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A C^*-algebra associated with dynamics on a graph of strings

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Abstract. An operator C^*-algebra $\mathfrak{E}$ associated with a dynamical system on a metric graph is introduced. The system is governed by the wave equation and controlled from boundary vertices. Algebra $\mathfrak{E}$ is generated by *eikonals*, which are self-adjoint operators related with reachable sets of the system. Its structure is studied. Algebra $\mathfrak{E}$ is determined by the boundary inverse data. This shows promise of its possible applications to inverse problems.

Introduction.

About the paper.

We introduce a self-adjoint (C^*-) operator algebra $\mathfrak{E}$ associated with a dynamical system on a metric graph. The system is governed by the wave equation and controlled from the boundary vertices. Algebra $\mathfrak{E}$ is generated by the so-called *eikonals*, which are the self-adjoint operators related with reachable sets of the system. The structure of reachable sets and the algebra is the main subject of the paper. We show that $\mathfrak{E}$ is a direct sum of "elementary blocks". Each block is an operator (sub)algebra, the operators multiplying $\mathbb{R}^n$-valued functions by continuous matrix-valued functions of special kind.

The eikonal algebra is determined by the boundary dynamical and/or spectral inverse data up to isometric isomorphism. It is an inspiring fact, which gives hope for its possible application to inverse problems. In particular, one can hope to extract information about geometry of the graph from algebra's spectrum $\widehat{\mathfrak{E}}$. Such a technique works well on manifolds [**6**]–[**8**].

The paper develops an algebraic version of the *boundary control method* in inverse problems [**1**]–[**8**]. Our approach reveals some new and hopefully prospective relations between inverse problems on graphs and C^*-algebras.

Contents.

In more detail, we deal with the dynamical system

$$\begin{aligned} &u_{tt} - \Delta u = 0 && \text{in } \Omega \times (0,T) \\ &u|_{t=0} = u_t|_{t=0} = 0 && \text{in } \Omega \\ &u = f && \text{on } \Gamma \times [0,T], \end{aligned}$$

where Ω is a finite compact metric graph, Γ is the set of its boundary vertices; Δ is

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the Laplace operator in Ω defined on a class of smooth functions satisfying the Kirchhoff conditions at the interior vertices; $T \leq \infty$; f is a *boundary control*. A solution $u = u^f(x,t)$ describes a wave, which is initiated at Γ by the control f and propagates into Ω.

With the system one associates the *reachable sets*

$$\mathcal{U}_\gamma^s = \{u^f(\,\cdot\,,s) \mid f \in L_2(\Gamma \times [0,T]), \operatorname{supp} f \in \gamma \times [0,T]\} \quad \gamma \in \Gamma,\, 0 \leq s \leq T.$$

Let P_γ^s be the orthogonal projection in $L_2(\Omega)$ onto $\mathcal{U}_\gamma^s$. A self-adjoint operator

$$E_\gamma^T = \int_0^T s\, dP_\gamma^s$$

is called an *eikonal* (corresponding to the boundary vertex γ).

Choose a subset $\Sigma \subseteq \Gamma$. The *eikonal algebra* $\mathfrak{E}_\Sigma^T$ is defined as the minimal norm-closed C^*-subalgebra of the bounded operator algebra $\mathfrak{B}(L_2(\Omega))$, which contains all E_γ^T as $\gamma \in \Sigma$. In other words, $\mathfrak{E}_\Sigma^T$ is generated by $\{E_\gamma^T \mid \gamma \in \Sigma\}$.

We provide characteristic description of the sets $\mathcal{U}_\gamma^s$ and projections P_γ^s. As a result, we clarify how the eikonals E_γ^T act. Thereafter, a structure of the eikonal algebra is revealed, and we arrive at the main result: the algebra is represented in the form of a finite direct sum

$$\mathfrak{E}_\Sigma^T = \bigoplus \sum_j \mathfrak{b}_j^T, \tag{0.1}$$

where $\mathfrak{b}_j^T$ are the so-called *block algebras*. Each $\mathfrak{b}_j^T$ is isometrically isomorphic to a subalgebra $\tilde{\mathfrak{b}}_j^T \subset \mathfrak{B}(L_2([0,\delta_j];\mathbb{R}^{M_j}))$ generated by the operators, which multiply elements of $L_2([0,\delta_j];\mathbb{R}^{M_j})$ (vector-valued functions of $r \in [0,\delta_j]$) by the matrix-functions of the form $B_{\gamma,j}^* D_{\gamma,j}(r) B_{\gamma,j}$ ($\gamma \in \Sigma$). Here each $B_{\gamma,j}$ is a *constant* projecting matrix; $D_{\gamma,j}$ is a diagonal matrix, its diagonal elements being the *linear functions* of the form $T_{\gamma,j} \pm r$ with $T_{\gamma,j} \in (0,T]$. These functions are continuous, and, hence, we have

$$\tilde{\mathfrak{b}}_j^T \subset C\big([0,\delta_j];\mathbb{M}^{M_j}\big),$$

where the latter is the algebra of continuous real $M_j \times M_j$-matrix valued functions on $[0,\delta_j]$.

Comments.

- Algebra $\mathfrak{E}_\Sigma^T$ associated with a graph is a straightforward analog of the eikonal algebras associated with a Riemannian manifold: see [**7**], [**8**]. These algebras possess two principal features, which enable one to apply them to solving inverse problems on manifolds:
  1. the eikonal algebra is determined (up to isometric isomorphism) by dynamical and/or spectral boundary inverse data
  2. its spectrum is, roughly speaking, identical to the manifold.

For this reason, one can solve the problem of reconstruction of the manifold via its inverse data by the plan [**6**]–[**8**]:

$$\text{data} \Rightarrow \text{relevant eikonal algebra } \mathfrak{E} \Rightarrow \text{its spectrum } \widehat{\mathfrak{E}} \equiv \text{manifold}.$$

It is so effective application, which has motivated to extend this approach to inverse problems on graphs. The hope was that a graph seems to be a simpler object than a manifold of arbitrary dimension and topology.

Surprisingly, the latter turns out to be an illusion. First of all, in contrast to the eikonal algebras on manifolds[1], the algebra $\mathfrak{E}^T_\Sigma$ is *noncommutative*. For this reason, in the general case, its spectrum $\widehat{\mathfrak{E}}^T_\Sigma$ endowed with the Jacobson topology is a *non-Hausdorff* space. Hence, $\widehat{\mathfrak{E}}^T_\Sigma$ is by no means identical to the (metric) graph Ω, so that property 2 fails.

However, property 1 does hold. Also, the known examples show that representation (0.1) and structure of the spectrum $\widehat{\mathfrak{E}}^T_\Sigma$ reflect some features of the graph geometry. Therefore, an attempt to extract information on Ω from the eikonal algebra and, eventually, to recover Ω seems quite reasonable. Hopefully, our paper is a step towards this goal.

- In view of big volume of the paper, we omit the proofs of some technical propositions.
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1. Graph.

1.1. Basic definitions.

1.1.1. Standard star.

Let $I_j := (0, a_j) = \{s \in \mathbb{R} \mid 0 < s < a_j < \infty\}$, $j = 1, \dots, m$ be finite intervals, each interval being regarded as a subspace of the metric space $\mathbb{R}$ with the distance $|s - s'|$. The set $S_m := \{0\} \cup I_1 \cup \cdots \cup I_m$ endowed with the metric

$$\operatorname{dist}(s, s') := \begin{cases} |s - s'| & s, s' \in I_j \\ s + s' & s \in I_i,\ s' \in I_j, \quad i \neq j \\ s' & s = 0,\ s' \in I_j \\ s & s \in I_i,\ s' = 0 \\ 0 & s = s' = 0 \end{cases}$$

is called a (standard) *m-star* (see Figure 1a). Note that a 2-star is evidently isometric to the interval $(-a_1, a_2)$.

[1]Somehow or other, these algebras are reduced to the algebra $C(\Omega)$ of continuous functions.

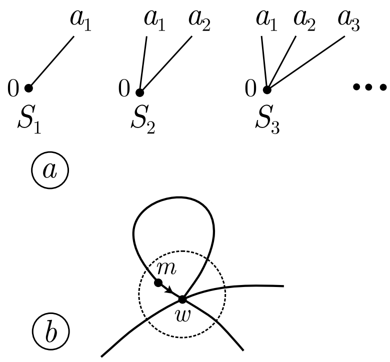

Figure 1. Stars and star neighborhoods.

1.1.2. Metric graph.

A compact connected metric space Ω with the metric $\tau : \Omega \times \Omega \to [0, \infty)$ is said to be a *homogeneous metric graph* if the following is fulfilled:

- $\Omega = E \cup V \cup \Gamma$, where $E = \{e_j\}_{j=1}^p$, e_j are the *edges*; $V = \{v_k\}_{k=1}^q$, v_k are the *interior vertices*; $\Gamma = \{\gamma_l\}_{l=1}^n$, γ_l are the *boundary vertices*, $\Gamma \neq \emptyset$
- each e_j is isometric to a finite interval $\{s \in \mathbb{R} \mid a_j < s < b_j\}$

  Convention 1. *In what follows, we assume that the isometries (parametrizations) $\eta_j : e_j \to (a_j, b_j)$ are fixed and write $\tilde{y}(s) := (y \circ \eta_j^{-1})(s)$, $s \in (a_j, b_j)$ for a function $y = y(x)$ on Ω restricted to the edge e_j.*

  So, for $x, x' \in e_j$ one has $\tau(x, x') = |\eta_j(x) - \eta_j(x')|$.
- every $w \in V \cup \Gamma$ has a neighborhood (in Ω) isometric to an S_m with $m \geq 3$ or $m = 1$, the number $m = m(w)$ being called a *valency* of w. The vertices of valency ≥ 3 constitute the set V. The vertices of valency 1 form the *boundary* Γ.

  The points of the open set $\Omega \backslash [\Gamma \cup V]$ are said to be *ordinary*. So, "ordinary" means "not a vertex".

  As was noted above, 2-stars are isometric to intervals (edges). For this reason, they do not take part in further considerations.

For a point $x \in \Omega$, by $\Omega^r[x] := \{x' \in \Omega \mid \tau(x, x') < r\}$ we denote its metric neighborhood of radius $r > 0$. For a subset $A \subset \Omega$, we put $\Omega^r[A] := \{x \in \Omega \mid \tau(x, A) < r\}$.

Let the points $x \neq x'$ belong to a (parametrized) edge e. The set $\eta^{-1}((\min\{\eta(x), \eta(x')\}, \max\{\eta(x), \eta(x')\})) \subset e$ is called an *interval* and denoted by $]x, x'[$.

1.1.3. Characteristic set.

A set introduced below is an intrinsic object of the graph geometry. Later on, in dynamics, it plays the role of the support of the wave singularities propagating into Ω.

In the space-time $\Omega \times \overline{\mathbb{R}}_+$, for a fixed (x_0, t_0) define a *characteristic cone*

$$\mathrm{ch}[(x_0, t_0)] := \{(x, t) \mid t - t_0 = \tau(x, x_0)\};$$

for a subset $A \subset \Omega \times \overline{\mathbb{R}}_+$ put

$$\mathrm{ch}[A] := \bigcup_{(x,t)\in A} \mathrm{ch}[(x,t)].$$

A *characteristic set* $\mathrm{Ch}[(x_0,t_0)]$ is introduced by the following recurrent procedure:

Step 0: put

$$C^0[(x_0,t_0)] := \mathrm{ch}[(x_0,t_0)]$$

and

$$W^0(x_0,t_0) := \{(w,t) \in C^0[(x_0,t_0)] \mid w \in V \cup \Gamma\};$$

Steps $j = 1, 2, \ldots$: put

$$C^j[(x_0,t_0)] := C^{j-1}[(x_0,t_0)] \bigcup \mathrm{ch}[W^{j-1}(x_0,t_0)]$$

and

$$W^j(x_0,t_0) := \{(w,t) \in C^{j-1}[(x_0,t_0)] \mid w \in V \cup \Gamma\};$$
$$\ldots\ldots\ldots\ldots\ldots\ldots .$$

At last, define

$$\mathrm{Ch}[(x_0,t_0)] := \bigcup_{j=0}^{\infty} C^j[(x_0,t_0)].$$

Note that $\mathrm{Ch}[(x_0,t_0)]$ can be also characterized as the minimal subset in $\Omega \times \overline{\mathbb{R}}_+$ satisfying the conditions:

- $\mathrm{ch}[(x_0,t_0)] \subset \mathrm{Ch}[(x_0,t_0)]$
- if $w \in V \cup \Gamma$ and $t_w \in \mathbb{R}_+$ are such that $(w,t_w) \in \mathrm{Ch}[(x_0,t_0)]$ then $\mathrm{ch}[(w,t_w)] \subset \mathrm{Ch}[(x_0,t_0)]$.

The characteristic set can be regarded as a space-time graph: see Figure 2. Such a graph is also a metric space: it is endowed with the length element

$$d\nu^2 := d\tau^2 + dt^2, \tag{1.1}$$

i.e., for the close points $(x,t), (x',t') \in \mathrm{Ch}[(x_0,t_0)]$ one puts $\nu((x,t),(x',t')) = [\tau^2(x,x') + (t-t')^2]^{1/2}$. For arbitrary points, the distance ν is defined as the length of the shortest curves lying in $\mathrm{Ch}[(x_0,t_0)]$ and connecting the points.

1.2. Spaces and operators.

1.2.1. Derivatives.

For an edge $e \in E$ parametrized by $\eta : e \to (a,b)$, a function y on Ω, and a point $x \in e$, we define

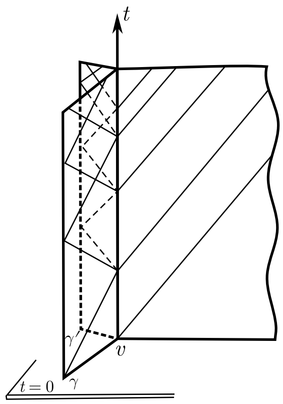

Figure 2. Characteristic set: the case $(x_0, t_0) = (\gamma, 0)$.

$$\frac{dy}{de}(x) := \frac{d\tilde{y}}{ds}\bigg|_{s=\eta(x)} = \lim_{\substack{x'\to x\\ \eta(x')>\eta(x)}} \frac{y(x') - y(x)}{\tau(x', x)}$$

(recall that $\tilde{y} := y \circ \eta^{-1}$).

Fix a vertex $w \in V \cup \Gamma$ and choose its neighborhood $\omega \subset \Omega$ isometric to S_m. We say an edge e to be *incident* to w if $\overline{e} \ni w$ or, equivalently, if $e \cap \omega \neq \emptyset$. Note that $e \cap \omega$ can consist of two components (see Figure 1b) and settle that, in this case, each component is regarded as a single edge (of the subgraph ω) incident to w.

For every e incident to w, define an *outward derivative*

$$\frac{dy}{de_+}(w) := \lim_{e\ni m\to w} \frac{y(m) - y(w)}{\tau(m, w)}.$$

For an interior vertex $v \in V$ and a function y, define an *outward flow*

$$\Pi_v[y] := \sum_{\overline{e}\ni v} \frac{dy}{de_+}(v),$$

the sum being taken over all edges incident to v in a star neighborhood $\omega \ni v$.

1.2.2. Spaces.

Introduce a (real) Hilbert space $\mathcal{H} := L_2(\Omega)$ of functions on Ω with the inner product

$$(y, u)_{\mathcal{H}} = \int_\Omega yu\, d\tau = \sum_{e\in E} \int_e yu\, d\tau := \sum_{e\in E} \int_{\eta(e)} \tilde{y}(s)\tilde{u}(s) ds.$$

By $C(\Omega) \subset \mathcal{H}$ we denote the class of functions continuous on Ω.

We assign a function y on Ω to a class $\mathcal{H}^2$ if $y \in C(\Omega)$ and $\tilde{y}|_{\eta(e)} \in H^2(\eta(e))$ [2] for

[2] $H^s(a, b)$ is the standard Sobolev space.

each $e \in E$.

Also, define the *Kirchhoff class*

$$\mathcal{K} := \{y \in \mathcal{H}^2 \mid \Pi_v[y] = 0, \quad v \in V\}. \tag{1.2}$$

1.2.3. Operator.

On the graph, the *Laplace operator* $\Delta : \mathcal{H} \to \mathcal{H}$, $\operatorname{Dom}\Delta = \mathcal{K}$,

$$(\Delta y)|_e := \frac{d^2 y}{de^2}, \qquad e \in E \tag{1.3}$$

is well defined (does not depend on the parametrizations). It is a closed densely defined operator in $\mathcal{H}$.

2. Waves on graph.

2.1. Dynamical system.

An initial boundary value problem of the form

$$u_{tt} - \Delta u = 0 \qquad \text{in } \mathcal{H},\ 0 < t < T \tag{2.1}$$

$$u(\,\cdot\,, t) \in \mathcal{K} \qquad \text{for all } t \in [0, T] \tag{2.2}$$

$$u|_{t=0} = u_t|_{t=0} = 0 \qquad \text{in } \Omega \tag{2.3}$$

$$u = f \qquad \text{on } \Gamma \times [0, T] \tag{2.4}$$

is referred to as a dynamical system associated with the graph Ω. Here $T < \infty$; $f = f(\gamma, t)$ is a *boundary control*; the solution $u = u^f(x, t)$ describes a *wave* initiated at Γ and propagating into Ω.

Note that, by definition (1.2), the condition (2.2) provides the *Kirchhoff laws*:

$$u(\,\cdot\,, t) \in C(\Omega), \qquad \Pi_v[u(\,\cdot\,, t)] = 0 \tag{2.5}$$

for all $t \geq 0$ and $v \in V$. Also, note that by (1.3), on each edge $e \in E$ parametrized by $\eta : e \to (a, b)$, the pull-back function $\tilde{u}(\,\cdot\,, t) = u(\,\cdot\,, t) \circ \eta^{-1}$ satisfies the homogeneous string equation

$$\tilde{u}_{tt} - \tilde{u}_{ss} = 0 \qquad \text{in } (a, b) \times (0, T). \tag{2.6}$$

It is the reason, by which we regard Ω as a graph consisting of *homogeneous strings*. As is well known, for a C^2-smooth (with respect to t) control f vanishing near $t = 0$ the problem has a unique classical solution u^f.

A space of controls $\mathcal{F}^T := L_2(\Gamma \times [0, T])$ with the inner product

$$(f, g)_{\mathcal{F}^T} := \sum_{\gamma \in \Gamma} \int_0^T f(\gamma, t) g(\gamma, t)\, dt$$

is called an *outer space* of system (2.1)–(2.4). It contains the subspaces $\mathcal{F}^T_\gamma := \{f \in \mathcal{F}^T \mid \operatorname{supp} f \subset \{\gamma\} \times [0,T]\}$ of controls, which act from single boundary vertices $\gamma \in \Gamma$, so that

$$\mathcal{F}^T = \bigoplus \sum_{\gamma \in \Gamma} \mathcal{F}^T_\gamma \tag{2.7}$$

holds. Each $f \in \mathcal{F}^T_\gamma$ is of the form $f(\gamma', t) = \delta_\gamma(\gamma')\varphi(t)$ with $\varphi \in L_2(0,T)$.

The space $\mathcal{H}$ is an *inner space*; the waves $u^f(\,\cdot\,,t)$ are time-dependent elements of $\mathcal{H}$.

2.2. Fundamental solution.

2.2.1. Definition.

Consider the system (2.1)–(2.4) with $T = \infty$.

For $\gamma, \gamma' \in \Gamma$, we denote

$$\delta_\gamma(\gamma') := \begin{cases} 0, & \gamma' \neq \gamma \\ 1, & \gamma' = \gamma \end{cases};$$

let $\delta(t)$ be the Dirac delta-function of time.

Fix a boundary vertex γ. Taking the (generalized) control $f(\gamma', t) = \delta_\gamma(\gamma')\delta(t)$, one can define the generalized solution $u^{\delta_\gamma \delta}$ to (2.1)–(2.4). A possible way is to use a smooth regularization $\delta^\varepsilon(t) \underset{\varepsilon \to 0}{\to} \delta(t)$ and then understand $u^{\delta_\gamma \delta}$ as a relevant limit of the classical solutions $u^{\delta_\gamma \delta^\varepsilon}$ as $\varepsilon \to 0$. Such a limit turns out to be a space-time distribution on $\Omega \times [0,T]$ of the class $C((0,T); H^{-1}(\Omega))$ (see, e.g., [**5**]).

The distribution $u^{\delta_\gamma \delta}$ is called a *fundamental* solution to (2.1)–(2.4) (corresponding to the given γ). It describes the wave initiated by an instantaneous source supported at γ. Let us consider its properties in more detail. Recall that τ is the distance in Ω.

2.2.2. First edge.

Let e be the edge incident to γ and parametrised by $s = \eta(x) := \tau(x, \gamma) \in (0, \tau(\gamma, v))$. Let $v \in V$ be the second vertex incident to e. For times $0 < t \le \tau(\gamma, v)$, by (2.6) one has

$$\begin{aligned} &\tilde{u}_{tt} - \tilde{u}_{ss} = 0 && \text{in } (0, \tau(\gamma, v)) \times (0,T) \\ &\tilde{u}|_{t=0} = \tilde{u}_t|_{t=0} = 0 && \text{in } [0, \tau(\gamma, v)] \\ &\tilde{u}|_{s=0} = \delta(t), && 0 \le t \le \tau(\gamma, v), \end{aligned}$$

which implies $\tilde{u}(s,t) = \delta(t - s)$. This evidently leads to the representation

$$u^{\delta_\gamma \delta}(\,\cdot\,, t) = \delta_{x(t)}(\,\cdot\,), \qquad 0 \le t \le \tau(\gamma, v), \tag{2.8}$$

where $x(t)$ belongs to e and satisfies $\tau(x(t), \gamma) = t$, $\delta_p \in H^{-1}(\Omega)$ is the Dirac measure supported at $p \in \Omega$. It means that the δ-singularity, which is injected into the graph from γ, moves along e towards v with velocity 1 (see Figure 3a).

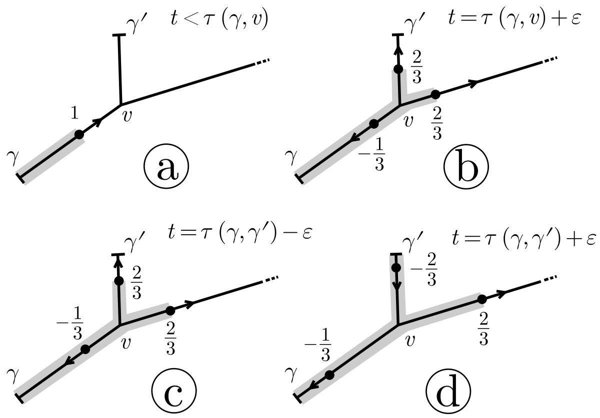

Figure 3. Propagation of singularities.

2.2.3. Passing through interior vertex.

At the time $t = \tau(\gamma, v)$ the singularity reaches v and then passes through v. Simple analysis using (2.5) and (2.6) provides

$$u^{\delta_\gamma\delta}(\,\cdot\,,t) = \sum_{e':\overline{e'}\ni v} a(x_{e'}(t))\delta_{x_{e'}(t)}(\,\cdot\,), \quad \tau(\gamma,v) < t \le \tau(\gamma,v)+\varepsilon, \tag{2.9}$$

as $\varepsilon > 0$ is small (namely, $\varepsilon < \tau(v,(V\cup\Gamma)\backslash\{v\})$), where $x_{e'}(t)$ belongs to e' and satisfies $\tau(x_{e'}(t),v) = t - \tau(\gamma,v)$. The function (*amplitude*) a is

$$a(x) = \begin{cases} -\dfrac{m(v)-2}{m(v)} & \text{as } x\in e \\ \dfrac{2}{m(v)} & \text{as } x\in e' : e'\neq e,\ \overline{e'}\ni v \end{cases}. \tag{2.10}$$

Hence, in passing through v, the singularity splits onto $m(v)$ parts (singularities), the first one is reflected back into e, the others are injected into the other $m(v)-1$ edges e' incident to v. The reflected singularity has the negative amplitude. The process is illustrated by Figure 3b. Note that the 'conservation law'

$$-\frac{m(v)-2}{m(v)} + (m(v)-1)\frac{2}{m(v)} = 1, \tag{2.11}$$

is valid, so that the total amplitude after the passage through v is equal to the amplitude of the incident singularity.

In what follows, we refer to (2.9)–(2.11) as a *splitting rule*.

2.2.4. Reflection from boundary.

Let $\gamma' \in \Gamma$ be a boundary vertex nearest to v:

$$\tau(\gamma',v) = \min_{\gamma''\in\Gamma} \tau(\gamma'',v)$$

(may be $\gamma = \gamma'$), so that $\tau(\gamma, \gamma') = \tau(\gamma, v) + \tau(v, \gamma')$ holds. Let e' be the edge incident to γ'.

As $t \to \tau(\gamma, \gamma') - 0$, one of the singularities, which have appeared as a result of passing through v (and, perhaps, through another vertices or reflected from v back to γ), approaches to γ' (see Figure 3c). Then this singularity is reflected from γ'. Simple analysis with the use of the condition $u^{\delta_\gamma \delta}(\gamma', t) = \delta_\gamma(\gamma')\delta(t) = 0$, $t > 0$ leads to the representation

$$u^{\delta_\gamma \delta}(\,\cdot\,, t) = \begin{cases} a\, \delta_{x(t)}(\,\cdot\,) & \text{as } t \in (\tau(\gamma, \gamma') - \varepsilon, \tau(\gamma, \gamma')) \\ -a\, \delta_{x(t)}(\,\cdot\,) & \text{as } t \in (\tau(\gamma, \gamma'), \tau(\gamma, \gamma') + \varepsilon), \end{cases} \tag{2.12}$$

where $x(t) \in e'$ satisfies $\tau(x(t), \gamma') = |t - \tau(\gamma, \gamma')|$, $a = \text{const} \neq 0$.

Thus, as a result of reflection from a boundary vertex, the singularity moves *from* it and changes the sign of the amplitude (see Figure 3d). This is a *reflection rule*.

The splitting and reflection rules are well known and derived in the framework of quite standard analysis: see [**1**], [**2**], [**4**], [**9**], [**10**]. As is easy to recognize, they uniquely determine the fundamental solution $u^{\delta_\gamma \delta}$ for all $t \geq 0$.

2.2.5. Hydra.

Definition. Return to the fundamental solution of (2.1)–(2.4) with $T = \infty$ and consider $u^{\delta_\gamma \delta}$ as a space-time distribution in $\Omega \times \overline{\mathbb{R}_+}$. In what follows, its support

$$H_\gamma := \operatorname{supp} u^{\delta_\gamma \delta}$$

plays important role and is called a *hydra*, the point $(\gamma, 0)$ being its *root*. The reason to introduce the characteristic set (see 1.1.3) is that it consists of the characteristic lines of the wave equation (2.1). As is seen from (2.8)–(2.12), singularities propagate along the characteristics that leads to the relation

$$H_\gamma \subseteq \operatorname{Ch}[(\gamma, 0)].$$

In particular, it shows that the space projections of singularities propagate along a homogeneous graph with velocity 1. This implies

$$\operatorname{supp} u^{\delta_\gamma \delta}(\,\cdot\,, t) \subset \overline{\Omega^t[\gamma]}, \qquad t > 0, \tag{2.13}$$

where the left hand side is understood as a time-depended element of $H^{-1}(\Omega)$. Also, note that the hydra is a connected set in $\Omega \times [0, T]$: as is obvious, any point $(x, t) \in H_\gamma$ is connected with the root $(\gamma, 0)$ through a path in H_γ.

Let

$$\pi : H_\gamma \to \Omega,\ \pi((x, t)) := x, \qquad \rho : H_\gamma \to \overline{\mathbb{R}_+},\ \rho((x, t)) := t$$

be the *space* and *time projection* respectively. For $A \subset \Omega$ and $B \subset \overline{\mathbb{R}_+}$, denote

$$\pi^{-1}(A) := \{(x,t) \in H_\gamma \mid x \in A\}, \quad \rho^{-1}(B) := \{(x,t) \in H_\gamma \mid t \in B\}.$$

Choose an edge $e \in E$ parametrized by $\eta : e \to (a,b)$. Its pre-image $\pi^{-1}(e) \subset H_\gamma$ consists of the sets

$$\tilde{e}_j := \{(\eta^{-1}(s), t_j + \sigma_j(s-a)) \mid a < s < b\}, \qquad \sigma_j = \pm 1,\ j = 1, 2, \dots,$$

which are the edges of the hydra as a space-time graph. There is a part of the fundamental solution of the form

$$u_j^{\delta_\gamma \delta}(\,\cdot\,, t) = a_j\, \delta_{x(t)}(\,\cdot\,), \qquad t \in \rho(\tilde{e}_j) \tag{2.14}$$

supported on $\tilde{e}_j$, where $a_j = \text{const} \neq 0$, $x(t) := \eta^{-1}(a + \sigma_j(t - t_j))$. In dynamics, (2.14) describes the singularity moving along e with velocity 1 as t runs over the time interval $\rho(\tilde{e}_j)$, the sign σ_j determining the direction of motion. The value of the amplitude a_j is determined by the prehistory of $u^{\delta_\gamma \delta}$ as $t < \inf \rho(\tilde{e}_j)$ and can be derived from the splitting and reflection rules.

Amplitude. The aforesaid (see (2.8), (2.9), (2.12), (2.14)) enables one to endow the hydra with a function (*amplitude*) a as follows. Take a point $(x,t) \in H_\gamma$ provided $x \in \Omega \backslash \{V \cup \Gamma\}$, so that there is a graph edge $e \ni x$. Hence, there is a hydra edge $\tilde{e}_j \subset \pi^{-1}(e)$ such that $(x,t) \in \tilde{e}_j$ and (2.14) does hold. In the mean time, as is easy to see, there may be at most one more edge $\tilde{e}_i \in \pi^{-1}(e)$, $\tilde{e}_i \neq \tilde{e}_j$, which contains the given (x,t) (so that $(x,t) = \tilde{e}_i \cap \tilde{e}_j$: see the point p on Figure 4). Then,

- (*generic case*) if $\tilde{e}_j$ is a unique edge, which contains (x,t), we set $a(x,t) := a_j$
- (*exclusive case*) if there is the second $\tilde{e}_i \ni (x,t)$, we define $a(x,t) := a_i + a_j$

(on Figure 4, $a(p) = -4/9 + 1/3 = -1/9$). To extend the amplitude to the whole hydra we settle the following:

- for points $(\gamma', t) \in H_\gamma$ with $\gamma' \in \Gamma$, we put $a(\gamma', t) := 0$ as $t > 0$, and $a(\gamma, 0) := 1$, $a(\gamma', 0) := 0$ for $\gamma' \neq \gamma$
- if $(v,t) \in H_\gamma$ and $v \in V$, we define $a(v,t) = a_1 + \cdots + a_k$, where a_i are the amplitudes (2.14) on the hydra edges $\tilde{e}_i \subset H_\gamma \cap \{(x',t') \mid t' < t\}$ incident to (v,t). Note that in the generic case such an $\tilde{e}_i$ is unique.

Corner points. Thus, the amplitude a is a well-defined piece-wise constant function on H_γ. Moreover, as a function on the metric space-time graph (see (1.1)), it is piece-wise continuous, the continuity being broken only in some exceptional points. Namely, we say $(x,t) \in H_\gamma$ to be a *corner point* if either $x \in \Gamma \cup V$ or there are an edge $e \ni x$ and the hydra edges $\tilde{e}_i, \tilde{e}_j \subset \pi^{-1}(e)$ such that $(x,t) = \tilde{e}_i \cap \tilde{e}_j$ (see the point p on Figure 4).

2.3. Generalized solutions.

Here we list some results on solutions of the problem (2.1)–(2.4), which can be easily derived from the above mentioned properties of the fundamental solution. Unless otherwise specified, we deal with $T < \infty$.

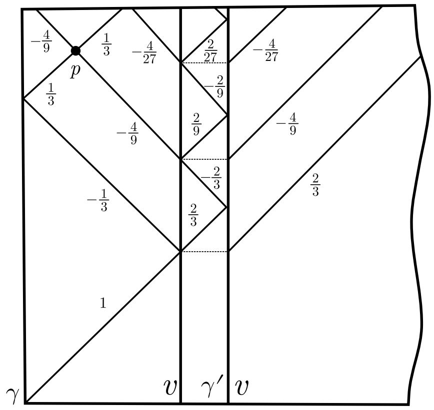

Figure 4. Hydra and amplitude function.

2.3.1. Definition.

Fix a $\gamma \in \Gamma$. Let a control $f \in \mathcal{F}^T_\gamma$ (see (2.7)) be such that u^f is a classical solution. Then the *Duhamel representation*

$$u^f = u^{\delta_\gamma \delta} * f, \qquad \text{in } \Omega \times [0,T] \tag{2.15}$$

(the convolution with respect to time) holds and motivates the following. For an $f \in \mathcal{F}^T$, we *define* a (generalized) solution to (2.1)–(2.4) by

$$u^f := \sum_{\gamma \in \Gamma} u^{\delta_\gamma \delta} * f, \qquad \text{in } \Omega \times [0,T]. \tag{2.16}$$

2.3.2. Properties.

1. As can be shown, solution (2.16) belongs to the class $C([0,T];\mathcal{H})$, i.e., is a continuous $\mathcal{H}$-valued function of time.
2. For $f \in \mathcal{F}^T_\gamma$, relation (2.13) implies

$$\operatorname{supp} u^f(\,\cdot\,,t) \subset \overline{\Omega^t[\gamma]}, \qquad t > 0, \tag{2.17}$$

which means that the waves propagate in Ω with velocity 1. Let $\Sigma \subseteq \Gamma$ be a set of boundary vertices and $f \in \bigoplus \sum_{\gamma \in \Sigma} \mathcal{F}^T_\gamma$. As a consequence of (2.17), we have the relation

$$\operatorname{supp} u^f(\,\cdot\,,t) \subset \bigcup_{\gamma \in \Sigma} \overline{\Omega^t[\gamma]} = \overline{\Omega^t[\Sigma]}, \qquad t > 0, \tag{2.18}$$

which is usually referred to as a *finiteness of domain of influence*.
For the rest of the paper we accept the following.

Convention 2. *All functions depending on time $t \geq 0$ are extended to $t < 0$ by zero.*

3. For $f \in \mathcal{F}^T$, denote by $f_s(\gamma, t) := f(\gamma, t-s)$ the delayed control. Since the graph and operator Δ, which governs the evolution of system (2.1)–(2.4), do not depend on time, one has the relation (*steady-state property*)

$$u^{f_s}(\,\cdot\,, t) = u^f(\,\cdot\,, t-s). \tag{2.19}$$

2.3.3. Point-wise values of wave.
Here we describe a "mechanism", which forms the values of waves u^f.
Fix a $\gamma \in \Gamma$. In $\Omega \times [0,T]$ define the *truncated* and *delayed hydras*

$$\begin{aligned} H^T_\gamma &:= \{(x,t) \in H_\gamma \mid 0 \le t \le T\}, \\ H^{T,s}_\gamma &:= \big\{(x, t+s) \in \Omega \times [0,T] \mid (x,t) \in H^T_\gamma\big\}, \end{aligned} \tag{2.20}$$

where $s \in (0,T)$ is a *delay*. Also, we put $H^{T,0}_\gamma := H^T_\gamma$ and $H^{T,T}_\gamma := (\gamma, T)$. Each $H^{T,s}_\gamma$ is endowed with an amplitude function by

$$a^{T,s}(x,t) := a(x, t-s), \tag{2.21}$$

where a is the amplitude on H_γ. The point (γ, s) is a *root* of $H^{T,s}_\gamma$.
A set

$$\kappa^{T,s} := \big\{x \in \Omega \mid (x,T) \in H^{T,s}_\gamma\big\} \qquad (0 \le s \le T)$$

consists of finite number of points in Ω, which we call *heads* of the hydra $H^{T,s}_\gamma$. The heads and delays are related by

$$\kappa^{T,s} = \pi(\rho^{-1}(T-s)). \tag{2.22}$$

The heads move into Ω as s varies.
Fix a point $x \in \Omega$. We say that a hydra $H^{T,s}_\gamma$ influences on x and write

$$H^{T,s}_\gamma \rhd x$$

if $x \in \kappa^{T,s}$, i.e., x is one of the heads of $H^{T,s}_\gamma$. The value $a^{T,s}(x,T)$ is referred to as amplitude of influence. As is easy to see, for any $x \in \overline{\Omega^T[\gamma]}$ there is at least one hydra, which influences on x. The number of the hydras influencing on x is always finite and equal to $\sharp\rho(\pi^{-1}(x))$.

Convention 3. *Here and in what follows, dealing with the truncated hydra H^T_γ, we understand $\pi^{-1}(A)$ as $\pi^{-1}(A) \cap H^T_\gamma$.*

Take a control $f \in \mathcal{F}^T_\gamma$ of the form $f(\gamma', t) = \delta_\gamma(\gamma')\varphi(t)$ with $\varphi \in C[0,T]$. Fix an $x \in \Omega\backslash\Gamma$. The structure of the fundamental solution and representation (2.15) easily imply that the value $u^f(x,T)$ can be calculated by the following procedure:

- take all $s \in [0, T)$ such that $H_\gamma^{T,s} \rhd x$ and determine the corresponding amplitudes $a^{T,s}(x, T)$
- find

$$u^f(x,T) = \sum_{s : H_\gamma^{T,s} \rhd x} a^{T,s}(x,T)\varphi(s) \overset{(2.21),(2.22)}{=} \sum_{t \in \rho(\pi^{-1}(x))} a(x,t)\varphi(T-t). \quad (2.23)$$

So, only the hydras, which influence on x, do contribute to the value of the wave. Each hydra contributes to $u^f(x, T)$ with the corresponding amplitude weighted by the value $f(\gamma, s) = \varphi(s)$ of the control at the hydra root (γ, s).

For any $f \in \mathcal{F}^T$ of the form $f = \sum_{\gamma \in \Gamma} \delta_\gamma(\,\cdot\,)\varphi_\gamma$ with $\varphi_\gamma \in C[0, T]$, relation (2.23) evidently implies

$$u^f(x,T) = \sum_{\gamma \in \Gamma} \sum_{s : H_\gamma^{T,s} \rhd x} a_\gamma^{T,s}(x,T)\varphi_\gamma(s), \quad (2.24)$$

where $a_\gamma^{T,s}(x, T)$ are the amplitudes on hydras $H_\gamma^{T,s}$. Also, representing $u^f(\,\cdot\,, t) = u^{f_{T-t}}(\,\cdot\,, T)$ by (2.19), one can find the value of the wave for any intermediate $t \in (0, T)$ via (2.23), (2.24).

2.4. Reachable sets.

2.4.1. Definition.

In dynamical system (2.1)–(2.4), a set of waves

$$\mathcal{U}^s := \left\{u^f(\,\cdot\,, s) \mid f \in \mathcal{F}^T\right\} \qquad (0 < s \le T)$$

is said to be *reachable* (from the boundary at the moment $t = s$). On graphs, $\mathcal{U}^s$ is a closed subspace in $\mathcal{H}$. Its structure is of principal importance for many applications, in particular to inverse problems: see [**5**]–[**9**], [**14**].

By (2.7), we have

$$\mathcal{U}^s = \sum_{\gamma \in \Gamma} \mathcal{U}_\gamma^s$$

(algebraic sum), where

$$\mathcal{U}_\gamma^s := \left\{u^f(\,\cdot\,, s) \mid f \in \mathcal{F}_\gamma^T\right\} \qquad (0 < s \le T)$$

are the sets reachable from single boundary vertices.

By (2.19), to study $\mathcal{U}_\gamma^s$ is to study $\mathcal{U}_\gamma^T$, and we deal mainly with the latter set. Its structure will be described in detail. However, the description requires certain preliminary considerations in 2.4.2–2.4.4.

2.4.2. Lattices and determination set.

We say two different points $l' = (x', t')$, $l'' = (x'', t'')$ of H_γ^T to be *neighbors* and write $l' \simeq l''$, if either $x' = x''$ or $t' = t''$ (equivalently: either $\pi(l') = \pi(l'')$ or $\rho(l') = \rho(l'')$).

We write $l' \cong l''$ if there are l_k such that $l' \simeq l_1 \simeq l_2 \simeq \cdots \simeq l_p \simeq l''$. As is easy to check, $\cong$ is an equivalence on the hydra. For an $l \in H^T_\gamma$, its equivalence class is called a *lattice* and denoted by $\mathcal{L}[l]$ [3].

For a $B \subset H^T_\gamma$ we set

$$\mathcal{L}[B] := \bigcup_{l\in B} \mathcal{L}[l] \subset H^T_\gamma .$$

We omit simple proofs of the following facts, which can be derived from the above-accepted definitions:

- for any $B \subset H^T_\gamma$, one has $\pi^{-1}(\pi(\mathcal{L}[B])) = \rho^{-1}(\rho(\mathcal{L}[B])) = \mathcal{L}[B]$
- the operation $\mathcal{L} : B \mapsto \mathcal{L}[B]$ satisfies the Kuratovski axioms:
  (i) (*extensiveness*) $\mathcal{L}[B] \supset B$,
  (ii) (*idempotency*) $\mathcal{L}[\mathcal{L}[B]] = \mathcal{L}[B]$,
  (iii) (*additivity*) $\mathcal{L}[B' \cup B''] = \mathcal{L}[B'] \cup \mathcal{L}[B'']$
  and, hence, is a topological closure. More precisely, there is a unique topology on the hydra, in which the closure coincides with $\mathcal{L}$ (see, e.g., [**13**]).
- each $\mathcal{L}[l]$ is a finite set; it is a closure of the single point set $\{l\}$ in the above mentioned topology. Any point $l' \in \mathcal{L}[l]$ determines the whole set $\mathcal{L}[l]$.

For a point $x \in \overline{\Omega^T[\gamma]}\backslash\Gamma$, define its *determination set* by

$$\Lambda^T_\gamma[x] := \pi\big(\mathcal{L}[\pi^{-1}(x)]\big) \subset \Omega. \tag{2.25}$$

The *alternating property* holds: for $x \neq x'$, one has either $\Lambda^T_\gamma[x] = \Lambda^T_\gamma[x']$ or $\Lambda^T_\gamma[x] \cap \Lambda^T_\gamma[x'] = \emptyset$.

Determination set $\Lambda^T_\gamma[x]$ consists of the heads of the delayed hydras H^{T,s_i}_γ, which satisfy $T - s_i \in \rho(\mathcal{L}[\pi^{-1}(x)])$. It is the hydras, which enter in representation (2.23).

On Figure 5,

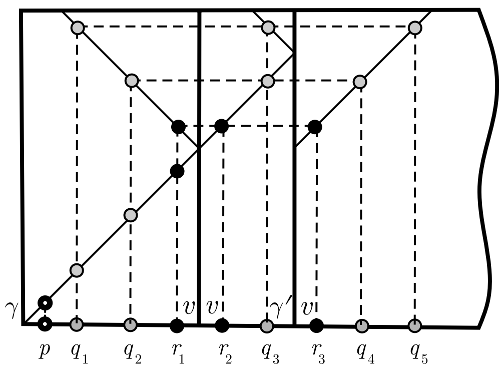

Figure 5. Lattices and determination sets.

[3]Here we regard $\mathcal{L}[l]$ as a subset of H^T_γ but not as an element of the factor-set $H^T_\gamma/\cong$.

- $\mathcal{L}[\pi^{-1}(p)]$ is the point of H_γ^T above p, $\Lambda_\gamma^T[p] = \{p\}$
- $\mathcal{L}[\pi^{-1}(q_m)]$ is the grey points on H_γ^T, $\Lambda_\gamma^T[q_m] = \{q_1, q_2, q_3, q_4, q_5\}$
- $\mathcal{L}[\pi^{-1}(r_m)]$ is the black points on H_γ^T, $\Lambda_\gamma^T[r_m] = \{r_1, r_2, r_3\}$.

2.4.3. Amplitude vectors.

Return to representation (2.23) and modify it as follows.

Fix an ordinary $x \in \Omega^T[\gamma]$; let

$$\rho\big(\mathcal{L}[\pi^{-1}(x)]\big) = \{t_i\}_{i=1}^N, \quad 0 \le t_1 < t_2 < \cdots < t_N \le T,$$

so that $N = \sharp\rho(\mathcal{L}[\pi^{-1}(x)])$. On the determination set, introduce N functions $\alpha^{T,t_i} : \Lambda_\gamma^T[x] \to \mathbb{R}$ of the form

$$\alpha^{T,t_i}(h) := \begin{cases} a(h,t_i) & \text{if } (h,t_i) \in H_\gamma^T \\ 0 & \text{otherwise} \end{cases} \tag{2.26}$$

and call them *amplitude vectors.*

So, the set $\Lambda_\gamma^T[x]$ is endowed with amplitude vectors $\alpha^{T,t_1}, \dots, \alpha^{T,t_N}$. Let $\boldsymbol{l}_2(\Lambda_\gamma^T[x])$ be Euclidean space of functions on $\Lambda_\gamma^T[x]$ with the standard product

$$\langle \alpha, \beta \rangle = \sum_{h \in \Lambda^T[x]} \alpha(h)\beta(h).$$

By $\mathcal{A}^T[x]$ we denote the subspace in $\boldsymbol{l}_2(\Lambda_\gamma^T[x])$ generated by amplitude vectors:

$$\mathcal{A}^T[x] := \operatorname{span}\big\{\alpha^{T,t_1}, \dots, \alpha^{T,t_N}\big\}.$$

Now, we are able to clarify the meaning of the definition (2.25) and the term "determination set". In accordance with (2.23), given $f = \delta_\gamma\varphi$, for all $x' \in \Lambda_\gamma^T[x]$ the values $u^f(x',T)$ *are determined* by hydras H_γ^{T,s_i}, whose heads constitute $\Lambda_\gamma^T[x]$. Moreover, each value is a combination of components of the amplitude vectors $\alpha^{T,t_1}, \dots, \alpha^{T,t_N}$. Hence, one can regard the set of values $\{u^f(x',T) \mid x' \in \Lambda_\gamma^T[x]\}$ as an element of $\boldsymbol{l}_2(\Lambda_\gamma^T[x])$, whereas (2.23) can be written in the form

$$u^f(\,\cdot\,,T)|_{\Lambda_\gamma^T[x]} = \sum_{i=1}^N \varphi(T-t_i)\alpha^{T,t_i} \in \mathcal{A}^T[x]. \tag{2.27}$$

Also, one has to keep in mind the dependence $t_i = t_i(x)$, $N = N(x)$.

2.4.4. Partition Π_γ^T.

Recall that $\Lambda_\gamma^T[x]$ is defined for *ordinary* points x, so that there is an edge $e \ni x$.

Under the conditions, which we are going to specify now, small variations of position of x on e lead to small variations of the set $\Lambda_\gamma^T[x]$ in Ω, which do not change its "staff".

Namely, the number

$$M := \sharp\Lambda^T_\gamma[x] = \dim \boldsymbol{l}_2(\Lambda^T_\gamma[x]),$$

the number N of amplitude vectors α^{T,t_i}, and the values of the amplitude vectors remain the same (do not depend on x). The question arises: What are the bounds for such variations? In this section the answer is given.

Critical points. Recall that the corner points on the complete hydra H_γ are introduced at the end of 2.2.5. Dealing with the truncated hydra H^T_γ, it is also convenient to assign its top $\{(x,t) \in H^T_\gamma \mid t = T\} = \rho^{-1}(T)$ to corner points. So, we say a space-time point $(x,t) \in H^T_\gamma$ to be a *corner point* if it is a corner point of H_γ or belongs to the top of H^T_γ. The set of corner points is denoted by $\operatorname{Corn} H^T_\gamma$. Note that, for any vertex $w \in [V \cup \Gamma] \cap \overline{\Omega^T[\gamma]}$, its pre-image $\pi^{-1}(w)$ consists of corner points.

Introduce the lattice $\mathcal{L}[\operatorname{Corn} H^T_\gamma]$, which is a finite set of points on H^T_γ. This lattice divides hydra H^T_γ so that the set $H^T_\gamma \backslash \mathcal{L}[\operatorname{Corn} H^T_\gamma]$ consists of a finite number of open space-time intervals, which do not contain corner points. By the latter, on these intervals the amplitude $a(\cdot)$ takes *constant values*.

Points of the set

$$\Theta^T_\gamma := \pi\big(\mathcal{L}[\operatorname{Corn} H^T_\gamma]\big) \subset \overline{\Omega^T[\gamma]} \tag{2.28}$$

are called *critical*.

Critical points divide neighborhood $\overline{\Omega^T[\gamma]}$ into parts. Namely, the set $\overline{\Omega^T[\gamma]}\backslash\Theta^T_\gamma$ is a collection of open intervals, each interval lying into an edge of Ω, whereas the critical points are the endpoints of these intervals. We refer to this collection as a *partition* Π^T_γ.

Families and cells. Intervals of partition Π^T_γ are joined in the *families* as follows. Let $c, c' \in e$ be critical points such that the interval $\omega =]c, c'[\subset e$ contains no critical points. This means that $\omega \in \Pi^T_\gamma$. As one can easily see, the preimage $\pi^{-1}(\omega)$ consists of a finite number of connected components. Each component is a (space-time) interval on H^T_γ of the same (space-time) length $\sqrt{2}\tau(c,c')$ (see (1.1)), the interval being free of corner points. By the latter, the same is valid for the lattice $\mathcal{L}[\pi^{-1}(\omega)]$: it is also a finite collection of open intervals on H^T_γ of length $\sqrt{2}\tau(c,c')$, which do not contain corner points.

As a consequence, the set

$$\Phi := \pi\big(\mathcal{L}[\pi^{-1}(\omega)]\big) \supset \omega \tag{2.29}$$

turns out to be a finite collection of open intervals $\omega_1, \omega_2, \dots, \omega_M \subset \overline{\Omega^T[\gamma]}\backslash\Theta^T_\gamma$ (with ω among them) of the same length:

$$\Phi = \bigcup_{l=1}^{M} \omega_m, \quad \operatorname{diam} \omega_m = \tau(c,c') =: \delta_\Phi. \tag{2.30}$$

We say this collection to be a *family*, intervals ω_m are called *cells* of Φ.

Comparing definitions (2.25) and (2.29), (2.30), one can easily conclude the following. For any $x \in \Phi$, the set $\Lambda^T_\gamma[x] \subset \Phi$ consists of the points $x_1, \dots, x_M$, each cell ω_m containing one (and only one) point x_m. Hence, we have

$$\sharp\Lambda^T_\gamma[x] = \dim \boldsymbol{l}_2(\Lambda^T_\gamma[x]) = M, \quad \Phi = \bigcup_{x\in\omega} \Lambda^T_\gamma[x] \tag{2.31}$$

as x varies in any cell $\omega \subset \Phi$.

Starting with another interval $\omega' \not\subset \Phi$ bounded by critical points, we get another family Φ', which has no mutual cells or points with Φ. Going on this way, we get a finite set of families $\Phi^1, \Phi^2, \dots, \Phi^J$ and conclude that partition Π^T_γ corresponds to the representation

$$\overline{\Omega^T[\gamma]}\backslash\Theta^T_\gamma = \bigcup_{j=1}^{J} \Phi^j = \bigcup_{j=1}^{J} \bigcup_{m=1}^{M_j} \omega^{(j)}_m \tag{2.32}$$

in the form of disjoint sums,

$$\operatorname{diam} \omega^{(j)}_1 = \dots = \operatorname{diam} \omega^{(j)}_{M_j} = \delta_{\Phi^j}.$$

On Figure 6,

- the set $\operatorname{Corn} H^T_\gamma$ is the black points with holes, the lattice $\mathcal{L}[\operatorname{Corn} H^T_\gamma]$ is the grey points along with the corner points

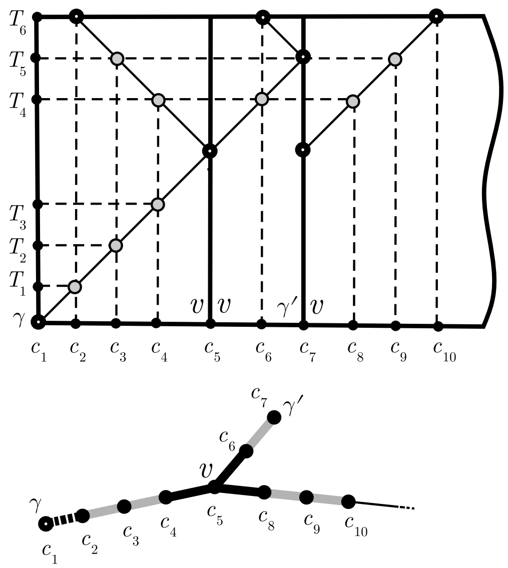

Figure 6. Partition $\Pi^T_\gamma[x]$.

- the critical point set is $\Theta^T_\gamma = \bigcup_{k=1}^{10} c_k$, $c_1=\gamma$, $c_5=v$, $c_7=\gamma'$
- the families and cells are

$$\Phi^1 = \omega_1^{(1)} =]c_1, c_2[\quad \text{(dotted line)},$$

$$\Phi^2 = \bigcup_{m=1}^{5} \omega_m^{(2)} =]c_2, c_3[\cup]c_3, c_4[\cup]c_6, c_7[\cup]c_8, c_9[\cup]c_9, c_{10}[\quad \text{(grey intervals)},$$

$$\Phi^3 = \bigcup_{m=1}^{3} \omega_m^{(3)} =]c_4, c_5[\cup]c_5, c_6[\cup]c_5, c_8[\quad \text{(black intervals)}.$$

Variations and bounds. Return to the question on the bounds at the beginning of 2.4.4.

Take a cell $\omega =]c, c'[\subset \Phi = \bigcup_{l=1}^{M} \omega_m$ and choose an $x \in \omega$. The determination set $\Lambda^T_\gamma[x]$ consists of the points $x_1, \dots, x_M$ (x among them), $x_m \in \omega_m$. Varying x, one varies the set $\Lambda^T_\gamma[x]$ (the points x_m).

Parametrize

$$\omega \ni x = x(r), \qquad r = \tau(x, c) \in (0, \delta_\Phi); \tag{2.33}$$

simultaneously, all $x_m(r) \in \Lambda^T_\gamma[x(r)]$ turn out to be also parametrized. When r varies from 0 to δ_Φ, each $x_m(r)$ runs over $\omega_m =]c_m, c'_m[$ (from c_m to c'_m or in the opposite direction) and sweeps the cell ω_m. Correspondingly, $\Lambda^T_\gamma[x(r)]$ varies continuously on the graph and sweeps the given family Φ.

An important fact is that, in process of such varying, the amplitude vectors $\alpha^{T,t_1(x(r))}, \dots, \alpha^{T,t_N(x(r))} \in \boldsymbol{l}_2(\Lambda^T_\gamma[x(r)])$ *do not vary.* This follows from definition (2.26): the points $(x_m(r), t_i(x(r)))$ of the "horizontal layer" $\rho^{-1}(t_i)$ move along the hydra but do not leave the intervals of H^T_γ, on which there are no corner points, and hence amplitude a takes constant values (does not depend on r):

$$\alpha^{T,t_i(x(r))}(x_m(r)) = a(x_m(r), t_i(x(r))) = \text{const} =: \alpha_m^{T,i} \tag{2.34}$$

as $i = 1, \dots, N$; $m = 1, \dots, M$. Therefore, it is natural to associate amplitude vectors not with the set $\Lambda^T_\gamma[x]$ but the given family $\Phi \ni x$ and regard them as piece-wise constant functions on Φ defined by

$$\alpha^{T,i}(x) := \alpha_m^{T,i} \qquad \text{as } x \in \omega_m \subset \Phi.$$

We do it in what follows.

If x passes through a critical point c and enters a cell of another family, the picture of amplitude vectors changes. So, it is the set Θ^T_γ, which provides the bounds for variations which do not disturb $\boldsymbol{l}_2(\Lambda^T_\gamma[x])$ and $\mathcal{A}^T[x]$.

Varying T, one varies the set of critical points. Some of them (in particular, the vertices $V \cup \Gamma$) do not change the position in Ω, the others are moving along the graph *with velocity* 1. For this reason, the set varies continuously in the following sense: for a given T, there is a positive ε_0 such that

$$\Theta_\gamma^{T+\varepsilon} \subset \overline{\Omega^{|\varepsilon|}[\Theta_\gamma^T]} \tag{2.35}$$

holds for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$.

2.4.5. Local and global structure of wave.

Return to (2.27) and recall that, in such a general representation, the amplitude vectors and delays are determined by position of x:

$$u^f(\,\cdot\,, T)|_{\Lambda_\gamma^T[x]} = \sum_{i=1}^{N(x)} \psi_i(x)\alpha^{T,t_i(x)} \tag{2.36}$$

with $\psi_i(x) := \varphi(T - t_i(x))$. Now, choose an x in a cell ω of a family $\Phi = \bigcup_{m=1}^M \omega_m$ and parametrize by (2.33): $x_m(r) \in \omega_m$, $r \in (0, \delta_\Phi)$. Taking into account (2.34), we arrive at basic representation of the wave u^f *on the family* Φ:

$$u^f(x_m(r), T) = \sum_{i=1}^{N} \alpha_m^{T,i}\psi_i(r), \qquad r \in (0, \delta_\Phi),\ m = 1, \dots, M, \tag{2.37}$$

where $\psi_i(r) := \psi_i((x(r)))$.

By (2.32) and (2.37), we represent the wave everywhere in $\Omega^T[\gamma]$ except of critical points and vertices, i.e., almost everywhere on the graph. Such a representation clarifies a local structure of waves in the cells of families.

Recall that we deal with a control of the form $f(\gamma', t) = \delta_\gamma(\gamma')\varphi(t)$ with $\varphi \in C[0, T]$. However, one can extend the representation to all controls $f \in \mathcal{F}_\gamma^T$ just by taking $\psi_i \in L_2(0, \delta_\Phi)$ in (2.37).

Representation (2.37) provides a characteristic description of waves on families. A function $y \in \mathcal{H} = L_2(\Omega)$, $\operatorname{supp} y \subset \overline{\Phi}$ is a wave (i.e., $y \in \mathcal{U}_\gamma^T$ does hold) if and only if y can be represented in the form of the right hand side of (2.37) with $\psi_i \in L_2(0, \delta_\Phi)$. Functions ψ_i play the role of the *independent* function parameters, which determine a wave supported in Φ.

Taking into account (2.32), we get the global characteristic description of the reachable set $\mathcal{U}_\gamma^T$: to be a wave a function y supported in $\overline{\Omega^T[\gamma]}$ has to admit the representation (2.37) on each $\Phi^j \subset \Omega^T[\gamma]$.

2.4.6. Projection P_γ^T.

Let P_γ^T be the (orthogonal) projection in $\mathcal{H} = L_2(\Omega)$ onto the reachable subspace $\mathcal{U}_\gamma^T$. Here we provide a constructive description of this projection by the use of representation (2.37).

For a subset $B \subset \Omega$, by χ_B we denote its *indicator* (the characteristic function) and introduce the subspace

$$\mathcal{H}\langle B\rangle := \chi_B\mathcal{H} = \{\chi_B y \mid y \in \mathcal{H}\}$$

of functions supported on B. In accordance with (2.32) and results of 2.4.5, one has

$$\mathcal{H}\langle\Omega^T[\gamma]\rangle = \bigoplus \sum_{\Phi\in\Pi^T_\gamma} \mathcal{H}\langle\Phi\rangle, \quad \mathcal{U}^T_\gamma = \bigoplus \sum_{\Phi\in\Pi^T_\gamma} \mathcal{U}^T_\gamma\langle\Phi\rangle, \tag{2.38}$$

where $\mathcal{U}^T_\gamma\langle\Phi\rangle \subset \mathcal{H}\langle\Phi\rangle$ is the subspace of waves supported in Φ and represented by (2.37). Therefore,

$$P^T_\gamma = \sum_{\Phi\in\Pi^T_\gamma} Q_\Phi, \tag{2.39}$$

where Q_Φ project in $\mathcal{H}\langle\Phi\rangle$ onto subspaces $\mathcal{U}^T_\gamma\langle\Phi\rangle$. Hence, to characterize P^T_γ is to describe projections Q_Φ.

Parametrize Φ by (2.33) and introduce an isometry U by

$$\mathcal{H}\langle\Phi\rangle \ni y \overset{U}{\mapsto} \left.\begin{pmatrix} y(x_1(r)) \\ \dots \\ y(x_M(r)) \end{pmatrix}\right|_{r\in(0,\delta_\Phi)} \in L_2((0,\delta_\Phi);\mathbb{R}^M). \tag{2.40}$$

Since ψ_i in (2.37) can be arbitrary, this representation implies

$$U\mathcal{U}^T_\gamma\langle\Phi\rangle = L_2((0,\delta_\Phi);\mathbb{A}_\Phi), \quad \mathbb{A}_\Phi := \operatorname{span}\left\{\begin{pmatrix}\alpha^{T,1}_1 \\ \dots \\ \alpha^{T,1}_M\end{pmatrix}, \dots, \begin{pmatrix}\alpha^{T,N}_1 \\ \dots \\ \alpha^{T,N}_M\end{pmatrix}\right\} \subset \mathbb{R}^M.$$

A substantial fact is that the vectors, which span $\mathbb{A}_\Phi$, are constant (do not depend on r). Let $p_\Phi = \{p^{mm'}_\Phi\}_{m,m'=1,\dots,M}$ be the (matrix) projection in $\mathbb{R}^M$ onto $\mathbb{A}_\Phi$. Projection $\tilde{Q}_\Phi$ in the space $L_2((0,\delta_\Phi);\mathbb{R}^M)$ onto its subspace $L_2((0,\delta_\Phi);\mathbb{A}_\Phi)$ acts *point wise* by the rule

$$(\tilde{Q}_\Phi v)(r) = p_\Phi \begin{pmatrix} v_1(r) \\ \dots \\ v_M(r)\end{pmatrix}, \qquad r\in(0,\delta_\Phi).$$

In the mean time, one has $Q_\Phi = U^*\tilde{Q}_\Phi U$. Summarizing, we arrive at the representation

$$(Q_\Phi y)(x_m(r)) = \sum_{m'=1}^{M} p^{mm'}_\Phi\, y(x_{m'}(r)), \quad r\in(0,\delta_\Phi), \quad m=1,\dots,M \tag{2.41}$$

with the *constant matrix* p_Φ, which characterizes the action of Q_Φ.

System $\check{\beta}^T_\gamma$. The latter representation can be written in more transparent form as follows.

Redesign the system of amplitude vectors $\{\alpha^{T,1},\dots,\alpha^{T,N}\}$ by the Schmidt procedure:

$$\beta^{T,i}:=\begin{cases}\dfrac{\alpha^{T,i}-\sum_{j=1}^{i-1}\langle\alpha^{T,i},\beta^{T,j}\rangle\beta^{T,j}}{\left\|\alpha^{T,i}-\sum_{j=1}^{i-1}\langle\alpha^{T,i},\beta^{T,j}\rangle\beta^{T,j}\right\|} & \text{if } \alpha^{T,i}\notin\operatorname{span}\{\alpha^{T,1},\dots,\alpha^{T,i-1}\}\\ 0 & \text{otherwise}\end{cases}, \tag{2.42}$$

and get a system $\check\beta^T:=\{\beta^{T,1},\dots,\beta^{T,N}\}$. Its nonzero elements satisfy $\langle\beta^{T,i},\beta^{T,j}\rangle=\delta_{ij}$, and $\operatorname{span}\check\beta^T=\mathcal{A}^T[x]$ holds.

By analogy with original vectors $\alpha^{T,i}$, it is convenient to regard new amplitude vectors as piece-wise constant functions on the family Φ:

$$\beta^{T,i}(x):=\beta^{T,i}_m, \qquad x\in\omega_m\subset\Phi. \tag{2.43}$$

Expressing the projection matrix p_Φ via system $\beta^{T,1},\dots,\beta^{T,N}$ in (2.41), one can represent the action of Q_Φ in the following final form

$$(Q_\Phi y)(x)=\begin{cases}\sum\limits_{i=1}^{N}\big\langle y|_{\Lambda^T_\gamma[x]},\beta^{T,i}\big\rangle\beta^{T,i}(x), & x\in\Phi,\\ 0, & x\in\Omega\backslash\Phi\end{cases} \tag{2.44}$$

which is valid for any $y\in\mathcal{H}$.

At last, recalling (2.39), we conclude that P^T_γ is characterized.

Note in addition that representations (2.39), (2.44) provide a look at controllability of a graph. Recall a version of the *boundary control problem*: given $y\in\mathcal{H}$ to find $f\in\mathcal{F}^T_\gamma$ such that $u^f(\,\cdot\,,T)=y$ holds. *Controllability from* γ means that $\mathcal{U}^T_\gamma=\mathcal{H}$, i.e., this problem is well solvable. In our terms, the latter is equivalent to the relations $\mathbb{A}^T_\Phi=\mathbb{R}^{M(\Phi)}$, $\Phi\in\Pi^T_\gamma$.

Dependence on T. Varying T, one changes the neighborhood $\Omega^T[\gamma]$ filled with waves, reachable set $\mathcal{U}^T_\gamma$ and projection P^T_γ. As is evident, $\mathcal{U}^T_\gamma$ is increasing in $\mathcal{H}$ as T grows. The following arguments show that P^T_γ varies continuously.

Take a small $\Delta T>0$. The lattice $\mathcal{L}[\rho^{-1}([T-\Delta T,T])]\subset H^T_\gamma$ is also "small": it consists of a finite set of (closed) intervals on the hydra, the total length of the intervals vanishing as $\Delta T\to 0$. The same holds for the intervals in Ω, which constitute the set $\pi(\mathcal{L}[\rho^{-1}([T-\Delta T,T])])$. The latter set is located in the small neighborhood $\Omega^{\Delta T}[\Theta^T_\gamma]$ of critical points (see (2.35)).

As is easy to see, for an ordinary point $x\notin\pi(\mathcal{L}[\rho^{-1}([T-\Delta T,T])])$ one has $\Lambda^T_\gamma[x]=\Lambda^{T-\Delta T}_\gamma[x]$, whereas the amplitude vectors, which take part in projecting (2.37), are the same: $\alpha^{T,i}=\alpha^{T-\Delta T,i}$. Therefore, for any function $y\in\mathcal{H}$ we have $(P^T_\gamma y)(x)=(P^{T-\Delta T}_\gamma y)(x)$. Hence, the difference $P^T_\gamma y-P^{T-\Delta T}_\gamma y$ has to be supported on the complement to such points:

$$\operatorname{supp}\big(P^T_\gamma-P^{T-\Delta T}_\gamma\big)y\subset\pi\big(\mathcal{L}\big[\rho^{-1}([T-\Delta T,T])\big]\big)\subset\overline{\Omega^{\Delta T}[\Theta^T_\gamma]}. \tag{2.45}$$

As $\Delta T \to 0$, the neighborhood $\Omega^{\Delta T}[\Theta^T_\gamma]$ shrinks to the finite set Θ^T_γ. This leads to $\|(P^T_\gamma - P^{T-\Delta T}_\gamma)y\| \to 0$, i.e., $P^{T-\Delta T}_\gamma \to P^T_\gamma$ in the strong operator topology in $\mathcal{H}$.

Quite analogous arguments with regard to the property (2.35) show that $P^{T+\Delta T}_\gamma \to P^T_\gamma$ as $\Delta T \to 0$. Hence, $\{P^T_\gamma\}_{T\geq 0}$ is an increasing *continuous* family of projections in $\mathcal{H}$.

3. Eikonal algebra.

3.1. Single eikonal.

Fix $\gamma \in \Gamma$ and $T > 0$. Let $\Xi = \{\xi_k\}_{k=0}^K$, $0 = \xi_0 < \xi_1 < \cdots < \xi_K = T$ be a partition of $[0, T]$ of the range $r(\Xi) = \max(\xi_k - \xi_{k-1})$; denote $\Delta P^{\xi_k}_\gamma := P^{\xi_k}_\gamma - P^{\xi_{k-1}}_\gamma$. With each boundary vertex $\gamma \in \Gamma$ we associate a bounded self-adjoint operator in $\mathcal{H}$ of the form

$$E^T_\gamma := \int_0^T \xi \, dP^\xi_\gamma = \lim_{r(\Xi)\to 0} \sum_{k=1}^K \xi_k \, \Delta P^{\xi_k}_\gamma \tag{3.1}$$

and call it an *eikonal*. Our nearest purpose is to describe how it acts. For the theory of such operator integrals see, e.g., [**11**].

3.1.1. Small T.

Begin with the case $T \leq \tau(\gamma, V)$. In outer space $\mathcal{F}^T$, choose a control $f(\gamma', t) = \delta_\gamma(\gamma')\varphi(t)$. By (2.8) and (2.15), one has

$$u^f(x, t) = \varphi(t - \tau(x, \gamma))$$

(recall the Convention 2). Therefore, the reachable sets are

$$\mathcal{U}^\xi_\gamma = \{\varphi(\xi - \tau(\,\cdot\,, \gamma)) \mid \varphi \in L_2(0, T)\} = \mathcal{H}\langle\Omega^\xi[\gamma]\rangle, \quad 0 \leq \xi \leq T.$$

Correspondingly, projection P^ξ_γ in $\mathcal{H}$ onto $\mathcal{U}^\xi_\gamma$ cuts off functions on the part $\Omega^\xi[\gamma]$ of the edge e incident to γ, i.e., multiplies functions by the indicator $\chi_{\Omega^\xi[\gamma]}$. Therefore, for a $y \in \mathcal{H}$, the summands in (3.1) are

$$\big(\xi_k \Delta P^{\xi_k}_\gamma y\big)(x) = \begin{cases} \xi_k y(x) \approx \tau(x, \gamma) y(x) & \text{for } x \in \Omega^{\xi_k}[\gamma] \backslash \Omega^{\xi_{k-1}}[\gamma] \\ 0 & \text{for other } x \in \Omega \end{cases}$$

($\approx$ means that $\tau(x, \gamma) = \xi_k + O(r(\Xi))$ for $x \in \Omega^{\xi_k}[\gamma] \backslash \Omega^{\xi_{k-1}}[\gamma]$). Summing up the terms and passing to the limit as $r(\Xi) \to 0$, we easily obtain

$$\big(E^T_\gamma y\big)(x) = \begin{cases} \tau(x, \gamma) y(x) & \text{for } x \in \Omega^T[\gamma] \\ 0 & \text{for other } x \in \Omega \end{cases}.$$

Thus, for small enough T's, the eikonal cuts off functions on $\Omega^T[\gamma]$ and multiplies by the distance to γ.

3.1.2. Functions τ_i.

Let $T > 0$ be arbitrary. Choose a family $\Phi \in \Pi_\gamma^T$. The set

$$\rho\big(\mathcal{L}[\pi^{-1}(\Phi)]\big) = \bigcup_{i=1}^{N} I_i \tag{3.2}$$

consists of $N = N(\Phi)$ disjoined open intervals $I_i \subset (0, T)$ of the same length δ_Φ. Note that the segments $\overline{I_i}$ may intersect at the endpoints. On Figure 6, there is $\rho(\mathcal{L}[\pi^{-1}(\Phi^2)]) = \bigcup_{i=1}^4 I_i$, where $I_1 = (T_1, T_2)$, $I_2 = (T_2, T_3)$, $I_3 = (T_4, T_5)$, $I_4 = (T_5, T_6)$ (see also Figure 7).

With representation (3.2) one associates N functions on Φ of the form

$$\begin{aligned} \tau_i(x) &:= \begin{cases} I_i \cap \rho(\pi^{-1}(x)) & \text{if } I_i \cap \rho(\pi^{-1}(x)) \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} t & \text{if } (x,t) \in \rho^{-1}(I_i) \\ 0 & \text{otherwise} \end{cases}, \end{aligned} \tag{3.3}$$

which take values in the corresponding intervals I_i. They are of clear geometric meaning in terms of the distance on the hydra: if $\tau_i(x) \neq 0$ then $(x, \tau_i(x)) \in H_\gamma^T$ and

$$\tau_i(x) = \frac{1}{\sqrt{2}} \nu((x, \tau_i(x)), (\gamma, 0)) \tag{3.4}$$

holds (see (1.1)). On Figure 7, the functions $\tau_1, \tau_2, \tau_3, \tau_4$, which correspond to family Φ^2 (see the grey part of Ω on Figure 6), are shown. The space between the supports and graphs of the functions is shaded.

3.1.3. Arbitrary T.

Fix a $T > \tau(\gamma, V)$. Let $y \in \mathcal{H}$ be such that $\operatorname{supp} y \subset \Phi \in \Pi_\gamma^T$ (i.e., $y \in \mathcal{H}\langle\Phi\rangle$: see (2.38)). Choose a $\xi \in (0, T]$ and a small $\Delta\xi > 0$.

- Assume that

$$(\xi - \Delta\xi, \xi) \subset [0, T] \setminus \bigcup_{i=1}^{N} \overline{I}_i.$$

In such a case, one has $\pi(\mathcal{L}[\rho^{-1}((\xi - \Delta\xi, \xi))]) \cap \Phi = \emptyset$. Therefore, by (2.45), the supports of the functions $(P_\gamma^\xi - P_\gamma^{\xi-\Delta\xi})y$ and y do not intersect, and we have $(P_\gamma^\xi y - P_\gamma^{\xi-\Delta\xi} y, y) = 0$, i.e., $\|\Delta P_\gamma^\xi y\| = 0$, where $\Delta P_\gamma^\xi = P_\gamma^\xi - P_\gamma^{\xi-\Delta\xi}$. By the latter, the interval $(\xi - \Delta\xi, \xi)$ contributes nothing to the integral (3.1), which defines $E_\gamma^T y$.

By the aforesaid and with regard to continuity of P_γ^ξ, for $y \in \mathcal{H}\langle\Phi\rangle$ the integral (3.1) can be taken over the intervals I_i only:

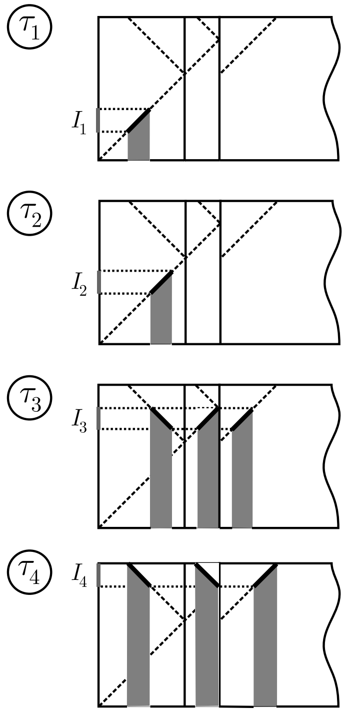

Figure 7. Functions τ_i.

$$E^T_\gamma y = \int_0^T \xi \, dP^\xi_\gamma y = \bigoplus \sum_{i=1}^N \int_{I_i} \xi \, dP^\xi_\gamma y. \tag{3.5}$$

The summands are pairwise orthogonal since the subspace $P^\xi_\gamma \mathcal{H}$ is increasing as ξ grows.

Let Φ and $y \in \mathcal{H}\langle\Phi\rangle$ be the same as before. For what follows, it is convenient to renumber the endpoints of the intervals so that $I_i = (T_i, T_i + \delta_\Phi)$. The amplitude vectors $\beta^{T,i}$ are regarded as piece-wise constant functions on Φ.

- Now, assume that $(\xi - \Delta\xi, \xi) \subset I_1$. In this case, the only amplitude vector, which contributes to the values of $P^\xi_\gamma y$ on Φ, is $\beta^{T,1}$. Therefore, in accordance with (2.44) and (2.45), we have

$$\big(\Delta P^\xi_\gamma y\big)(x) = \Delta\chi^\xi(x)\big\langle y|_{\Lambda^T_\gamma[x]}, \beta^{T,1}\big\rangle \beta^{T,1}(x), \qquad x \in \Phi,$$

where $\Delta\chi^\xi$ is the indicator of the set $\pi(\mathcal{L}[\rho^{-1}((\xi - \Delta\xi, \xi))]) \subset \Phi$. Correspondingly,

$$\big(\xi\Delta P^\xi_\gamma y\big)(x) \approx \tau_1(x)\Delta\chi^\xi(x)\big\langle y|_{\Lambda^T_\gamma[x]}, \beta^{T,1}\big\rangle \beta^{T,1}(x), \qquad x \in \Phi,$$

where $\tau_i(x)$ are introduced by (3.3). Summing up the terms of this form, one can easily justify the limit passage as $r(\Xi) \to 0$ and get the equality

$$\left(\int_{T_1}^{T_1+\delta_\Phi} \xi \, dP_\gamma^\xi y\right)(x) = \tau_1(x)\big\langle y|_{\Lambda_\gamma^T[x]}, \beta^{T,1}\big\rangle \beta^{T,1}(x), \qquad x \in \Phi.$$

- Assume that $(\xi - \Delta\xi, \xi) \subset I_2$. In this case, In this case, for any $u \in \mathcal{U}_\gamma^{T_1+\delta_\Phi}\langle\Phi\rangle$ one has $P_\gamma^{T_1+\delta_\Phi} u = u$, which leads to

$$\begin{aligned}\big(\Delta P_\gamma^\xi y, u\big)_{\mathcal{H}\langle\Phi\rangle} &= \big(\Delta P_\gamma^\xi y, P_\gamma^{T_1+\delta_\Phi} u\big)_{\mathcal{H}\langle\Phi\rangle} \\ &= \big(\big[P_\gamma^{T_1+\delta_\Phi} P_\gamma^\xi - P_\gamma^{T_1+\delta_\Phi} P_\gamma^{\xi-\Delta\xi}\big] y, u\big)_{\mathcal{H}\langle\Phi\rangle} \\ &= \big(\big[P_\gamma^{T_1+\delta_\Phi} - P_\gamma^{T_1+\delta_\Phi}\big] y, u\big)_{\mathcal{H}\langle\Phi\rangle} = 0\end{aligned}$$

since $P_\gamma^r P_\gamma^s = P_\gamma^{\min\{r,s\}}$ by monotonicity of P_γ^ξ, whereas $\xi > \xi - \Delta\xi > T_1 + \delta_\Phi$ holds. In the mean time, the only amplitude vectors, which contribute to the values of $P_\gamma^\xi y$ and $P_\gamma^{\xi-\Delta\xi} y$ on Φ, are $\beta^{T,1}$ and $\beta^{T,2}$. Therefore, in accordance with (2.36), (2.44) and (2.37), one represents

$$(\Delta P_\gamma^\xi y)(x_m(r)) = \eta_1(r)\beta_m^{T,1} + \eta_2(r)\beta_m^{T,2}, \quad u(x_m(r)) = \theta(r)\beta_m^{T,1},$$
$$m = 1, \dots, M, \quad r \in (0, \delta_\Phi).$$

This implies

$$\begin{aligned}0 = (\Delta P_\gamma^\xi y, u)_{\mathcal{H}\langle\Phi\rangle} &= \int_\Phi (\Delta P_\gamma^\xi y)(x) u(x) dx \\ &= \sum_{m=1}^M \int_{\omega_m} (\Delta P_\gamma^\xi y)(x) u(x) dx = \sum_{m=1}^M \int_0^{\delta_\Phi} (\Delta P_\gamma^\xi y)(x(r)) u(x(r)) dr \\ &= \int_0^{\delta_\Phi} \big\langle \eta_1(r)\beta^{T,1} + \eta_2(r)\beta^{T,2}, \theta(r)\beta^{T,1}\big\rangle_{\mathbb{R}^M} dr = \int_0^{\delta_\Phi} \eta_1(r)\theta(r) dr\end{aligned}$$

by normalization of $\beta^{T,i}$. Since θ can be arbitrary, we conclude that $\eta_1 = 0$. Hence, $\Delta P_\gamma^\xi y\big|_{\Lambda_\gamma^T[x]}$ has to be proportional to $\beta^{T,2}$ and we easily get

$$\big(\Delta P_\gamma^\xi y\big)(x) = \Delta\chi^\xi(x)\big\langle y|_{\Lambda_\gamma^T[x]}, \beta^{T,2}\big\rangle \beta^{T,2}(x), \qquad x \in \Phi.$$

Correspondingly,

$$\big(\xi \Delta P_\gamma^\xi y\big)(x) \approx \tau_2(x)\Delta\chi^\xi(x)\big\langle y|_{\Lambda_\gamma^T[x]}, \beta^{T,2}\big\rangle \beta^{T,2}(x), \qquad x \in \Phi.$$

Summing up such terms and passing to the limit, we obtain

$$\left(\int_{T_2}^{T_2+\delta_\Phi} \xi \, dP_\gamma^\xi y\right)(x) = \tau_2(x)\big\langle y|_{\Lambda_\gamma^T[x]}, \beta^{T,2}\big\rangle \beta^{T,2}(x), \qquad x \in \Phi.$$

- Going on in the same way, we easily get

$$\left(\int_{T_i}^{T_i+\delta_\Phi} \xi\, dP^\xi_\gamma y\right)(x) = \tau_i(x)\big\langle y|_{\Lambda^T_\gamma[x]}, \beta^{T,i}\big\rangle \beta^{T,i}(x), \qquad x \in \Phi,$$

whereas (3.5) leads to the representation

$$\big(E^T_\gamma y\big)(x) = \sum_{i=1}^N \tau_i(x)\big\langle y|_{\Lambda^T_\gamma[x]}, \beta^{T,i}\big\rangle \beta^{T,i}(x), \qquad x \in \Phi. \tag{3.6}$$

So, the eikonal projects functions $y \in \mathcal{H}\langle\Phi\rangle$ on amplitude vectors and multiplies by relevant distances.

3.1.4. Eikonal in parametric form.

Parametrize Φ by (2.33), (2.40): $x_m(r) \in \omega_m \subset \Phi$, $m = 1, 2, \ldots, M$, $0 < r < \delta_\Phi$. Denote

$$\vec{y}(r) := \begin{pmatrix} y(x_1(r)) \\ \ldots \\ y(x_M(r)) \end{pmatrix}, \quad B_\Phi := \begin{pmatrix} \beta^{T,1}_1 & \ldots & \beta^{T,1}_M \\ \beta^{T,2}_1 & \ldots & \beta^{T,2}_M \\ \ldots & \ldots & \ldots \\ \beta^{T,N}_1 & \ldots & \beta^{T,N}_M \end{pmatrix},$$

$$D_\Phi(r) = \{\tau_i(r)\delta_{ij}\}^N_{i,j=1},$$

where $\tau_i(r) := \tau_i(x(r))$ is either $T_i + r$ or $T_i + \delta_\Phi - r$ (see (3.3)). Note that $B^*_\Phi B_\Phi$ is the matrix of the projection p_Φ in $\mathbb{R}^M$ onto $\mathbb{A}_\Phi = \operatorname{span}\{\beta^{T,1}, \ldots, \beta^{T,N}\}$. In this notation, (3.6) takes the form

$$(\overrightarrow{E^T_\gamma y})(r) = [B^*_\Phi D_\Phi(r) B_\Phi]\vec{y}(r), \qquad r \in (0, \delta_\Phi). \tag{3.7}$$

Recalling the decomposition (2.38), we conclude that it reduces the eikonal and the representation

$$E^T_\gamma = \bigoplus \sum_{\Phi \in \Pi^T_\gamma} E^T_\gamma \chi_\Phi \tag{3.8}$$

holds, where χ_Φ is understood as an operator in $\mathcal{H}$ multiplying by the indicator. Each part $E^T_\gamma \chi_\Phi$ acts in $\mathcal{H}\langle\Phi\rangle$ by (3.6) (by (3.7) in the parametrized form).

Note in addition that (3.7) and (3.8) in fact provide a canonical representation (*diagonalization*) of the eikonal in the sense of the Spectral Theorem for self-adjoint operators: see, e.g., [**11**]. Also, the eikonal is reduced by the subspaces $\mathcal{U}^T_\gamma$ and $[\mathcal{U}^T_\gamma]^\perp = \mathcal{H}\langle\Omega^T[\gamma]\rangle \ominus \mathcal{U}^T_\gamma$, and annuls $[\mathcal{U}^T_\gamma]^\perp$. As one can show, the spectrum of its part $E^T_\gamma\big|_{\mathcal{U}^T_\gamma}$ is simple (of multiplicity 1) and absolutely continuous.

3.2. Algebra $\mathfrak{E}^T_\Sigma$.

3.2.1. Definition.

Let $\mathfrak{B}(\mathcal{H})$ be the normed algebra of bounded operators in $\mathcal{H}$. An *operator C^*-algebra* is a norm-closed subalgebra of $\mathfrak{B}(\mathcal{H})$ invariant with respect to the operator conjugation: see [**12**], [**15**].

Let $\Sigma \subseteq \Gamma$ be a subset of boundary vertices. For controls

$$f \in \mathcal{F}^T_\Sigma := \bigoplus \sum_{\gamma \in \Sigma} \mathcal{F}^T_\gamma,$$

the waves $u^f(\,\cdot\,, T)$ are supported in the metric neighborhood $\overline{\Omega^T[\Sigma]} \subset \Omega$ (see (2.18)). These waves constitute a reachable set

$$\mathcal{U}^T_\Sigma = \sum_{\gamma \in \Sigma} \mathcal{U}^T_\gamma \subset \mathcal{H}\langle \Omega^T[\Sigma] \rangle$$

(algebraic sum).

With each $\gamma \in \Sigma$ one associates the eikonal $E^T_\gamma \in \mathfrak{B}(\mathcal{H}\langle \Omega^T[\Sigma] \rangle)$. By $\mathfrak{E}^T_\Sigma$ we denote the *C^*-subalgebra of $\mathfrak{B}(\mathcal{H}\langle \Omega^T[\Sigma] \rangle)$ generated by eikonals* $\{E^T_\gamma\}_{\gamma \in \Sigma}$, i.e., the minimal C^*-subalgebra in $\mathfrak{B}(\mathcal{H}\langle \Omega^T[\Sigma] \rangle)$, which contains all these eikonals.

Our paper is written for the sake of introducing algebra $\mathfrak{E}^T_\Sigma$. It is defined by perfect analogy with the eikonal algebras associated with Riemannian manifolds: see [**7**], [**8**]. In the rest of the paper, we clarify the structure of $\mathfrak{E}^T_\Sigma$.

3.2.2. Partition Π^T_Σ.

The set

$$H^T_\Sigma := \bigcup_{\gamma \in \Sigma} H^T_\gamma \subset \Omega \times [0, T]$$

is also said to be a *hydra*. It is also a space-time graph.

The analogs of the objects, which were defined for H^T_γ, are introduced for H^T_Σ as follows.

- The projections $\pi : (x, t) \mapsto x$ and $\rho : (x, t) \mapsto t$ are now understood as the maps from H^T_Σ to Ω and $[0, T]$ respectively.
- An *amplitude* on H^T_Σ is

$$a(x, t) := \sum_{\gamma \in \Sigma : (x,t) \in H^T_\gamma} a_\gamma(x, t),$$

  where a_γ is the amplitude on H^T_γ.
- The equivalence $l' \cong l''$ and lattices $\mathcal{L}[B]$ are defined as in 2.4.2, just replacing H^T_γ by H^T_Σ.
- The set Corn H^T_Σ of *corner points* is defined as in 2.4.4, replacing H^T_γ by H^T_Σ. The evident relation

$$\operatorname{Corn} H^T_\Sigma \supset \bigcup_{\gamma\in\Sigma} \operatorname{Corn} H^T_\gamma$$

holds. However, the latter sum can be smaller than $\operatorname{Corn} H^T_\Sigma$ because additional corner points on H^T_Σ do appear owing to intersection of the space-time edges of H^T_γ with edges of $H^T_{\gamma'}$ for different $\gamma, \gamma' \in \Sigma$. It happens as $T > (1/2)\tau(\gamma,\gamma')$.

- *Critical points* are introduced in the same way as (2.28): they constitute a finite set

$$\Theta^T_\Sigma := \pi\big(\mathcal{L}[\operatorname{Corn} H^T_\Sigma]\big) \supset \bigcup_{\gamma\in\Sigma} \Theta^T_\gamma.$$

- Fix an $x \in \overline{\Omega^T[\Sigma]}\backslash\Theta^T_\Sigma$; let $]c, c'[\, \ni x$ be an open interval in the graph between the critical points c, c', which contains no critical points. By analogy with (2.25), define a *determination set*

$$\Lambda^T_\Sigma[x] := \pi\big(\mathcal{L}[\pi^{-1}(x)]\big) \ni x$$

(here $\mathcal{L}$ is a lattice on H^T_Σ!). As is evident, one has $\Lambda^T_\Sigma[x] \supset \Lambda^T_\gamma[x]$ as $\gamma \in \Sigma$. If x runs over $]c, c'[$ from c to c', the set $\Lambda^T_\Sigma[x]$ sweeps a *family* $\Phi = \bigcup_{m=1}^M \omega_m$ of open intervals ω_m (*cells*) of the same length δ_Φ.

By the aforesaid and analogy with (2.32), we have the representation

$$\overline{\Omega^T[\Sigma]}\backslash\Theta^T_\Sigma = \bigcup_{j=1}^J \Phi^j = \bigcup_{j=1}^J \bigcup_{m=1}^{M_j} \omega^{(j)}_m \tag{3.9}$$

in the form of disjoint sums, where $\operatorname{diam}\omega^{(j)}_1 = \cdots = \operatorname{diam}\omega^{(j)}_{M_j} = \delta_{\Phi^j}$. This representation is referred to as a *partition* Π^T_Σ.

Figure 8 illustrates the case $\Sigma = \{\gamma, \gamma'\}$ for $T = \tau(\gamma,\gamma') + \varepsilon$:

- $\operatorname{Corn} H^T_\Sigma$ is the black points with small holes at the center, $\mathcal{L}[\operatorname{Corn} H^T_\Sigma]$ is $\operatorname{Corn} H^T_\Sigma$ plus the grey points
- the critical points $\Theta^T_\Sigma = \bigcup_{k=1}^{16} c_k$, ($c_1 = \gamma$, $c_5 = v$, $c_{13} = \gamma'$) are denoted by $c_k \equiv k$
- the families and cells are

$$\Phi^1 = \bigcup_{m=1}^2 \omega^{(1)}_m \text{ (dashed)}, \quad \Phi^2 = \bigcup_{m=1}^8 \omega^{(2)}_m \text{ (grey)}, \quad \Phi^3 = \bigcup_{m=1}^5 \omega^{(3)}_m \text{ (black)}$$

- the neighborhoods $\Omega^T[\gamma]$ and $\Omega^T[\gamma']$ filled with waves are contoured with the dashed lines.

3.2.3. Functions $\tau^{\gamma,\Phi}_i$ and system $\breve{\beta}^T_{\gamma,\Phi}$.

Choose a family $\Phi = \bigcup_{m=1}^M \omega_m$, which is an element of the partition Π^T_Σ. Quite analogously to (3.2), the set

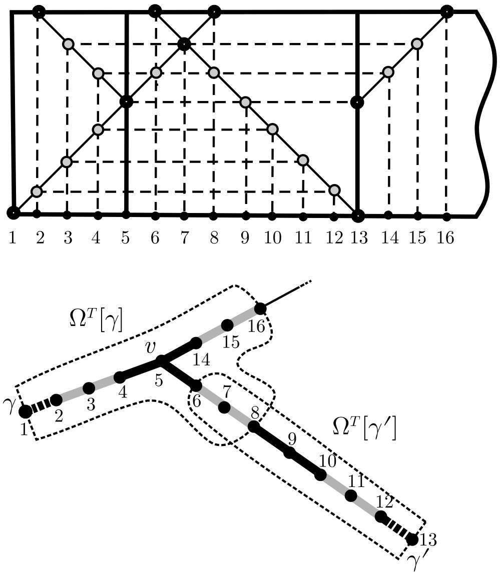

Figure 8. Partition Π^T_Σ.

$$\rho\big(\mathcal{L}[\pi^{-1}(\Phi)]\big) = \bigcup_{i=1}^{N} I_i \tag{3.10}$$

consists of $N = N(\Phi)$ disjoined open intervals $I_i = (T_i, T_i + \delta_\Phi) \subset (0, T)$ of the same length δ_Φ. We number them so that $0 \le T_1 < T_2 < \cdots T_N < T_N + \delta_\Phi \le T$.

Fix a $\gamma \in \Sigma$. With representation (3.10) one associates N functions on Φ of the form

$$\begin{aligned}\tau_i^{\gamma,\Phi}(x) &:= \begin{cases} I_i \cap \rho\big(\pi^{-1}(x) \cap H^T_\gamma\big) & \text{if } I_i \cap \rho\big(\pi^{-1}(x) \cap H^T_\gamma\big) \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} t & \text{if } (x,t) \in \rho^{-1}(I_i) \cap H^T_\gamma \\ 0 & \text{otherwise} \end{cases}.\end{aligned} \tag{3.11}$$

As is easy to recognize, these functions are just a version of the functions (3.3), the version being relevant to partition Π^T_Σ.

For the family $\Phi \in \Pi^T_\Sigma$, a relevant version of the system $\check{\beta}^T$ (see (2.42)) is constructed as follows:

- For each $i = 1, \dots, N$, take a $t_i \in I_i$ and choose an $x \in \pi(\rho^{-1}(t_i)) \subset \Lambda^T_\Sigma[x] = \{x_1, \dots, x_M\}$, where $x_m \in \omega_m$. Define a vector $\alpha^i_\gamma \in \boldsymbol{l}_2(\Lambda^T_\Sigma[x])$ by

$$\alpha^i_\gamma(x_m) := \begin{cases} a_\gamma(x_m, t_i) & \text{as } (x_m, t_i) \in H^T_\gamma \\ 0 & \text{otherwise} \end{cases}.$$

- Redesign the system $\alpha^1_\gamma, \dots, \alpha^N_\gamma$ by the Schmidt process (see (2.42)) and get the system $\breve{\beta}^T_{\gamma,\Phi} := \{\beta^{T,1}_{\gamma,\Phi}, \dots, \beta^{T,N}_{\gamma,\Phi}\}$. The amplitude subspace is

$$\mathcal{A}^T_{\gamma,\Phi} := \operatorname{span} \breve{\beta}^T_{\gamma,\Phi} \subset \boldsymbol{l}_2(\Lambda^T_\Sigma[x]).$$

  Also, with each vector $\beta^{T,i}_{\gamma,\Phi}$ we associate a piece-wise constant function on Φ of the form

$$\beta^{T,i}_{\gamma,\Phi}(x) := \big(\beta^{T,i}_{\gamma,\Phi}\big)_m, \qquad x \in \omega_m \subset \Phi \tag{3.12}$$

  (see (2.43)).
- The total amplitude subspace corresponding to the family $\Phi \in \Pi^T_\Sigma$ is

$$\mathcal{A}^T_\Phi = \sum_{\gamma\in\Sigma} \mathcal{A}^T_{\gamma,\Phi} = \operatorname{span}\big\{\breve{\beta}^T_{\gamma,\Phi} \mid \gamma \in \Sigma\big\} \subset \boldsymbol{l}_2(\Lambda^T_\Sigma[x]).$$

3.2.4. Projections and eikonals.

Recall that P^T_γ projects in $\mathcal{H}$ onto $\mathcal{U}^T_\gamma$. Repeating the arguments, which have led to the representations (2.39) and (2.44), one can modify them to the following form relevant to the complete hydra H^T_Σ:

$$P^T_\gamma = \bigoplus \sum_{\Phi\in\Pi^T_\Sigma} Q^\gamma_\Phi, \tag{3.13}$$

where

$$\big(Q^\gamma_\Phi y\big)(x) = \begin{cases} \sum\limits_{i=1}^{N(\Phi)} \big\langle y|_{\Lambda^T_\Sigma[x]},\, \beta^{T,i}_{\gamma,\Phi}\big\rangle \beta^{T,i}_{\gamma,\Phi}(x), & x \in \Phi \\ 0, & x \in \Omega^T[\Sigma]\backslash\Phi \end{cases}. \tag{3.14}$$

Quite analogously, the relevant version of the representations (3.6) and (3.8) takes the form

$$E^T_\gamma = \bigoplus \sum_{\Phi\in\Pi^T_\Sigma} E^T_\gamma \chi_\Phi, \tag{3.15}$$

where

$$\big(E^T_\gamma \chi_\Phi y\big)(x) = \begin{cases} \sum\limits_{i=1}^{N(\Phi)} \tau^{\gamma,\Phi}_i(x) \big\langle y|_{\Lambda^T_\Sigma[x]}, \beta^{T,i}_{\gamma,\Phi}\big\rangle \beta^{T,i}_{\gamma,\Phi}(x), & x \in \Phi \\ 0, & x \in \Omega^T[\Sigma]\backslash\Phi \end{cases}. \tag{3.16}$$

Fix a $\gamma \in \Sigma$ and choose a family $\Phi = \bigcup_{m=1}^M \omega_m \in \Pi^T_\Sigma$; recall that the number N is defined in (3.10). Parametrize Φ by (2.33), (2.40): $x_m(r) \in \omega_m$, $m = 1, 2, \dots, M$,

$0 < r < \delta_\Phi$. Denote

$$\vec{y}(r) := \begin{pmatrix} y(x_1(r)) \\ \dots \\ y(x_M(r)) \end{pmatrix}, \quad B_{\gamma,\Phi} := \begin{pmatrix} (\beta^{T,1}_{\gamma,\Phi})_1 & \dots & (\beta^{T,1}_{\gamma,\Phi})_M \\ (\beta^{T,2}_{\gamma,\Phi})_1 & \dots & (\beta^{T,2}_{\gamma,\Phi})_M \\ \dots & \dots & \dots \\ (\beta^{T,N}_{\gamma,\Phi})_1 & \dots & (\beta^{T,N}_{\gamma,\Phi})_M \end{pmatrix},$$

$$D_{\gamma,\Phi}(r) = \left\{D^{ij}_{\gamma,\Phi}(r)\right\}^N_{i,j=1}: \quad D^{ij}_{\gamma,\Phi}(r) = \tau^{\gamma,\Phi}_i(r)\delta_{ij},$$

where $\tau^{\gamma,\Phi}_i(r) := \tau^{\gamma,\Phi}_i(x(r))$ is either $T_i + r$ or $T_i + \delta_\Phi - r$. Note that $B^*_{\gamma,\Phi}B_{\gamma,\Phi}$ is the matrix of the projection $p_{\gamma,\Phi}$ in $\mathbb{R}^M$ onto $\mathbb{A}^T_{\gamma,\Phi} = \operatorname{span}\check{\beta}^T_{\gamma,\Phi} \subset \mathbb{A}^T_\Phi$. In this notation, the first line in the right hand side of (3.16) takes the form

$$\left(\overrightarrow{E^T_\gamma y}\right)(r) = [B^*_{\gamma,\Phi}D_{\gamma,\Phi}(r)B_{\gamma,\Phi}]\vec{y}(r), \quad r \in (0, \delta_\Phi) \ \ (\gamma \in \Sigma), \tag{3.17}$$

which is just the relevant form of (3.7).

A key feature of the representations (3.16) and (3.17) is the following. They represent eikonals E^T_γ in the form, which is the same for all the vertices $\gamma \in \Sigma$ and available for any family Φ of partition Π^T_Σ of the complete hydra H^T_Σ.

3.2.5. Block algebras. Structure of $\mathfrak{E}^T_\Sigma$.

In accordance with (3.9), we have the decomposition

$$\mathcal{H}\langle\Omega^T[\Sigma]\rangle = \bigoplus \sum_{\Phi\in\Pi^T_\Sigma} \mathcal{H}\langle\Phi\rangle, \tag{3.18}$$

which reduces each eikonal E^T_γ for $\gamma \in \Sigma$:

$$E^T_\gamma = \bigoplus \sum_{\Phi\in\Pi^T_\Sigma} E^T_\gamma\Big|_{\mathcal{H}\langle\Phi\rangle} \tag{3.19}$$

(compare with (2.38) and (3.8)). Each part $E^T_\gamma|_{\mathcal{H}\langle\Phi\rangle}$ acts in the subspace $\mathcal{H}\langle\Phi\rangle$ by (3.16) or, equivalently, by (3.17) in the parametrized form.

Let $\mathfrak{b}^T_\Phi \subset \mathfrak{B}(\mathcal{H}\langle\Phi\rangle)$ be the C^*-subalgebra generated by the system $\{E^T_\gamma|_{\mathcal{H}\langle\Phi\rangle} \mid \gamma \in \Sigma\}$ of the eikonal parts. We say $\mathfrak{b}^T_\Phi$ to be a *block algebra*.

By (3.17), each $\mathfrak{b}^T_\Phi$ is isometrically isomorphic to the subalgebra $\tilde{\mathfrak{b}}^T_\Phi \subset \mathfrak{B}(L_2((0,\delta_\Phi);\mathbb{R}^{M(\Phi)}))$ generated by the operators, which multiply elements (vector-functions) $\vec{y}(\cdot)$ by the matrix-functions $B^*_{\gamma,\Phi}D_{\gamma,\Phi}(\cdot)B_{\gamma,\Phi}$ $(\gamma \in \Sigma)$. These functions are continuous[4], and hence we have

$$\tilde{\mathfrak{b}}^T_\Phi \subset C\big([0,\delta_\Phi];\mathbb{M}^{M(\Phi)}\big), \tag{3.20}$$

[4]Moreover, the matrix elements are the linear functions of $r \in [0, \delta_\Phi]$.

where the algebra in the right hand side is understood as the subalgebra of $\mathfrak{B}(L_2((0,\delta_\Phi);\mathbb{R}^{M(\Phi)}))$, whose elements multiply $\vec{y} \in L_2((0,\delta_\Phi);\mathbb{R}^{M(\Phi)})$ by real continuous $M(\Phi)\times M(\Phi)$-matrix-valued functions on $[0,\delta_\Phi]$.

Just summarizing these considerations, we arrive at the main result of the paper: the decomposition (3.18) reduces eikonal algebra $\mathfrak{E}^T_\Sigma$, and the representation

$$\mathfrak{E}^T_\Sigma = \bigoplus \sum_{\Phi\in\Pi^T_\Sigma} \mathfrak{b}^T_\Phi \tag{3.21}$$

holds.

3.3. Discussion.

3.3.1. Noncommutativity.

As we noted in Introduction, algebra $\mathfrak{E}^T_\Sigma$ is noncommutative. The reason is that the matrix projections $p_{\gamma,\Phi}$ (in $\mathbb{R}^{M(\Phi)}$ onto $\mathbb{A}^T_{\gamma,\Phi}$) corresponding to different $\gamma\in\Sigma$ do not have to commute. As a result, eikonal parts $E^T_\gamma|_{\mathcal{H}\langle\Phi\rangle}$ and $E^T_{\gamma'}|_{\mathcal{H}\langle\Phi\rangle}$ do not commute as $\gamma\neq\gamma'$, so that the block-algebra $\mathfrak{b}^T_\Phi$ turns out to be noncommutative.

Moreover, the eikonal algebra on a graph is, in a sense, *strongly* noncommutative. We mean the following. In the Maxwell dynamical system on a Riemannian manifold, a straightforward analog of $\mathfrak{E}^T_\Sigma$ is also a noncommutative algebra but its factor over the ideal of compact operators turns out to be commutative [**8**] (and, moreover, isometric to the algebra $C(\Omega)$). This may be referred to as a *weak* noncommutativity. On graphs, it is definitely not the case: simple examples show that no factorization eliminates noncommutativity of a generic block-algebra.

3.3.2. Spectrum.

A *spectrum* $\widehat{\mathfrak{E}}^T_\Sigma$ of the algebra $\mathfrak{E}^T_\Sigma$ is the set of its irreducible representations endowed with the Jacobson topology (see [**12**], [**15**]). By (3.21), one has

$$\widehat{\mathfrak{E}}^T_\Sigma = \bigcup_{\Phi\in\Pi^T_\Sigma} \widehat{\mathfrak{b}}^T_\Phi,$$

so that to study a structure of $\widehat{\mathfrak{E}}^T_\Sigma$ is to analyze $\widehat{\mathfrak{b}}^T_\Phi$. In the general case of arbitrary graph, such an analysis is an open and difficult problem.

In particular, noncommutativity implies that the (topologized) spectrum $\widehat{\mathfrak{b}}^T_\Phi$ is not necessarily a Hausdorff space: it may contain the clusters. We say a subset $\boldsymbol{c}\subset\widehat{\mathfrak{b}}^T_\Phi$ to be a *cluster*, if its points are not separable, i.e., for any point $p\in\boldsymbol{c}$ and arbitrary neighborhood $U\ni p$ one has $\boldsymbol{c}\subset U$. In examples, clusters do appear as a consequence of Kirchhoff laws (2.5). By these laws, the amplitude vectors, which correspond to different families Φ,Φ' are not quite independent.

3.3.3. An example.

Here we discuss some "experimental material" provided by examples on simple graphs. The discussion is short since we plan to devote a separate paper to these examples. Here we just announce some preliminary results. The example under consideration is, probably, the simplest one.

Let Ω be a 3-star with the edges e_1, e_2, e_3, boundary $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$, and interior vertex v. Let all e_i be of the same length 1. We put $T = 2$ and, in what follows, omit the superscript T in notation.

Let $\mathfrak{E}_\Gamma$ be the algebra generated by the eikonals $E_{\gamma_1}, E_{\gamma_2}, E_{\gamma_3}$ in $\mathfrak{B}(L_2(\Omega))$. As one can show, there is an isometry $I : \mathfrak{E}_\Gamma \to C([0,1]; \mathbb{M}^3)$ such that

$$IE_{\gamma_1} = \begin{pmatrix} 1-r & 0 & 0 \\ 0 & \frac{1+r}{2} & \frac{1+r}{2} \\ 0 & \frac{1+r}{2} & \frac{1+r}{2} \end{pmatrix}, \quad IE_{\gamma_2} = \begin{pmatrix} \frac{1+r}{2} & 0 & \frac{1+r}{2} \\ 0 & 1-r & 0 \\ \frac{1+r}{2} & 0 & \frac{1+r}{2} \end{pmatrix},$$

$$IE_{\gamma_3} = \begin{pmatrix} \frac{1+r}{2} & \frac{1+r}{2} & 0 \\ \frac{1+r}{2} & \frac{1+r}{2} & 0 \\ 0 & 0 & 1-r \end{pmatrix},$$

where $r \in [0,1]$ [5], whereas the image of the eikonal algebra is

$$I\mathfrak{E}_\Gamma = \left\{ b \in C([0,1]; \mathbb{M}^3) \;\middle|\; \exists \lambda \in \mathbb{R} : b(0) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}. \tag{3.22}$$

With each $r \in (0,1]$ one associates a 3-dimensional irreducible representation

$$I\mathfrak{E}_\Gamma \ni b \overset{\varkappa^3(r)}{\longmapsto} b(r) \in \mathbb{M}^3.$$

Also, there are two irreducible representations associated with $r = 0$. To describe them, introduce the matrix

$$u := \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix},$$

and denote $\tilde{b}(0) = ub(0)u^*$. Then

$$\varkappa_0^1 : b \mapsto \tilde{b}_{11}(0) \in \mathbb{M}^1 = \mathbb{R} \quad \text{and} \quad \varkappa_0^2 : b \mapsto \begin{pmatrix} \tilde{b}_{22}(0) & \tilde{b}_{23}(0) \\ \tilde{b}_{32}(0) & \tilde{b}_{33}(0) \end{pmatrix} \in \mathbb{M}^2$$

[5] r is the distance from v to a point in Ω: see (2.33).

provide the 1- and 2-dimensional representations respectively. As a result, the spectrum

$$\widehat{\mathfrak{E}_\Gamma} = \bigcup_{r\in(0,1]} \varkappa^3(r) \bigcup \varkappa_0^1 \bigcup \varkappa_0^2 \tag{3.23}$$

can be identified with the semi-segment $0 < r \leq 1$ joined with the split point (cluster) $r = 0$, which consists of two points 0_1 and 0_2 corresponding to $\varkappa_0^1$ and $\varkappa_0^2$ respectively.

Analysis shows that the condition in (3.22), which specifies elements b at $r = 0$, appears as a consequence of the Kirchhoff law (2.5) at the interior vertex v. Also, note that the "massive part" of the spectrum (3.23) consists of the elements $\varkappa^3(r)$ corresponding to the points of Ω. Surely, this example is too degenerate. However, it demonstrates that the structure of $\widehat{\mathfrak{E}}^T_\Sigma$ is correlated with the graph geometry.

3.4. Comments.

- In the known examples, the clusters in $\widehat{\mathfrak{E}}^T_\Sigma$ do appear if $\Omega[\gamma] \cap \Omega[\gamma'] \ni v$ occurs for the different $\gamma, \gamma' \in \Sigma$ and an interior vertex $v \in V$. Therefore, it would be reasonable to suggest that the number of interior vertices n_V and the number of clusters $n_{\boldsymbol{c}}$ are related through an inequality: presumably, $n_V \geq n_{\boldsymbol{c}}$ holds. Since $n_{\boldsymbol{c}}$ is a topological invariant of spectrum $\widehat{\mathfrak{E}}^T_\Sigma$, this relation could be helpful in inverse problems on graphs, in which the inverse data do determine the eikonal algebra $\mathfrak{E}^T_\Sigma$ up to an isometric isomorphism (see [**6**]–[**10**]). Therefore, the data determine the spectrum $\widehat{\mathfrak{E}}^T_\Sigma$ up to a homeomorphism, and the external observer, which possesses the data, can hope for getting information about the graph from the spectrum.
- An intriguing question is whether another geometric characteristics of the graph (number of edges and cycles, valency of interior vertices, etc) are related with topological invariants of the spectrum $\widehat{\mathfrak{E}}^T_\Sigma$. A prospective (but rather far) goal is *to recover the graph* from the boundary inverse data via its eikonal algebra.

Hopefully, our approach relates inverse problems on graphs with the C^*-algebra theory. The answers on the above posed questions could confirm productivity of these relations.

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