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Adjunction and singular loci of hyperplane sections

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Abstract. Let (X, L) be a smooth polarized variety of dimension n. Let $A \in |L|$ be an effective irreducible divisor, and let Σ be the singular locus of A. We assume that Σ is a smooth subvariety of dimension $k \geq 2$, and codimension $c \geq 3$, consisting of non-degenerate quadratic singularities. We study positivity conditions for adjoint bundles $K_X + tL$ with $t \geq n-3$. Several explicit examples motivate the discussion.

Introduction.

Let (X, L) be a smooth polarized variety of dimension n. Let $A \in |L|$ be an effective irreducible divisor. Let $\Sigma = \mathrm{Sing}(A)$ be the singular locus of A. We assume that Σ is a smooth subvariety of dimension $k \geq 2$, consisting of non-degenerate quadratic singularities. For instance, in the study of projective manifolds with degenerate dual variety, the defect and the tangency locus of a general tangent hyperplane section provide examples for the role of k and of Σ respectively. In that specific case, however, (Σ, L_Σ) is forced to be $(\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(1))$, according to a classical result of Bertini [**4**, Chapter 9, Number 13, p. 200]. We set $c := \mathrm{codim}_X(\Sigma)$ and we further assume that $c \geq 3$ (whence, in particular, $n \geq 5$).

A good motivation to consider this setting comes e.g. from the results of [**5**], [**2**] and [**12**]. In these articles, under suitable conditions on some basic projective invariants, the geometry of a polarized pair (X, L) is studied in the case when the divisor A is reducible. Precisely, A is union of $r \geq 2$ smooth normal crossing divisors $A_1, \ldots, A_r$. In our setting, forgetting the irreducibility of A, this situation would correspond to $c = 2$. By the way, let us just mention that $c = 1$ would correspond to the case of a non-reduced divisor A.

Our aim is to study positivity conditions for adjoint bundles $K_X + tL$ with $t \geq n-3$. We follow (as done in [**5**], [**2**]) the adjunction theory approach [**1**]. To this purpose, let us present an explicit example to suggest the connection between the singular loci of ample hypersurfaces we are dealing with in the paper and adjunction theory.

Let $X \subset \mathbb{P}^r$ be a smooth fivefold, let $L = (\mathcal{O}_{\mathbb{P}^r}(1))_X$, and suppose that $|L|$ contains an irreducible hypersurface A whose singular locus $\Sigma = \mathrm{Sing}(A)$ consists of non-degenerate quadratic singularities and is isomorphic to the projective plane $\mathbb{P}^2$. Let N be the normal bundle of Σ in X. By adapting a result of Ein we produce an isomorphism $N \cong N^* \otimes L_\Sigma$, which implies the relation $2 \det N = 3L_\Sigma$. Therefore L_Σ is divisible by 2 in the Picard group. Suppose that $L_\Sigma = \mathcal{O}_{\mathbb{P}^2}(2)$. Then $K_\Sigma = -(3/2)L_\Sigma$. We thus see, by adjunction, that $K_X + 3L$ restricts trivially to Σ, hence it fails to be ample. Moreover,

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$K_X + (3 - \epsilon)L$ is not nef for any $\epsilon > 0$. In other words, our fivefold (X, L) fits into the range of polarized manifolds whose nefvalue τ is $\geq 3 = \dim(X) - 2$. Then, by applying [**1**, Chapter 7], we get a short list of possibilities; moreover, a case-by-case analysis shows that in fact $\tau = \dim(X) - 2$, which makes the list very short. E.g., the "Mukai fivefold" $(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(2))$ offers the concrete example where A is a quadric hypersurface of rank 3.

Here is an outline of the paper. In Section 1 we discuss a generalization of a result due to Ein (see [**6**, Theorem 2.2], [**11**, Theorem 1.2]) on which the article is based. Such a result implies the key property $N \cong N^* \otimes L_\Sigma$ mentioned in the above example. In particular, it turns out that if Σ contains a line ℓ with respect to L_Σ, then c must be even (a generalization of Landman's parity theorem).

In Section 2 we discuss a consequence of the key result above, which says that if Σ contains a line ℓ with respect to L such that $N_{\ell/\Sigma}$ is ample, then $(\Sigma, L_\Sigma) = (\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(1))$ and (X, L) is covered by lines. Moreover, we provide several examples enlightening the setting we are dealing with.

Section 3 is devoted to the general case, with no assumptions on the parity of c. We prove in Theorem 3.1 that $K_X + (n-1)L$ is nef and big unless, possibly, $c = 4$ and $(\Sigma, L_\Sigma) = (\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(1))$. Moreover, Σ maps down isomorphically under the first reduction map, $\varphi : X \to X'$, of (X, L). We also show that in the special remarkable case in which $(\Sigma, L_\Sigma) = (\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(1))$ and L is very ample the nefvalue morphism of (X, L) is a Mori contraction which maps Σ to a point.

In Section 4 we assume c to be odd. From the generalized Ein result recalled above it thus follows the crucial fact that Σ does not contain 1-cycles of odd degree with respect to L (see Corollary 1.2). In Theorem 4.1, we prove that there exists the second reduction $(\widehat{X}, \mathscr{D})$, $\psi : X' \to \widehat{X}$, of (X, L). Moreover, Σ meets no exceptional divisors of ψ pulled-back to X via the first reduction morphism $\varphi : X \to X'$. In Theorem 4.4 we show that the third adjoint bundle $K_{\widehat{X}} + (n-3)\mathscr{D}$ is nef unless $(\widehat{X}, \mathscr{D}) = (\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2))$, in which case, necessarily, $(\Sigma, L_\Sigma) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$. Furthermore, $K_{\widehat{X}} + (n-3)\mathscr{D}$ is also big, provided that $k \geq 4$.

Finally, it is worth to recall a classical bound for the maximum number of double points. Let X be a reduced irreducible hypersurface in $\mathbb{P}^{n+1}$ of degree d with finitely many singular points, all of which are non-degenerate quadratic singularities. Let δ be their number. It is then a classical result (see [**15**] and the recent nice survey paper [**7**]) that, for any $d \geq n$,

$$\delta \leq \frac{1}{2} d\big((d-1)^n - 1\big).$$

We also recall the best known asymptotical bound for the number δ for a surface of degree $d \geq 4$ in $\mathbb{P}^3$,

$$\delta \leq \frac{4}{9} d(d-1)^2,$$

due to Miyaoka [**13**] (see also [**14**, p. 164]).

Notation and terminology.

We work on the complex field $\mathbb{C}$ and use the standard terminology in algebraic geometry.

In particular, we often use the additive notation for the tensor product of line bundles on a projective variety X. Moreover, for any $\mathbb{Q}$-line bundle $\mathcal{L}$ on X we denote by $\mathcal{L}_V$ the restriction of $\mathcal{L}$ to a subvariety V of X, and by K_X the canonical bundle of X, for X smooth. For any morphism $f : X \to W$ we denote by $f|_Y$ the restriction of f to Y. We will denote by $\equiv$ the numerical equivalence of line bundles.

For all adjunction theoretic terminology (in particular for the notions of scrolls, quadric fibrations, special varieties, reductions, nefvalue, nefvalue morphisms) and results used throughout the paper we refer to [**1**].

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1. Non-degenerate quadratic singularities and generalized Ein's theorem.

Let X be a smooth complex projective variety of dimension $n \geq 2$ and let L be a line bundle on X. Assume that the complete linear system $|L|$ contains an irreducible reduced divisor A and let s be a section of L defining A. We say that a point $x \in A$ is an *isolated non-degenerate quadratic singularity* if, with local coordinates $x_1, \dots, x_n$, around x, s can be written in the form

$$s = \sum_{i,j} a_{ij} x_i x_j + (\text{degree} > 2 \text{ terms}),$$

where $a_{ij} = a_{ji}$ and the Hessian matrix (a_{ij}) satisfies $\det(a_{ij}) \neq 0$. Now let P be a smooth subvariety of A of dimension $k > 0$. We say that P is a *locus of non-degenerate quadratic singularities* of A if for every $x \in P$ there exist k smooth hypersurfaces $H_1, \dots, H_k$ of X meeting transversally along a submanifold $Y \subset X$ such that $x = P \cap Y$ is an isolated non-degenerate quadratic singularity of $A \cap Y$. This can be rephrased by requiring that for any point $x \in P$, with an appropriate choice of local affine coordinates $(u_1, \dots, u_n)$ on X around x (such that P is defined by $u_{k+1} = \dots = u_n = 0$), A is described by

$$u_{k+1}^2 + u_{k+2}^2 + \dots + u_n^2 = 0.$$

For instance, if L is the hyperplane line bundle of $X \subset \mathbb{P}^r$ and $A \in |L|$ is a general tangent hyperplane section, then its contact locus is a locus of non-degenerate quadratic singularities.

For a smooth projective variety $X \subset \mathbb{P}^r$, whose general contact locus P is positive dimensional, Ein discovered a key property of the normal bundle $N_{P/X}$ in connection with the hyperplane bundle [**6**, Theorem 2.2]. More generally, the same property holds in the line bundle setting for the normal bundle of any smooth locus of non-degenerate quadratic singularities.

Theorem 1.1 (Generalized Ein Theorem). *Let X be a smooth projective variety and let L be a line bundle on X with a section defining an irreducible hypersurface A. Assume that A has only non-degenerate quadratic singularities, constituting a smooth subvariety $P \subset X$ of positive dimension. Then there is an isomorphism*

$$N_{P/X} \cong N^*_{P/X} \otimes L_P.$$

Here and in the following the star stands for the dual. Furthermore, the claimed isomorphism is symmetric (see [**11**, Theorem 1.2]), but we do not need this property in the present paper. We will refer to it as the Ein isomorphism: it will play a crucial role in the sequel. The proof runs essentially as in [**1**, Theorem 14.4.1], but we include it for the convenience of the reader.

Proof. Let $s \in H^0(L)$ be the section defining A; then its first jet $j_1(s)$ is zero on P. Consider the exact sequence

$$0 \to T_X^{*(2)} \otimes L \to J_2(L) \xrightarrow{\alpha} J_1(L) \to 0,$$

where $T_X^{*(2)}$ is the second symmetric power of the cotangent bundle, $J_m(L)$ is the m-th jet bundle of L, and α is the natural surjection given by truncation (see [**10**] for more details). Let $\dim(X) = n$. Then around every point $x \in P$ we can choose local coordinates $u_1, \dots, u_n$ on X such that P is defined by $u_{k+1} = \cdots = u_n = 0$, where $k = \dim(P) > 0$. Now look at the second jet $j_2(s)$. Since $\alpha(j_2(s)) = j_1(s)$ is trivial on P we have $(j_2(s))_P \in H^0(P, (T_X^{*(2)} \otimes L)_P)$. On the other hand, since s vanishes on P all partial derivatives in the u_i directions ($i = 1, \dots, k$) are zero. This shows that in fact

$$(j_2(s))_P \in H^0\big(P, N_{P/X}^{*(2)} \otimes L_P\big) \subseteq H^0\big(P, (T_X^{*(2)} \otimes L)_P\big).$$

Recalling that

$$N_{P/X}^{*(2)} \otimes L_P \subseteq N^*_{P/X} \otimes N^*_{P/X} \otimes L_P \cong \mathcal{H}om(N_{P/X}, N^*_{P/X} \otimes L_P),$$

we thus see that $j_2(s)$ defines a homomorphism $h(s) : N_{P/X} \to N^*_{P/X} \otimes L_P$. Note that $h(s)$ is represented at every point $x \in P$ by the Hessian matrix of s with respect to the coordinates $u_{k+1}, \dots, u_n$. But this matrix has maximal rank, since x is a non-degenerate quadratic singularity. Hence $h(s)$ is an isomorphism. □

Theorem 1.1 has some immediate consequences, crucial for our purpose.

Corollary 1.2. *Assumptions and notation as in Theorem* 1.1. *Let c be the codimension of P in X. Then we have*

1. $cL_P = 2 \det N_{P/X}$;

2. $K_P = (K_X + (c/2)L)_P$ *in* $\mathrm{Pic}(P)$;
3. *If* c *is odd then* P *does not contain* 1*-cycles of odd degree* (*w.r.t.* L).

PROOF. Since $N_{P/X}$ is a rank c vector bundle on P, Theorem 1.1 yields

$$\det N_{P/X} = \det(N^*_{P/X}) + cL_P$$

which proves 1), whence 3). Now, adjunction formula $K_P = (K_X)_P + \det N_{P/X}$ gives 2). □

The following observation can be regarded as a generalization of the so-called Landman's parity theorem.

REMARK 1.3. Notation as in Theorem 1.1. Suppose that P contains a line ℓ with respect to L (that is, ℓ is a smooth rational curve with $L \cdot \ell = 1$). Then the restriction of the normal bundle $N_{P/X}$ to ℓ has the following form

$$(N_{P/X})_\ell = \bigoplus_{i=1}^{u} \mathcal{O}_\ell(-x_i) \oplus \mathcal{O}_\ell^{\oplus v} \oplus \mathcal{O}_\ell(1)^{\oplus v} \oplus \bigoplus_{i=1}^{u} \mathcal{O}_\ell(1+x_i),$$

for positive integers i, $i = 1, \dots, u$. In particular, the codimension of P in X must be even.

In fact, letting $c := \mathrm{codim}_X(P)$ we can write $(N_{P/X})_\ell = \oplus_{j=1}^{c} \mathcal{O}_\ell(\alpha_j)$. Then Theorem 1.1 implies that for every j there exists another index i such that $\alpha_j = 1 - \alpha_i$. In particular, this says that the number, say v, of summands of type $\mathcal{O}_\ell(1)$ equals that of the summands of type $\mathcal{O}_\ell$. Moreover, for every summand of degree different from 0 and 1, say $-x_i$, there is another summand of degree $1 + x_i$. This proves the assertion. In particular, $c = 2(u + v)$.

2. Set-up, preliminary results and examples.

First, let us fix the context of work.

2.1. Let (X, L) be a smooth polarized variety of dimension n. Let $A \in |L|$ be an effective irreducible divisor. Let $\Sigma = \mathrm{Sing}(A)$ be the singular locus of A. We assume that Σ is a smooth subvariety of dimension $k \geq 2$, consisting of non-degenerate quadratic singularities. We set $c := \mathrm{codim}_X(\Sigma)$ and we further assume that $c \geq 3$ (whence, in particular, $n \geq 5$). Then, according to what we said before, for any point $x \in \Sigma$, A can be described by

$$u_1^2 + u_2^2 + \cdots + u_c^2 = 0,$$

where $(u_1, u_2, \dots, u_n)$ are suitable local affine coordinates on X around x (with Σ defined by $u_1 = \cdots = u_c = 0$ at x).

Let us point out the following fact.

LEMMA 2.2. *Let $\pi : X \to Y$ be a surjective morphism from a smooth projective variety X to a normal projective variety Y, let L be an ample line bundle on X and suppose that $(F, L_F) = (\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1))$ with $r \geq 1$ for the general fiber F of π. Let $A \in |L|$ be any divisor. Then A cuts a general fiber F along a smooth element. In particular, $\pi(\mathrm{Sing}(A))$ is a proper algebraic subset of Y.*

PROOF. Just note that A cannot contain the general (hence every) fiber of π. □

When L is very ample we can prove the following consequence of Theorem 1.1, which will be applied in the proof of Proposition 3.4 when $(\Sigma, L_\Sigma) = (\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(1))$.

PROPOSITION 2.3. *Assume that L is very ample. If Σ contains a line ℓ with respect to L such that $N_{\ell/\Sigma}$ is ample, then (X, L) is covered by lines.*

PROOF. By adjunction, $K_\Sigma \cdot \ell = -2 - \deg(N_{\ell/\Sigma}) \leq -2 - (k-1) = -(k+1)$, since the normal bundle $N_{\ell/\Sigma}$ is ample, of rank $k-1$. Thus $(K_\Sigma + (k+1)L_\Sigma) \cdot \ell \leq 0$. This implies that $K_\Sigma + (k+1)L_\Sigma$ is not ample and then $(\Sigma, L_\Sigma) = (\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(1))$ (e.g., see [**1**, Theorem 7.2.1]). Therefore, if L embeds Σ into a projective space, say $\mathbf{P}$, the variety Σ becomes a linear space in $\mathbf{P}$. Thus $N^*_{\Sigma/X}(1)$ is spanned, being a quotient of the trivial bundle $N^*_{\Sigma/\mathbf{P}}(1)$. Theorem 1.1 then applies to give spannedness of $N_{\Sigma/X}$. Since $H^1(\ell, N_{\ell/\Sigma}) = 0$, from the normal bundle sequence

$$0 \to N_{\ell/\Sigma} \to N_{\ell/X} \to (N_{\Sigma/X})_\ell \to 0$$

it thus follows that $N_{\ell/X}$ is spanned, too. From the tangent-normal bundle sequence

$$0 \to T_\ell \to (T_X)_\ell \to N_{\ell/X} \to 0$$

we then conclude that $(T_X)_\ell$ is spanned. Therefore ℓ induces a covering family on X [**9**, II, Section 3, IV, (1.9)]. □

We provide now several examples illustrating the hypotheses made. Assumptions and notation as in Paragraph 2.1.

EXAMPLE 2.4 (A general construction). Let X be a smooth projective variety of dimension n, choose ample and spanned line bundles $H_1, H_2, \dots, H_c$ on X such that $2(H_i - H_j) = 0$, for every i, j, and set $L := 2H_i$, $i = 1, \dots, c$. Take global sections $s_i \in \Gamma(X, H_i)$ such that the divisors $s_i^{-1}(0)$ intersect transversally everywhere. Then consider the linear system defined by the vector subspace $\langle s_1^2, \dots, s_c^2 \rangle \subseteq \Gamma(X, L)$. Let

$$s = \lambda_1 s_1^2 + \cdots + \lambda_c s_c^2,$$

$\lambda_i \in \mathbb{C}$, $i = 1, \dots, c$. By Bertini's theorem, for general $(\lambda_1, \dots, \lambda_c) \in \mathbb{C}^c$, we have that the divisor $A := s^{-1}(0)$ is smooth away from the set-theoretic base locus, say $\Sigma = \{s_1 = \cdots = s_c = 0\}$, of this linear system. Clearly, $\mathrm{codim}_X(\Sigma) = c$ and $\Sigma \subset \mathrm{Sing}(A)$.

Note that for $c = 2$, $s = \lambda_1 s_1^2 + \lambda_2 s_2^2 = (\alpha s_1 + i\beta s_2)(\alpha s_1 - i\beta s_2)$, where $\alpha^2 = \lambda_1$

and $\beta^2 = \lambda_2$. Hence any A as above is reducible. Therefore, to have an irreducible hypersurface A we need $c \geq 3$.

We claim that each point $x \in \Sigma$ is a non-degenerate quadratic singularity for A, and $\Sigma = \mathrm{Sing}(A)$. Indeed, if $(u_1, u_2, \ldots, u_n)$ are local coordinates on X at x, we may assume that $s_1 = u_1, \ldots, s_c = u_c$ (since the divisors $s_i^{-1}(0)$ intersect transversally). Then, locally near the point x, the divisor A can be expressed by

$$\lambda_1 u_1^2 + \cdots + \lambda_c u_c^2 = 0,$$

showing the claim.

For example, let $X = \mathbb{P}^3$, $H_i = \mathcal{O}_{\mathbb{P}^3}(m)$, $i = 1, 2, 3$, and $L = \mathcal{O}_{\mathbb{P}^3}(2m)$ for some positive integer m. Consider a general element $A \in |L|$. Thus the singular locus $\mathrm{Sing}(A)$ consists of $m^3 = H_1 \cdot H_2 \cdot H_3$ isolated non-degenerate quadratic singularities. Let us point out that the value m^3 is considerably less than the upper bound

$$\frac{4}{9}(2m(2m-1)^2) = \frac{32}{9}m^3 + (\text{degree} < 3 \text{ terms})$$

given by Miyaoka's inequality recalled in the introduction.

The general construction as in the Example 2.4 specializes to the following basic cases.

Example 2.5 ($(\Sigma, L_\Sigma) = (\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(2))$). Let $X = \mathbb{P}^n$, $H_i \in |\mathcal{O}_{\mathbb{P}^n}(1)|$, $i = 1, \ldots, n-k$, and $L = \mathcal{O}_{\mathbb{P}^n}(2)$. Then choose $n-k$ general sections $s_i \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, let $s = \sum_{i=1}^{n-k} \lambda_i s_i^2$ and set $A := s^{-1}(0) \in |\mathcal{O}_{\mathbb{P}^n}(2)|$. Thus A is a quadric hypersurface of rank $n-k$ and $\mathrm{Sing}(A)$ is a smooth $\mathbb{P}^k$ consisting of non-degenerate quadratic singularities, with $L_{\mathbb{P}^k} = \mathcal{O}_{\mathbb{P}^k}(2)$.

Example 2.6 ($(\Sigma, L_\Sigma) = (\mathbb{Q}^k, \mathcal{O}_{\mathbb{Q}^k}(2))$). Let X be a smooth n-dimensional quadric $\mathbb{Q}^n$ in $\mathbb{P}^{n+1}$ and $H \in |\mathcal{O}_{\mathbb{Q}^n}(1)|$. Choose $n-k$ general sections $s_i \in \Gamma(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$, let $s = \sum_{i=1}^{n-k} \lambda_i s_i^2$ and set $A := s^{-1}(0) \in |\mathcal{O}_{\mathbb{Q}^n}(2)|$. Then A is a complete intersection of type $(2,2)$ in $\mathbb{P}^{n+1}$ and the singular locus of A is

$$\mathrm{Sing}(A) = \bigcap_{i=1}^{n-k} s_i^{-1}(0) = \mathbb{P}^{k+1} \cap \mathbb{Q}^n,$$

that is, a k-dimensional smooth quadric $\mathbb{Q}^k$, consisting of non-degenerate quadratic singularities.

Example 2.7 ((Σ, L_Σ) a $\mathbb{P}^{k-1}$-bundle over a smooth curve C, $L_{\mathbb{P}^{k-1}} = \mathcal{O}_{\mathbb{P}^{k-1}}(2)$). Let X be a $\mathbb{P}^{n-1}$-bundle over a smooth curve Y, and let H be the tautological line bundle. Suppose that H is ample and spanned and take $n-k$ general divisors H_i in the linear system $|H|$. The transversal intersection of such divisors H_i, $i = 1, \ldots, n-k$, intersects each fiber $F = \mathbb{P}^{n-1}$ of the bundle in a linear $\mathbb{P}^{k-1} \subset F$. Let now $L = 2H$ and consider a general element A in the sublinear system of $|L|$ generated by $2H_1, \ldots, 2H_{n-k}$. The intersection $A \cap F$ is therefore a quadric $\mathbb{Q}^{n-2} \subset F$ which is singular along the $\mathbb{P}^{k-1} \subset F$, the locus of non-degenerate quadratic singularities for the quadric $\mathbb{Q}^{n-2}$.

More generally, if C has positive genus and H is sufficiently ample, we can choose $H_i \in |H + \eta_i|$, where η_i is a 2-torsion element in $\mathrm{Pic}(X)$. In this case, the smooth hypersurfaces H_i define distinct line bundles but $2H_i \in |L|$ for every i.

Note that for $k = 1$ we get a $\mathbb{Q}^{n-2}$-fibration $A \to Y$ which is singular along a section of $X \to Y$.

Note also that this construction extends to the case when Σ is a $\mathbb{P}^d$-bundle over a smooth projective variety Z of dimension $k - d$, and $L_F = \mathcal{O}_{\mathbb{P}^d}(2)$ for each fiber F.

Example 2.8 ((Σ, L_Σ) a quadric fibration over a smooth curve, $L_{\mathbb{Q}^{k-1}} = \mathcal{O}_{\mathbb{Q}^{k-1}}(2)$). Let $(P, \mathcal{H})$ be an $(n+1)$-dimensional scroll over a smooth curve Y, with $\mathcal{H}$ ample and spanned. Let $X \in |2\mathcal{H}|$ be a smooth element, so that (X, H) is an n-dimensional quadric fibration over Y, where $H = \mathcal{H}_X$. Take $n - k$ general divisors H_i in the linear system $|H|$. The transversal intersection of such divisors H_i, $i = 1, \ldots, n-k$, intersects any general fiber $F = \mathbb{Q}^{n-1}$ of the quadric fibration along a quadric $Q := \mathbb{Q}^{k-1} \subset F$. Let now $L = 2H$ and consider a general element A in the linear subsystem of $|L|$ generated by $2H_1, \ldots, 2H_{n-k}$. Therefore $A \cap F$ is a complete intersection V of type $(2, 2)$ in F which is singular along the quadric Q, the locus of non-degenerate quadratic singularities for V.

Finally we point out that the construction in Example 2.4 is stable under general finite coverings.

Example 2.9. Assumptions and notation as in Example 2.4. Let $\pi : X' \to X$ be a finite covering branched along a smooth hypersurface $\Delta \subset X$ transversal to the zero loci $s_i^{-1}(0)$, $i = 1, 2 \ldots, c$. Let $L' := \pi^* L$, $A' := \pi^* A$ and $\Sigma' := \pi^* \Sigma$. Then the same conclusions as in Example 2.4 hold true for X', L', and Σ', provided that $\Sigma \cap \Delta = \emptyset$.

3. The general case: positivity of the first adjoint bundle.

The first result we prove in this section deals with the nefness and bigness of the first adjoint bundle for a polarized pair (X, L) satisfying the assumptions as in Paragraph 2.1. Compare with Theorem 4.1, concerned with the second adjoint bundle for $c = \mathrm{codim}_X(\Sigma)$ odd.

Theorem 3.1. *Let (X, L) be a smooth polarized variety of dimension n. Let A be an irreducible member of $|L|$ whose singular locus $\Sigma = \mathrm{Sing}(A)$ consists of non-degenerate quadratic singularities. Assume that Σ is smooth of dimension $k \geq 2$ and that $c = \mathrm{codim}_X(\Sigma) \geq 3$. Then the first reduction (X', L'), $\varphi : X \to X'$, exists, unless, possibly, if $c = 4$ and $(\Sigma, L_\Sigma) = (\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(1))$. Moreover, Σ meets no exceptional divisors of φ.*

Proof. We systematically use the adjunction process described in [**1**, Chapter 7]. By Corollary 1.2(2) we have

$$(K_X)_\Sigma = K_\Sigma - \frac{c}{2} L_\Sigma. \tag{1}$$

Let τ be the nefvalue of (X, L). If $\tau = n+1$, i.e., if $K_X + (n+1)L$ is trivial, then we get from (1)

$$\mathcal{O}_\Sigma = (K_X + (n+1)L)_\Sigma = K_\Sigma + \left(k + 1 + \frac{c}{2}\right) L_\Sigma,$$

which contradicts the ampleness of $K_\Sigma + (k+1+(c/2))L_\Sigma$ since $k+1+(c/2) > \dim(\Sigma)+1$ (see [**1**, Theorem 7.2.1]).

Now suppose that $n-1 < \tau \le n$. Since $n = k + c \ge 5$, it cannot be $(X, L) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$, so that (X, L) is either $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$ or a scroll over a smooth curve (see [**1**, Proposition 7.2.2 and Theorem 7.2.4]). In both cases, as $k \ge 2$, there exists a curve $C \subset \Sigma$ such that $K_X + nL$ restricts trivially to C. Therefore by (1) we get

$$\mathcal{O}_C = (K_X + nL)_C = \left(K_\Sigma + \left(k + \frac{c}{2}\right) L_\Sigma\right)_C,$$

which is a contradiction again since $c \ge 3$. We thus conclude that $K_X + (n-1)L$ is nef, i.e., $\tau \le n-1$. If it is not big then (X, L) is either a Del Pezzo manifold, or a quadric fibration over smooth curve, or a scroll over a smooth surface, according to whether the image of the nefvalue morphism ϕ of (X, L) has dimension 0, 1, or 2 (see [**1**, Theorem 7.3.2]). We know that in all these cases there exists a curve $C \subset \Sigma$ to which $K_X + (n-1)L$ restricts trivially. This is obvious in the first two cases, while in the third one it follows from the fact that $\phi(\Sigma)$ has dimension ≤ 1 according to Lemma 2.2. By using (1) again, we get

$$\mathcal{O}_C = (K_X + (n-1)L)_C = \left(K_\Sigma + \left(k + \frac{c}{2} - 1\right) L_\Sigma\right)_C.$$

Once again, this is a contradiction if $c \ge 5$, so that $3 \le c \le 4$. Moreover, since $c \ge 3$, the above relation allows us to conclude that $K_\Sigma + kL_\Sigma$ is not nef, which implies that $(\Sigma, L_\Sigma) = (\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(1))$ (see [**1**, Proposition 7.2.2]). Therefore case $c = 3$ cannot occur by Corollary 1.2(3). Hence $c = 4$ and in addition we see that Σ must be contained in a fiber of the morphism ϕ. In conclusion, $K_X + (n-1)L$ is nef and big unless $c = 4$ and $(\Sigma, L_\Sigma) = (\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(1))$.

Thus the first reduction morphism, $\varphi : X \to X'$, exists. Let $E = \mathbb{P}^{n-1}$ be any exceptional divisor. If Σ meets E, then, since E is a divisor and $k \ge 2$, we can find a curve C contained in $\Sigma \cap E$ such that $(K_X + (n-1)L)_C = \mathcal{O}_C$. Once again, relation (1) yields

$$\left(K_\Sigma + \left(k + \frac{c}{2} - 1\right) L_\Sigma\right)_C = \mathcal{O}_C.$$

This leads to the usual contradiction as soon as $c \ge 5$. Moreover, if $c = 3$, the above equality shows that $K_\Sigma + (k + (1/2))L_\Sigma$ is not ample. On the other hand it is nef being the restriction of a nef line bundle. This says that the nefvalue of the polarized pair

(Σ, L_Σ) is equal to $k+(1/2)$, contradicting [**1**, Proposition 7.2.2]. We thus conclude that $\Sigma \cap E = \emptyset$, as we want. $\square$

The following example explains why we required $c \geq 3$ in Theorem 3.1.

EXAMPLE 3.2. Let's focus on the case $(X, L) = (\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$ in the proof of Theorem 3.1. Here

$$\mathcal{O}_\Sigma = (K_X + nL)_\Sigma = K_\Sigma + \left(k + \frac{c}{2}\right) L_\Sigma,$$

by Corollary 1.2(2). Note that, if $c = 2$, then $K_\Sigma + (k+1)L_\Sigma$ is not ample. Therefore $(\Sigma, L_\Sigma) = (\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(1))$. Recalling that $k \geq 2$ one has $n \geq 4$. On the other hand, linear spaces contained in a smooth quadric $\mathbb{Q}^n$ cannot have dimension exceeding $n/2$. Thus $n - 2 = k \leq n/2$, i.e. $n \leq 4$, whence $n = 4$. In this case an element $A \in |L|$ is a quadric 3-fold; since it has to be singular along a $\mathbb{P}^2$ we infer that A is reducible (compare this with what we said in the introduction concerning the case $c = 2$).

Motivated by the exception in Theorem 3.1, we now discuss the case when $(\Sigma, L_\Sigma) = (\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(1))$.

First, note the following fact. Let ℓ be a line in $\Sigma = \mathbb{P}^k$. Then, by using Corollary 1.2(1), we infer that

$$-K_X \cdot \ell = (-K_\Sigma + \det N_{\Sigma/X}) \cdot \ell = -K_\Sigma \cdot \ell + \frac{c}{2} = k + 1 + \frac{c}{2} = \frac{n}{2} + 1 + \frac{k}{2}. \tag{2}$$

REMARK 3.3. If $(\Sigma, L_\Sigma) = (\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(1))$, then (X, L) cannot be $(\mathbb{P}^{n/2} \times \mathbb{P}^{n/2}, \mathcal{O}(1,1))$. Otherwise, since any line ℓ of this product is contained in one of the factors, we would get

$$-K_X \cdot \ell = \mathcal{O}\left(\frac{n}{2} + 1, \frac{n}{2} + 1\right) \cdot \ell = \frac{n}{2} + 1.$$

This is not possible in view of relation (2), since k is positive.

PROPOSITION 3.4. *Let (X, L) be a smooth polarized variety, with L very ample. Let τ and $\phi : X \to Y$ be the nefvalue and the nefvalue morphism of the pair (X, L) respectively. Let A be an irreducible member of $|L|$ whose singular locus $\Sigma = \operatorname{Sing}(A)$ consists of non-degenerate quadratic singularities. Assume that $(\Sigma, L_\Sigma) = (\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(1))$, $k \geq 1$. Then*

$$\tau = n + 1 - \frac{c}{2},$$

where $c = \operatorname{codim}_X(\Sigma)$; moreover, ϕ is a Mori contraction which maps Σ to a point.

PROOF. Let $\ell \subset \Sigma$ be a line. As $N_{\ell/\Sigma} = \mathcal{O}_\ell(1)^{\oplus(k-1)}$ and L is very ample, the assumptions of Proposition 2.3 are satisfied. Therefore the family $\mathscr{F}$ defined by ℓ covers

X; moreover, $\mathscr{F}$ is a non-breaking family since $L \cdot \ell = 1$, and $K_X \cdot \ell < 0$ by (2). All this says that the first two of the three assumptions in [**3**, (2.0)] are satisfied.

Let $\mathscr{F}_x$ denote the set of curves from the family $\mathscr{F}$ passing through x and observe that in our case

$$\dim(\mathscr{F}_x) = h^0(N_{\ell/X}(-1)) = \deg(N_{\ell/X}) = -K_X \cdot \ell - 2,$$

for any point $x \in X$. This says that the third assumptions in [**3**, (2.0)] is satisfied as well. Furthermore, relation (2) shows that $-K_X \cdot \ell - 2 > (n-2)/2$. Thus [**3**, Theorem (2.3)], combined with (2) again, applies to give the expression of τ and to conclude that ϕ contracts each line $\ell \subset \Sigma = \mathbb{P}^k$ to a point. This clearly implies that ϕ contracts Σ to a point, as we want. $\square$

The previous result agrees with the "obvious" expectation that for high values of c the polarized pair (X, L) presents a good behaviour for the adjoint bundles. This is illustrated by the following statement (recall that τ is the smallest rational number such that $K_X + \tau L$ is nef).

Corollary 3.5. *Notation as in Proposition* 3.4. *Then*:

1. $K_X + nL$ *is nef.*
2. $K_X + (n-1)L$ *is nef unless* $c \leq 3$.
3. $K_X + (n-2)L$ *is nef unless* $c \leq 5$.

On the other hand, for small values of c, the pair (X, L) tends to be one of the special varieties arising from adjunction theory. In particular, for $c = 2$, (X, L) is either $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$ or a scroll over a smooth curve; for $c = 3$ we get $\tau = n - (1/2)$ which is impossible since $n \geq 4$ (see [**1**, Theorem 7.2.4]). If $c = 4$, then $\tau = n - 1$, so that (X, L) is one of the pairs described in [**1**, Theorem 7.3.2].

4. Odd codimension: positivity of the second and third adjoint bundle.

Whenever the locus Σ of non-degenerate quadratic singularities is of odd codimension more can be said. This section is devoted to this case.

First note that, as a consequence of Theorem 3.1, the first reduction morphism $\varphi : X \to X'$ is an isomorphism in a neighborhood of Σ. In particular, we can identify Σ with its image $\Sigma' := \varphi(\Sigma)$ in the first reduction (X', L') of (X, L), and all properties of $N_{\Sigma/X}$ expressed by Corollary 1.2 hold true for the normal bundle of Σ' in X' w.r.t. L'. Moreover, $\Sigma' \subseteq \mathrm{Sing}(\varphi(A))$. We have the following result.

Theorem 4.1. *Let (X, L) be a smooth polarized variety of dimension n. Let A be an irreducible member of $|L|$ whose singular locus $\Sigma = \mathrm{Sing}(A)$ consists of non-degenerate quadratic singularities. Assume that Σ is smooth of dimension $k \geq 3$, with odd codimension $c := \mathrm{codim}_X(\Sigma) \geq 3$. Then there exists the second reduction $(\widehat{X}, \mathscr{D})$, $\psi : X' \to \widehat{X}$, of (X, L). Moreover, Σ meets no exceptional divisors of ψ pulled-back to X via the first reduction morphism $\varphi : X \to X'$.*

Proof. By what we said before we can assume that $\tau < n-1$, where τ denotes now the nefvalue of the first reduction (X', L'). Let $\varphi : X \to X'$ and $\Sigma' = \varphi(\Sigma)$ be as before. We continue with the adjunction process according to [**1**, Chapter 7]. By Corollary 1.2(2) and the above discussion we have

$$(K_{X'})_{\Sigma'} = K_{\Sigma'} - \frac{c}{2} L'_{\Sigma'}. \tag{3}$$

Since $n \geq 6$, the range $n-2 < \tau < n-1$ is ruled out by [**1**, Theorem 7.3.4]. We thus conclude that $K_{X'} + (n-2)L'$ is nef, i.e., $\tau \leq n-2$. If it is not big then (X', L') is either a Mukai manifold, or a Del Pezzo fibration over smooth curve, or a quadric fibration over a normal surface, or a scroll over a normal threefold, according to whether the image of the nefvalue morphism ϕ' of (X', L') has dimension 0, 1, 2, or 3 (see [**1**, Theorem 7.5.3]).

We claim that there exists a curve $C \subset \Sigma'$ to which $K_{X'} + (n-2)L'$ restricts trivially. This is obvious in the first two cases. In the third case, let F be a general fiber of the nefvalue morphism ϕ'. From the inequality

$$\dim(\Sigma' \cap F) \geq k + (n-2) - n = k - 2$$

we are done since $k \geq 3$. In the fourth case the assertion follows from the fact that $\phi'(\Sigma')$ has dimension ≤ 3 according to Lemma 2.2, recalling that $\Sigma' \subseteq \mathrm{Sing}(\varphi(A))$. This proves the claim. Therefore from (3) we get

$$\mathcal{O}_C = (K_{X'} + (n-2)L')_C = \left(K_{\Sigma'} + \left(k - 2 + \frac{c}{2} \right) L'_{\Sigma'} \right)_C .$$

This contradicts the ampleness of $K_{\Sigma'} + (\dim(\Sigma') + 1) L'_{\Sigma'}$ as soon as $k - 2 + (c/2) > k + 1$, i.e. $c \geq 7$. Therefore, either $c = 5$ or $c = 3$. If $c = 5$, then $K_{\Sigma'} + kL'_{\Sigma'}$ is not ample, whence $(\Sigma', L'_{\Sigma'}) = (\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(1))$ (see [**1**, Theorem 7.2.1]). But this is impossible by Corollary 1.2(3). If $c = 3$, then $K_{\Sigma'} + (k-1)L'_{\Sigma'}$ is not nef. Therefore [**1**, Theorem 7.2.4] applies to rule out this case.

We thus conclude that $K_{X'} + (n-2)L'$ is nef and big; so the second reduction $(\widehat{X}, \mathscr{D})$, $\psi : X' \to \widehat{X}$, of (X, L) exists. In general, $\widehat{X}$ is singular with moderate singularities, and $\mathscr{D} := \psi_*(L')^{**}$ (the double dual) is a 2-Cartier divisor [**1**, Section 7.5].

As to the second part of the statement, recall that (see [**1**, Theorem 7.5.3, 5)] any irreducible component of the exceptional locus of $\psi : X' \to \widehat{X}$ is one of the following:

1. a divisor E which contracts to a point;
2. a $\mathbb{P}^{n-2}$-bundle $E \to B$ via $\psi|_E$ onto some curve B in $\widehat{X}$.

Suppose, by contradiction, that $\Sigma' \cap E \neq \emptyset$. Then we have $\dim(\Sigma' \cap E) \geq k - 1$ in the former case. In the latter case, Σ' must intersect a fiber f of $\psi|_E$ so that $\dim(\Sigma' \cap f) \geq k + (n-2) - n = k - 2$. Since $k \geq 3$, in both cases we find a curve $C \subset \Sigma'$ such that $\mathcal{O}_C = (K_{X'} + (n-2)L')_C$, which leads to the same contradiction as above. □

Remark 4.2 ($\dim(\Sigma) = 2$). By looking over the proof above, we see that Theorem 4.1 holds true in case $k = 2$ unless, possibly, when the nefvalue morphism ϕ' gives to

(X', L') the structure of either a quadric fibration over a normal surface or a scroll over a normal 3-fold. Actually, in these cases we cannot grant that $\dim(\Sigma' \cap F) \geq 1$ for some fiber F of ϕ'.

As a consequence of Theorem 4.1, when studying the structure of Σ, we may assume to work on the second reduction $(\widehat{X}, \mathscr{D})$ of (X, L) for appropriate c, namely, c odd and ≥ 3. This is exactly what we do in the remaining part of this section, where we study positivity conditions for the third adjoint bundle.

First let us note the following general fact (which we will use for c odd and $t = k-2$).

LEMMA 4.3. *Let (X, L) be a smooth polarized variety of dimension n. Let A be an irreducible member of $|L|$ whose singular locus $\Sigma = \mathrm{Sing}(A)$ consists of non-degenerate quadratic singularities. Assume that Σ is smooth of dimension k, with codimension $c := \mathrm{codim}_X(\Sigma) \geq 3$. Suppose that $K_\Sigma + tL_\Sigma$ is nef for some positive rational number t. Then $(K_X + (t + c - 1)L) \cdot C > 0$ for every curve $C \subset \Sigma$.*

PROOF. By Corollary 1.2(2) we can write

$$\begin{aligned}(K_X + (t + c - 1)L) \cdot C &= \left(K_X + \frac{c}{2}L + \left(t + \frac{c}{2} - 1\right)L\right) \cdot C \\ &= (K_\Sigma + tL_\Sigma) \cdot C + \left(\frac{c}{2} - 1\right)L_\Sigma \cdot C \geq \left(\frac{c}{2} - 1\right)L_\Sigma \cdot C > 0,\end{aligned}$$

where the first inequality comes from the nefness assumption and the latter from the ampleness of L and the fact that $c \geq 3$. $\square$

By Theorem 4.1 we know that, if Σ has dimension $k \geq 3$ and odd codimension $c \geq 3$, then there exists the second reduction of (X, L). Now, we study nefness and bigness of the third adjoint bundle.

THEOREM 4.4. *Under the assumptions as in Theorem 4.1, the following hold true.*

1. *The third adjoint bundle $K_{\widehat{X}} + (n-3)\mathscr{D}$ is nef unless $(\widehat{X}, \mathscr{D}) = (\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2))$, in which case $(\Sigma, L_\Sigma) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$.*
2. *If $K_\Sigma + (k-2)L_\Sigma$ is nef, then $K_{\widehat{X}} + (n-3)\mathscr{D}$ is also big, provided that $k \geq 4$.*

PROOF. Set $\mathscr{D} = \psi_*(L')^{**}$ and recall that $\mathscr{D}$ is a 2-Cartier divisor on $\widehat{X}$ such that $K_{X'} + (n-2)L' = \psi^*\mathscr{K}$, where $\mathscr{K}$ is an ample line bundle $\mathbb{Q}$-linearly equivalent to $K_{\widehat{X}} + (n-2)\mathscr{D}$ [**1**, Lemma 7.5.8]. Moreover, $\mathscr{D}$ is $\widehat{\phi}$-ample, where $\widehat{\phi}$ is now the nefvalue morphism of $(\widehat{X}, \mathscr{K})$. More precisely, the following isomorphism of rank 1 reflexive sheaves

$$(n-2)(K_{\widehat{X}} + (n-3)\mathscr{D}) \cong (K_{\widehat{X}} + (n-3)\mathscr{K}) \tag{4}$$

plays a key role [**1**, Corollary 7.6.2]. In particular, it shows that $K_{\widehat{X}} + (n-3)\mathscr{D}$ is nef if and only if $K_{\widehat{X}} + (n-3)\mathscr{K}$ is nef. Thus [**1**, Section 7.7] says that $K_{\widehat{X}} + (n-3)\mathscr{D}$ is nef unless $(\widehat{X}, \mathscr{K}) = (\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(1))$, in which case $\mathscr{D} = \mathcal{O}_{\mathbb{P}^6}(2)$, in view of the above relations.

By Theorem 4.1 we also know that the composite map $\psi \circ \varphi : X \to \widehat{X}$ is an isomorphism in a neighborhood U of Σ in X, hence $L_U \cong \mathscr{D}_{(\psi\circ\varphi)(U)}$. Letting $\widehat{\Sigma} := (\psi \circ \varphi)(\Sigma)$, and taking into account that $6 = n = k + c$, with $k, c \geq 3$, we thus get $(\widehat{\Sigma}, \mathscr{D}_{\widehat{\Sigma}}) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$, leading to the first assertion.

We prove now the second assertion. As $n \geq 6$ (in fact $n \geq 7$ as $k \geq 4$) we know that $K_{\widehat{X}} + (n-3)\mathscr{K}$ is also big except for five exceptional cases listed in [**1**, Proposition 7.7.6], which correspond to the values $m = 0, \dots, 4$ of the dimension of the image of the nefvalue morphism $\widehat{\phi}$ of $(\widehat{X}, \mathscr{K})$. Note that the bigness of $K_{\widehat{X}} + (n-3)\mathscr{K}$ is equivalent to that of $K_{\widehat{X}} + (n-3)\mathscr{D}$ in view of [**1**, Corollary 7.6.2]. Let F be the pullback to X via $\psi \circ \varphi$ of a general fiber of $\widehat{\phi}$. If $0 \leq m \leq 3$, due to the assumption $k \geq 4$, we see that

$$\dim(\Sigma \cap F) = k + n - m - n = k - m \geq 1.$$

Hence there is a curve $C \subset \Sigma$ such that $K_{\widehat{X}} + (n-3)\mathscr{K}$ is trivial on $\widehat{C} = (\psi \circ \varphi)(C)$. Then $K_{\widehat{X}} + (n-3)\mathscr{D}$ is trivial on $\widehat{C}$ as well, by (4). Coming back to X, this implies

$$(K_X + (n-3)L)_C = \mathcal{O}_C. \tag{5}$$

If $m = 4$ we know from [**1**, Proposition 7.7.6] that $(\widehat{X}, \mathscr{K})$ is a scroll over a normal 4-fold W. In particular, the scroll projection $p : \widehat{X} \to W$ induces a $\mathbb{P}^{n-4}$-bundle over a Zariski dense open subset W_0 of W and $\mathscr{K}$ restricts as $\mathcal{O}_{\mathbb{P}^{n-4}}(1)$ to any fiber of it. Thus Lemma 2.2 (used for $t = k-2$) allows us to conclude that there is a fiber of $\widehat{\phi}$ such that for the corresponding F on X, $\Sigma \cap F$ has positive dimension even in this case. Therefore we can find a curve $C \subset \Sigma$ satisfying (5) for $m = 4$ as well. On the other hand, equation (5) contradicts Lemma 4.3. This proves the second assertion. $\square$

Remark 4.5. Notation as above. By studying the adjunction mapping on Σ we can produce complete effective lists of all the exceptions to the nefness and bigness of $K_\Sigma + (k-2)L_\Sigma$ for $k \geq 3$. This would allow us to make concrete the assumption made in Lemma 4.3 and used in Theorem 4.4(2).

Note added in proof (March 1, 2015). For further progress relying on a generalization of Theorem 1.1, we refer to: M.C. Beltrametti, A. Lanteri and A.J. Sommese, Adjunction and singular loci of hyperplane sections, II, Rend. Circ. Mat. Palermo, **63** (2014), 247–255.

References

[1] M. C. Beltrametti and A. J. Sommese, The Adjunction Theory of Complex Projective Varieties, de Gruyter Exp. Math., **16**, Walter de Gruyter, Berlin, 1995.

[2] M. C. Beltrametti, K. A. Chandler and A. J. Sommese, Reducible hyperplane sections. II, Kodai Math. J., **25** (2002), 139–150.

[3] M. C. Beltrametti, A. J. Sommese and J. A. Wiśniewski, Results on varieties with many lines and their applications to adjunction theory (with an appendix by M. C. Beltrametti and A. J. Sommese), In: Complex Algebraic Varieties, Bayreuth, 1990 (eds. K. Hulek, T. Peternell, M. Schneider and F.-O. Schreyer), Lecture Notes in Math., **1507**, Springer-Verlag, New York, 1992, pp. 16–38.

[4] E. Bertini, Introduzione alla geometria proiettiva degli iperspazi, Spoerri, Pisa, 1907.

[5] K. A. Chandler, A. Howard and A. J. Sommese, Reducible hyperplane sections. I, J. Math. Soc. Japan, **51** (1999), 887–910.

[6] L. Ein, Varieties with small dual varieties. I, Invent. Math., **86** (1986), 63–74.

[7] D. Gallarati, Superficie algebriche con molti punti singolari isolati, Bull. Math. Soc. Sci. Math. Roumanie (N.S.), **55(103)** (2012), 249–274.

[8] R. Hartshorne, Ample Subvarieties of Algebraic Varieties, Lecture Notes in Math., **156**, Springer-Verlag, New York, 1970.

[9] J. Kollár, Rational Curves on Algebraic Varieties, Ergeb. Math. Grenzgeb. (3), **32**, Springer-Verlag, Berlin, 1996.

[10] A. Kumpera and D. Spencer, Lie Equations. Vol. I: General Theory, Ann. of Math. Stud., **73**, Princeton University Press, Princeton, NJ, 1972.

[11] A. Lanteri, M. Palleschi and A. J. Sommese, Discriminant loci of varieties with smooth normalization, Comm. Algebra, **28** (2000), 4179–4200.

[12] A. Lanteri and A. L. Tironi, On reducible hyperplane sections of 4-folds, J. Math. Soc. Japan, **53** (2001), 559–563.

[13] Y. Miyaoka, The maximal number of quotient singularities on surfaces with given numerical invariants, Math. Ann., **268** (1984), 159–171.

[14] Y. Miyaoka, Theme and variations—inequalities between Chern numbers, Sugaku Expositions, **4** (1991), 157–176.

[15] F. Severi, Sul massimo numero di nodi di una superficie di dato ordine dello spazio ordinario o di una forma di un iperspazio, Ann. Mat. Pura Appl. (4), **25** (1946), 1–41.

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