

On the geometry of sets satisfying the sequence selection property

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Abstract. In this paper we study fundamental directional properties of sets under the assumption of condition (SSP) (introduced in [3]). We show several transversality theorems in the singular case and an (SSP)-structure preserving theorem. As a geometric illustration, our transversality results are used to prove several facts concerning complex analytic varieties in 3.3. Also, using our results on sets with condition (SSP), we give a classification of spirals in the appendix 5.

The (SSP)-property is most suitable for understanding transversality in the Lipschitz category. This property is shared by a large class of sets, in particular by subanalytic sets or by definable sets in an o-minimal structure.

1. Introduction.

The notions of tangent cone and direction set have taken a very important role in the study of several equisingularity type problems, in particular, after the pioneering works of H. Whitney [8], [9], to the topological equisingularity problem. For instance, I. Nakai discussed and used directional properties in [6] in order to show the appearance of topological moduli in a family of polynomial map-germs $(\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$ for $n \geq 3$, $p \geq 2$. On the other hand, the authors showed in [3] that the dimension of the common direction set of two subanalytic subsets, called the *directional dimension*, is preserved by a bi-Lipschitz homeomorphism provided that their images are also subanalytic. In order to prove this result we introduced and employed in an essential way the notion of sequence selection property ((SSP) for short). (SSP) is a notion based on the direction set. It takes an important role in the study of Lipschitz equisingularity. Using the aforementioned theorem in [3], we can see that the Oka family [7] is not Lipschitz trivial as a family of zero-sets of real polynomial function germs. Our aim in this paper is to study the geometry of sets satisfying (SSP), their behaviour under bi-Lipschitz transformations and to point out applications to complex singularities and also other fields.

In order to do this, we introduce the notions of transversality and weak transversality, using the real cone (half-cone) of the direction set, essential tools for understanding the sets satisfying condition (SSP). Our main concern is to decide under which conditions the transversality of sets is preserved by (bi-Lipschitz) homeomorphisms. In particular we show that the transversality for complex analytic sets is preserved by bi-Lipschitz

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homeomorphisms (Theorem 3.2), provided that their images are also complex analytic sets, and that the weak transversality for general sets is preserved by bi-Lipschitz homeomorphisms, provided that one of them and its image satisfy the sequence selection property (Theorems 3.5 and 3.11). In fact the weak transversality is preserved for arbitrary sets if the bi-Lipschitz homeomorphism satisfies the condition *semiline-(SSP)*, simply a corollary of Theorem 2.25.

In addition, we introduce and study the notion of (SSP) mappings. We show that the (SSP) structure is preserved by (SSP) bi-Lipschitz homeomorphisms (Theorem 4.7). In general the behaviour of a merely bi-Lipschitz homeomorphism can be very wild in respect to the direction sets. We show that whenever a bi-Lipschitz homeomorphism is also an (SSP) mapping, this is no longer the case. Indeed, we are able to control this behaviour by either considering it in regard to sets satisfying condition (SSP) or by considering bi-Lipschitz homeomorphisms endowed with extra properties. In particular, we look for those homeomorphisms with a good directional behaviour and we single out two large classes of examples.

2. Directional properties of sets.

Let us recall our notion of direction set. For simplicity in this paper we only consider the direction sets at the origin.

DEFINITION 2.1. Let A be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$. We define the *direction set* $D(A)$ of A at $0 \in \mathbb{R}^n$ by

$$D(A) := \left\{ a \in S^{n-1} \mid \exists \{x_i\} \subset A \setminus \{0\}, x_i \rightarrow 0 \in \mathbb{R}^n \text{ s.t. } \frac{x_i}{\|x_i\|} \rightarrow a, i \rightarrow \infty \right\}.$$

Here S^{n-1} denotes the unit sphere centred at $0 \in \mathbb{R}^n$.

For a subset $A \subset S^{n-1}$, we denote by $L(A)$ a half-cone of A with the origin $0 \in \mathbb{R}^n$ as the vertex:

$$L(A) := \{ta \in \mathbb{R}^n \mid a \in A, t \geq 0\}.$$

In the case A is a point (not the origin) we call $L(A)$ a *semiline*. For a set-germ A at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$, we put $LD(A) := L(D(A))$, and call it the *real tangent cone* at $0 \in \mathbb{R}^n$.

Let $U, V \subset \mathbb{R}^n$ such that $0 \in \overline{U} \cap \overline{V}$. The following are true:

- (1) $D(\overline{U}) = D(U)$.
- (2) $D(U \cup V) = D(U) \cup D(V)$.
- (3) $\overline{\bigcup_i D(U_i)} \subseteq D(\bigcup U_i)$.
- (4) If U_i are half-cones then $\overline{\bigcup_i D(U_i)} = D(\bigcup U_i)$.
- (5) $D(U \cap V) \subseteq D(U) \cap D(V)$.

2.1. Condition (SSP).

In [3] sea-tangle properties and directional properties of sets with the sequence selection property played an essential role in the proof of the main theorem (cf. Theorem 2.2). For the reader's convenience let us recall the main theorem in [3]. See H. Hironaka [2] for the definition of subanalyticity.

THEOREM 2.2 (Main Theorem in [3]). *Let $A, B \subset \mathbb{R}^n$ be subanalytic set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}$, and let $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism. Suppose that $h(A), h(B)$ are also subanalytic. Then we have the equality of dimensions,*

$$\dim(D(h(A)) \cap D(h(B))) = \dim(D(A) \cap D(B)).$$

We denote by (SSP) the sequence selection property for short. Here we introduce a generalised notion of (SSP) relatively to a subset of \mathbb{R}^n .

DEFINITION 2.3. Let A, B be two set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}, D(A) \subseteq D(B)$. We say that A satisfies *condition (SSP)-relative to B* , if for any sequence of points $\{a_m\}$ of B tending to $0 \in \mathbb{R}^n$ such that $\lim_{m \rightarrow \infty} a_m / \|a_m\| \in D(A)$, there is a sequence of points $\{b_m\} \subset A$ such that

$$\|a_m - b_m\| \ll \|a_m\|, \|b_m\|.$$

In the case $B = \mathbb{R}^n$ we will not mention B (it is the usual (SSP) condition).

Concerning this relative condition (SSP), we can easily show the following:

PROPOSITION 2.4. *The relative condition (SSP) is transitive, namely if A satisfies condition (SSP)-relative to B and B satisfies condition (SSP)-relative to C , then A satisfies condition (SSP)-relative to C .*

We give some remarks on the relative condition (SSP) ((2) and (3) follow from the above proposition).

REMARK 2.5.

- (1) A (resp. \overline{A}) satisfies condition (SSP)-relative to \overline{A} (resp. A).
- (2) A satisfies condition (SSP) if and only if A satisfies condition (SSP)-relative to $LD(A)$.
- (3) A satisfies condition (SSP) if and only if \overline{A} satisfies condition (SSP).
- (4) Let $A \subset \mathbb{R}^n$ be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$, and let $d, C > 0$. The *sea-tangle neighbourhood* $ST_d(A; C)$ of A , of degree d and width C , is defined by:

$$ST_d(A; C) := \{x \in \mathbb{R}^n \mid \text{dist}(x, A) \leq C|x|^d\}.$$

Then, A satisfies condition (SSP)-relative to $ST_d(A; C), d > 1$.

In this paper we consider also the notion of weak sequence selection property, denoted by (WSSP) for short.

DEFINITION 2.6. Let A, B be two set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}$, $D(A) \subseteq D(B)$. We say that A satisfies *condition (WSSP)-relative to B* , if for any sequence of points $\{a_m\}$ of B tending to $0 \in \mathbb{R}^n$ such that $\lim_{m \rightarrow \infty} a_m / \|a_m\| \in D(A)$, there is a subsequence $\{m_j\}$ of $\{m\}$ with $\{b_{m_j}\} \subset A$ such that

$$\|a_{m_j} - b_{m_j}\| \ll \|a_{m_j}\|, \|b_{m_j}\|.$$

We have the following characterisation of condition (SSP). The proof in the relative case is similar to the non-relative case for which we gave a detailed proof in [4]. We sketch a slightly rough proof here.

PROPOSITION 2.7. *Let A, B be two set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}$. If A satisfies condition (WSSP)-relative to B , then it satisfies condition (SSP)-relative to B . Namely, the conditions relative (SSP) and relative (WSSP) are equivalent.*

PROOF. Assume that A does not satisfy condition (SSP). Then there is a sequence of points $\{a_m \in B\}$ tending to $0 \in \mathbb{R}^n$ such that $\lim_{m \rightarrow \infty} a_m / \|a_m\| \in D(A)$ and $\lim_{m \rightarrow \infty} d(a_m, A) / \|a_m\| = \alpha > 0$, where $d(a_m, A)$ denotes the distance between a_m and A . This implies that there does not exist a sequence of points $\{b_m\} \subset A$ such that $\|a_m - b_m\| \ll \|a_m\|$. Therefore A does not satisfy condition (WSSP). \square

We make some remarks on (SSP):

REMARK 2.8.

- (1) In fact one can easily see that A satisfies condition (SSP)-relative to B if and only if for any sequence of points $\{a_m\}$ of B tending to $0 \in \mathbb{R}^n$ such that $\lim_{m \rightarrow \infty} a_m / \|a_m\| \in D(A)$, then $d(a_m, A) / \|a_m\|$ tends to $0 \in \mathbb{R}$. (Or there is a subsequence which tends to zero.)
- (2) Condition (SSP) is C^1 invariant, but not bi-Lipschitz invariant (cf. Section 5 in [3]). Note that condition (SSP) is invariant under a bi-Lipschitz homeomorphism $h : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$. We leave the proof of this fact to the interested reader.

As stated in the above remark, the condition (SSP) is not bi-Lipschitz invariant. However if a map h is bi-Lipschitz, we have the following:

LEMMA 2.9. *Let $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism, and let A, B be two set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}$. Then A satisfies condition (SSP)-relative to B and $D(B) = D(A)$ if and only if $h(A)$ satisfies condition (SSP)-relative to $h(B)$ and $D(h(B)) = D(h(A))$. From this we can conclude that if A satisfies condition (SSP), then $D(h(A)) = D(h(LD(A)))$ and $h(A)$ satisfies condition (SSP)-relative to $h(LD(A))$ ($B = LD(A)$).*

PROOF. Use (1) of Remark 2.8. \square

Below we give several examples of sets satisfying the condition (SSP).

REMARK 2.10. Let $A, B \subseteq \mathbb{R}^n$ be set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}$.

- (1) The cone $LD(A)$ satisfies condition (SSP).
- (2) If A is subanalytic or definable in an o-minimal structure, then it satisfies condition (SSP).
- (3) If A is a finite union of sets, all of which satisfy condition (SSP), then A satisfies condition (SSP).
- (4) If $0 \in A$, a C^1 manifold, then it satisfies condition (SSP) and $LD(A) = T_0(A)$. (This is not necessarily true for C^0 manifolds or if $0 \notin A$.)
- (5) If $A \subseteq B$, $D(A) = D(B)$, A satisfies condition (SSP), then B satisfies condition (SSP).
- (6) If $A \cup \{0\}$ is path connected with $D(A)$ a point, then A satisfies condition (SSP). The trajectories of the gradient flow of an analytic function satisfy this property; this is the famous gradient conjecture of R. Thom, proven in [5]. They may not be always subanalytic.
- (7) If $D(A) = \{a_1, \dots, a_k\}$ and there are subsets $A_i \subseteq A$, $D(A_i) = \{a_i\}$ and $A_i \cup \{0\}$, $i = 1, \dots, k$, are path connected, then A satisfies condition (SSP).

We give one more important example satisfying condition (SSP).

PROPOSITION 2.11 (Proposition 6.5 in [3]). *Let $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism, and let $A, h(A) \subset \mathbb{R}^n$ be subanalytic set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$. Then the set $h(LD(A))$ satisfies condition (SSP).*

Concerning the condition (SSP) it is important to remember that $LD(A)$ satisfies condition (SSP) for any subset $A, 0 \in \overline{A}$. Accordingly we will try to replace A by its real tangent cone $LD(A)$ whenever possible and convenient. The remaining results of this subsection are in this spirit. We recall the following lemma.

LEMMA 2.12 (Lemma 5.6 in [3]). *Let $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism, and let $A \subset \mathbb{R}^n$ be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$. Then $D(h(A)) \subset D(h(LD(A)))$. If A satisfies condition (SSP) or if h is a C^1 -diffeomorphism the equality holds.*

Using the above lemmas we can improve Proposition 2.11. In fact, we gave an improvement in the non-relative case in [4]. Here we generalise it to the relative case.

THEOREM 2.13. *Let $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism, and let $A \subset \mathbb{R}^n$ be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$, and $B \subset \mathbb{R}^n$ a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{B}$. Assume that A satisfies condition (SSP). Then $h(A)$ satisfies condition (SSP)-relative to B if and only if $h(LD(A))$ satisfies condition (SSP)-relative to B .*

PROOF. Let us assume that $h(A)$ satisfies condition (SSP)-relative to B . By assumption, A satisfies condition (SSP). Therefore it follows from Lemma 2.12 that $D(h(LD(A))) = D(h(A))$. Let $\{y_m\}$ be an arbitrary sequence of points of B tending to $0 \in \mathbb{R}^n$ such that

$$\lim_{m \rightarrow \infty} \frac{y_m}{\|y_m\|} \in D(h(LD(A))) = D(h(A)).$$

Let $y_m = h(x_m)$ for each m . Since $h(A)$ satisfies condition (SSP)-relative B , there is a sequence of points $\{z_m\} \subset A$ such that

$$\|h(x_m) - h(z_m)\| \ll \|h(x_m)\|, \|h(z_m)\|.$$

On the other hand, there is a subsequence $\{z_{m_j}\}$ of $\{z_m\}$ such that $\lim_{m_j \rightarrow \infty} z_{m_j}/\|z_{m_j}\| \in D(A)$. Since $LD(A)$ satisfies condition (SSP), there is a sequence of points $\{\theta_{m_j}\} \subset LD(A)$ such that

$$\|z_{m_j} - \theta_{m_j}\| \ll \|z_{m_j}\|, \|\theta_{m_j}\|.$$

It follows from h being bi-Lipschitz that

$$\|h(z_{m_j}) - h(\theta_{m_j})\| \ll \|h(z_{m_j})\|, \|h(\theta_{m_j})\|.$$

Then we have

$$\|h(x_{m_j}) - h(\theta_{m_j})\| \leq \|h(x_{m_j}) - h(z_{m_j})\| + \|h(z_{m_j}) - h(\theta_{m_j})\| \ll \|h(z_{m_j})\|.$$

Therefore we have

$$\|h(x_{m_j}) - h(\theta_{m_j})\| \ll \|h(x_{m_j})\|, \|h(\theta_{m_j})\|.$$

Thus $h(LD(A))$ satisfies condition (WSSP)-relative to B , and also condition (SSP)-relative to B by Proposition 2.7. The other claim can be proved in a similar way. \square

Note that even if both $h(A)$ and $h(LD(A))$ satisfy condition (SSP), it does not imply that A satisfies condition (SSP) (the spiral example, Figure 1 below).

PROPOSITION 2.14. *Let $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism, and let $A \subset \mathbb{R}^n$ be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \bar{A}$. Then $LD(h(A)) = LD(h(LD(A)))$ and $h(LD(A))$ satisfy condition (SSP) if and only if $LD(h^{-1}(LD(h(A)))) = LD(A)$ and $h^{-1}(LD(h(A)))$ satisfy condition (SSP).*

PROOF. As our conditions are symmetric in h (our bi-Lipschitz homeomorphism) it suffices to prove only the “if” part implication. Since $h^{-1}(LD(h(A)))$ satisfies condition (SSP) it follows that

$$LD(h(A)) = LD(h(h^{-1}(LD(h(A)))))) = LD(h(LD(h^{-1}(LD(h(A)))))),$$

and because we always have

$$LD(h(LD(A))) = LD(h(LD(h^{-1}(h(A)))))) \subseteq LD(h(LD(h^{-1}(LD(h(A))))))$$

it follows that $LD(h(A)) = LD(h(LD(A)))$.

Assume that $\{h(y_m)\}$ is an arbitrary sequence of points of \mathbb{R}^n tending to $0 \in \mathbb{R}^n$ such that

$$\lim_{m \rightarrow \infty} \frac{h(y_m)}{\|h(y_m)\|} \in D(h(LD(A))) = D(h(A)).$$

As cones satisfy condition (SSP) we can assume that $h(y_m) \in LD(h(LD(A))) = LD(h(A))$, so $y_m \in h^{-1}(LD(h(A)))$. Passing to a subsequence, if necessary, we may assume that in fact $\lim_{m \rightarrow \infty} y_m/\|y_m\| \in D(h^{-1}(LD(h(A)))) = D(A)$. Again as cones satisfy condition (SSP) we can claim the existence of a sequence $x_i \in LD(A)$ such that

$$\|y_i - x_i\| \ll \|x_i\|, \|y_i\|.$$

The fact that h is bi-Lipschitz implies that

$$\|h(x_i) - h(y_i)\| \ll \|h(x_i)\|, \|h(y_i)\|.$$

As $h(x_i) \in h(LD(A))$ we proved that $h(LD(A))$ satisfies condition (SSP). □

REMARK 2.15. In order to show $h(LD(A))$ satisfies condition (SSP), we cannot drop the assumption $LD(h^{-1}(LD(h(A)))) = LD(A)$. Indeed if h is the spiral bi-Lipschitz homeomorphism of Example 3.3 in [3], we put $A = \mathbb{R} \times 0$ so that $h(LD(A)) = h(A)$ is a spiral which does not satisfy condition (SSP) (Figure 1 below). Clearly $h^{-1}(LD(h(A))) = \mathbb{R}^2$ so it satisfies condition (SSP), and $LD(h^{-1}(LD(h(A)))) = \mathbb{R}^2 \neq LD(A)$.

In the same spirit we have the following.

PROPOSITION 2.16. *Let $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism, and let $A \subset \mathbb{R}^n$ be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$. The following are equivalent:*

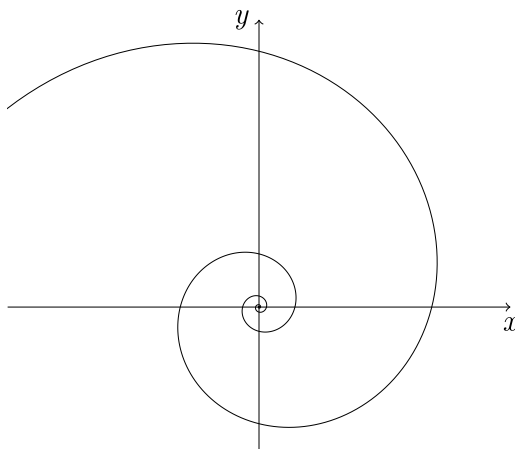


Figure 1.

- (1) $A, h(A)$ both satisfy condition (SSP).
- (2) $A, h^{-1}(LD(h(A)))$ both satisfy condition (SSP) and $LD(h^{-1}(LD(h(A)))) = LD(A)$.
- (3) $h(A), h(LD(A))$ both satisfy condition (SSP) and $LD(h(LD(A))) = LD(h(A))$.

EXAMPLE 2.17. For instance, the situation in the above result happens in the following two general cases.

- (1) If both $A, h(A)$ are subanalytic or definable in an o -minimal structure over \mathbb{R} .
- (2) If A satisfies condition (SSP) and h is a C^1 -diffeomorphism.

2.2. Condition semiline-(SSP).

Our general purpose is to provide a large class of examples of homeomorphisms which preserve the condition (SSP). In this subsection we introduce the condition semiline-(SSP), and we use it to give some characterisations of the condition (SSP). In particular, in the bi-Lipschitz case, we prove that the condition semiline-(SSP) is equivalent to preserving the condition (SSP) (Corollary 2.23). Furthermore we prove that a semiline-(SSP) bi-Lipschitz homeomorphism h , induces a “positive homogeneous” bi-Lipschitz homeomorphism which corresponds the real cones of arbitrary sets A and their images $h(A)$ (Theorem 2.25).

DEFINITION 2.18. We say that a homeomorphism $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ satisfies condition *semiline*-(SSP), if $h(\ell)$ has a unique direction for all semilines ℓ .

REMARK 2.19. The bi-Lipschitz homeomorphisms satisfying condition *semiline*-(SSP) are bi-Lipschitz homeomorphisms which are Gateaux right differentiable.

PROPOSITION 2.20. *Let $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism. Suppose that $h^{-1}(\tau)$ satisfies condition (SSP) for all semilines τ . Then $LD(h(\ell))$ is a semiline for all semilines ℓ , that is h satisfies condition *semiline*-(SSP). (In particular $h(\ell)$ satisfies condition (SSP).)*

PROOF. Indeed, take a semiline ℓ and sequences of points $\{b_i\}, \{c_i\} \subset \ell$ tending to $0 \in \mathbb{R}^n$ such that $LD(h(\{b_i\})) = \ell_1$ and $LD(h(\{c_i\})) = \ell_2$, where ℓ_1, ℓ_2 are semilines. Since ℓ_1 (resp. ℓ_2) satisfies condition (SSP), there is a sequence of points $\{b'_i\}$ with $\{h(b'_i)\} \subset \ell_1$ (resp. $\{c'_i\}$ with $\{h(c'_i)\} \subset \ell_2$) such that

$$\|h(b_i) - h(b'_i)\| \ll \|h(b_i)\|, \|h(b'_i)\| \quad (\text{resp. } \|h(c_i) - h(c'_i)\| \ll \|h(c_i)\|, \|h(c'_i)\|).$$

It follows that

$$\|b_i - b'_i\| \ll \|b_i\|, \|b'_i\| \quad (\text{resp. } \|c_i - c'_i\| \ll \|c_i\|, \|c'_i\|). \quad (2.1)$$

On the other hand, we have

$$\left\{ \lim_{i \rightarrow \infty} \frac{b_i}{\|b_i\|} \right\} = \left\{ \lim_{i \rightarrow \infty} \frac{c_i}{\|c_i\|} \right\} = D(\ell)$$

and $\{c'_i\} \subset h^{-1}(\ell_2)$. By (2.1), we have

$$\left\{ \lim_{i \rightarrow \infty} \frac{b'_i}{\|b'_i\|} \right\} = \left\{ \lim_{i \rightarrow \infty} \frac{b_i}{\|b_i\|} \right\} = D(\ell) \subset D(h^{-1}(\ell_2)).$$

Since $h^{-1}(\ell_2)$ satisfies condition (SSP), there is a sequence of points $\{c''_i\}$ with $h(\{c''_i\}) \subset \ell_2$ such that

$$\|b'_i - c''_i\| \ll \|b'_i\|, \|c''_i\|.$$

This implies that

$$D(\ell_1) = \left\{ \lim_{i \rightarrow \infty} \frac{h(b'_i)}{\|h(b'_i)\|} \right\} = \left\{ \lim_{i \rightarrow \infty} \frac{h(c''_i)}{\|h(c''_i)\|} \right\} = D(\ell_2),$$

that is $\ell_1 = \ell_2$. □

We have the following corollaries.

COROLLARY 2.21. *In the case of a bi-Lipschitz homeomorphism, the condition semiline-(SSP) is equivalent with asking that $h(\ell)$ satisfies condition (SSP) for all semilines ℓ . Moreover in the bi-Lipschitz case it follows that h satisfies condition semiline-(SSP) is equivalent to h^{-1} satisfies condition semiline-(SSP).*

PROOF. Indeed assume that $h(\ell)$ satisfies condition (SSP) for all semilines ℓ . From the result above it follows that h^{-1} satisfies condition semiline-(SSP), and therefore it satisfies condition (SSP) as well. This in turn shows that h satisfies condition semiline-(SSP). □

COROLLARY 2.22. *Let $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism, and let $A \subset \mathbb{R}^n$ be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$. Suppose that A satisfies condition (SSP), and h satisfies condition semiline-(SSP). Then $h(A)$ satisfies condition (SSP).*

PROOF. Let ℓ be an arbitrary semiline contained in $LD(h(A))$. Then there is a sequence of points $\{a_i\} \subset A$ tending to $0 \in \mathbb{R}^n$ such that $LD(\{h(a_i)\}) = \ell$. Since ℓ satisfies condition (SSP), there is a sequence of points $\{c_i\}$ with $\{h(c_i)\} \subset \ell$ such that

$$\|h(a_i) - h(c_i)\| \ll \|h(a_i)\|, \|h(c_i)\|.$$

It follows that

$$\|a_i - c_i\| \ll \|a_i\|, \|c_i\|.$$

Therefore we have $LD(\{a_i\}) = LD(\{c_i\}) \subset LD(h^{-1}(\ell))$. We can use the previous proposition to claim that $LD(\{a_i\}) = LD(\{c_i\}) = LD(h^{-1}(\ell))$ is a semiline $\ell_1 \subset LD(A)$.

Let $\{b_i\}$ be an arbitrary sequence of points tending to $0 \in \mathbb{R}^n$ such that $LD(\{h(b_i)\}) = \ell \subset LD(h(A))$. Since ℓ satisfies condition (SSP), there is a sequence

of points $\{b'_i\}$ with $\{h(b'_i)\} \subset \ell$ such that

$$\|h(b_i) - h(b'_i)\| \ll \|h(b_i)\|, \|h(b'_i)\|.$$

It follows that

$$\|b_i - b'_i\| \ll \|b_i\|, \|b'_i\|. \tag{2.2}$$

Note that $\{b'_i\} \subset h^{-1}(\ell)$. Therefore we have

$$LD(\{b'_i\}) = LD(h^{-1}(\ell)) = \ell_1 \subset LD(A).$$

Since A satisfies condition (SSP), there is a sequence of points $\{b''_i\} \subset A$ such that

$$\|b'_i - b''_i\| \ll \|b'_i\|, \|b''_i\|. \tag{2.3}$$

By (2.2) and (2.3), we have

$$\|b_i - b''_i\| \ll \|b_i\|, \|b''_i\|.$$

It follows that

$$\|h(b_i) - h(b''_i)\| \ll \|h(b_i)\|, \|h(b''_i)\|.$$

Since $\{h(b''_i)\} \subset h(A)$, $h(A)$ satisfies condition (SSP). □

Using the above corollary, we can see the following:

COROLLARY 2.23. *Let $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism. Then the following are equivalent:*

- (1) h has the property that for any set-germ at $0 \in \mathbb{R}^n$, $A \subset \mathbb{R}^n$ such that $0 \in \bar{A}$, we have that A satisfies condition (SSP) if and only if $h(A)$ satisfies condition (SSP).
- (2) h (so h^{-1}) satisfies condition semiline-(SSP).

REMARK 2.24. Take a germ of a semiarc $\gamma : ([0, \epsilon), 0) \rightarrow (\mathbb{R}^n, 0)$ with a unique direction, say $\ell = LD(\gamma)$. (It is not difficult to see that γ satisfies condition (SSP).) It follows from Proposition 2.20 that for a bi-Lipschitz homeomorphism $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ where h^{-1} satisfies condition semiline-(SSP), we do have that $h(\gamma)$ has also a unique direction. Indeed, we can easily see that $LD(h(\gamma)) = LD(h(LD(\gamma))) = LD(h(\ell))$ is also a semiline. Let

$$\mathcal{S}\mathcal{L} := \{\gamma : ([0, \epsilon), 0) \rightarrow (\mathbb{R}^n, 0) \mid LD(\gamma) \text{ is a semiline}\}.$$

The above argument implies that if h^{-1} satisfies condition semiline-(SSP), then the map $h : \mathcal{S}\mathcal{L} \rightarrow \mathcal{S}\mathcal{L}$ induces a map $\bar{h} : S^{n-1} \rightarrow S^{n-1}$ defined by $\bar{h}(D(\gamma)) = D(h(\gamma))$ for

$\gamma \in \mathcal{S}\mathcal{L}$. If both h, h^{-1} satisfy condition semiline-(SSP), then $\bar{h} : S^{n-1} \rightarrow S^{n-1}$ is a one-to-one correspondence, in other words, $\bar{h} : S^{n-1} \rightarrow S^{n-1}$ is bijective.

Note that in the case where $\gamma : ([0, \epsilon), 0) \rightarrow (\mathbb{C}^n, 0)$, $\gamma \in \mathcal{S}\mathcal{L}$, we have that the complex tangent cone, $LD^*(\gamma)$, is a complex line, and all complex lines can be obtained in this way (see 2.39 for a definition of the complex tangent cone).

THEOREM 2.25. *Let $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism such that h (so h^{-1}) satisfies condition semiline-(SSP). Then the induced map $\bar{h} : S^{n-1} \rightarrow S^{n-1}$ given in Remark 2.24 extends to a bi-Lipschitz homeomorphism $\bar{h} : \mathbb{R}^n \rightarrow \mathbb{R}^n$, and for any set-germ at $0 \in \mathbb{R}^n$, $A \subset \mathbb{R}^n$ such that $0 \in \bar{A}$, we have $\bar{h}(D(A)) = D(h(LD(A))) = D(h(A))$. In particular we have $\dim D(A) = \dim D(h(A))$.*

PROOF. First we prove the result for A which satisfies condition (SSP). Indeed $D(A) = D(LD(A))$ and the latest satisfies condition (SSP).

Let us put $\ell_a := \{ta \mid t \geq 0\}$ for $a \in S^{n-1}$. Then we have $LD(A) = \bigcup_{a \in D(A)} \ell_a$. Let us assume that A satisfies condition (SSP), then we have the following:

$$\begin{aligned} \bar{h}(D(A)) &= \left(\bigcup_{a \in D(A)} LD(h(\ell_a)) \right) \cap S^{n-1} \\ &\subseteq LD\left(\bigcup_{a \in D(A)} h(\ell_a) \right) \cap S^{n-1} \\ &= LD\left(h\left(\bigcup_{a \in D(A)} \ell_a \right) \right) \cap S^{n-1} \\ &= LD(h(LD(A))) \cap S^{n-1} = D(h(A)). \end{aligned}$$

By Corollary 2.22, $h(A)$ also satisfies condition (SSP). Using the same argument as above, we have

$$(\bar{h})^{-1}(D(h(A))) \subset D(h^{-1}(h(A))) = D(A).$$

It follows that

$$D(h(A)) \subset \bar{h}(D(A)) \subset D(h(A)).$$

Therefore we have $\bar{h}(D(A)) = D(h(A))$.

Since $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ is a bi-Lipschitz homeomorphism, there are positive numbers $K_1, K_2 \in \mathbb{R}$ with $0 < K_1 \leq K_2$, called *Lipschitz constants*, such that

$$K_1 \|x_1 - x_2\| \leq \|h(x_1) - h(x_2)\| \leq K_2 \|x_1 - x_2\|$$

in a small neighbourhood of $0 \in \mathbb{R}^n$. Let $\bar{h} : S^{n-1} \rightarrow S^{n-1}$, $S^{n-1} \subset \mathbb{R}^n$, be the mapping defined by

$$\bar{h}(a) = \lim_{t \rightarrow 0} \frac{h(ta)}{\|h(ta)\|}.$$

Let $a, b \in S^{n-1}$. Then for sufficiently small, arbitrary $t > 0$, we have

$$\begin{aligned} \left\| \frac{h(ta)}{\|h(ta)\|} - \frac{h(tb)}{\|h(tb)\|} \right\| &\leq \frac{\|h(ta) - h(tb)\|}{\min(\|h(ta)\|, \|h(tb)\|)} \\ \frac{\|h(ta) - h(tb)\|}{\min(\|h(ta)\|, \|h(tb)\|)} &\leq \frac{K_2\|ta - tb\|}{\min(K_1\|ta\|, K_1\|tb\|)} \leq \frac{K_2}{K_1}\|a - b\|. \end{aligned}$$

Taking the limit as $t \rightarrow 0^+$, we have $\|\bar{h}(a) - \bar{h}(b)\| \leq (K_2/K_1)\|a - b\|$. Therefore it follows that \bar{h} is a bi-Lipschitz homeomorphism. It is not difficult to extend \bar{h} to a global bi-Lipschitz homeomorphism, we put $\bar{h}(tx) = t\bar{h}(x)$, $x \in S^{n-1}$ (its radial extension).

Our proof shows that in fact $D(h(A)) \subseteq D(h(LDA)) = \bar{h}(D(A))$ for any A . Because $\bar{h}^{-1} = \overline{h^{-1}}$, the equality $D(h(A)) = D(h(LDA)) = \bar{h}(D(A))$ holds in general. \square

REMARK 2.26. In particular the above property holds for any definable bi-Lipschitz homeomorphism, and for any subanalytic bi-Lipschitz homeomorphism (for the subanalytic case see [1]).

REMARK 2.27. The assumption on h cannot be much relaxed. Indeed, consider a bi-Lipschitz zig-zag homeomorphism $h : \mathbb{R} \rightarrow \mathbb{R}$ (in particular it preserves the (SSP) property) whose graph is like in Example 4.11, Figure 3 below. Then $H := 1 \times h : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$ is a bi-Lipschitz homeomorphism and for the semiline $A = \{(t, t) \mid t \geq 0\}$, $H(A)$ is exactly that part of the graph of h which is a zigzag. Therefore $\dim D(H(A)) = 1$ (even A satisfies condition (SSP)), but $D(A)$ is only a point. Clearly H does not satisfy semiline-(SSP).

COROLLARY 2.28. Let $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism such that $h(h^{-1})$ satisfies condition semiline-(SSP), and let $A \subset \mathbb{R}^n$ be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \bar{A}$. Then $LD(A)$ and $LD(h(A))$ are bi-Lipschitz homeomorphic.

PROOF. Indeed by the previous result we have that $D(A)$ and $D(h(A))$ are bi-Lipschitz homeomorphic, and the radial extension of \bar{h} gives the result. \square

2.3. Directional properties of intersection sets.

In this subsection we treat some directional properties of intersections. Even if A, B satisfy condition (SSP), $A \cap B$ does not always satisfy condition (SSP).

PROPOSITION 2.29. Let $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism, and let $U, V \subset \mathbb{R}^n$ be closed cones with $0 \in \mathbb{R}^n$ as the vertex. Suppose that $h(U)$ satisfies condition (SSP). Then $D(h(U \cap V)) = D(h(U)) \cap D(h(V))$.

PROOF. Since the inclusion \subset is obvious, we show \supset here. Let α be an arbitrary element of $D(h(U)) \cap D(h(V))$. Then there is a sequence of points $\{a_m\} \subset V$ tending to $0 \in \mathbb{R}^n$ such that $\lim_{m \rightarrow \infty} h(a_m)/\|h(a_m)\| = \alpha$. Since $h(U)$ has condition (SSP), there

is a sequence of points $\{b_m\} \subset U$ tending to $0 \in \mathbb{R}^n$ such that

$$\|h(a_m) - h(b_m)\| \ll \|h(a_m)\|, \|h(b_m)\|.$$

It follows that

$$\|a_m - b_m\| \ll \|a_m\|, \|b_m\|. \quad (2.4)$$

On the other hand, there is a subsequence $\{a_{m_j}\}$ of $\{a_m\}$ such that

$$\lim_{m_j \rightarrow \infty} \frac{a_{m_j}}{\|a_{m_j}\|} = \beta \in D(V).$$

By (2.4) we have

$$\lim_{m_j \rightarrow \infty} \frac{b_{m_j}}{\|b_{m_j}\|} = \beta \in D(U).$$

Since U, V are closed cones, $\beta \in D(U) \cap D(V) \subset U \cap V$. Let $\tilde{\beta}$ denote the real half line through 0 and β . Then $\tilde{\beta} \subset U \cap V$. Note that $\tilde{\beta}$ satisfies condition (SSP). Therefore there is a sequence of points $\{c_{m_j}\} \subset \tilde{\beta}$ tending to $0 \in \mathbb{R}^n$ such that

$$\|a_{m_j} - c_{m_j}\| \ll \|a_{m_j}\|, \|c_{m_j}\|.$$

This implies

$$\|h(a_{m_j}) - h(c_{m_j})\| \ll \|h(a_{m_j})\|, \|h(c_{m_j})\|.$$

Thus

$$\lim_{m_j \rightarrow \infty} \frac{h(c_{m_j})}{\|h(c_{m_j})\|} = \lim_{m_j \rightarrow \infty} \frac{h(a_{m_j})}{\|h(a_{m_j})\|} = \alpha.$$

It follows that $\alpha \in D(h(U \cap V))$. Thus $D(h(U)) \cap D(h(V)) \subset D(h(U \cap V))$. \square

Using a similar argument to the above proposition, we can generalise it as follows:

THEOREM 2.30. *Let $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism, and let $U, V \subset \mathbb{R}^n$ be set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \bar{U} \cap \bar{V}$. Suppose that $D(U \cap V) = D(U) \cap D(V)$, and $U \cap V$ and $h(U)$ satisfy condition (SSP). Then $D(h(U \cap V)) = D(h(U)) \cap D(h(V))$.*

REMARK 2.31. We cannot drop any assumption from the above theorem.

- (1) $D(U \cap V) = D(U) \cap D(V)$: Let $h : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ be the identity map, and let $V = \{1/m \mid m \in \mathbb{N}\}$ and $U = \mathbb{R} \setminus V$. Then $D(U \cap V) \neq D(U) \cap D(V)$, and $U \cap V = \emptyset$ and $h(U) = U$ satisfy condition (SSP). But $D(h(U) \cap h(V)) \neq D(h(U)) \cap D(h(V))$.

- (2) (SSP) of $U \cap V$: Let $h : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ be the inverse of the slow spiral bi-Lipschitz homeomorphism given in Example 3.3 of [3] (see Figure 1 and Remark 2.15), and let A, B be spirals on the source space mapped by h to two lines ℓ_1, ℓ_2 through the origin on the target space, respectively. We set $U = A \cup (B \setminus m)$ and $V = A \cup (B \cap m)$, where m is a half line with $0 \in \mathbb{R}^2$ as an end point. Then $D(U \cap V) = D(U) \cap D(V) = S^1, h(U) = \ell_1 \cup (\ell_2 \setminus C)$, where C is a sequence of points on ℓ_2 convergent to $0 \in \mathbb{R}^2$, satisfies condition (SSP) and $U \cap V = A$ does not satisfy condition (SSP). On the other hand, we can see that $D(h(U) \cap h(V)) = \ell_1 \cap S^1$ and $D(h(U)) \cap D(h(V)) = (\ell_1 \cup \ell_2) \cap S^1$.
- (3) (SSP) of $h(U)$: Let $h : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ be the zigzag bi-Lipschitz homeomorphism given in Example 3.4 of [3], and let $U = \{y = 0\}$ and $V = \{y = ax\}$ for a sufficiently small positive number $a > 0$. Then $D(U \cap V) = D(U) \cap D(V) = \emptyset, U \cap V = \{0\}$ satisfies condition (SSP) and $h(U)$ does not satisfy condition (SSP) (see Remark 5.4 in [3]). On the other hand, we can see that $D(h(U \cap V)) = \emptyset$ but $D(h(U)) \cap D(h(V)) \neq \emptyset$.

REMARK 2.32 (Example 5.2 (2) in [3]). Let T be an angle with vertex at $O \in \mathbb{R}^2$. We choose sequences of points $\{P_m\}$ and $\{Q_m\}$ on the edges of T such that $\overline{OP_m} = 1/m^2$ and $\overline{OQ_m} = (1/2)(1/m^2 + 1/(m+1)^2)$, and let C_2 be the zigzag curve connecting P_m 's and Q_m 's. Then C_2 satisfies condition (SSP).

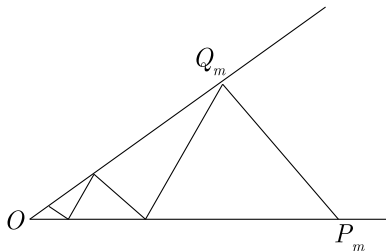


Figure 2.

Suppose that there are a subanalytic curve U and a bi-Lipschitz homeomorphism $h : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ such that $h(U) = C_2$ satisfies condition (SSP). Let V be a half line arbitrarily close to $LD(U)$, therefore close to U as well, such that $U \cap V = \{0\}$ and $D(U \cap V) = D(U) \cap D(V) = \emptyset$. On the other hand, $D(h(U \cap V)) = \emptyset$ and $D(h(U)) \cap D(h(V)) \neq \emptyset$, as the image $h(V)$ has to be arbitrarily close to the zigzag C_2 . By Theorem 2.30, we see that C_2 cannot be the image of any subanalytic curve by any bi-Lipschitz homeomorphism.

2.4. Directional properties of product sets.

We give some elementary set-theoretical properties concerning the condition (SSP).

PROPOSITION 2.33 (Product). *Let $A \subset \mathbb{R}^m$ be a set-germ at $0 \in \mathbb{R}^m$ such that $0 \in \overline{A}$ and let $B \subset \mathbb{R}^n$ be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{B}$. Then A, B satisfy condition (SSP) at $0 \in \mathbb{R}^m, 0 \in \mathbb{R}^n$ respectively if and only if $A \times B$ satisfies condition (SSP) at $(0, 0) \in \mathbb{R}^m \times \mathbb{R}^n$.*

PROOF. We first show the “only if” part. Let $\{(a_k, b_k)\}$ be an arbitrary sequence of points of $\mathbb{R}^m \times \mathbb{R}^n$ tending to $(0, 0) \in \mathbb{R}^m \times \mathbb{R}^n$ such that

$$\lim_{k \rightarrow \infty} \frac{(a_k, b_k)}{\|(a_k, b_k)\|} = (a, b) \in D(A \times B).$$

In the case where $\|a\|, \|b\| \neq 0$, $\lim_{k \rightarrow \infty} a_k/\|a_k\| = a/\|a\| \in D(A)$ and $\lim_{k \rightarrow \infty} b_k/\|b_k\| = b/\|b\| \in D(B)$. Therefore it is easy to see that there exists a sequence of points $\{(c_k, d_k)\}$ of $A \times B$ tending to $(0, 0) \in \mathbb{R}^m \times \mathbb{R}^n$ such that

$$\|(a_k, b_k) - (c_k, d_k)\| \ll \|(a_k, b_k)\|, \|(c_k, d_k)\|.$$

Let us assume that $\|a\| = 0$ and $\|b\| = 1$. Then $\|a_k\| \ll \|b_k\|$ and $\lim_{k \rightarrow \infty} b_k/\|b_k\| \in D(B)$. Since B satisfies condition (SSP) at $0 \in \mathbb{R}^n$, there is a sequence of points $\{d_k\}$ of B tending to $0 \in \mathbb{R}^n$ such that

$$\|b_k - d_k\| \ll \|b_k\|, \|d_k\|.$$

Let $\{c_j\}$ be a sequence of points of A tending to $0 \in \mathbb{R}^m$ such that $\lim_{j \rightarrow \infty} c_j/\|c_j\| \in D(A)$. Take a subsequence $\{c_{j_k}\}$ of $\{c_j\}$ so that $\|c_{j_k}\| < (1/k)\|d_k\|$. Then $\{(c_{j_k}, d_k)\}$ is a sequence of points of $A \times B$ tending to $(0, 0) \in \mathbb{R}^m \times \mathbb{R}^n$ such that

$$\|(a_k, b_k) - (c_{j_k}, d_k)\| \ll \|b_k\|, \|d_k\|.$$

It follows that

$$\|(a_k, b_k) - (c_{j_k}, d_k)\| \ll \|(a_k, b_k)\|, \|(c_{j_k}, d_k)\|.$$

The case where $\|a\| = 1$ and $\|b\| = 0$ follows similarly to the above. Thus $A \times B$ satisfies condition (SSP) at $(0, 0) \in \mathbb{R}^m \times \mathbb{R}^n$.

We next show the “if” part. Since the proof of the other part is the same, it suffices to show that A satisfies condition (SSP) at $0 \in \mathbb{R}^m$. Let $\{a_k\}$ be an arbitrary sequence of points of \mathbb{R}^m tending to $0 \in \mathbb{R}^m$ such that

$$\lim_{k \rightarrow \infty} \frac{a_k}{\|a_k\|} = a \in D(A).$$

We take a sequence of points $\{b_k\}$ of \mathbb{R}^n tending to $0 \in \mathbb{R}^n$ such that

$$\lim_{k \rightarrow \infty} \frac{b_k}{\|b_k\|} = b \in D(B).$$

Taking a subsequence if necessary, we may assume that $\|b_k\| \leq \|a_k\|$ for any $k \in \mathbb{N}$, and

$$\lim_{k \rightarrow \infty} \frac{(a_k, b_k)}{\|(a_k, b_k)\|} = (pa, \sqrt{1-p^2}b) \in D(A \times B)$$

where $0 < p \leq 1$. Since $A \times B$ satisfies condition (SSP) at $(0, 0) \in \mathbb{R}^m \times \mathbb{R}^n$, there is a sequence of points $\{(c_k, d_k)\}$ of $A \times B$ tending to $(0, 0) \in \mathbb{R}^m \times \mathbb{R}^n$ such that

$$\|(a_k, b_k) - (c_k, d_k)\| \ll \|(a_k, b_k)\|, \|(c_k, d_k)\|.$$

It follows that

$$\|a_k - c_k\| \ll \|a_k\|, \|c_k\|.$$

Thus A satisfies condition (SSP) at $0 \in \mathbb{R}^m$. □

PROPOSITION 2.34. *Let $A \subseteq \mathbb{R}^m, B \subseteq \mathbb{R}^n$ be set-germs at $0 \in \mathbb{R}^m$ and $0 \in \mathbb{R}^n$ respectively, such that $0 \in \overline{A}, 0 \in \overline{B}$. Then*

$$D(A \times B) \subseteq \{(ta, \sqrt{1-t^2}b) \mid a \in D(A), b \in D(B), t \in [0, 1]\}.$$

Moreover if both A and B satisfy condition (SSP), then the equality holds.

PROOF. Let $\{(a_k, b_k)\} \in A \times B$ be an arbitrary sequence of points tending to $(0, 0) \in \mathbb{R}^m \times \mathbb{R}^n$ such that

$$\lim_{k \rightarrow \infty} \frac{(a_k, b_k)}{\|(a_k, b_k)\|} = (a, b) \in D(A \times B).$$

We must have at least one of $\|a\| \neq 0$ or $\|b\| \neq 0$, hence we get that $\lim_{k \rightarrow \infty} a_k/\|a_k\| = a/\|a\| \in D(A)$ or $\lim_{k \rightarrow \infty} b_k/\|b_k\| = b/\|b\| \in D(B)$. In any case we take $t = \|a\| = \sqrt{1 - \|b\|^2}$. In the case $a = 0$ then $t = 0$ and $b \in D(B)$ so we can write $(0, b)$ as required.

For the other inclusion, let $(ta, \sqrt{1-t^2}b)$, $a \in D(A)$, $b \in D(B)$, for some $t \in [0, 1]$. If $t \neq 0, 1$, then take $s = \sqrt{1-t^2}/t$ and consider a sequence of points $(t_i a, st_i b)$, $t_i \rightarrow 0$, such that $(t_i a, st_i b)/\|(t_i a, st_i b)\| \rightarrow (ta, stb)$. Using the fact that A and B satisfy the condition (SSP) we can find $a_i \in A, b_i \in B$ such that $\|a_i - t_i a\| \ll t_i, \|b_i - st_i b\| \ll st_i$ and this implies that $(a_i, b_i)/\|(a_i, b_i)\| \rightarrow (ta, \sqrt{1-t^2}b)$. The case when $t = 0, 1$ is trivial. (We can always reduce the (SSP) property to the case when the points are on a line.) □

2.5. Complex sequence selection property.

We next consider the complex tangent cone and introduce a complex analogue for the condition (SSP). Let $A \subset \mathbb{C}^n$ be a set-germ at $0 \in \mathbb{C}^n$ such that $0 \in \overline{A}$. The *complex tangent cone* of A is defined as follows:

$$LD^*(A) := \left\{ v \in \mathbb{C}^n \mid \begin{array}{l} \exists \{c_i\} \subset \mathbb{C}, \exists \{v_i\} \subset A \setminus \{0\} \rightarrow 0 \in \mathbb{C}^n \\ \text{s.t. } \lim_{i \rightarrow \infty} c_i v_i = v \end{array} \right\}.$$

Note that if A is a real (resp. complex) vector space, then $LD(A) = A$, $LD^*(A) = A + iA$ (resp. $LD^*(A) = A$).

Let $CD(A) = \{cv \in \mathbb{C}^n \mid c \in \mathbb{C}, v \in D(A)\}$. Then we have following:

LEMMA 2.35. $LD^*(A) = CD(A)$.

PROOF. Since the inclusion $LD^*(A) \supset CD(A)$ is obvious, we show the converse inclusion. Note that $0 \in LD^*(A) \cap CD(A)$. Take an element $v \in LD^*(A) \setminus \{0\}$. By definition, there exist $\{c_i\} \subset \mathbb{C}$ and $\{v_i\} \subset A \setminus \{0\} \rightarrow 0 \in \mathbb{C}^n$ such that $\lim_{i \rightarrow \infty} c_i v_i = v$. Then there are subsequences $\{c_{i_j}\}$ of $\{c_i\}$ and $\{v_{i_j}\}$ of $\{v_i\}$ such that $\lim_{j \rightarrow \infty} c_{i_j} / \|c_{i_j}\| = c \in \mathbb{C} \setminus \{0\}$ and $\lim_{j \rightarrow \infty} v_{i_j} / \|v_{i_j}\| = w \in D(A)$. Then we have $v = c\|v\|w \in CD(A)$. It follows that $LD^*(A) \subset CD(A)$. \square

We define the *complex projective direction set* $D^*(A) \subset PC^{n-1}$ of A as the quotient set of $LD^*(A) \setminus \{0\}$ by $\mathbb{C} \setminus \{0\}$. Then we have

LEMMA 2.36 (Lemma 8.1 in H. Whitney [9]). *Let $A \subset \mathbb{C}^n$ be an analytic variety such that $0 \in A$. Then $LD^*(A)$ is also an analytic variety in \mathbb{C}^n and $D^*(A)$ is a projective variety. In addition, we have*

$$\dim_{\mathbb{C}} A = \dim_{\mathbb{C}} LD^*(A) = \dim_{\mathbb{C}} D^*(A) + 1.$$

The next lemma follows also from Remark 8.2 and Theorem 11.8 in [9]:

LEMMA 2.37. *For an analytic variety $0 \in A \subset \mathbb{C}^n$, $LD^*(A) = LD(A)$.*

REMARK 2.38. Let $S^1 = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$. For $A \subset \mathbb{C}^n$ such that $0 \in \overline{A}$, $LD^*(A) = LD(A)$ if and only if $S^1 D(A) = D(A)$. Note that $D^*(A)$ is the quotient of $D(A)$ by S^1 .

One can also consider the sequence selection property over the complex numbers, which we denote by (CSSP).

DEFINITION 2.39. Let $A \subset \mathbb{C}^n$ be a set-germ at $0 \in \mathbb{C}^n$ such that $0 \in \overline{A}$. We say that A satisfies *condition (CSSP)*, if for any sequence of points $\{a_m\}$ of \mathbb{C}^n tending to $0 \in \mathbb{C}^n$ such that $\lim_{m \rightarrow \infty} a_m / \|a_m\| \in LD^*(A)$, there is a sequence of points $\{b_m\} \subset A$ such that

$$\|a_m - b_m\| \ll \|a_m\|, \|b_m\|.$$

REMARK 2.40. If A satisfies *condition (CSSP)*, then $LD^*(A) = LD(A)$, and it is clear that it also satisfies *condition (SSP)*. In particular, Lemma 2.12 implies that $D(h(A)) = D(h(LD(A))) = D(h(LD^*(A)))$ and $LD^*(h(A)) = LD^*(h(LD^*(A)))$. In general it is not true that $D(h(A)) = D(h(LD^*(A)))$ implies A satisfies *condition (SSP)*. Amongst the examples of sets satisfying *condition (CSSP)* we mention the complex tangent cones $LD^*(A)$ and the complex analytic varieties.

PROPOSITION 2.41. *Let $A \subset \mathbb{C}^n$ be a set-germ at $0 \in \mathbb{C}^n$ such that $0 \in \overline{A}$. Then A satisfies *condition (CSSP)* if and only if A satisfies *condition (SSP)* and $S^1 D(A) = D(A)$. Consequently if A satisfies *condition (SSP)*, then both $S^1 A$ and $\mathbb{C}A$ satisfy *condition (CSSP)*.*

PROOF. The direct implication is clear from the comments above. For the other

implication let us consider a sequence $\{a_m\}$ of \mathbb{C}^n tending to $0 \in \mathbb{C}^n$, such that $\lim_{m \rightarrow \infty} a_m / \|a_m\| \in LD^*(A) = CD(A)$. It follows that $\lim_{m \rightarrow \infty} a_m / \|a_m\| = ca \in S^1 D(A) = D(A)$ by assumption. Because A satisfies *condition* (SSP) it follows that there are $b_m \in A$ such that $\|a_m - b_m\| \ll \|a_m\|, \|b_m\|$, that is A satisfies *condition* (CSSP). \square

3. Transversality.

3.1. Transversality for singular sets.

Let us define the notion of transversality for complex analytic varieties, using the complex tangent cones.

DEFINITION 3.1. Let $0 \in A, B \subset \mathbb{C}^n$ be analytic varieties. Then we say that A and B are *transverse at* $0 \in \mathbb{C}^n$ if the following equality holds:

$$\dim_{\mathbb{C}} LD^*(A) + \dim_{\mathbb{C}} LD^*(B) - n = \dim_{\mathbb{C}}(LD^*(A) \cap LD^*(B)).$$

Concerning this transversality, we have

THEOREM 3.2. Let $h : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ be a bi-Lipschitz homeomorphism, and let $0 \in A, B, h(A), h(B) \subset \mathbb{C}^n$ be analytic varieties. Then A and B are transverse at $0 \in \mathbb{C}^n$ if and only if $h(A)$ and $h(B)$ are transverse at $0 \in \mathbb{C}^n$.

PROOF. We show only the “only if” part. The “if” part follows similarly. By assumption,

$$\dim_{\mathbb{C}} LD^*(A) + \dim_{\mathbb{C}} LD^*(B) - n = \dim_{\mathbb{C}}(LD^*(A) \cap LD^*(B)).$$

By Lemma 2.36, we see that

$$\dim_{\mathbb{C}} LD^*(A) = \dim_{\mathbb{C}} LD^*(h(A)), \quad \dim_{\mathbb{C}} LD^*(B) = \dim_{\mathbb{C}} LD^*(h(B)).$$

Then, using Lemma 2.37 and Theorem 2.2, we can compute $\dim_{\mathbb{C}}(LD^*(A) \cap LD^*(B))$ as follows:

$$\begin{aligned} 2 \dim_{\mathbb{C}}(LD^*(A) \cap LD^*(B)) &= \dim_{\mathbb{R}}(LD^*(A) \cap LD^*(B)) \\ &= \dim_{\mathbb{R}}(LD(A) \cap LD(B)) \\ &= \dim_{\mathbb{R}}(LD(h(A)) \cap LD(h(B))) \\ &= \dim_{\mathbb{R}}(LD^*(h(A)) \cap LD^*(h(B))) \\ &= 2 \dim_{\mathbb{C}}(LD^*(h(A)) \cap LD^*(h(B))). \end{aligned}$$

Therefore we have

$$\dim_{\mathbb{C}} LD^*(h(A)) + \dim_{\mathbb{C}} LD^*(h(B)) - n = \dim_{\mathbb{C}}(LD^*(h(A)) \cap LD^*(h(B))).$$

Thus $h(A)$ and $h(B)$ are transverse at $0 \in \mathbb{C}^n$. □

3.2. Weak transversality.

When dealing with singular sets in the real set up, we find more convenient to use a weaker form of transversality, in terms of real tangent cones. This is analogous to the use of semi-arcs in Real Algebraic Geometry.

DEFINITION 3.3. Let $A, B \subset \mathbb{R}^n$ be set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}$. We say that A and B are *weakly transverse at $0 \in \mathbb{R}^n$* if $D(A) \cap D(B) = \emptyset$ (if and only if $LD(A)$ and B are weakly transverse at $0 \in \mathbb{R}^n$).

Concerning this weak transversality, we have the following:

LEMMA 3.4. Let A, B be two set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}$, and let $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism. Suppose that $h(A)$ (or $h(B)$) satisfies condition (SSP). If $D(A) \cap D(B) = \emptyset$, then $D(h(A)) \cap D(h(B)) = \emptyset$.

As a corollary of this we have the following.

THEOREM 3.5. Let A, B be two set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}$, and let $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism. Suppose that A or B satisfies condition (SSP), and $h(A)$ or $h(B)$ satisfies condition (SSP). Then A and B are weakly transverse at $0 \in \mathbb{R}^n$ if and only if $h(A)$ and $h(B)$ are weakly transverse at $0 \in \mathbb{R}^n$.

PROOF OF LEMMA. By hypothesis, $LD(A) \cap LD(B) = \{0\}$.

Assume that $h(A)$ and $h(B)$ are not weakly transverse at $0 \in \mathbb{R}^n$. Namely, there are a half line $\ell \subset LD(h(A)) \cap LD(h(B))$ and a sequence of points $\{b_m\} \subset B$ tending to $0 \in \mathbb{R}^n$ such that $\lim_{m \rightarrow \infty} h(b_m) / \|h(b_m)\| = D(\ell)$. Here $LD(\ell) = \ell \subset LD(h(A)) \cap LD(h(B))$.

Since $h(A)$ satisfies condition (SSP), there is a sequence of points $\{a_m\} \subset A$ such that

$$\|h(a_m) - h(b_m)\| \ll \|h(a_m)\|, \|h(b_m)\|.$$

It follows that

$$\|a_m - b_m\| \ll \|a_m\|, \|b_m\|. \tag{3.1}$$

Taking a subsequence of $\{b_m\}$ if necessary, we may assume that $\lim_{m \rightarrow \infty} b_m / \|b_m\| = \hat{b} \in D(B)$. By (3.1), $\hat{b} = \lim_{m \rightarrow \infty} a_m / \|a_m\| \in D(A)$. Thus it follows that $D(A) \cap D(B) \neq \emptyset$, which contradicts the hypothesis. Thus it follows that $h(A)$ and $h(B)$ are weakly transverse at $0 \in \mathbb{R}^n$. □

REMARK 3.6. We cannot drop the assumption of (SSP) from the above theorem. For instance, consider Figure 1, the “slow spiral” bi-Lipschitz homeomorphism pictured before.

As a corollary of Theorem 3.5, we have the following:

COROLLARY 3.7. *Let A, B be two set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}$, and let $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism. Suppose that $h(LD(A))$ satisfies condition (SSP). Then A and B are weakly transverse at $0 \in \mathbb{R}^n$ if and only if $h(LD(A))$ and $h(B)$ are weakly transverse at $0 \in \mathbb{R}^n$.*

The following is a simple corollary of Theorem 2.25.

COROLLARY 3.8. *Let $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism such that h satisfies condition semiline-(SSP) and $A, B \subset \mathbb{R}^n$ two arbitrary set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}$. Then A and B are weakly transverse at $0 \in \mathbb{R}^n$ if and only if $h(A)$ and $h(B)$ are weakly transverse at $0 \in \mathbb{R}^n$.*

3.3. Applications to complex analytic varieties.

Having developed our transversality theory specifically to deal with the singular situations, let us apply (illustratively) the above results to arbitrary complex analytic varieties. We first give an important proposition.

PROPOSITION 3.9. *Let $A, B \subset \mathbb{C}^n$ be set-germs at $0 \in \mathbb{C}^n$ such that $0 \in \overline{A} \cap \overline{B}$. If $LD(A) \cap LD^*(B) = \{0\}$, then $LD^*(A) \cap LD^*(B) = \{0\}$.*

PROOF. Assume that there exists $v \in LD^*(A) \cap LD^*(B)$ such that $v \neq 0 \in \mathbb{C}^n$. Then, by Lemma 2.35, there is a non-zero $c \in \mathbb{C}$ such that $cv \in LD(A) \cap LD^*(B)$. This contradicts the hypothesis. Thus the statement follows. \square

As a corollary of Proposition 3.9 and Lemma 2.37, we have

COROLLARY 3.10. *Let $0 \in V \subset \mathbb{C}^n$ be an analytic variety, and let $A \subset \mathbb{C}^n$ such that $0 \in \overline{A}$. Then $LD^*(A) \cap LD^*(V) = \{0\}$ if and only if $LD(A) \cap LD(V) = \{0\}$.*

Let $0 \in V, W \subset \mathbb{C}^n$ be analytic varieties, and let A be a subset of \mathbb{C}^n such that $0 \in \overline{A}$. Suppose that there exists a bi-Lipschitz homeomorphism $h : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ such that $h(V) = W$. Then, by Lemma 3.4, Lemma 2.37 and Proposition 3.9, we can see the following:

THEOREM 3.11. *$LD^*(A) \cap LD^*(V) = \{0\}$ if and only if $LD^*(h(A)) \cap LD^*(W) = \{0\}$.*

We consider also the application to the singular points sets of complex analytic varieties. Let V, W, A and h be the same as above. Let us denote by $\Sigma(V)$ (resp. $\Sigma(W)$) the singular points set of V (resp. W). Note that $h(\Sigma(V)) = \Sigma(W)$.

By Lemma 3.4, we can easily see the following:

PROPOSITION 3.12. *A and $\Sigma(V)$ are weakly transverse at $0 \in \mathbb{C}^n$ if and only if $h(A)$ and $\Sigma(W)$ are weakly transverse at $0 \in \mathbb{C}^n$.*

Let us apply our Proposition 2.29 to complex analytic hypersurfaces. Let $0 \in V, W \subset \mathbb{C}^n$ be analytic hypersurfaces, and let the ideals $I(V)$ and $I(W)$ of V and W be generated by complex analytic functions f and g , respectively. Let f_d and g_k be the

initial homogeneous forms of f and g , respectively.

We note that for a hypersurface $V = \{f = 0\}$, as above, we have $LD(V) = LD^*(V) = \{f_d = 0\}$. Suppose that there exists a bi-Lipschitz homeomorphism $h : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ such that $h(V) = W$. Then, by Lemma 2.12, we have

OBSERVATION 1. $LD(h(LD(V))) = LD(h(V))$.

In addition, by Proposition 2.11, we have

OBSERVATION 2. $h(LD(V))$ satisfies condition (SSP).

Using these facts, we can show the following:

COROLLARY 3.13. *Let $A \subset \mathbb{C}^n$ be a set-germ at $0 \in \mathbb{C}^n$ such that $0 \in \bar{A}$. Then we have*

$$LD(h(LD(A) \cap LD(V))) = LD(h(LD(A))) \cap LD(W).$$

PROOF. By Observation 2, $h(LD(V))$ satisfies condition (SSP). Then it follows from Proposition 2.29 and Observation 1 that

$$\begin{aligned} LD(h(LD(A) \cap LD(V))) &= LD(h(LD(A))) \cap LD(h(LD(V))) \\ &= LD(h(LD(A))) \cap LD(W). \end{aligned} \quad \square$$

We end this section with an application to analytic curves.

Let W_1, W_2 be the set-germs of two analytic curves at $0 \in \mathbb{C}^n$.

Then $LD(W_1) = \bigcup_{j=1}^s m_j$, $m_i \cap m_j = \{0\}$, $i \neq j$, $LD(W_2) = \bigcup_{j=1}^t l_j$, $l_i \cap l_j = \{0\}$, $i \neq j$, $s, t \in \mathbb{N}$, where m_j, l_j are complex lines through $0 \in \mathbb{C}^n$.

PROPOSITION 3.14. *Suppose that there is a bi-Lipschitz homeomorphism $h : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ such that $h(W_1) = W_2$. Then $s = t$.*

PROOF. We are going to use the known fact that the tangent cone of an irreducible complex curve is just a complex line. We know that

$$\bigcup_{j=1}^t l_j = LD^*(W_2) = LD(W_2) = LD(h(LD(W_1))) = \bigcup_{j=1}^s LD^*(h(m_j)).$$

This shows that for any $j, 1 \leq j \leq s$, $LD^*(h(m_j))$ consists of some lines l_k . We will show that we cannot have more than one l_k . Indeed assume that l_1, l_2 are in $LD^*(h(m_1))$, for convenience. This would imply that there are sequences $a_i, b_i \in W_1$ realising the direction m_1 so that their images $h(a_i), h(b_i)$ realise l_1 and l_2 respectively. As l_1, l_2 are distinct directions, following the cited result it follows that the sequences $h(a_i)$ and $h(b_i)$ are in different irreducible components of W_2 , say in V_1 and in V_2 respectively. As h is a homeomorphism it follows that $a_i \in h^{-1}(V_1)$ and $b_i \in h^{-1}(V_2)$ are also on different irreducible components of W_1 . This contradicts our Theorem 3.11. It follows that each

$LD^*(h(m_j))$ consists exactly of one line and therefore $s \geq t$. By symmetry we conclude our proof. \square

REMARK 3.15. It is not difficult to see that the above result does not hold for h merely a homeomorphism.

4. (SSP) mappings.

In this section we introduce and investigate the notion of (SSP) mappings.

DEFINITION 4.1. Let $A \subset \mathbb{R}^m$ be a set-germ at $0 \in \mathbb{R}^m$ such that $0 \in \overline{A}$ and $B \subset \mathbb{R}^n$ a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{B}$. Let $h : (A, 0) \rightarrow (B, 0)$ be an arbitrary map (or a homeomorphism) germ. We say that h is an (SSP) map ((SSP) homeomorphism) if the graph of h satisfies condition (SSP) at $(0, 0) \in \mathbb{R}^m \times \mathbb{R}^n$.

Subanalytic maps and definable maps in an o-minimal structure are examples of (SSP) maps. Also the Cartesian product of two (SSP) maps is an (SSP) map. By Theorem 4.19 weak diffeomorphisms are also (SSP) homeomorphisms. A function $h : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ whose graph is a zigzag given in Example 2.32 is also an (SSP) map. (Of course, the zigzag should be expanded to the negative part.)

We next consider the image of a set satisfying condition (SSP) by an (SSP) map. Let $\pi : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^{n-1}, 0)$ be the projection on the first $(n - 1)$ coordinates, and let A be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$. Then the following result holds:

PROPOSITION 4.2. *Suppose that $\ker \pi$ and A are weakly transverse at $0 \in \mathbb{R}^n$. Then we have*

- (1) $\pi(LD(A)) = LD(\pi(A))$.
- (2) *If A satisfies condition (SSP), then so does $\pi(A)$.*

PROOF. (1) For the first inclusion \subseteq , it suffices to show $\pi(D(A)) \subseteq LD(\pi(A))$. Take $a \in D(A)$. Then there is a sequence of points $\{a_m\} \subset A \setminus \{0\}$ tending to $0 \in \mathbb{R}^n$ such that $\lim_{m \rightarrow \infty} a_m / \|a_m\| = a$. By the weak transversality, $a \notin \ker \pi$. Since $\pi(a_m) \neq 0$ for sufficiently large m , we may assume that $\pi(a_m) \neq 0$ for any m . Then we have

$$\frac{\pi(a)}{\|\pi(a)\|} = \lim_{m \rightarrow \infty} \frac{\pi(a_m)}{\|\pi(a_m)\|} \in D(\pi(A)).$$

Hence $\pi(a) \in LD(\pi(A))$.

For the second inclusion it suffices to show $D(\pi(A)) \subseteq \pi(LD(A))$. Take $b \in D(\pi(A))$. Then there is a sequence of points $\{a_m\} \subset A \setminus \{0\}$ tending to $0 \in \mathbb{R}^n$ such that $\lim_{m \rightarrow \infty} \pi(a_m) / \|\pi(a_m)\| = b$. Because of the same reason as above, we may assume that $\pi(a_m) \neq 0$ for any m . Then, by the weak transversality, there is a subsequence $\{a_{m_j}\}$ of $\{a_m\}$ such that

$$\lim_{m_j \rightarrow \infty} \frac{a_{m_j}}{\|a_{m_j}\|} = a \in D(A) \text{ and } \pi(a) \neq 0.$$

Then we have

$$b = \lim_{m_j \rightarrow \infty} \frac{\pi(a_{m_j})}{\|\pi(a_{m_j})\|} = \frac{\pi(a)}{\|\pi(a)\|} = \pi\left(\frac{a}{\|\pi(a)\|}\right) \in \pi(LD(A)).$$

(2) Let $\{b_m\}$ be an arbitrary sequence of points of \mathbb{R}^{n-1} tending to $0 \in \mathbb{R}^{n-1}$ such that

$$\lim_{m \rightarrow \infty} \frac{b_m}{\|b_m\|} = b \in D(\pi(A)).$$

Let $\ell = \{tb \mid t \geq 0\} \subset LD(\pi(A))$. Then by (1), there is a half line $L \subset LD(A)$ such that $\pi(L) = \ell$. Let us express L as $\{t(b, c) \mid t \geq 0\}$ for some $c \in \mathbb{R}$. Let $\alpha_m = (b_m, \|b_m\|c)$ for each m . Then we have

$$\lim_{m \rightarrow \infty} \frac{\alpha_m}{\|\alpha_m\|} = \lim_{m \rightarrow \infty} \frac{(b_m, \|b_m\|c)}{\|(b_m, \|b_m\|c)\|} = \lim_{m \rightarrow \infty} \frac{(b_m/\|b_m\|, c)}{\|(b_m/\|b_m\|, c)\|} = \frac{(b, c)}{\|(b, c)\|} \in D(A).$$

Since A satisfies condition (SSP), there is a sequence of points $\{\beta_m\} \subset A$, where $\beta_m = (a_m, d_m) \in \mathbb{R}^{n-1} \times \mathbb{R}$, tending to $0 \in \mathbb{R}^n$ such that

$$\|\beta_m - \alpha_m\| \ll \|\beta_m\|, \|\alpha_m\|.$$

It follows that

$$\|\pi(\beta_m) - \pi(\alpha_m)\| = \|\pi(\beta_m - \alpha_m)\| \leq \|\beta_m - \alpha_m\| \ll \|\beta_m\|.$$

Then, by the weak transversality,

$$\|\pi(\beta_m) - \pi(\alpha_m)\| \ll \|\pi(\beta_m)\| \quad (\text{and also } \|\pi(\alpha_m)\|).$$

This means

$$\|a_m - b_m\| \ll \|a_m\|, \|b_m\|.$$

Since $a_m = \pi(\beta_m) \in \pi(A)$, $\pi(A)$ satisfies condition (SSP). □

REMARK 4.3. We cannot drop the assumption of the weak transversality in the above theorem.

Let $\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the projection defined by $\pi(x, y, z) = (x, y)$, and let $A = \{z^4 = x^2 + y^2\} \cap \pi^{-1}(S)$, where S is a slow spiral on (x, y) -plane. Then we can see that A satisfies condition (SSP), but $\pi(A) = S$ does not satisfy condition (SSP). In addition, $\pi(LD(A)) = \{0\}$ but $LD(\pi(A)) = \mathbb{R}^2$.

Concerning the weak transversality assumption of Proposition 4.2, we have the following lemma.

LEMMA 4.4. *Let $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$ be a map such that there is $c > 0$ with $|f(x)| \leq c|x|$ in a neighbourhood of the origin. Let $\pi : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^n$ be the projection on the first n -coordinates. Then $\ker \pi$ and the graph of f are weakly transverse at $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^p$.*

Using Proposition 4.2 and Lemma 4.4, we can show the following theorem on the (SSP) structure:

THEOREM 4.5. *Let $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be a Lipschitz homeomorphism such that $c_1|x| \leq |h(x)|$, for some $c_1 > 0$, in a neighbourhood of $0 \in \mathbb{R}^n$, and let $A \subset \mathbb{R}^n$ be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$. Suppose that A satisfies condition (SSP) and h is an (SSP) map. Then $h(A)$ also satisfies condition (SSP).*

PROOF. Let $\pi_2 : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the projection on the second n -coordinates, and let

$$G_A := \{(a, h(a)) \in \mathbb{R}^n \times \mathbb{R}^n \mid a \in A\}.$$

Suppose that G_A satisfies condition (SSP) as a set-germ at $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^n$. Since $h^{-1} : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ satisfies $|h^{-1}(x)| \leq (1/c_1)|x|$ in a neighbourhood of $0 \in \mathbb{R}^n$, it follows from Proposition 4.2 and Lemma 4.4 that $h(A) = \pi_2(G_A)$ satisfies condition (SSP). Therefore it suffices to show that G_A satisfies condition (SSP).

Let $\pi_1 : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the projection on the first n -coordinates, and let G be the graph of h . Since $\ker \pi_1$ and G are weakly transverse at $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^n$, so are $\ker \pi_1$ and G_A .

Let us show that G_A satisfies condition (SSP). Let $\{\alpha_m\}$ be an arbitrary sequence of points of $\mathbb{R}^n \times \mathbb{R}^n$ tending to $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$\lim_{m \rightarrow \infty} \frac{\alpha_m}{\|\alpha_m\|} = \alpha \in D(G_A) \subset D(G),$$

where $\alpha_m = (b_m, c_m) \in \mathbb{R}^n \times \mathbb{R}^n$ for $m \in \mathbb{N}$. Let $L = \{t\alpha \mid t \geq 0\} \subset LD(G_A)$. Since G satisfies condition (SSP), there is a sequence of points of $\{\beta_m\} \subset G$ such that

$$\|\alpha_m - \beta_m\| \ll \|\alpha_m\|, \|\beta_m\|, \tag{4.1}$$

where $\beta_m = (d_m, h(d_m)) \in \mathbb{R}^n \times \mathbb{R}^n$ for $m \in \mathbb{N}$. By the weak transversality of $\ker \pi_1$ and G_A , $\pi_1(L) = \ell \subset LD(A)$. Note that $\ell = \{tb \mid t \geq 0\}$ for $b = \lim_{m \rightarrow \infty} b_m / \|b_m\| \in D(A)$. Therefore it follows from the weak transversality that

$$\|b_m - d_m\| \ll \|b_m\|, \|d_m\|. \tag{4.2}$$

On the other hand, since A satisfies condition (SSP), there is a sequence of points $\{a_m\} \subset A$ tending to $0 \in \mathbb{R}^n$ such that

$$\|a_m - b_m\| \ll \|a_m\|, \|b_m\|. \tag{4.3}$$

It follows from (4.2) and (4.3) that

$$\|a_m - d_m\| \ll \|a_m\|, \|d_m\|. \quad (4.4)$$

Because h is Lipschitz, (4.4) implies that,

$$\|h(a_m) - h(d_m)\| \ll \|a_m\|, \|d_m\|. \quad (4.5)$$

Consequently our assumption on h implies that

$$\|h(a_m) - h(d_m)\| \ll \|h(a_m)\|, \|h(d_m)\|. \quad (4.6)$$

Let $\gamma_m = (a_m, h(a_m)) \in G_A$ for $m \in \mathbb{N}$. It follows from (4.4) and (4.6) that

$$\|\gamma_m - \beta_m\| \ll \|\gamma_m\|, \|\beta_m\|. \quad (4.7)$$

By (4.1) and (4.7) we have

$$\|\alpha_m - \gamma_m\| \ll \|\alpha_m\|, \|\gamma_m\|.$$

Therefore G_A satisfies condition (SSP). This completes the proof of Theorem 4.5. \square

DEFINITION 4.6. We call a homeomorphism $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ an (SSP) *bi-Lipschitz homeomorphism* if it is bi-Lipschitz and an (SSP) map.

Obviously a C^1 diffeomorphism $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ is an (SSP) bi-Lipschitz homeomorphism.

As a special case of the above theorem we have the following preserving (SSP) structure Theorem.

THEOREM 4.7. *Let $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be an (SSP) bi-Lipschitz homeomorphism, and let $A \subset \mathbb{R}^n$ be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$. Then A satisfies condition (SSP) if and only if $h(A)$ satisfies condition (SSP).*

We have a corollary of the proof of Theorem 4.5.

COROLLARY 4.8. *Let $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be a Lipschitz homeomorphism as in Theorem 4.5, and let $A \subset \mathbb{R}^n$ be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$. Suppose that h is an (SSP) map and A satisfies condition (SSP). Then the restriction $h|_A$ is an (SSP) map.*

We can give a characterisation of an (SSP) map as follows:

PROPOSITION 4.9. *Let $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be a Lipschitz homeomorphism as in Theorem 4.5. Then h is an (SSP) map if and only if its restrictions to any semiline ℓ are (SSP) maps.*

PROOF. The corollary above gives the necessity. Let G be the graph of h . To prove the sufficiency, let us consider a sequence of points $\{(a_m, b_m)\}$ of $\mathbb{R}^n \times \mathbb{R}^n$ tending to $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$\lim_{m \rightarrow \infty} \frac{(a_m, b_m)}{\|(a_m, b_m)\|} = (a, b) \in D(G).$$

We put $l := \{(ta, tb) \mid t \geq 0\}$ and $l_1 := \{ta \mid t \geq 0\}$. Then there is a sequence of points $\{(c_i, h(c_i))\}$ of G such that $\lim_{i \rightarrow \infty} (c_i, h(c_i)) / \|(c_i, h(c_i))\| = (a, b)$. Since l satisfies condition (SSP), there are positive numbers $s_i \in \mathbb{R}$ so that

$$\|s_i(a, b) - (c_i, h(c_i))\| \ll \|c_i\|, \quad s_i.$$

This shows that the direction l is also attained by the sequence $\{(s_i a, h(s_i a))\}$, namely it appears as a direction of the graph of the restriction of h to l_1 , and we can apply the hypothesis to end the proof. \square

REMARK 4.10. Unfortunately a homeomorphism which is merely an (SSP) homeomorphism, does not always preserve the condition (SSP). We can construct an (SSP) homeomorphism $h : \mathbb{R} \rightarrow \mathbb{R}$, which also satisfies semiline-(SSP), such that there is a set A satisfying condition (SSP) but $h(A)$ does not.

Concerning Theorem 4.7, it may be natural to ask the following question:

QUESTION 1. Let $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism. Suppose that if A satisfies condition (SSP), so does $h(A)$ for any set-germ A at $0 \in \mathbb{R}^n$ such that $0 \in \bar{A}$. Then is h an (SSP) map?

We have a negative example to the above question.

EXAMPLE 4.11. Let $h : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ be a zig-zag function whose graph is drawn below (Figure 3). (Note that the zigzag in Figure 2 is not the graph of a function!)

Then h is a bi-Lipschitz homeomorphism. As stated in Remark 2.8 (2), h satisfies the (SSP) assumption in Question 1. But the graph of h does not satisfy condition (SSP). Therefore h is not an (SSP) map, moreover h also satisfies condition semiline-(SSP).

REMARK 4.12. We can consider a similar question to Question 1 in the semialgebraic category or in the subanalytic one. Namely, we consider the question, replacing condition (SSP) with semialgebraic or subanalytic. Indeed, let $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism. Suppose that if A is semialgebraic (or subanalytic), then so does $h(A)$ (for any set-germ A at $0 \in \mathbb{R}^n$ such that $0 \in \bar{A}$). Does this property imply that h is a semialgebraic map (subanalytic respectively)? The above example provides a negative answer.

This kind of phenomenon is not particular to the one-dimensional case. For instance, let $T := \{(x, y) \in \mathbb{R}^2; x > 0 \text{ and } \exp(-1/x^2) < y < 2 \exp(-1/x^2)\}$. Then we can define a homeomorphism germ $h : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ by identity outside T , and, on T , we can take

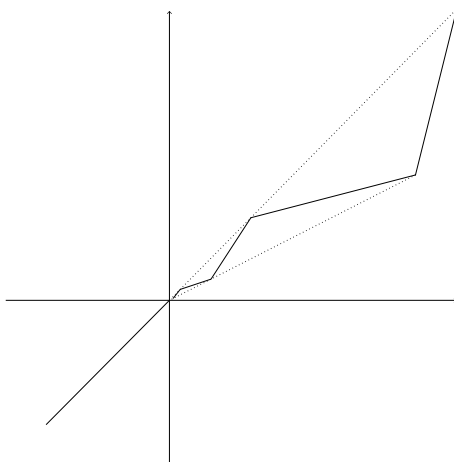


Figure 3.

any extension so that h is a non-semialgebraic homeomorphism. However this kind of h takes semialgebraic set-germs to semialgebraic set-germs. Indeed, for any 1-dimensional semialgebraic set A such that $0 \in \overline{A}$, $A \cap T$ is empty as a set-germ at $0 \in \mathbb{R}^2$, and obviously its image $h(A) = A$ is semialgebraic. If B is an arbitrary 2-dimensional semialgebraic set such that $0 \in \overline{B}$, then the boundary of B does not intersect T as set-germs at $0 \in \mathbb{R}^2$. Therefore we can see that $h(B)$ is also a semialgebraic set-germ.

The subanalytic case is similar.

Concerning the above phenomenon we mention the following results.

PROPOSITION 4.13.

- (1) Both $h_i : (\mathbb{R}^{n_i}, 0) \rightarrow (\mathbb{R}^{n_i}, 0)$, $i = 1, 2$, are (SSP) bi-Lipschitz homeomorphisms if and only if $h_1 \times h_2 : (\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, 0 \times 0) \rightarrow (\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, 0 \times 0)$ is an (SSP) bi-Lipschitz homeomorphism.
- (2) Let $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism. Then $I_n \times h : (\mathbb{R}^n \times \mathbb{R}^n, 0 \times 0) \rightarrow (\mathbb{R}^n \times \mathbb{R}^n, 0 \times 0)$ (or $I_n \times h^{-1}$) satisfies condition semiline-(SSP) if and only if $I_n \times h : (\mathbb{R}^n \times \mathbb{R}^n, 0 \times 0) \rightarrow (\mathbb{R}^n \times \mathbb{R}^n, 0 \times 0)$ is an (SSP) map.
- (3) Let $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism. Then h is an (SSP) map if and only if $I_n \times h : (\mathbb{R}^n \times \mathbb{R}^n, 0 \times 0) \rightarrow (\mathbb{R}^n \times \mathbb{R}^n, 0 \times 0)$ (or $I_n \times h^{-1}$) satisfies condition semiline-(SSP).

(Here $I_n : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ represents the identity map.)

PROOF. Note that the graph of $h_1 \times h_2$ is the Cartesian product of the graphs of h_1 and h_2 . Then (1) follows from Proposition 2.33.

In (2) we already know the sufficiency by Theorem 4.5. For necessity, in our set up, it follows that $I_n \times h$ takes (SSP) sets to (SSP) sets, see Corollary 2.23. In particular the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$ is taken to the graph of h , so h is an (SSP) map and by (1) so is $I_n \times h$.

Now (3) clearly follows from (1) and (2). □

REMARK 4.14. Note that if $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ is an (SSP) bi-Lipschitz homeomorphism, then for any semiline ℓ the cone $LD(G_\ell)$ is also a semiline. This fact also explains Example 4.11. (Here G_ℓ is the graph of the restriction of h to ℓ .)

REMARK 4.15.

(1) There are bi-Lipschitz homeomorphisms $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$, $n \geq 2$, which are not (SSP) bi-Lipschitz homeomorphisms.

For instance, let $h : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ be a zigzag bi-Lipschitz homeomorphism in Example 3.4 of [3] or a slow spiral bi-Lipschitz homeomorphism, and let A be the positive x -axis. Clearly A satisfies condition (SSP) and $h(A)$ does not satisfy condition (SSP). Then, by Theorem 4.7, h is not an (SSP) map.

(2) The homeomorphism \bar{h} associated to a bi-Lipschitz homeomorphism which satisfies condition semiline-(SSP) is an (SSP) map.

In order to give another large class of examples of (SSP) homeomorphisms, let us consider a category of homeomorphisms $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ called *weak diffeomorphisms*, namely those h and h^{-1} which admit derivative (= linear approximation) at $0 \in \mathbb{R}^n$.

We will point out some directional and (SSP) properties for the class of weak diffeomorphisms, namely we will show that the weak diffeomorphisms are also (SSP) homeomorphisms.

REMARK 4.16. Note that a weak diffeomorphism is not necessarily Lipschitz. For instance we may have $h(x, y, z) = (x, y, z + (x^5 + y^5)^{1/3})$.

Let $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ denote a weak diffeomorphism. Then h can be expressed in a neighbourhood of $0 \in \mathbb{R}^n$ as follows:

$$h(x) = M_h(x) + O_h(x),$$

where M_h is a regular linear map from \mathbb{R}^n to \mathbb{R}^n , and $\lim_{x \rightarrow 0} \|O_h(x)\|/\|x\| = 0$. Note that $M_{\bar{h}^{-1}} \circ M_h = \text{Id}$.

LEMMA 4.17. Let $A \subset \mathbb{R}^n$ be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \bar{A}$, and let G and G_A be the graphs of the weak diffeomorphism h and $h|_A$ respectively. Then we have

- (1) $LD(M_h(A)) = M_h(LD(A)) = LD(h(A))$.
- (2) $LD(G_A) = LD(\text{graph}(M_h|_A))$. In particular $LD(G) = \text{graph}(M_h)$ is an n -dimensional linear subspace of $\mathbb{R}^n \times \mathbb{R}^n$.

PROOF. (1) Since we can easily see the first equality, we show only the second one. Moreover, interchanging h and h^{-1} , it suffices to show $M_h(LD(A)) \subset LD(h(A))$.

Let α be an arbitrary element of $D(A)$. Then there is a sequence of points $\{a_m\} \subset A$ tending to $0 \in \mathbb{R}^n$ such that $\lim_{m \rightarrow \infty} a_m/\|a_m\| = \alpha$. Therefore

$$\begin{aligned}
\frac{M_h(\alpha)}{\|M_h(\alpha)\|} &= \lim_{m \rightarrow \infty} \frac{M_h(a_m/\|a_m\|)}{\|M_h(a_m/\|a_m\|)\|} \\
&= \lim_{m \rightarrow \infty} \frac{(1/\|a_m\|)(M_h(a_m) + O_h(a_m))}{(1/\|a_m\|)\|M_h(a_m) + O_h(a_m)\|} \\
&= \lim_{m \rightarrow \infty} \frac{M_h(a_m) + O_h(a_m)}{\|M_h(a_m) + O_h(a_m)\|} \\
&= \lim_{m \rightarrow \infty} \frac{h(a_m)}{\|h(a_m)\|} \in D(h(A)).
\end{aligned}$$

It follows that $M_h(LD(A)) \subset LD(h(A))$.

(2) The proof is similar to the above and it is omitted. \square

REMARK 4.18. It is also worth mentioning that there are (SSP) homeomorphisms which do not satisfy condition semiline-(SSP). For example one may consider the function f which has a zig-zag graph and the associated homeomorphism $h : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$, $h(x, y) = (x, y + f(x))$. This shows that outside the bi-Lipschitz category there is no direct implication between the (SSP) homeomorphisms and those satisfying condition semiline-(SSP) (see also 4.11).

The following theorem shows that the weak diffeomorphisms are also suitable for the (SSP) category.

THEOREM 4.19. *A weak diffeomorphism is an (SSP) homeomorphism and satisfies condition semiline-(SSP) as well.*

PROOF. Let h be a weak diffeomorphism. In fact it is an easy consequence of Lemma 4.17 that for any $A \subset \mathbb{R}^n$ satisfying condition (SSP), G_A satisfies condition (SSP), where G_A is the graph of the restriction of h to A . Therefore G satisfies condition (SSP) at $0 \in \mathbb{R}^{2n}$. \square

As a corollary of the proof above and Lemma 4.4 we have the following corollary.

COROLLARY 4.20. *Let $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be a weak diffeomorphism and let $A \subset \mathbb{R}^n$ be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \bar{A}$. Then A satisfies condition (SSP) if and only if $h(A)$ satisfies condition (SSP).*

5. Appendix – Geometric applications to spirals.

We consider polar coordinates (r, θ) , $0 < r$, $\theta < \infty$. Let $R : (0, \infty) \rightarrow (0, \infty)$ be a continuous function. We say that $S_0 : r = R(\theta)$ is a spiral at $0 \in \mathbb{R}^2$ if R is strictly monotone and

$$\lim_{\theta \rightarrow \infty} R(\theta) = 0 \quad \text{or} \quad \lim_{\theta \rightarrow \infty} R(\theta) = \infty.$$

In the first case we write $R(\infty) = 0$ and note that the extension $R : (0, \infty] \rightarrow [0, \infty)$ is continuous and injective. In the second case we write $R(0) = 0$ and note that also the

extension $R : [0, \infty) \rightarrow [0, \infty)$ is continuous and injective.

Let us introduce the homeomorphism germ induced by a spiral, defined in polar coordinates by:

$$h_{S_0} : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0), \quad h_{S_0}(r, \alpha) = (r, R^{-1}(r) + \alpha), \quad 0 \leq \alpha < 2\pi.$$

For $0 \leq \alpha < 2\pi$ we put $L_\alpha := \{(r, \alpha) | 0 \leq r < \infty\}$ and $S_\alpha := h_{S_0}(L_\alpha)$. Note that S_0 is just the spiral $r = R(\theta)$ together with $0 \in \mathbb{R}^2$.

If $0 \leq \alpha < 2\pi$ we denote by R_α the rotation of \mathbb{R}^2 centred at the origin and of angle α . Then the following is obvious:

REMARK 5.1. $S_\alpha = R_\alpha(S_0)$ and $D(h_{S_0}(L_\alpha)) = R_\alpha(D(S_0))$.

For applications to spirals we need Proposition 2.29 modified to the following:

PROPOSITION 5.2. *Let $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be a homeomorphism, and let $U, V \subset \mathbb{R}^n$ be set-germs $0 \in \mathbb{R}^n$ such that $0 \in \bar{U} \cap \bar{V}$. Suppose that*

- (1) $D(U \cap V) = D(U) \cap D(V)$,
- (2) $U \cap V$ satisfies condition (SSP),
- (3) $h(U)$ satisfies condition (SSP), and
- (4) h is bi-Lipschitz.

Then $D(h(U \cap V)) = D(h(U)) \cap D(h(V))$.

We will use the above proposition to give a classification of spirals. (We note that in general is quite tedious to test the property of being bi-Lipschitz.) Firstly note that for $\alpha \neq \beta, \alpha, \beta \in [0, 2\pi)$ we have $D(L_\alpha \cap L_\beta) = D(L_\alpha) \cap D(L_\beta) = \emptyset$ and $L_\alpha \cap L_\beta = \{0\}$ satisfies condition (SSP). That is, L_α and L_β satisfy the first two conditions in Proposition 5.2.

We first consider the case when $\#(D(S_0)) > 1$, which is equivalent with the following condition: There are $\alpha \neq \beta, \alpha, \beta \in [0, 2\pi)$ such that $D(h_{S_0}(L_\alpha)) \cap D(h_{S_0}(L_\beta)) \neq \emptyset$.

On the other hand, as $D(h_{S_0}(L_\alpha \cap L_\beta)) = \emptyset$, the conclusion of Proposition 5.2 does not hold; this may happen only if one or both of conditions (3) and (4) fail. We can therefore divide the case $\#(D(S_0)) > 1$ in three classes as follows:

- (A) S_0 satisfies condition (SSP) at $0 \in \mathbb{R}^2$. In this case the induced homeomorphism h_{S_0} is not bi-Lipschitz. For example, this is the case for the hyperbolic spiral, $r = a/\theta$, $a > 0$. Note that the length of the spiral is infinite (even for $c \leq \theta < \infty, c > 0$). On the other hand the spiral $r = a/\theta^2$ also satisfies (SSP) so, although its length is finite for $c \leq \theta < \infty, c > 0$, the induced homeomorphism is, yet again, not bi-Lipschitz.
- (B) The induced homeomorphism h_{S_0} is bi-Lipschitz, and therefore S_0 does not satisfy (SSP). This is the case for the logarithmic spiral $r = ae^{-b\theta}$, $a, b > 0$.
- (C) In this case S_0 does not satisfy (SSP) and h_{S_0} is not bi-Lipschitz.

Finally we have the remaining case when $\#D(S_0) = 1$. This condition is equivalent to the condition $\forall \alpha \neq \beta, \alpha, \beta \in [0, 2\pi)$ we have $D(h_{S_0}(L_\alpha)) \cap D(h_{S_0}(L_\beta)) = \emptyset$. This is the case for the Archimedean spiral $r = a\theta, a > 0$.

Let us recall some examples on (SSP) analysed in [3].

EXAMPLE 5.3. (1) Let $0 < r < 1$, and let $A = \{a_m | m \in \mathbb{N}\}$ be a sequence of points of \mathbb{R} , defined by $a_m := r^m$. Then A does not satisfy (SSP) at $0 \in \mathbb{R}$.

(2) Let $B = \{b_m | m \in \mathbb{N}\}$ and $C = \{c_m | m \in \mathbb{N}\}$ be sequences of points of \mathbb{R} defined by $b_m := 1/m$ and $c_m := (1/m)^2$, respectively. Then B and C satisfy (SSP) at $0 \in \mathbb{R}$.

The first example above can be used to construct spirals belonging to the class (C), whilst the second one can be used to explain the examples given in the class (A).

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