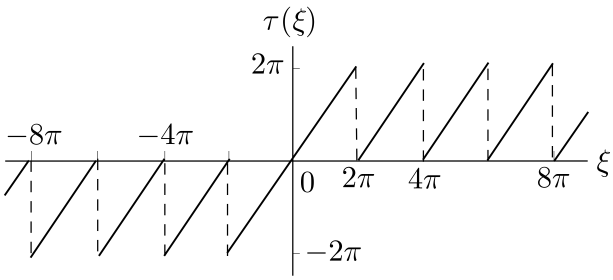

Figure 4. $\tau(\xi)$.

If f is real-valued, then $T_c^\dagger f$ is also real-valued. The family $\{T_c^\dagger\}_{c\in\mathbb{R}}$ constitutes a one-parameter group of unitary operators: $T_c^\dagger T_d^\dagger = T_{c+d}^\dagger$, $T_0^\dagger = I$. Further, $T_c^\dagger$ are commutative with the translations (but not with the dilations): $T_b T_c^\dagger = T_c^\dagger T_b$, $b, c \in \mathbb{R}$.

Remark 5.4. If $c = k$ is an integer, then $e^{-ik\tau(\xi)} = e^{-ik\xi}$, and hence $T_k^\dagger$ is just the translation: $T_k^\dagger = T_k$, $k \in \mathbb{Z}$. If $\operatorname{supp}\widehat{f} \subset [-2\pi, 2\pi]$, then $T_c^\dagger f = T_c f$, $c \in \mathbb{R}$. So, in a sense, $T_c^\dagger$ is the translation in a low frequency domain.

At the end of the next section, we give several graphs of $T_{1/2}^{\dagger}\phi$ and related functions.

6. Main results.

In this section, we consider general scaling functions. We assume the following.

Assumption : ϕ is a scaling function, and ψ is the wavelet function naturally associated with ϕ.

By the commutativity (5.4), the following is almost obvious, though a proof is given in Section 8.

PROPOSITION 6.1. *For every $c \in \mathbb{R}$, we have the followings.*

(1) *$\mathcal{H}_c\phi$ is a scaling function.*
(2) *$\mathcal{H}_c\psi$ is the wavelet function naturally associated with $\mathcal{H}_c\phi$.*

Unfortunately, $\mathcal{H}_c\phi$ has bad localization in general. In particular, if $\phi \in L^1(\mathbb{R})$ and $c \notin \mathbb{Z}$, then $\mathcal{H}_c\phi \notin L^1(\mathbb{R})$. In fact, $\widehat{\phi}$ is continuous and $\widehat{\phi}(0) \neq 0$, hence $\widehat{\mathcal{H}_c\phi}(\xi)$ has a jump at $\xi = 0$. Figures 5, 6, 7 illustrate the graphs of $\mathcal{H}_{1/2}\phi = \mathcal{H}\phi$.

The following is the main result, whose proof is given in Section 8. Note that $T_c\phi$ ($c \notin \mathbb{Z}$) is not necessarily a scaling function.

THEOREM 6.2. *For every $c \in \mathbb{R}$, we have the following.*

(1) *$T_c^{\dagger}\phi$ is a scaling function.*
(2) *$\mathcal{H}_c\psi$ is the wavelet function naturally associated with $T_c^{\dagger}\phi$.*

COROLLARY 6.3. *If $\operatorname{supp}\widehat{\phi} \subset [-2\pi, 2\pi]$, then $T_c\phi$ is a scaling function. Further, $\mathcal{H}_c\psi$ is the wavelet function naturally associated with $T_c\phi$.*

The scaling function $T_c^{\dagger}\phi$ does not have so bad localization in many cases. In particular, if ϕ is a Meyer scaling function, then $T_c^{\dagger}\phi = T_c\phi \in \mathcal{S}$. We give more properties of $T_c^{\dagger}$ in Section 7.

This theorem gives answers to the main questions in Section 4.

[Ans1] In the case of Meyer wavelets, $\mathcal{H}_c\psi_b = \mathcal{H}_{c+b}\psi$ is naturally associated with $T_c\phi_b = T_{c+b}\phi$, $c, b \in \mathbb{R}$.
[Ans2] $\operatorname{supp}\widehat{\phi} \subset [-2\pi, 2\pi]$ implies that $T_c\phi$ is a scaling function, and $\mathcal{H}_c\psi$ is associated with $T_c\phi$. (Corollary 6.3.)
[Ans3] In general, $\mathcal{H}_c\psi$ is naturally associated with $T_c^{\dagger}\phi$. (Theorem 6.2.)

In Figures 5–7, we show the graphs of ϕ, $\mathcal{H}\phi = \mathcal{H}_{1/2}\phi$, $T_{1/2}^{\dagger}\phi$, ψ, and $\mathcal{H}\psi = \mathcal{H}_{1/2}\psi$ for the case of the Meyer wavelets and the Daubechies wavelets ([**8**]). ${}_N\phi$ and ${}_N\psi$ denotes the Daubechies scaling function and wavelet function where the wavelet function has N vanishing moments. In the case of Meyer wavelets, we have $T_{1/2}^{\dagger}\phi = T_{1/2}\phi$. In the case of Daubechies wavelets, $T_{1/2}^{\dagger}{}_N\phi$ approaches $T_{1/2}{}_N\phi$ as $N \to \infty$, since $\widehat{{}_N\phi}$ concentrate in $[-2\pi, 2\pi]$. In both cases, the scaling functions $\mathcal{H}\phi \notin L^1(\mathbb{R})$ have bad localization. $T_{1/2}^{\dagger}\phi$ and $\mathcal{H}\psi$ have far better localization than $\mathcal{H}\phi$, as we explain in Section 7.

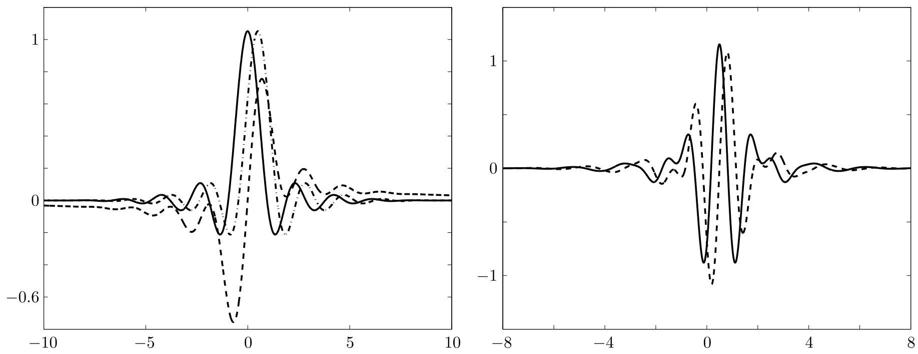

Figure 5. Case of Meyer wavelets. Left: ϕ (solid), $\mathcal{H}\phi$ (broken), $T^{\dagger}_{1/2}\phi$ (dash-dot). Right: ψ (solid), $\mathcal{H}\psi$ (broken).

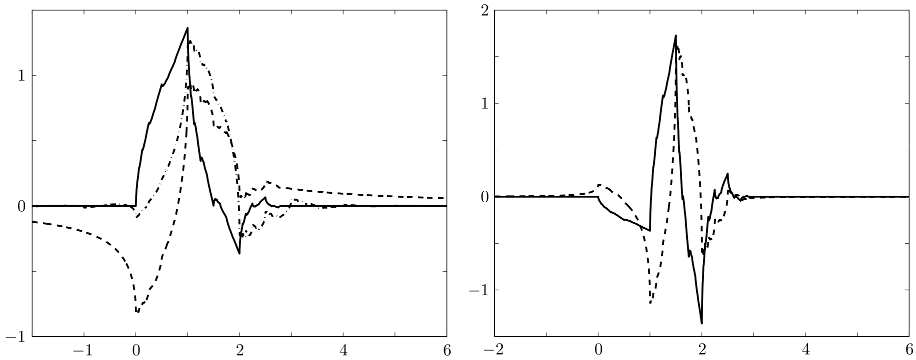

Figure 6. Case of Daubechies wavelets $N = 2$. Left: ${}_2\phi$ (solid), $\mathcal{H}_2\phi$ (broken), $T^{\dagger}_{1/2}\,{}_2\phi$ (dash-dot). Right: ${}_2\psi$ (solid), $\mathcal{H}_2\psi$ (broken).

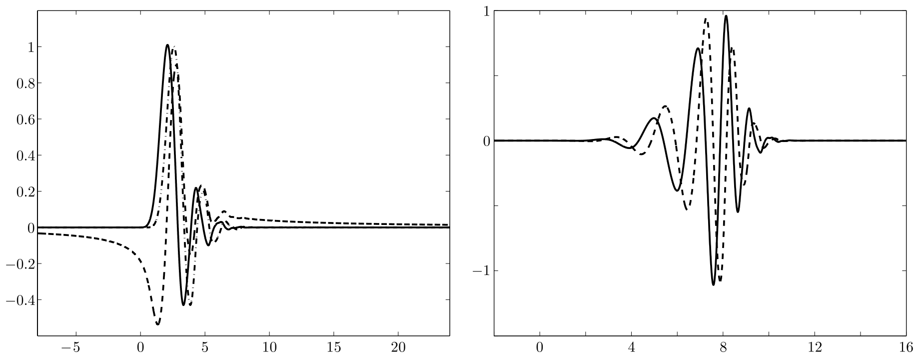

Figure 7. Case of Daubechies wavelets $N = 8$. Left: ${}_8\phi$ (solid), $\mathcal{H}_8\phi$ (broken), $T^{\dagger}_{1/2}\,{}_8\phi$ (dash-dot). Right: ${}_8\psi$ (solid), $\mathcal{H}_8\psi$ (broken).

7. Properties of $T_c^\dagger$.

In this section, we give several properties of $T_c^\dagger$. Let $\mathcal{S}'$ be the space of tempered distributions on $\mathbb{R}$. As for the distributions, see [**18**] (in this book $\mathcal{S}'$ is called the space of distributions of slow growth) for example. The operator $(1+|D|^2)^{s/2}$, $s \in \mathbb{R}$, is defined as $\{(1+|D|^2)^{s/2}f\}^\wedge(\xi) = (1+|\xi|^2)^{s/2}\widehat{f}(\xi)$ for $f \in \mathcal{S}'$. From now on, the derivatives are in the distribution sense.

As for smoothness, $T_c^\dagger f$ and $\mathcal{H}_c f$ have the same smoothness as f in the following sense. For $s \in \mathbb{R}$, set $H^s = \{f \in \mathcal{S}' \mid (1+|D|^2)^{s/2} f \in L^2(\mathbb{R})\}$, which is the Sobolev space of order s. The following is almost trivial by the boundedness of $e^{-ic\tau(\xi)}$ and $\operatorname{sgn}\xi$.

PROPOSITION 7.1. *Let $s \geq 0$. If $f \in H^s$, then $T_c^\dagger f \in H^s$ and $\mathcal{H}_c f \in H^s$.*

Next, we measure the localization of $f(x)$ by the integer index p such that $(1+|\cdot|)^p f \in L^2(\mathbb{R})$, which is equivalent to $\widehat{f}^{(j)} \in L^2(\mathbb{R})$, $0 \leq j \leq p$.

For $r \in \mathbb{N} \cup \{0\}$ and $s \in \mathbb{R}$, we set

$$\begin{aligned} H_r^s &:= \{f \in \mathcal{S}' \mid (1+|\cdot|)^r (1+|D|^2)^{s/2} f \in L^2(\mathbb{R})\} \\ &= \{f \in \mathcal{S}' \mid (\cdot)^j (1+|D|^2)^{s/2} f \in L^2(\mathbb{R}) \ \text{ for } \ 0 \leq j \leq r\} \\ &= \big\{f \in \mathcal{S}' \mid \partial_\xi^j \{(1+|\xi|^2)^{s/2}\widehat{f}(\xi)\} \in L^2(\mathbb{R}) \ \text{ for } \ 0 \leq j \leq r\big\} \\ &= \big\{f \in \mathcal{S}' \mid (1+|\cdot|^2)^{s/2}\widehat{f}^{(j)} \in L^2(\mathbb{R}) \ \text{ for } \ 0 \leq j \leq r\big\}. \end{aligned} \tag{7.1}$$

Note that if $r \in \mathbb{N}$ and $f \in H_r^0$, then $(1+|\cdot|)^{r-1} f \in L^1(\mathbb{R})$ and hence $\widehat{f} \in C^{r-1}(\mathbb{R})$, which allows us to talk about $\int_{\mathbb{R}} x^j f(x)\,dx$ and $\widehat{f}^{(j)}(0)$ for $0 \leq j < r$.

The vanishing moments property of ψ is closely relevant to the localization of $T_c^\dagger \phi$ and $\mathcal{H}_c \psi$. For $r \in \mathbb{N}$, we say that a wavelet function ψ has r *vanishing moments* if $(1+|x|)^{r-1}\psi(x) \in L^1(\mathbb{R})$ and

$$\int_{\mathbb{R}} x^j \psi(x)\,dx = 0, \quad 0 \leq j < r.$$

The following is a variant of a well-known result, and it can be proved in the same way as in [**4**], though the assumptions are a little different.

THEOREM 7.2. *Assume that $r \in \mathbb{N}$ and $\phi, \psi \in H_r^0$. Then, $\widehat{\phi}$ and $\widehat{\psi}$ are of C^{r-1} class. Also assume that*

$$\text{there exists } l_0 \in \mathbb{Z} \text{ such that } \ \widehat{\phi}(\pi + 2l_0\pi) \neq 0. \tag{7.2}$$

Then, m_0 is also of C^{r-1} class. Further, ψ has r vanishing moments if and only if each of the following conditions is satisfied.

(1) $\widehat{\psi}^{(j)}(0) = 0$, $0 \leq j < r$.
(2) $m_0^{(j)}(\pi) = 0$, $0 \leq j < r$.

(3) $\widehat{\phi}^{(j)}(2k\pi) = 0$, $0 \le j < r$, $k \in \mathbb{Z} \setminus \{0\}$.

Remark 7.3. It is well-known that if ϕ is a scaling function, then we have

$$\sum_{k\in\mathbb{Z}} |\widehat{\phi}(\xi + 2k\pi)|^2 = 1 \quad \text{a.e. on } \mathbb{R}. \tag{7.3}$$

But this holds only a.e. in ξ, and it does not necessarily imply (7.2), even if $\widehat{\phi} \in C^0(\mathbb{R})$. If we further impose some conditions which imply the local uniform convergence of the series in (7.3), then we can show that (7.3) holds for every ξ, and hence (7.2) holds. For example, the condition that there exist a constant $\epsilon > 0$ such that $\phi \in H^{1/2+\epsilon}_{1/2+\epsilon}$ implies the local uniform convergence.

Here, we fix $r \in \mathbb{N}$ and $s \in \mathbb{R}$ with $s \ge 0$. We show that the localization condition $\phi \in H^s_r$ together with the moment condition (3) in Theorem 7.2 are preserved by $T^\dagger_c$. We also give a similar result about $\mathcal{H}_c$, whose proof is similar and simpler. As for $\mathcal{H} = \mathcal{H}_{1/2}$, a similar result on localization was obtained in [**6**].

Theorem 7.4. *Let $r \in \mathbb{N}$ and $s \in \mathbb{R}$, $s \ge 0$.*

(1) *If $f \in H^s_r$ and if $\widehat{f}^{(j)}(2k\pi) = 0$ for $0 \le j < r$, $k \in \mathbb{Z} \setminus \{0\}$, then $T^\dagger_c f$ also satisfies the same conditions, that is, $T^\dagger_c f \in H^s_r$ and $(\widehat{T^\dagger_c f})^{(j)}(2k\pi) = 0$ for $0 \le j < r$, $k \in \mathbb{Z} \setminus \{0\}$.*
(2) *If $f \in H^s_r$ and if $\widehat{f}^{(j)}(0) = 0$ for $0 \le j < r$, then $\mathcal{H}_c f$ also satisfies the same conditions, that is, $\mathcal{H}_c f \in H^s_r$ and $(\widehat{\mathcal{H}_c f})^{(j)}(0) = 0$ for $0 \le j < r$.*

Proofs are given in the next section.

Remark 7.5. (1) Note that $(1+|\cdot|^2)^{-1/4-\epsilon} \in L^2(\mathbb{R})$ for every $\epsilon > 0$, and hence $f \in H^s_r$ implies that $(1+|\cdot|^2)^{s/2-1/4-\epsilon}\widehat{f}^{(j)} \in L^1(\mathbb{R})$ for $0 \le j \le r$. This is equivalent to $\partial^j_\xi\{(1+|\xi|^2)^{s/2-1/4-\epsilon}\widehat{f}(\xi)\} \in L^1(\mathbb{R})$ for $0 \le j \le r$, which implies $(\cdot)^j(1+|D|^2)^{s/2-1/4-\epsilon} f \in L^\infty(\mathbb{R})$ for $0 \le j \le r$. Thus, there exists a constant C such that

$$|(1+|D|^2)^{s/2-1/4-\epsilon} f(x)| \le \frac{C}{(1+|x|)^r}, \quad x \in \mathbb{R}.$$

In particular, if $s > 1/2$, then $f \in H^s_r$ implies

$$|f(x)| \le \frac{C}{(1+|x|)^r}, \quad x \in \mathbb{R}.$$

(2) We can also show the following by similar (and easier) proofs.

(i) If $\widehat{f} \in C^{r-1}(\mathbb{R})$ and if $\widehat{f}^{(j)}(2k\pi) = 0$ for $0 \le j < r$, $k \in \mathbb{Z}\setminus\{0\}$, then $\widehat{T^\dagger_c f} \in C^{r-1}(\mathbb{R})$ and $(\widehat{T^\dagger_c f})^{(j)}(2k\pi) = 0$ for $0 \le j < r$, $k \in \mathbb{Z} \setminus \{0\}$.
(ii) If $\widehat{f} \in C^{r-1}(\mathbb{R})$ and if $\widehat{f}^{(j)}(0) = 0$ for $0 \le j < r$, then $\widehat{\mathcal{H}_c f} \in C^{r-1}(\mathbb{R})$ and $(\widehat{\mathcal{H}_c f})^{(j)}(0) = 0$ for $0 \le j < r$.

(3) We restricted ourselves to the case $s \geq 0$ since we defined the operators $T_c^\dagger$ and $\mathcal{H}_c$ only on $L^2(\mathbb{R})$. We can extend the results to the case $s < 0$ by extending the operators $T_c^\dagger$ and $\mathcal{H}_c$ on H^s.

Example 7.6. (1) In the case of Meyer wavelets, we can apply our theorem for all $r, s \in \mathbb{N}$, and hence we have $T_c^\dagger \phi, \mathcal{H}_c \psi \in \mathcal{S}$ by Remark 7.5 (1), although this is almost trivial by the definition.

(2) If $\phi = {}_N\phi$ and $\psi = {}_N\psi$ are the Daubechies scaling function and wavelet function for which ${}_N\psi$ has N vanishing moments, then we can apply our theorem for $r = N$ and $s = 0$. In particular, $\mathcal{H}_c\, {}_N\psi$ has also N vanishing moments.

If $N \geq 3$, then we can apply our theorem for $r = N$ and $s = 1$, since it is known that ${}_N\phi, {}_N\psi \in C^1(\mathbb{R})$ for $N \geq 3$. In particular, there exists a constant C such that

$$|(T_c^\dagger\, {}_N\phi)(x)| \leq \frac{C}{(1+|x|)^N}, \quad |(\mathcal{H}_c\, {}_N\psi)(x)| \leq \frac{C}{(1+|x|)^N},$$

by Remark 7.5 (1).

For $N = 2$, it is known ([**8**]) that there exists $\epsilon > 0$ such that $\phi := {}_2\phi \in H^{1/2+\epsilon}$. Since ϕ has a compact support, we can show that $f\phi \in H^{1/2+\epsilon}$ for every $f \in C^\infty(\mathbb{R})$, in particular, for $f(x) = 1, x, x^2$. This implies $(1+|\xi|^2)^{1/4+\epsilon/2}\widehat{\phi}^{(j)} \in L^2(\mathbb{R})$, $j = 0, 1, 2$, and hence we have ${}_2\phi \in H_2^{1/2+\epsilon}$. By the same way, we have ${}_2\psi \in H_2^{1/2+\epsilon}$. Thus, we can use our results for $r = 2$ and $s = 1/2 + \epsilon$. This implies that there exists a constant C such that

$$|(T_c^\dagger\, {}_2\phi)(x)| \leq \frac{C}{(1+|x|)^2}, \quad |(\mathcal{H}_c\, {}_2\psi)(x)| \leq \frac{C}{(1+|x|)^2},$$

by Remark 7.5 (1).

For $N = 1$ (Haar), we can have only that $(1+|x|)T_c^\dagger\, {}_1\phi$, $(1+|x|)\mathcal{H}_c\, {}_1\psi \in L^2(\mathbb{R})$, which implies $T_c^\dagger\, {}_1\phi, \mathcal{H}_c\, {}_1\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

8. Proofs of the results.

We give a proof of Proposition 6.1.

Proof of Proposition 6.1. Since $\mathcal{H}_c$ is a unitary operator which commutes with T_b and D_a, $(b, a) \in \mathbb{R} \times \mathbb{R}_+$, $\widetilde{V_j} := \mathcal{H}_c(V_j)$ constitute an MRA with the scaling function $\widetilde{\phi} := \mathcal{H}_c\phi$. Since

$$(\mathcal{H}_c\phi)^\wedge(2\xi) = e^{-ic\pi\,\mathrm{sgn}(2\xi)}\,\widehat{\phi}(2\xi) = e^{-ic\pi\,\mathrm{sgn}\,\xi}\, m_0(\xi)\,\widehat{\phi}(\xi) = m_0(\xi)\,(\mathcal{H}_c\phi)^\wedge(\xi),$$

the low-pass filter $\widetilde{m_0}$ for $\widetilde{\phi}$ is the same as m_0.

Let m_1 be the high-pass filter naturally associated with ϕ: $m_1(\xi) = e^{-i\xi}\,\overline{m_0(\xi+\pi)}$. We have $\widehat{\psi}(\xi) = m_1(\xi/2)\,\widehat{\phi}(\xi/2)$. Then,

$$(\mathcal{H}_c\psi)^\wedge(\xi) = e^{-ic\pi(\mathrm{sgn}\,\xi)}\,\widehat{\psi}(\xi) = e^{-ic\pi(\mathrm{sgn}\,\xi)}\, m_1(\xi/2)\,\widehat{\phi}(\xi/2) = m_1(\xi/2)\,(\mathcal{H}_c\phi)^\wedge(\xi/2).$$

This means that $\widetilde{\psi} := \mathcal{H}_c\psi$ is the wavelet function naturally associated with $\widetilde{\phi} = \mathcal{H}_c\phi$ with the high-pass filter $\widetilde{m_1} = m_1$. □

Before giving the proof of Theorem 6.2, we give known conditions for a function to be a scaling function.

Theorem 8.1. *Let $\phi \in L^2(\mathbb{R})$. Then, ϕ is a scaling function if and only if the following three conditions hold* ([**9**, Chapter 7, Theorem 5.2]).

(A1) *The equality*

$$\sum_{k\in\mathbb{Z}} |\widehat{\phi}(\xi + 2k\pi)|^2 = 1 \qquad \text{a.e. on } \mathbb{R} \tag{8.1}$$

is satisfied. This condition is equivalent to that $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ is an orthonormal system.

(A2) *There exists a 2π-periodic function $m_0(\xi)$ such that $\widehat{\phi}(2\xi) = m_0(\xi)\widehat{\phi}(\xi)$ a.e. on $\mathbb{R}$.*

(A3) $\lim_{j\to\infty} |\widehat{\phi}(2^{-j}\xi)| = 1$ *a.e. on $\mathbb{R}$.*

We have the following basic relation between $\mathcal{H}_c$ and $T_c^\dagger$. Let $\lfloor z \rfloor := \max\{m \in \mathbb{Z} \mid m \le z\}$.

Proposition 8.2. *Set $\rho(\xi) := \tau(\xi) - \pi \operatorname{sgn} \xi = \xi - \pi - 2\pi\lfloor \xi/(2\pi)\rfloor$ (Figure 8). Then, ρ is a 2π-periodic function and $\rho(2\xi) = \rho(\xi) + \rho(\xi + \pi)$, $\rho(-\xi) = -\rho(\xi)$. Further, for $f \in L^2(\mathbb{R})$, we have*

$$\widehat{T_c^\dagger f}(\xi) = e^{-ic\rho(\xi)}\widehat{\mathcal{H}_c f}(\xi). \tag{8.2}$$

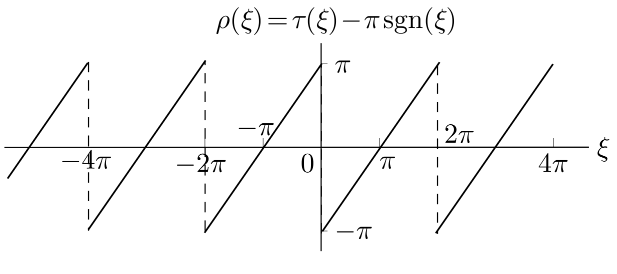

Figure 8. $\rho(\xi) = \tau(\xi) - \pi \operatorname{sgn} \xi = \xi - \pi - 2\pi\lfloor(\xi/2\pi)\rfloor$.

Proof. It is a straight verification to show $\tau(\xi) - \pi \operatorname{sgn} \xi = \xi - \pi - 2\pi\lfloor\xi/(2\pi)\rfloor$.

Since $\rho(\xi + 2\pi) = \xi + \pi - 2\pi\lfloor(\xi + 2\pi)/(2\pi)\rfloor = \xi + \pi - 2\pi(\lfloor\xi/(2\pi)\rfloor + 1) = \xi - \pi - 2\pi\lfloor\xi/(2\pi)\rfloor = \rho(\xi)$, ρ is 2π-periodic.

If $0 \le \xi < \pi$, then $\rho(2\xi) - \rho(\xi) - \rho(\xi + \pi) = 2\xi - \pi - \xi + \pi - (\xi + \pi) + \pi = 0$. If $\pi \le \xi < 2\pi$, then $\rho(2\xi) - \rho(\xi) - \rho(\xi + \pi) = 2\xi - \pi - 2\pi - \xi + \pi - (\xi + \pi) + \pi + 2\pi = 0$. Also, if $0 \le \xi < \pi$, then $\rho(\xi) + \rho(-\xi) = \xi - \pi + (-\xi) - \pi + 2\pi = 0$. If $\pi \le \xi < 2\pi$, then $\rho(\xi) + \rho(-\xi) = \xi - \pi + (-\xi) - \pi + 2\pi = 0$. (8.2) is easily obtained by $\tau(\xi) = \rho(\xi) + \pi \operatorname{sgn} \xi$. □

Fix $c \in \mathbb{R}$. By Proposition 6.1, we know that if ϕ is a scaling function, then $\mathcal{H}_c\phi$ is a scaling function. Note that the low-pass filters are the same for ϕ and $\mathcal{H}_c\phi$. By Theorem 8.1 and Proposition 8.2, we have the following.

PROPOSITION 8.3. *If ϕ is a scaling function, then $T_c^\dagger\phi$ is a scaling function which defines the same MRA as $\mathcal{H}_c\phi$ does.*

PROOF. Since $e^{-ic\rho(\xi)}$ is 2π-periodic, (8.2) for $f = \phi$ implies that both $\{(T_c^\dagger\phi)(\cdot - k)\}_{k\in\mathbb{Z}}$ and $\{(\mathcal{H}_c\phi)(\cdot - k)\}_{k\in\mathbb{Z}}$ are the orthonormal bases of the same V_0. Hence, $T_c^\dagger\phi$ and $\mathcal{H}_c\phi$ defines the same MRA. □

Now, we give a proof of the main theorem.

PROOF OF THEOREM 6.2. (1) is already proved by Proposition 8.3.
(2) We have

$$(T_c^\dagger\phi)^\wedge(2\xi) = e^{-ic\tau(2\xi)}\widehat{\phi}(2\xi) = e^{-ic\tau(2\xi)}m_0(\xi)\widehat{\phi}(\xi) = e^{-ic(\tau(2\xi)-\tau(\xi))}m_0(\xi)(T_c^\dagger\phi)^\wedge(\xi).$$

Hence the low-pass filter $m_0^\dagger$ associated with $T_c^\dagger\phi$ is

$$m_0^\dagger(\xi) = e^{-ic(\tau(2\xi)-\tau(\xi))}m_0(\xi) = e^{-ic(\rho(2\xi)-\rho(\xi))}m_0(\xi) = e^{-ic\rho(\xi+\pi)}m_0(\xi).$$

The high-pass filter naturally associated with $T_c^\dagger\phi$ is

$$m_1^\dagger(\xi) = e^{-i\xi}\,\overline{m_0^\dagger(\xi+\pi)} = e^{-i\xi}e^{ic\rho(\xi)}\,\overline{m_0(\xi+\pi)} = e^{ic\rho(\xi)}m_1(\xi),$$

and the wavelet function $\psi^\dagger$ naturally associated with $T_c^\dagger\phi$ is given by

$$\widehat{\psi^\dagger}(\xi) = m_1^\dagger(\xi/2)(T_c^\dagger\phi)^\wedge(\xi/2) = e^{ic\rho(\xi/2)}m_1(\xi/2)e^{-ic\tau(\xi/2)}\widehat{\phi}(\xi/2) = e^{-ic\pi\,\mathrm{sgn}\,\xi}\widehat{\psi}(\xi).$$

Thus, we have $\psi^\dagger = \mathcal{H}_c\psi$. □

Before giving a proof of Theorem 7.4, we prepare the following lemma.

LEMMA 8.4. *Let $a < b < c$. If $f, f' \in L^2(a,c)$, which implies $f \in C^0(a,c)$, if $f(b) = 0$, and if ν is a constant function on $(a,b)\cup(b,c)$, then $g := \nu f \in L^2(\mathbb{R})$ satisfies $g' = \nu f' \in L^2(\mathbb{R})$.*

PROOF. Since $f' \in L^2(a,c) \subset L^1(a,c)$, the antiderivative f in the sense of distribution is absolutely continuous on (a,c). For any $\varphi \in C^1(a,c)$, $f\varphi$ is also absolutely continuous on (a,c). Hence for $a < p < q < c$, we have

$$\int_p^q f(\xi)\varphi'(\xi)\,d\xi = \big[f(\xi)\varphi(\xi)\big]_p^q - \int_p^q f'(\xi)\varphi(\xi)\,d\xi.$$

Let $\nu(\xi) = \nu_1$ on (a,b) and $\nu(\xi) = \nu_2$ on (b,c). Then, for any $\varphi \in \mathcal{D}(a,c) = C_0^\infty(a,c)$, we have the following, with (f,φ) denoting the duality between $\mathcal{D}'(a,c)$ and $\mathcal{D}(a,c)$,

$$\begin{aligned}(g',\varphi)&=-(g,\varphi')=-\int_a^c \nu(\xi)f(\xi)\varphi'(\xi)\,d\xi\\&=-\nu_1\int_{a+\epsilon}^b f(\xi)\varphi'(\xi)\,d\xi-\nu_2\int_b^{c-\epsilon} f(\xi)\varphi'(\xi)\,d\xi,\end{aligned}$$

where $\epsilon>0$ is sufficiently small. Thus, by $f(b)=0$, we have

$$\begin{aligned}(g',\varphi)&=-\nu_1\big[f(\xi)\varphi(\xi)\big]_{a+\epsilon}^b+\nu_1\int_{a+\epsilon}^b f'(\xi)\varphi(\xi)\,d\xi\\&\quad-\nu_2\big[f(\xi)\varphi(\xi)\big]_b^{c-\epsilon}+\nu_2\int_b^{c-\epsilon} f'(\xi)\varphi(\xi)\,d\xi\\&=\int_a^c \nu(\xi)f'(\xi)\varphi(\xi)\,d\xi=(\nu f',\varphi),\end{aligned}$$

which means $g'=\nu f'$. □

Proof of Theorem 7.4. (1) We have only to show that $g(\xi):=\widehat{T_c^\dagger f}(\xi)=e^{-ic\tau(\xi)}\widehat{f}(\xi)$ satisfies

$$\begin{aligned}&(1+|\cdot|^2)^{s/2}g^{(j)}\in L^2(\mathbb{R})\ \text{ for }\ 0\le j\le r,\ \text{ and}\\&g^{(j)}(2k\pi)=0\ \text{ for }\ 0\le j<r,\ k\in\mathbb{Z}\setminus\{0\}.\end{aligned}$$

Set $\nu(\xi):=e^{-ic(\tau(\xi)-\xi)}$, which is a constant function on each interval $(2k\pi,2(k+1)\pi)$, $k\in\mathbb{Z}\setminus\{-1,0\}$, and $(-2\pi,2\pi)$. Set $g_1(\xi)=\nu(\xi)\widehat{f}(\xi)$. Since $g(\xi)=e^{-ic\xi}g_1(\xi)$, we have only to show

$$\begin{aligned}&(1+|\cdot|^2)^{s/2}g_1^{(j)}\in L^2(\mathbb{R})\ \text{ for }\ 0\le j\le r,\ \text{ and}\\&g_1^{(j)}(2k\pi)=0\ \text{ for }\ 0\le j<r,\ k\in\mathbb{Z}\setminus\{0\}.\end{aligned}\tag{8.3}$$

By repeated use of Lemma 8.4, we have that $g_1^{(j)}(\xi)=\nu(\xi)\widehat{f^{(j)}}(\xi)$ for $0\le j\le r$. This shows (8.3) by the assumption on f.

(2) Since $\mathcal{H}_c$ is a linear combination of I and $\mathcal{H}$, we have only to show that $h(\xi):=\widehat{\mathcal{H}f}(\xi)=-i(\operatorname{sgn}\xi)\widehat{f}(\xi)$ satisfy

$$\begin{aligned}&(1+|\cdot|^2)^{s/2}h^{(j)}\in L^2(\mathbb{R})\ \text{ for }\ 0\le j\le r,\ \text{ and}\\&h^{(j)}(0)=0\ \text{ for }\ 0\le j<r.\end{aligned}$$

Just in the same way as above, we can show that $h^{(j)}(\xi)=-i(\operatorname{sgn}\xi)\widehat{f^{(j)}}(\xi)$, $0\le j\le r$, which implies the result. □

9. A generalization of the Meyer scaling functions.

If $\operatorname{supp}\widehat{\phi} \subset [-2\pi, 2\pi]$, then we have $T_c^\dagger \phi = T_c \phi$. In this last section, we give a class of scaling functions with this property, which generalizes the Meyer scaling functions.

DEFINITION 9.1. A scaling function $\phi \in L^2(\mathbb{R})$ is called a *generalized Meyer* scaling function if $\operatorname{supp}\widehat{\phi} \subset [-a_1, a_2]$, $0 < a_1 < 2\pi$, $0 < a_2 < 2\pi$, $a_1/2 + a_2 \leq 2\pi$, $a_1 + a_2/2 \leq 2\pi$. A wavelet function associated with a generalized Meyer scaling function is also called a generalized Meyer wavelet function. Note that the condition (A1) in Theorem 8.1 implies $a_1 + a_2 \geq 2\pi$, and the equality holds only if $|\widehat{\phi}| = \chi_{[-a_1, a_2]}$. The region of possible (a_1, a_2) is illustrated as the gray region in Figure 9.

Note that the Meyer scaling functions are the case when $a_1 = a_2 = (4/3)\pi$, and the Shannon scaling function is the case when $a_1 = a_2 = \pi$.

PROPOSITION 9.2. *A function $\phi \in L^2(\mathbb{R})$ is a generalized Meyer scaling function if and only if the following three conditions hold* (*Figure* 10).

(gM1) $\operatorname{supp}\widehat{\phi} \subset [-a_1, a_2]$, $0 < a_1 < 2\pi$, $0 < a_2 < 2\pi$, $a_1/2 + a_2 \leq 2\pi$, $a_1 + a_2/2 \leq 2\pi$, $a_1 + a_2 \geq 2\pi$.
(gM2) $|\widehat{\phi}(\xi)| = 1$ *a.e. on* $[a_2 - 2\pi, 2\pi - a_1]$.
(gM3) $|\widehat{\phi}(\xi)|^2 + |\widehat{\phi}(\xi - 2\pi)|^2 = 1$ *a.e. on* $[2\pi - a_1, a_2]$.

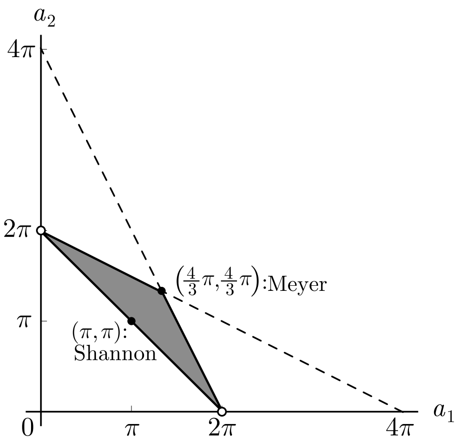

Figure 9. The region of (a_1, a_2). The boundary is included except $(2\pi, 0)$, $(0, 2\pi)$.

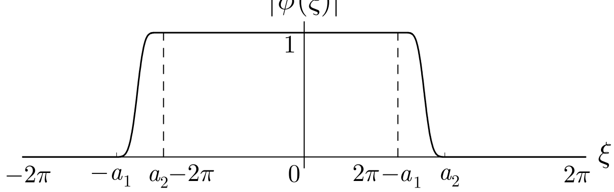

Figure 10. Graph of $|\widehat{\phi}(\xi)|$ for a generalized Meyer scaling function.

Note that (gM1) implies $-2\pi < -a_1 \le a_2 - 2\pi < 2\pi - a_1 \le a_2 < 2\pi$, and the width of the support is not greater than $a_1 + a_2 \le (8/3)\pi$. Also note that the conditions depend only on the absolute value of $\widehat{\phi}$, and hence if ϕ is a generalized Meyer scaling function and if $|\alpha(\xi)| = 1$, then $\alpha(D)\phi$ is also a generalized Meyer scaling function. In particular, if ϕ is a generalized Meyer scaling function, then $T_c\phi$ is also a generalized Meyer scaling function.

Proof. We omit "a.e.". Assume that ϕ satisfies the conditions (gM1)–(gM3).

We first show (A1). Set $F(\xi) := \sum_{k\in\mathbb{Z}} |\widehat{\phi}(\xi + 2k\pi)|^2$. On $[a_2 - 2\pi, 2\pi - a_1]$, we have $F(\xi) = |\widehat{\phi}(\xi)|^2 = 1$ by (gM1) and (gM2). On $[2\pi - a_1, a_2]$, we have $F(\xi) = |\widehat{\phi}(\xi)|^2 + |\widehat{\phi}(\xi - 2\pi)|^2 = 1$ by (gM1), (gM3), and by that $a_2 < 4\pi - a_1$, $a_2 - 2\pi \le 2\pi - a_1$. Since F is 2π-periodic, we have $F(\xi) = 1$ on $\mathbb{R}$.

Next, we show (A2). Since $\operatorname{supp}\widehat{\phi}(2\cdot) \subset [-a_1/2, a_2/2]$ where $|\widehat{\phi}(\xi)| = 1$ by $a_2 - 2\pi \le -a_1/2$ and $a_2/2 \le 2\pi - a_1$, there exists $\nu(\xi)$ such that $\widehat{\phi}(2\xi) = \nu(\xi)\widehat{\phi}(\xi)$ and $\operatorname{supp}\nu \subset [-a_1/2, a_2/2]$. Set $m_0(\xi) := \sum_{k\in\mathbb{Z}} \nu(\xi + 2k\pi)$, which is 2π-periodic. Then, we have $\widehat{\phi}(2\xi) = m_0(\xi)\widehat{\phi}(\xi)$ on $\mathbb{R}$. In fact, we have

$$m_0(\xi)\widehat{\phi}(\xi) = \sum_{k\in\mathbb{Z}} \nu(\xi + 2k\pi)\widehat{\phi}(\xi) = \nu(\xi)\widehat{\phi}(\xi) = \widehat{\phi}(2\xi),$$

since (gM1) holds, $\operatorname{supp}\nu \subset [-a_1/2, a_2/2]$, $a_2 \le 2\pi - a_1/2$ and $a_2/2 - 2\pi \le -a_1$.

Since (A3) is trivially satisfied, ϕ is a scaling function by Theorem 8.1.

Conversely, assume that ϕ is a generalized Meyer scaling function. (gM1) is trivial. Since $F(\xi) = 1$, we have

$$|\widehat{\phi}(\xi)|^2 = 1 - \sum_{k\neq 0} |\widehat{\phi}(\xi + 2k\pi)|^2.$$

On $[a_2 - 2\pi, 2\pi - a_1]$, we have $\widehat{\phi}(\xi + 2k\pi) = 0$ if $k \neq 0$ by (gM1), and hence $|\widehat{\phi}(\xi)|^2 = 1$.

Finally, since $a_2 < 4\pi - a_1$ and $a_2 - 2\pi < 2\pi - a_1$, we have $\widehat{\phi}(\xi + 2k\pi) = 0$ on $[2\pi - a_1, a_2]$ if $k \neq 0, -1$, and hence we have (gM3) by $F(\xi) = 1$. □

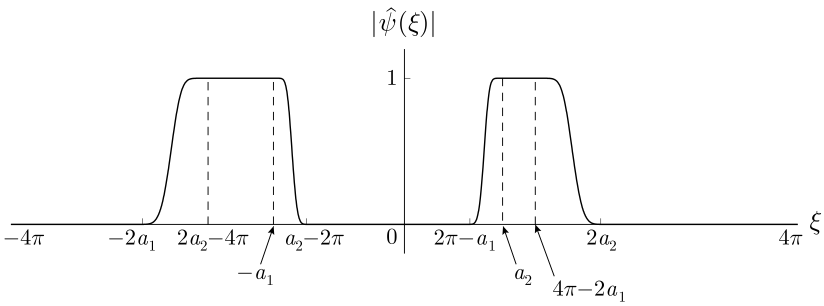

Figure 11. Graph of $|\widehat{\psi}(\xi)|$ for a generalized Meyer wavelet function.

Proposition 9.3. *If ϕ is a generalized Meyer scaling function, then any associated wavelet function ψ has the following properties* (*Figure* 11).

(gMw1) $\operatorname{supp} \widehat{\psi} \subset [-2a_1, a_2 - 2\pi] \cup [2\pi - a_1, 2a_2]$.
(gMw2) $|\widehat{\psi}(\xi)| = 1$ *a.e. on* $[2a_2 - 4\pi, -a_1] \cup [a_2, 4\pi - 2a_1]$,
(gMw3) $|\widehat{\psi}(2\xi + 4\pi)| = |\widehat{\psi}(\xi)|$ *a.e. on* $[-a_1, a_2 - 2\pi]$, $|\widehat{\psi}(2\xi - 4\pi)| = |\widehat{\psi}(\xi)|$ *a.e. on* $[2\pi - a_1, a_2]$, $|\widehat{\psi}(\xi)|^2 + |\widehat{\psi}(\xi - 2\pi)|^2 = 1$ *a.e. on* $[2\pi - a_1, a_2]$.

This proposition easily follows from the fact that $\widehat{\psi}(\xi) = e^{-i\xi/2}\nu(\xi)\,\overline{m_0(\xi/2 + \pi)} \cdot \widehat{\phi}(\xi/2)$, where ν is a 2π-periodic function with $|\nu(\xi)| = 1$ a.e. on $\mathbb{R}$.

Let ϕ be a generalized Meyer scaling function, and ψ be the wavelet function naturally associated with ϕ. If $\phi \in \mathcal{S}$, then the three functions $T_c^\dagger \phi = T_c\phi$, ψ, and $\mathcal{H}_c\psi$ also belong to $\mathcal{S}$, while $\mathcal{H}_c\phi \notin L^1(\mathbb{R})$ unless $c \in \mathbb{Z}$.

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Ryuichi Ashino
Mathematical Sciences
Osaka Kyoiku University
Kashiwara
Osaka 582-8582, Japan
E-mail: ashino@cc.osaka-kyoiku.ac.jp

Takeshi Mandai
Department of Engineering Science
Osaka Electro-Communication University
Neyagawa
Osaka 572-8530, Japan
E-mail: mandai@isc.osakac.ac.jp

Akira Morimoto
Information Science
Osaka Kyoiku University
Kashiwara
Osaka 582-8582, Japan
E-mail: morimoto@cc.osaka-kyoiku.ac.jp