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Distributive modules and Armendariz modules

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Abstract. Motivated by a recent result of Mazurek and Ziembowski in [**21**] that every left distributive ring is Armendariz, in this paper we present methods of constructing Armendariz modules using a distributive module. We prove that, for a bimodule ${}_RV_A$, ${}_RV$ being distributive implies that V_A is Armendariz, and that every right module over a right distributive ring is Armendariz. These results can be used to construct new Armendariz rings. Examples are provided to illustrate and delimit the results obtained.

1. Introduction.

In this paper we will consider Armendariz rings and Armendariz modules. Following Rege and Chhawchharia [**22**], a ring R is called Armendariz if $f(t)g(t) = 0$ in $R[t]$ where $f(t) = \sum_{i=0}^{n} a_i t^i$ and $g(t) = \sum_{j=0}^{k} b_j t^j$ always implies $a_i b_j = 0$ for all i and j. By Anderson and Camillo [**3**], a right R-module V is called Armendariz if $v(t)f(t) = 0$ with $v(t) = \sum_{i=0}^{n} v_i t^i \in V[t]$ and $f(t) = \sum_{j=0}^{k} a_j t^j \in R[t]$ implies $v_i a_j = 0$ for all i and j. Thus, a ring R is Armendariz if and only if the regular module R_R is Armendariz. These notions are useful in understanding the annihilator conditions of polynomial rings and polynomial modules, and have been studied in many publications (see, for example, [**1**], [**2**], [**3**], [**4**], [**10**], [**11**], [**12**], [**13**], [**14**], [**15**], [**16**], [**17**], [**18**], [**19**], [**21**], [**22**] and [**28**]). In the literature, several important families of Armendariz rings are constructed via reduced rings and Armendariz modules: Every reduced ring is Armendariz (see Armendariz [**4**]); for $n \geq 2$, $R[t]/(t^n)$ is Armendariz if and only if R is a reduced ring (see Anderson and Camillo [**3**]); one can construct Armendariz trivial extensions via Armendariz rings and Armendariz modules (see Anderson and Camillo [**3**] and Lee and Zhou [**15**]). Similarly, Armendariz modules can be constructed using reduced modules, where a module V_R is reduced if $va = 0$ with $v \in V$ and $a \in R$ implies $vR \cap Va = 0$ (see [**16**]).

The objective of this paper is to explore how distributivity is related to Armendariz property of rings and modules. This is motivated by a recent result of Mazurek and Ziembowski in [**21**] that every right (or left) distributive ring is Armendariz. Let us first recall two key notions needed in this paper. A module V is distributive if the lattice of its submodules is distributive, i.e., $(X+Y) \cap Z = (X \cap Z) + (Y \cap Z)$ for any submodules X, Y, Z of V, and a ring R is said to be right distributive if the module R_R is distributive (see [**24**]). Left distributive rings are defined similarly. A left and right distributive ring is called a distributive ring. For detailed information on distributive rings and modules,

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the reader is referred to [**27**]. Our observation seems to indicate that a good way to understand the relation between distributivity and Armendariz condition is to work in the context of a bimodule. In fact, one of our main results says that, for a bimodule ${}_RV_A$, if ${}_RV$ is distributive then V_A is Armendariz (indeed, a more general result is proved without assuming that R is a ring) (Theorem 2 and Corollary 3). Using this result, together with the characterization of a right distributive ring obtained by Tuganbaev in [**25**], we can show that every right module over a right distributive ring is Armendariz (Corollary 7). These results can be used to construct new Armendariz rings (Corollary 8).

Throughout, rings are associative with unity and modules are unitary. Homomorphisms of modules are written on the opposite side of their arguments. For a module V over a ring R, by $R[t]$ (resp., $V[t]$) we denote the polynomial ring (resp., the polynomial module). The endomorphism ring of a module ${}_RV$ (resp., V_R) is denoted by $\mathrm{End}({}_RV)$ (resp., $\mathrm{End}(V_R)$). For a bimodule ${}_RV_R$, the trivial extension of R by V, denoted $R \propto V$, is the ring with additive abelian group $R \oplus V$ and with multiplication defined by $(a, x)(b, y) = (ab, ay + xb)$ for $a, b \in R$ and $x, y \in V$. For convenience, we let $I \propto X = \{(a, x) : a \in I, x \in X\}$ where I is a subset of R and X is a subset of V.

2. The results.

If $(A, +)$ is an abelian group, then $A[t]$, the abelian group of polynomials in t with coefficients in A, consists of all formal sums $\sum_{i=0}^{\infty} a_i t^i$, where $a_i = 0$ for all but finitely many values of i. For $\sum_{i=0}^{\infty} a_i t^i, \sum_{i=0}^{\infty} b_i t^i \in A[t]$, by writing $\sum_{i=0}^{\infty} a_i t^i = \sum_{i=0}^{\infty} b_i t^i$, we mean that $a_i = b_i$ for all i. Here polynomials are added componentwise. For $\alpha := \sum_{i=0}^{\infty} a_i t^i \in A[t]$, define the support of α by $supp(\alpha) = \{i : a_i \neq 0\}$.

Let $(V, +)$ and $(A, +)$ be abelian groups. Suppose that there is a scalar multiplication of V by A given by the mapping $V \times A \to V$, $(x, a) \mapsto xa$. Then there is an induced mapping $V[t] \times A[t] \mapsto V[t]$, $(v(t), a(t)) \mapsto v(t)a(t)$. Precisely, if $v(t) = \sum_{i=0}^{\infty} v_i t^i$ and $a(t) = \sum_{i=0}^{\infty} a_i t^i$, then $v(t)a(t) = \sum_{i=0}^{\infty} (v_0 a_i + v_1 a_{i-1} + \cdots + v_i a_0) t^i$. The scalar multiplication of V by A is said to be Armendariz if, whenever $v(t)a(t) = 0$ where $v(t) = \sum_{i=0}^{\infty} v_i t^i \in V[t]$ and $a(t) = \sum_{i=0}^{\infty} a_i t^i \in A[t]$, we have $v_i a_j = 0$ for all i and j.

LEMMA 1 ([**8**, Proposition 3.1]). *A right module V over a ring R is distributive if and only if, for any $x, y \in V$ and for any maximal right ideal I of R, there exists $d \in R \backslash I$ such that either $xd \in yR$ or $yd \in xR$.*

THEOREM 2. *Let $(V, +)$ and $(A, +)$ be abelian groups for which there is a mapping $V \times A \to V$, $(x, a) \mapsto xa$. Suppose that V is a distributive left module over a ring R such that $(rx)a = r(xa)$ for all $r \in R, x \in V$ and $a \in A$. Then the scalar multiplication of V by A is Armendariz.*

PROOF. Assume on the contrary that the claim does not hold. Then there exist $v(t) = \sum_{i=0}^{\infty} v_i t^i \in V[t]$ and $a(t) = \sum_{i=0}^{\infty} a_i t^i \in A[t]$ such that

$$(*) \qquad v(t)a(t) = 0 \ \text{ with } \ v_i a_j \neq 0 \ \text{ for some } i \text{ and, } j.$$

Choose $v(t) \in V[t]$ and $a(t) \in A[t]$ with $(*)$ such that $|supp(v(t))| + |supp(a(t))|$ is smallest. Let $k = |supp(v(t))| + |supp(a(t))|$ and let $\alpha(v(t), a(t)) = |\{(i,j) : v_i a_j = 0, v_i \neq 0 \text{ and } a_j \neq 0\}|$. Then $\alpha(v(t), a(t)) < |supp(v(t))| \cdot |supp(a(t))|$. We can further assume that $\alpha(v(t), a(t)) \geq \alpha(v'(t), a'(t))$ for all pairs $(v'(t), a'(t))$ with $(*)$ and with $|supp(v'(t))| + |supp(a'(t))| = k$.

Without loss of generality, we can assume that $v_0 \neq 0$ and $a_0 \neq 0$. From $v(t)a(t) = 0$, we have $v_0 a_0 = 0$. If $v_i a_0 = 0$ for all $i \geq 0$, then $v(t)a_1(t) = 0$ where $a_1(t) = \sum_{i=1}^{\infty} a_i t^i$. As $|supp(v(t))| + |supp(a_1(t))| < k$, by the choice of the pair $(v(t), a(t))$, we must have $v_i a_j =$ for all $i \geq 0$ and all $j \geq 1$; but this shows that $(*)$ does not hold for $v(t)$ and $a(t)$. Hence $v_l a_0 \neq 0$ for some $l > 0$. On the other hand, if $v_0 a_j = 0$ for all $j \geq 0$, then $v_1(t)a(t) = 0$ where $v_1(t) = \sum_{i=1}^{\infty} v_i t^i$. As $|supp(v_1(t))| + |supp(a(t))| < k$, by the choice of the pair $(v(t), a(t))$, we must have $v_i a_j = 0$ for all $i \geq 1$ and all $j \geq 0$; but this also shows that $(*)$ does not hold for $v(t)$ and $a(t)$. Hence $v_0 a_s \neq 0$ for some $s > 0$. Let $P = \{r \in R : r(v_0 a_j) = 0 \text{ for } j = 0, 1, \ldots\})$. Then P is a proper left ideal of R. Take a maximal left ideal M of R with $P \subseteq M$. As ${}_RV$ is a distributive module, by Lemma 1 there exists $d \in R \backslash M$ such that either $dv_0 \in Rv_l$ or $dv_l \in Rv_0$.

Case 1: $dv_0 \in Rv_l$. Write $dv_0 = cv_l$ with $c \in R$. Then $0 = c(v(t)a(t)) = (cv(t))a(t)$. Note that $(cv_l)a_0 = (dv_0)a_0 = d(v_0 a_0) = 0$. If $(*)$ would hold for the pair $(cv(t), a(t))$, then we would have $|supp(cv(t))| + |supp(a(t))| = k$ and $\alpha(cv(t), a(t)) > \alpha(v(t), a(t))$. This would be impossible by the choice of $v(t)$ and $a(t)$. So $(*)$ does not hold for $cv(t)$ and $a(t)$. Hence $cv_i a_j = 0$ for all i and j. In particular, we have $dv_0 a_j = cv_l a_j = 0$ for all $j = 0, 1, \ldots$. But then $d \in P \cap (R \backslash M)$, a contradiction.

Case 2: $dv_l \in Rv_0$. Write $dv_l = cv_0$ with $c \in R$. Then $0 = d(v(t)a(t)) = (dv(t))a(t)$. As $(dv_l)a_0 = (cv_0)a_0 = c(v_0 a_0) = 0$, an argument as in Case 1 shows $dv_0 a_j = 0$ for all $j = 0, 1, \ldots$, and this again leads to a contradiction. $\square$

Corollary 3. *Let R, A be rings and ${}_RV_A$ be a bimodule. If ${}_RV$ is distributive, then V_A is Armendariz.*

Let ${}_RV$ be a module. A function $f : V \to V$ is called homogeneous if $(rx)f = r(xf)$ for all $r \in R$ and $x \in V$. The set of all homogeneous functions of V is denoted by $M(V)$. With respect to function addition, $M(V)$ is an abelian group and it is a near-ring with respect to function addition and composition. The endomorphism ring $\mathrm{End}({}_RV)$ is contained in $M(V)$, but, in general, $M(V)$ is not a ring.

Corollary 4. *Let ${}_RV$ be a distributive module. Then the action of $M(V)$ on V is Armendariz. In particular, V_S is an Armendariz module where $S = \mathrm{End}({}_RV)$.*

It is natural to ask if the module V_R being distributive also implies that V_R is Armendariz. This is not the case by the next example.

Example 5. Let $R = \mathbb{M}_2(\mathbb{Z}_2)$ and $V = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 \end{pmatrix}$. As V is a minimal right ideal of R, V_R is simple and so distributive. For $v(t) = \begin{pmatrix} 0 & \bar{1} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \bar{1} & 0 \\ 0 & 0 \end{pmatrix} t + \begin{pmatrix} \bar{1} & \bar{1} \\ 0 & 0 \end{pmatrix} t^2 \in V[t]$ and $a(t) = \begin{pmatrix} \bar{1} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \bar{1} & 0 \end{pmatrix} t + \begin{pmatrix} \bar{1} & 0 \\ \bar{1} & 0 \end{pmatrix} t^2 \in R[t]$, we have $v(t)a(t) = 0$, but $\begin{pmatrix} 0 & \bar{1} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \bar{1} & 0 \end{pmatrix} \neq 0$. So V_R is not Armendariz.

A module is called endodistributive if it is a distributive module over its endomorphism ring.

LEMMA 6 ([**25**, Theorem 1]). *A ring R is right distributive if and only if every injective right R-module is endodistributive.*

COROLLARY 7. *If R is a right distributive ring, then every right R-module is Armendariz.*

PROOF. Let V be a right R-module, and let E be the injective hull of V. Then ${}_TE$ is a distributive module by Lemma 6, where $T = \mathrm{End}(E_R)$. Since ${}_TE_R$ is a bimodule, E_R is an Armendariz module by Corollary 3, and hence V_R is Armendariz (being a submodule of E_R). $\square$

For a right R-module V and for its endomorphism ring $T := \mathrm{End}(V_R)$, as above ${}_TV_R$ is a bimodule. The endomorphism ring of ${}_TV$ is called the biendomorphism ring of V_R, and is abbreviated $B := \mathrm{BiEnd}(V_R) = \mathrm{End}({}_TV)$. Then ${}_TV_B$ is a bimodule. As R is isomorphic to a subring of B, V_B being Armendariz implies that V_R is Armendariz. The proof of Corollary 7 actually shows a strengthening that if R is a right distributive ring, then every right R-module V is Armendariz as a right module over its biendomorphism ring.

In [**15**, Corollary 2.7], it was shown that, for any module V over $\mathbb{Z}$, $\mathbb{Z} \propto V$ is an Armendariz ring. Corollary 7 can be used to construct more Armendariz rings through trivial extensions. A left (resp., right) chain ring is a ring whose left (resp., right) ideals are linearly ordered by set inclusion. A ring R is said to be a right Prüfer ring if, for any maximal right ideal P of R, the ring of fractions R_P exists and is a right chain ring. Left Prüfer ring is defined similarly. A left and right Prüfer ring is called a Prüfer ring.

COROLLARY 8. *Let R be a domain* (*not necessarily commutative*) *that is a Prüfer ring, and let V be an (R,R)-bimodule. Then $R \propto V$ is an Armendariz ring.*

PROOF. By Brungs [**5**, Theorem 1], a domain is right Prüfer if and only if it is right distributive (by Gilmer [**9**], the commutative distributive domains are the Prüfer domains). Hence R is a distributive ring, and so V is Armendariz as a left and right R-module by Corollary 7. By [**15**, Corollary 2.3], $R \propto V$ is an Armendariz ring. $\square$

COROLLARY 9 ([**21**]). *If R is a left distributive ring, then R is Armendariz.*

One may ask if the converse of Corollary 7 holds. This is not the case by the next example.

EXAMPLE 10. There exists a hereditary ring R that is not left distributive, but every right R-module is Armendariz and every left R-module is Armendariz.

PROOF. By [**6**, Theorem 1.4], the Cozzens domains are all left and right hereditary. By [**29**, Example 5.5], some Cozzens domains are not left distributive. Thus, there exists a domain R that are left and right hereditary and that is not left distributive. We show that every right R-module is Armendariz. As R is right hereditary, every right R-module is projective and hence a direct summand of a free R-module. So to show every right R-module is Armendariz, it suffices to show that every right free R-module is Armendariz. As R is a domain, R_R is an Armendariz module, and hence every free right R-module is Armendariz by [**15**, Lemma 2.5]. □

Remark 11. A right module V over a ring R is called McCoy if, whenever $v(t)f(t) = 0$ with $v(t) \in V[t]$ and $f(t) \in R[t]$, there exists $0 \neq r \in R$ such that $v(t)r = 0$ (see [**7**]). Left McCoy modules are defined similarly. A left (resp., right) duo ring is a ring whose left (resp., right) ideals are ideals. The authors of [**7**] proved that every cyclic right module over a right duo ring is McCoy, and asked whether the ring R is right duo in case every cyclic right R-module is McCoy, or equivalently, whether the ring R is left duo in case every cyclic left R-module is McCoy (see [**7**, Question 1]). The answer to the question is negative. Indeed, let R be the ring as in Example 10. Then every left and every right R-module is Armendariz, so McCoy. As R is left hereditary but not left distributive, R is not left duo by [**26**, Theorem 1].

In view of Corollary 7, we may ask whether every left module over a right distributive ring is Armendariz. Next we give a negative answer to this question. To do so, we need to recall the construction of a generalized power series ring (see [**23**]).

Given a ring R and a strictly ordered monoid $(S, \leq)$, we consider the set A of all mappings $f : S \to R$ whose support $\mathrm{supp}(f) = \{s \in S \mid f(s) \neq 0\}$ is artinian (i.e. it does not contain any infinite strictly decreasing chains of elements) and narrow (i.e. it does not contain infinite subsets of pairwise order-incomparable elements). If $f, g \in A$ and $s \in S$, it turns out that the set

$$\mathrm{X}_s(f, g) = \{(x, y) \in \mathrm{supp}(f) \times \mathrm{supp}(g) : s = xy\}$$

is finite. Thus we can define the product fg: $S \to R$ of $f, g \in A$ as follows:

$$(fg)(s) = \sum_{(x,y)\in \mathrm{X}_s(f,g)} f(x)g(y) \quad \text{for any} \quad s \in S$$

(by convention, a sum over the empty set is 0). With pointwise addition and multiplication as defined above, A becomes a ring, called the ring of generalized power series with coefficients in R and exponents in S, and denoted by $R[[S]]$.

We will use the symbol 1 to denote the identity elements of the monoid S, the ring R and the ring $R[[S]]$. To each $r \in R$ and $s \in S$, we associate elements $\mathsf{c}_r, \mathsf{e}_s \in R[[S]]$ defined by

$$\mathsf{c}_r(x) = \begin{cases} r & \text{if } x = 1 \\ 0 & \text{if } x \in S \setminus \{1\}, \end{cases} \qquad \mathsf{e}_s(x) = \begin{cases} 1 & \text{if } x = s \\ 0 & \text{if } x \in S \setminus \{s\} \end{cases}$$

It is clear that $r \mapsto \mathsf{c}_r$ is a ring embedding of R into $R[[S]]$ and $s \mapsto \mathsf{e}_s$ is a monoid embedding of S into the multiplicative monoid of the ring $R[[S]]$. Furthermore, we have $\mathsf{e}_s\mathsf{c}_r = \mathsf{c}_r\mathsf{e}_s$ for any $r \in R$ and $s \in S$.

Recall that an ordered monoid $(S, \leq)$ is positively ordered if $1 \leq s$ for any $s \in S$. A monoid R is said to be a right chain monoid if the right ideals of R are totally ordered by set inclusion, i.e. if $aR \subseteq bR$ or $bR \subseteq aR$ for any $a, b \in R$. Left chain monoids are defined similarly. If R is left and right chain, then we say that R is a chain monoid.

Example 12. There exists a right distributive ring R such that some left R-module is not Armendariz.

Proof. Let S be the set of all pairs (n, z) such that n is a non-negative integer and $z \in \mathbb{Z}$, except for the pairs of the form $(0, z)$ with $z < 0$. We define a multiplication on S in the following way

$$(n, z)(m, c) = (n + m, 2^m z + c)$$

and order S lexicographically (i.e. $(n, z) \leq (m, c)$ if and only if either $n < m$ or $n = m$ and $z \leq c$). Then $(S, \leq)$ becomes a right chain positively strictly totally ordered (so also cancellative) monoid and since $(1, 0) \notin S(2, 1)$ and $(2, 1) \notin S(1, 0)$, S is not left chain.

By [**20**, Corollary 24] the generalized power series ring $R = \mathbb{Z}_2[[S]]$ is a right chain right duo domain which is not left chain. Obviously R being right chain is right distributive. We will construct a left R-module M which is not McCoy (so obviously also not Armendariz).

Let T be the following subset of S

$$T = \{(m, c) \in S : m \geq 2,\ c \text{ is odd number}\}.$$

Notice that for any $(n, z) \in S$ and $(m, c) \in T$, $(n, z)(m, c) = (n + m, 2^m z + c) \in T$. Thus T is a left ideal of S and considering the set

$$B = \{f \in R : \mathit{supp}(f) \subseteq T\}$$

it is easy to see that B is a left ideal of R.

We consider the cyclic left R-module $M = R/B$ and two non-zero elements

$$f = \mathsf{e}_{(2,3)} + \mathsf{e}_{(2,2)} x \in R[x], \quad m = \bar{\mathsf{e}}_{(0,2)} + \bar{\mathsf{e}}_{(0,1)} x \in M[x]$$

(where bars denote images of elements of R via the canonical homomorphism $R \to R/B$). For these elements we have

$$(**) \qquad fm = [\mathsf{e}_{(2,3)} + \mathsf{e}_{(2,2)} x][\bar{\mathsf{e}}_{(0,2)} + \bar{\mathsf{e}}_{(0,1)} x] = \bar{\mathsf{e}}_{(2,5)} + [\bar{\mathsf{e}}_{(2,4)} + \bar{\mathsf{e}}_{(2,4)}]x + \bar{\mathsf{e}}_{(2,3)} x^2 = 0.$$

Let $g \in R$ be an element such that $g\bar{\mathsf{e}}_{(0,2)} = 0$. Then we have $\mathrm{supp}(g\mathsf{e}_{(0,2)}) \subseteq T$. As $\mathrm{supp}(\mathsf{e}_{(0,2)})) = \{(0, 2)\}$ and our monoid S is cancellative, it is easy to see that $\mathrm{supp}(g\mathsf{e}_{(0,2)}) = \mathrm{supp}(g) \cdot \{(0, 2)\}$. Thus for any element $(n, z) \in \mathrm{supp}(g)$

$$(n, z + 2) = (n, z)(0, 2) \in T$$

which implies that $n \geq 2$ and z is odd number.

Using similar arguments as above we can see that if an element $h \in R$ annihilates $\bar{\mathsf{e}}_{(0,1)}$, then $\operatorname{supp}(h) \cdot \{(0,1)\} \subseteq T$. Thus for any $(n, z) \in \operatorname{supp}(h)$ we have $(n, z+1) = (n, z)(0,1) \in T$ which in this case implies that $n \geq 2$ and z is even number.

Thus there does not exist a non-zero element of R which annihilates $\bar{\mathsf{e}}_{(0,2)}$ and $\bar{\mathsf{e}}_{(0,1)}$ simultaneously. Since we have $(**)$ it follows that M as a left R-module is not McCoy. $\square$

Following [**16**], a ring R is called Armendariz of power series type if, whenever $(\sum_{i\geq 0} a_i t^i)(\sum_{j\geq 0} b_j t^j) = 0$ in $R[[t]]$, $a_i b_j = 0$ for all i and j. An Armendariz ring of power series type is also called a power-serieswise Armendariz ring in [**13**]. It is known that reduced rings are Armendariz of power series type, and Armendariz rings of power series type are Armendariz; but the converses are not true (see [**13**] or [**17**]). Because of Corollary 9, it is natural to ask if every right distributive ring is Armendariz of power series type. The answer is no by the next example.

Example 13. There exists a commutative distributive ring that is not Armendariz of power series type.

Proof. Let $R = \mathbb{Z} \propto \mathbb{Z}_{2^\infty}$, where $\mathbb{Z}_{2^\infty}$ is the Prüfer group. Then R is not an Armendariz ring of power series type by [**17**, Example 3]. Next we show that R is distributive. Let $(a, x), (b, y) \in R$ and let P be a maximal ideal of R. Then $P = p\mathbb{Z} \propto \mathbb{Z}_{2^\infty}$, where p is a prime. Write $a = p^r a_1$ and $b = p^s b_1$ with $r \geq s \geq 0$ and with $\gcd(p, a_1) = \gcd(p, b_1) = 1$.

Case 1: $p \neq 2$. As $x, y \in \mathbb{Z}_{2^\infty}$, there exists $k \geq 0$ such that $2^k x = 2^k y = 0$. We have $(a, x)(2^k b_1, 0) = (b, y)(2^k p^{r-s} a_1, 0) \in (b, y)R$ with $(2^k b_1, 0) \in R\backslash P$. So R is distributive by Lemma 1.

Case 2: $p = 2$. As $2^{r-s} a_1 y - x b_1 \in \mathbb{Z}_{2^\infty}$ and $\mathbb{Z}_{2^\infty}$ is divisible, there exists $z \in \mathbb{Z}_{2^\infty}$ such that $(2^r a_1) z = 2^{r-s} a_1 y - x b_1$. Thus, we have $(a, x)(b_1, z) = (b, y)(2^{r-s} a_1, 0) \in (b, y)R$ with $(b_1, z) \in R\backslash P$. So R is a distributive ring by Lemma 1. $\square$

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