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Classification of the fundamental groups of join-type curves of degree seven

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Abstract. We compute the fundamental groups $\pi_1(\mathbb{P}^2 \setminus C)$ for all *complex* curves C of degree 7 defined by an equation of the form

$$\prod_{j=1}^{\ell}(Y - \beta_j Z)^{\nu_j} = c \cdot \prod_{i=1}^{m}(X - \alpha_i Z)^{\lambda_i},$$

where $\sum_{j=1}^{\ell} \nu_j = \sum_{i=1}^{m} \lambda_i$ is the degree of the curve, $c \in \mathbb{R}\setminus\{0\}$, and $\beta_1, \ldots, \beta_\ell$ (respectively $\alpha_1, \ldots, \alpha_m$) mutually distinct *real* numbers.

1. Introduction.

The fundamental groups of plane curve complements are powerful tools to study ramified coverings and to distinguish the path-connected components (and, in many cases, the irreducible components) of equisingular moduli spaces. A systematic study of these groups was initiated by O. Zariski [**14**] and E. R. van Kampen [**6**] in the thirties and was later developped by many other mathematicians. Bibliographies covering the classical results as well as the latest advances can be found, for instance, in [**2**], [**7**], [**10**]. Among the pioneer and most remarkable results, let us mention the famous theorem of O. Zariski, W. Fulton and P. Deligne (cf. [**14**], [**16**], [**5**], [**3**]). This theorem says that if C is a curve with only simple points or node singularities, then the fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian. In the same vein, a result of M. V. Nori [**8**] says that if C is an irreducible curve having only nodes and cusps as singularities with

$$2 \times (\text{number of nodes}) + 6 \times (\text{number of cusps}) < (\deg(C))^2, \tag{1.1}$$

then $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian too. Note that when the inequality (1.1) is not satisfied, the group $\pi_1(\mathbb{P}^2 \setminus C)$ may be non-abelian, as shown by the famous Zariski's three-cuspidal quartic (cf. [**14**]). The present article is a sequel of the papers [**9**], [**4**] where a classification of the fundamental groups occuring in another family of curves (so-called 'join-type curves') was initiated.

Join-type curves are defined as follows. Consider positive integers $\nu_1, \ldots, \nu_\ell$, $\lambda_1, \ldots, \lambda_m$ with $\sum_{j=1}^{\ell} \nu_j = \sum_{i=1}^{m} \lambda_i$. A curve C in the complex projective plane $\mathbb{P}^2$

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is called a *join-type curve with exponents* $(\nu_1, \dots, \nu_\ell; \lambda_1, \dots, \lambda_m)$ if it is defined by an equation of the form

$$a \cdot \prod_{j=1}^{\ell} (Y - \beta_j Z)^{\nu_j} = b \cdot \prod_{i=1}^{m} (X - \alpha_i Z)^{\lambda_i},$$

where X, Y, Z are homogeneous coordinates in $\mathbb{P}^2$, a, b non-zero complex numbers, and $\beta_1, \dots, \beta_\ell$ (respectively $\alpha_1, \dots, \alpha_m$) mutually distinct complex numbers. In the chart $\mathbb{C}^2 := \mathbb{P}^2 \setminus \{Z = 0\}$, with coordinates $x = X/Z$ and $y = Y/Z$, the curve C is defined by the equation $f(y) = g(x)$, where

$$f(y) := a \cdot \prod_{j=1}^{\ell} (y - \beta_j)^{\nu_j} \quad \text{and} \quad g(x) := b \cdot \prod_{i=1}^{m} (x - \alpha_i)^{\lambda_i}.$$

(Note that the line defined by $Z = 0$ meets C at d distinct points, where d is the degree of C.) The singular points of C (i.e., the points (x, y) satisfying $f(y) = g(x)$ and $f'(y) = g'(x) = 0$) divide into two categories: the points (x, y) which also satisfy the equations $f(y) = g(x) = 0$, and those for which $f(y) \neq 0$ and $g(x) \neq 0$. Clearly, the singular points contained in the intersection of lines $f(y) = g(x) = 0$ are the points (α_i, β_j) with $\lambda_i, \nu_j \geq 2$. Hereafter, such singular points will be called *typical* singularities, while the singular points (x, y) with $f(y) \neq 0$ and $g(x) \neq 0$ will be called *exceptional* singularities.

By [**9**, Theorem (1.3)], we know that if C does not have any exceptional singularity, then the fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$ is isomorphic to the group $G(\nu; \lambda; d/\nu)$ defined by the presentation

$$G(\nu; \lambda; d/\nu) := \langle \omega,\ a_k\ (k \in \mathbb{Z}) \mid \omega = a_{\nu-1} a_{\nu-2} \cdots a_0,\ \omega^{d/\nu} = e,$$
$$a_{k+\lambda} = a_k,\ a_{k+\nu} = \omega a_k \omega^{-1}\ (k \in \mathbb{Z}) \rangle,$$

where ν (respectively λ) is the greatest common divisor of $\nu_1, \dots, \nu_\ell$ (respectively $\lambda_1, \dots, \lambda_m$), and e the unit element. For example, if C does not have any exceptional singularity and if λ or ν is equal to 1, then the fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian, isomorphic to $\mathbb{Z}_d$ (in particular, C is irreducible).

We say that C is an $\mathbb{R}$-*join-type curve* if the coefficients a, b, α_i $(1 \leq i \leq m)$ and β_j $(1 \leq j \leq \ell)$ are *real* numbers. Exceptional singularities of such curves are necessarily node singularities. In [**4**], we proposed the following conjecture.[1]

CONJECTURE 1.1. *Let C, $C' \subset \mathbb{P}^2$ be $\mathbb{R}$-join-type curves with exponents $(\nu_1, \dots, \nu_\ell; \lambda_1, \dots, \lambda_m)$ and with the same component type (see below for the definition). We suppose that C' does not have any exceptional singularity. Then,*

$$\pi_1(\mathbb{P}^2 \setminus C) \simeq \pi_1(\mathbb{P}^2 \setminus C') \simeq G(\nu; \lambda; d/\nu),$$

[1] The statement given in [**4**] is incorrect. It should be replaced by the statement given here.

where d is the (common) degree of C and C', and ν (respectively λ) the greatest common divisor of $\nu_1, \dots, \nu_\ell$ (respectively $\lambda_1, \dots, \lambda_m$). In particular, if C is irreducible and if λ or ν is equal to 1 (in which case C' is also irreducible), then $\pi_1(\mathbb{P}^2 \setminus C) \simeq \mathbb{Z}_d$.

Here, we say that C and C' have the same component type if $C = \bigcup_{i=1}^r C_i$ and $C' = \bigcup_{i=1}^r C'_i$, where C_i and C'_i are irreducible and $\deg(C_i) = \deg(C'_i)$ for each i. The r-ple $\{\deg(C_1), \dots, \deg(C_r)\}$ is called the *component type* of C (and C').

REMARK 1.2. Note that when C does have exceptional singularities, the condition '$\lambda = 1$ or $\nu = 1$' does not imply that C is irreducible.

In [**4**], we classified the fundamental groups of all $\mathbb{R}$-join-type curves of degree 6. In the present paper, we compute the fundamental groups of all $\mathbb{R}$-join-type curves of degree 7, the smallest degree for which still very little is known. (There is an abundant literature for degrees less than or equal to 6.) As an immediate corollary, we get that Conjecture 1.1 is true for all such curves.

THEOREM 1.3. *Suppose $C \subset \mathbb{P}^2$ is an $\mathbb{R}$-join-type curve of degree 7 defined by the (affine) equation $f(y) = g(x)$, where*

$$f(y) = a \cdot \prod_{j=1}^{\ell} (y - \beta_j)^{\nu_j} \quad \text{and} \quad g(x) = b \cdot \prod_{i=1}^{m} (x - \alpha_i)^{\lambda_i}.$$

(1) *If the set of exponents $\mathscr{E} := (\nu_1, \dots, \nu_\ell; \lambda_1, \dots, \lambda_m)$ of C is not the set $(7;7)$, then the fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian. When, in addition, $\mathscr{E}$ is neither the set $(2,2,2,1;2,2,2,1)$ nor the set $(1,\dots,1;1,\dots,1)$, the group $\pi_1(\mathbb{P}^2 \setminus C)$ is isomorphic to $\mathbb{Z}_7$ or $\mathbb{Z}$ depending on whether the curve is irreducible or has two irreducible components. When $\mathscr{E}$ is the set $(2,2,2,1;2,2,2,1)$ or the set $(1,\dots,1;1,\dots,1)$, the group $\pi_1(\mathbb{P}^2 \setminus C)$ is isomorphic to $\mathbb{Z}_7$, $\mathbb{Z}$ or $\mathbb{Z}^3$ depending on whether the curve has one, two or four irreducible components.*

(2) *If $\mathscr{E} = (7;7)$, then $\pi_1(\mathbb{P}^2 \setminus C)$ is non-abelian, isomorphic to the group given by the presentation*

$$\langle a_0, a_1, \dots, a_6 \mid a_6 a_5 \cdots a_0 = e \rangle.$$

Actually, when $\mathscr{E}$ is the set of exponents $(2,2,2,1;2,2,2,1)$ or the set $(1,\dots,1;1,\dots,1)$, the curve C has only node singularities. Moreover, it has at most 15 such singular points except when the polynomials f and g are as in Figures 48 or 50. In the latter cases, C has 18 nodes. When C has at most 15 nodes, the group $\pi_1(\mathbb{P}^2 \setminus C)$ is isomorphic to $\mathbb{Z}_7$ or $\mathbb{Z}$. When it has 18 nodes, $\pi_1(\mathbb{P}^2 \setminus C)$ is isomorphic to $\mathbb{Z}^3$.

When $\mathscr{E} = (7;7)$, exceptional singularities do not occur, and therefore the fundamental group is given by [**9**]. (Actually, in this special case, we can also simply observe that the curve is a union of seven concurrent lines, and therefore, its complement is $\mathbb{C} \times (\mathbb{C} \setminus \{6 \text{ points}\})$.) Still according to [**9**], when $\mathscr{E}$ is not the set $(7;7)$ and the curve C does not have any exceptional singularity, the group $\pi_1(\mathbb{P}^2 \setminus C)$ is always isomorphic to $\mathbb{Z}_7$. (In particular, this is the case when $\mathscr{E}$ is of the form $(7;\lambda_1,\dots,\lambda_m)$ with $m \geq 2$.)

Remark 1.4. Note that as soon as we know that the group $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian, we completely know its structure. Indeed, in this case, by the Hurewicz theorem, $\pi_1(\mathbb{P}^2 \setminus C)$ is isomorphic to first integral homology group $H_1(\mathbb{P}^2 \setminus C)$. Then, by Poincaré–Lefschetz duality, it is not difficult to see that $H_1(\mathbb{P}^2 \setminus C) \simeq \mathbb{Z}^{r-1} \times \mathbb{Z}_{d_0}$, where r is the number of irreducible components of C and $d_0 := \gcd(d_1, \ldots, d_r)$. (Here, d_i is the degree of the i-th irreducible component, $1 \leq i \leq r$.) See e.g. [**12**].

Table 1. Exponents and sets of singularities of pseudo-maximal curves.

#	Exponents	Pseudo-maximal sets of singularities
1.	$(6,1;6,1)$	$\boldsymbol{B}_{6,6} \oplus \boldsymbol{A}_1$
2.	$(6,1;5,2)$	$\boldsymbol{B}_{6,5} \oplus \boldsymbol{A}_5 \oplus \boldsymbol{A}_1$
3.	$(6,1;4,3)$	$\boldsymbol{B}_{6,4} \oplus \boldsymbol{B}_{6,3} \oplus \boldsymbol{A}_1$
4.	$(6,1;4,2,1)$	$\boldsymbol{B}_{6,4} \oplus \boldsymbol{A}_5 \oplus 2\,\boldsymbol{A}_1$
5.	$(6,1;3,2,2)$	$\boldsymbol{B}_{6,3} \oplus 2\,\boldsymbol{A}_5 \oplus 2\,\boldsymbol{A}_1$
6.	$(6,1;2,2,2,1)$	$3\,\boldsymbol{A}_5 \oplus 3\,\boldsymbol{A}_1$
7.	$(5,2;5,2)$	$\boldsymbol{B}_{5,5} \oplus 2\,\boldsymbol{A}_4 \oplus 2\,\boldsymbol{A}_1$
8.	$(5,2;4,3)$	$\boldsymbol{B}_{5,4} \oplus \boldsymbol{E}_8 \oplus \boldsymbol{A}_3 \oplus \boldsymbol{A}_2 \oplus \boldsymbol{A}_1$
9.	$(5,2;4,2,1)$	$\boldsymbol{B}_{5,4} \oplus \boldsymbol{A}_4 \oplus \boldsymbol{A}_3 \oplus 3\,\boldsymbol{A}_1$
10.	$(5,2;3,2,2)$	$\boldsymbol{E}_8 \oplus 2\,\boldsymbol{A}_4 \oplus \boldsymbol{A}_2 \oplus 4\,\boldsymbol{A}_1$
11.	$(5,2;2,2,2,1)$	$3\,\boldsymbol{A}_4 \oplus 6\,\boldsymbol{A}_1$
12.	$(4,3;4,3)$	$\boldsymbol{B}_{4,4} \oplus 2\,\boldsymbol{E}_6 \oplus \boldsymbol{D}_4 \oplus \boldsymbol{A}_1$
13.	$(4,3;4,2,1)$	$\boldsymbol{B}_{4,4} \oplus \boldsymbol{E}_6 \oplus \boldsymbol{A}_3 \oplus \boldsymbol{A}_2 \oplus 2\,\boldsymbol{A}_1$
14.	$(4,3;3,2,2)$	$\boldsymbol{D}_4 \oplus \boldsymbol{E}_6 \oplus 2\,\boldsymbol{A}_3 \oplus 2\,\boldsymbol{A}_2 \oplus 2\,\boldsymbol{A}_1$
15.	$(4,3;2,2,2,1)$	$3\,\boldsymbol{A}_3 \oplus 3\,\boldsymbol{A}_2 \oplus 3\,\boldsymbol{A}_1$
16.	$(4,2,1;4,2,1)$	$\boldsymbol{B}_{4,4} \oplus 2\,\boldsymbol{A}_3 \oplus 5\,\boldsymbol{A}_1$
17.	$(4,2,1;3,2,2)$	$\boldsymbol{E}_6 \oplus 2\,\boldsymbol{A}_3 \oplus \boldsymbol{A}_2 \oplus 6\,\boldsymbol{A}_1$
18.	$(4,2,1;2,2,2,1)$	$3\,\boldsymbol{A}_3 \oplus 9\,\boldsymbol{A}_1$
19.	$(3,2,2;3,2,2)$	$\boldsymbol{D}_4 \oplus 4\,\boldsymbol{A}_2 \oplus 8\,\boldsymbol{A}_1$
20.	$(3,2,2;2,2,2,1)$	$3\,\boldsymbol{A}_2 \oplus 12\,\boldsymbol{A}_1$

2. Proof of Theorem 1.3.

Throughout this section, we assume that C has at least one exceptional singularity. (When the curve does not have any exceptional singularity, the result is already proved in [**9**].)

For most of the fundamental groups, the commutativity is obtained by the degeneration principle. This principle says that if $\{C_t\}_{t \in U}$ is an analytic family of reduced curves in $\mathbb{P}^2$, where $U \subset \mathbb{C}$ is a connected open set containing the origin, if the family of curves $\{C_t\}_{t \in U \setminus \{0\}}$ is equisingular, and if the total Milnor number of C_t, $t \neq 0$, is less than the total Milnor number of C_0 (in which case one says that C_t degenerates to C_0), then there is a canonical epimorphism $\pi_1(\mathbb{P}^2 \setminus C_0) \twoheadrightarrow \pi_1(\mathbb{P}^2 \setminus C_t)$. In particular, if $\pi_1(\mathbb{P}^2 \setminus C_0)$ is abelian, so is $\pi_1(\mathbb{P}^2 \setminus C_t)$. See [**14**]. An $\mathbb{R}$-join-type septic is said to be *pseudo-maximal* if it does not degenerate, within the class of *all* join-type septics,[2] to any other $\mathbb{R}$-join-type septic whose exponents are not $(7;7)$. (Notice that an $\mathbb{R}$-join-type septic always degenerates to an $\mathbb{R}$-join-type septic with exponents $(7;7)$.) Thus, to determine the fundamental groups of $\mathbb{R}$-join-type septics, it suffices to find the groups of such pseudo-maximal curves. For any set of exponents in Table 1, there is a pseudo-maximal $\mathbb{R}$-join-type septic with this set of exponents and the mentioned set of singularities. (However, a curve may have a set of exponents listed in Table 1 without being pseudo-maximal.) For all the other sets of exponents, the curves are non-pseudo-maximal or have only node singularities (cf. Tables 2 & 3). By the Zariski–Fulton–Deligne theorem [**14**], [**16**], [**5**], [**3**], the fundamental groups associated with curves having only node singularities are abelian. Therefore, by Remark 1.4, to find their fundamental groups, it suffices to know their component types. The component types of the curves associated with the sets of exponents № 86–94 and № 98 can be easily determined (cf. Section 2.23). For the sets № 96, 97, 99–104, it is more difficult. However, for these exponents, it is easy to check that the curves are non-pseudo-maximal. The sets $(2,2,2,1;2,2,2,1)$ and $(1,\dots,1;1,\dots,1)$ are 'special' and need to be discussed separately (cf. Section 2.23).

The fundamental groups of pseudo-maximal $\mathbb{R}$-join-type septics whose exponents are in Table 1 are computed in Sections 2.1–2.20 below. To compute these groups, we use the Zariski–van Kampen theorem with the pencil $\mathscr{P}$ given by the horizontal lines $L_\delta\colon y=\delta$, $\delta\in\mathbb{C}$. This theorem says that $\pi_1(\mathbb{P}^2\setminus C)$ is isomorphic to $\pi_1(L_{\delta_0}\setminus C)/G$, where L_{δ_0} is a generic line of the pencil and G the normal subgroup of $\pi_1(L_{\delta_0}\setminus C)$ generated by the monodromy relations associated with the 'special' lines of the pencil (cf. [**14**], [**6**]).

A line L_δ of the pencil meets the curve C at a point (γ,δ) with intersection multiplicity greater than or equal to 2 (i.e., L_δ is a special line) if and only if $f(\delta)=g(\gamma)$ and $g'(\gamma)=0$. By considering the restriction of the function $g(x)$ to real numbers, we see immediately that the equation $g'(x)=0$ has at least one real root γ_i in the open interval (α_i,α_{i+1}) for each $i=1,\dots,m-1$ (we can assume that $\alpha_1<\dots<\alpha_m$ and $\beta_1<\dots<\beta_\ell$). Since the degree of

$$g'(x)\Big/\prod_{i=1}^{m}(x-\alpha_i)^{\lambda_i-1}$$

is $m-1$, it follows that the roots of $g'(x)=0$ are exactly $\gamma_1,\dots,\gamma_{m-1}$ and the α_i's with $\lambda_i\ge 2$. In particular, this shows that $\gamma_1,\dots,\gamma_{m-1}$ are *simple* roots of $g'(x)=0$.

Let $\delta_{i,1},\dots,\delta_{i,d}$ be the roots of $f(y)=g(\gamma_i)$ for $1\le i\le m-1$.

If $f'(\delta_{i,k})\neq 0$, then $(\gamma_i,\delta_{i,k})$ is a simple point of C, and in a small neighbourhood of it, C is topologically described by

$$y-\delta_{i,k}=c\,(x-\gamma_i)^2, \tag{2.1}$$

where $c\neq 0$. Indeed, by Taylor's formula, f and g can be written as

[2]Including those which are not $\mathbb{R}$-join-type septics, and of course those which have other exponents.

Table 2. Exponents of (some) non-pseudo-maximal curves.

#	Exponents	#	Exponents	#	Exponents
21.	$(6,1;5,1,1)$	43.	$(5,1,1;4,2,1)$	65.	$(3,3,1;2,1,1,1,1,1)$
22.	$(6,1;3,3,1)$	44.	$(5,1,1;3,3,1)$	66.	$(3,2,2;4,1,1,1)$
23.	$(6,1;4,1,1,1)$	45.	$(5,1,1;3,2,2)$	67.	$(3,2,2;3,2,1,1)$
24.	$(6,1;3,2,1,1)$	46.	$(5,1,1;4,1,1,1)$	68.	$(3,2,2;3,1,1,1,1)$
25.	$(6,1;3,1,1,1,1)$	47.	$(5,1,1;3,2,1,1)$	69.	$(3,2,2;2,2,1,1,1)$
26.	$(6,1;2,2,1,1,1)$	48.	$(5,1,1;2,2,2,1)$	70.	$(3,2,2;2,1,1,1,1,1)$
27.	$(6,1;2,1,1,1,1,1)$	49.	$(5,1,1;3,1,1,1,1)$	71.	$(4,1,1,1;4,1,1,1)$
28.	$(5,2;5,1,1)$	50.	$(5,1,1;2,2,1,1,1)$	72.	$(4,1,1,1;3,2,1,1)$
29.	$(5,2;3,3,1)$	51.	$(5,1,1;2,1,1,1,1,1)$	73.	$(4,1,1,1;2,2,2,1)$
30.	$(5,2;4,1,1,1)$	52.	$(4,2,1;3,3,1)$	74.	$(4,1,1,1;3,1,1,1,1)$
31.	$(5,2;3,2,1,1)$	53.	$(4,2,1;4,1,1,1)$	75.	$(4,1,1,1;2,2,1,1,1)$
32.	$(5,2;3,1,1,1,1)$	54.	$(4,2,1;3,2,1,1)$	76.	$(4,1,1,1;2,1,1,1,1,1)$
33.	$(5,2;2,2,1,1,1)$	55.	$(4,2,1;3,1,1,1,1)$	77.	$(3,2,1,1;3,2,1,1)$
34.	$(5,2;2,1,1,1,1,1)$	56.	$(4,2,1;2,2,1,1,1)$	78.	$(3,2,1,1;2,2,2,1)$
35.	$(4,3;5,1,1)$	57.	$(4,2,1;2,1,1,1,1,1)$	79.	$(3,2,1,1;3,1,1,1,1)$
36.	$(4,3;3,3,1)$	58.	$(3,3,1;3,3,1)$	80.	$(3,2,1,1;2,2,1,1,1)$
37.	$(4,3;4,1,1,1)$	59.	$(3,3,1;3,2,2)$	81.	$(3,2,1,1;2,1,1,1,1,1)$
38.	$(4,3;3,2,1,1)$	60.	$(3,3,1;4,1,1,1)$	82.	$(2,2,2,1;3,1,1,1,1)$
39.	$(4,3;3,1,1,1,1)$	61.	$(3,3,1;3,2,1,1)$	83.	$(3,1,1,1,1;3,1,1,1,1)$
40.	$(4,3;2,2,1,1,1)$	62.	$(3,3,1;2,2,2,1)$	84.	$(3,1,1,1,1;2,2,1,1,1)$
41.	$(4,3;2,1,1,1,1,1)$	63.	$(3,3,1;3,1,1,1,1)$	85.	$(3,1,1,1,1;2,1,1,1,1,1)$
42.	$(5,1,1;5,1,1)$	64.	$(3,3,1;2,2,1,1,1)$		

Table 3. Exponents of curves with only $\boldsymbol{A}_1$-singularities.

#	Exponents	#	Exponents
86.	$(6,1;1,\dots,1)$	96.	$(2,2,2,1;2,2,1,1,1)$
87.	$(5,2;1,\dots,1)$	97.	$(2,2,2,1;2,1,1,1,1,1)$
88.	$(4,3;1,\dots,1)$	98.	$(2,2,2,1;1,\dots,1)$
89.	$(5,1,1;1,\dots,1)$	99.	$(3,1,1,1,1;1,\dots,1)$
90.	$(4,2,1;1,\dots,1)$	100.	$(2,2,1,1,1;2,2,1,1,1)$
91.	$(3,3,1;1,\dots,1)$	101.	$(2,2,1,1,1;2,1,1,1,1,1)$
92.	$(3,2,2;1,\dots,1)$	102.	$(2,2,1,1,1;1,\dots,1)$
93.	$(4,1,1,1;1,\dots,1)$	103.	$(2,1,1,1,1,1;2,1,1,1,1,1)$
94.	$(3,2,1,1;1,\dots,1)$	104.	$(2,1,1,1,1,1;1,\dots,1)$
95.	$(2,2,2,1;2,2,2,1)$	105.	$(1,\dots,1;1,\dots,1)$

$$f(y) = \sum_{q=0}^{7} a_q (y - \delta_{i,k})^q \quad \text{and} \quad g(x) = \sum_{q=0}^{7} b_q (x - \gamma_i)^q. \tag{2.2}$$

The equality $f(\delta_{i,k}) = g(\gamma_i)$ implies $a_0 = b_0$. The relation $f'(\delta_{i,k}) \neq 0$ shows $a_1 \neq 0$. Finally, since γ_i is a simple root of $g'(x) = 0$, we have $b_1 = 0$ and $b_2 \neq 0$. Thus, the equation $f(y) = g(x)$ takes the form

$$a_1\,(y - \delta_{i,k}) + \text{higher order terms} = b_2\,(x - \gamma_i)^2 + \text{higher order terms},$$

and therefore, near $(\gamma_i, \delta_{i,k})$, the curve C is topologically given by (2.1). In particular, this says that the line $y = \delta_{i,k}$ is tangent to the curve at $(\gamma_i, \delta_{i,k})$ with intersection multiplicity 2.

If $f'(\delta_{i,k}) = 0$, then $(\gamma_i, \delta_{i,k})$ is an exceptional singularity of type $\boldsymbol{A}_1$, and near this point, the curve is topologically equivalent to

$$(y - \delta_{i,k})^2 = c\,(x - \gamma_i)^2. \tag{2.3}$$

Indeed, since $f(\delta_{i,k}) = g(\gamma_i) \neq 0$, the same argument used to prove that $\gamma_1, \ldots, \gamma_{m-1}$ are simple roots of $g'(x) = 0$ shows that, if $f'(\delta_{i,k}) = 0$, then $\delta_{i,k}$ is a simple root of $f'(y) = 0$. Now, if γ_i and $\delta_{i,k}$ are both simple roots, then the coefficients a_1, a_2, b_1 and b_2 in (2.2) satisfy $a_1 = b_1 = 0$, $a_2 \neq 0$, $b_2 \neq 0$, and the equation $f(y) = g(x)$ takes the form

$$a_2\,(y - \delta_{i,k})^2 + \text{higher order terms} = b_2\,(x - \gamma_i)^2 + \text{higher order terms}.$$

Thus, near $(\gamma_i, \delta_{i,k})$, the curve C is topologically described by (2.3).

For each α_i with $\lambda_i \geq 2$, the roots of $f(y) = g(\alpha_i)$ are $\beta_1, \ldots, \beta_\ell$. By Taylor's formula again, we write

$$f(y) = \sum_{q=0}^{7} a'_q (y - \beta_j)^q \quad \text{and} \quad g(x) = \sum_{q=0}^{7} b'_q (x - \alpha_i)^q. \tag{2.4}$$

Then, the same argument as above shows that, if $\nu_j = 1$, then (α_i, β_j) is a simple point of C, and near this point, C is topologically given by

$$y - \beta_j = c\,(x - \alpha_i)^{\lambda_i}. \tag{2.5}$$

In particular, the line $y = \beta_j$ is tangent to C at (α_i, β_j) with intersection multiplicity λ_i. Similarly, if $\nu_j \geq 2$, then (α_i, β_j) is a typical singularity of Brieskorn–Pham type $\boldsymbol{B}_{\nu_j,\lambda_i}$, and in a small neighbourhood of it, the curve is topologically equivalent to

$$(y - \beta_j)^{\nu_j} = c\,(x - \alpha_i)^{\lambda_i}. \tag{2.6}$$

Indeed, in this case, the coefficients a'_q and b'_q in (2.4) satisfy $a'_q = 0$ for $q \leq \nu_j - 1$, $b'_q = 0$

for $q \leq \lambda_i - 1$, and $a'_{\nu_j} \neq 0$, $b'_{\lambda_i} \neq 0$.

Let us now give a detailed calculation of the fundamental groups of the pseudo-maximal curves. We shall proceed case-by-case for each set of exponents in Table 1. We take the point $(1\colon 0\colon 0)$ as base point for all the groups. This point is nothing but the axis of the pencil $\mathscr{P}$, which is also the point at infinity of the lines L_δ. (Note that it does not belong to the curve.)

2.1. Exponents $(6,1;6,1)$.

For these exponents, we can suppose that the polynomials f and g are of the form $f(y) = a\,(y-\beta_1)^6\,(y-\beta_2)$ and $g(x) = b\,(x-\alpha_1)^6\,(x-\alpha_2)$ and that their real graphs are as in Figure 1, so that there exist real numbers $\theta \in (\beta_1, \beta_2)$ and $\gamma \in (\alpha_1, \alpha_2)$ satisfying $f'(\theta) = g'(\gamma) = 0$ and $f(\theta) = g(\gamma)$. Indeed, any $\mathbb{R}$-join-type septic with the set of exponents $(6,1;6,1)$ is topologically equivalent to the $\mathbb{R}$-join-type septic defined by such f and g. This can be shown as follows. For the given exponents, the possibilities for f and g are given in Figure 2. (In this figure, the numbers refer to the exponents.) The change of coordinates $(x,y) \mapsto (x,-y)$ shows that the curve $f_3(y) = g_i(x)$ (respectively $f_4(y) = g_i(x)$) is topologically equivalent to the curve $f_2(y) = g_i(x)$ (respectively $f_1(y) = g_i(x)$). Then, it is enough to consider f_1, f_2, and g_i for $1 \leq i \leq 4$. Similarly, the change of coordinates $(x,y) \mapsto (-x,y)$ shows that the curve $f_i(y) = g_3(x)$ (respectively $f_i(y) = g_4(x)$) is topologically equivalent to the curve $f_i(y) = g_2(x)$ (respectively $f_i(y) = g_1(x)$), and therefore it suffices to consider f_1, f_2, g_1 and g_2. Actually, we do not need to consider g_2 either. Indeed, the change of coordinates $(x,y) \mapsto (-x,-y)$ shows that the curve $f_1(y) = g_2(x)$ (respectively $f_2(y) = g_2(x)$) is topologically equivalent to the curve $-f_1(-y) = -g_2(-x)$ (respectively $-f_2(-y) = -g_2(-x)$), which is in turn topologically equivalent to the curve $f_2(y) = g_1(x)$ (respectively $f_1(y) = g_1(x)$). The curve $f_2(y) = g_1(x)$ does not have any exceptional singularity and can be eliminated as well. Finally, to find the fundamental group of pseudo-maximal curves with exponents $(6,1;6,1)$, it suffices to consider the polynomials $f := f_1$ and $g := g_1$ given in Figure 1. It is not necessary to find explicit expressions for these polynomials. It suffices to know that such f and g exist, and this is guaranteed by [**13**]. However, for this set of exponents, it is not difficult to see that the graphs in Figure 1 can be obtained, for instance, by taking $\beta_1 = \alpha_1 = 0$, $\beta_2 = \alpha_2 = 1$, and $a = b = -1$.

The set of singularities of the corresponding curve C, defined by the equation $f(y) - g(x) = 0$, is $\boldsymbol{B}_{6,6} \oplus \boldsymbol{A}_1$, while its component type is $\{6,1\}$ — actually, as we mentioned it above, we can take $f(y) = -y^6(y-1)$ and $g(x) = -x^6(x-1)$ so that

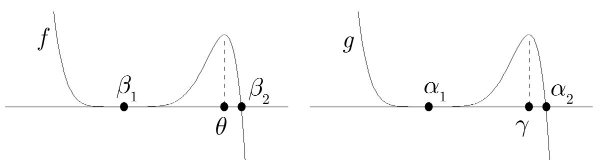

Figure 1. Real graphs of f and g (exponents $(6,1;6,1)$).

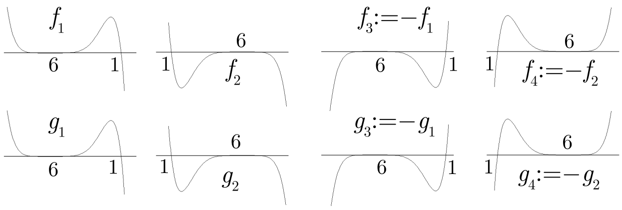

Figure 2.

$$
\begin{aligned}
f(y) - g(x) = (x-y)(&x^6 + yx^5 - x^5 + y^2x^4 - yx^4 + x^3y^3 \\
&- y^2x^3 + x^2y^4 - x^2y^3 + xy^5 - xy^4 + y^6 - y^5).
\end{aligned}
$$

Here, the sextic component C_6 has a singular point of type $\boldsymbol{B}_{5,5}$, and the line component intersects C_6 at $\boldsymbol{B}_{5,5}$ and at a smooth point. The pencil $\mathscr{P}$ has 8 special lines L_{β_1}, L_{β_2}, L_θ, $L_{\delta_1}, \dots, L_{\delta_5}$ with respect to C. These lines are given by the vertices of the 'dessin d'enfants' $f^{-1}([0, f(\theta)])$ associated with the polynomial f (cf. Figure 3). In this figure, the black vertices correspond to the roots β_1, β_2 of the equation $f(y) = 0$, and the white ones to the roots θ, $\delta_1, \dots, \delta_5$ of the equation $f(y) = f(\theta) = g(\gamma)$.

Consider the generic line $L_{\beta_2-\varepsilon}$, and choose generators $\xi_1, \dots, \xi_7$ of the fundamental group $\pi_1(L_{\beta_2-\varepsilon} \setminus C)$ as in Figure 4, where $\varepsilon > 0$ is sufficiently small. (In the figure, we do not respect the numerical scale; we even zoom on the part that collapses to α_1 when $\varepsilon \to 0$.) The ξ_k's are 'lassos' oriented counterclockwise around the intersection points of the line $L_{\beta_2-\varepsilon}$ with the curve. To find the monodromy relations around the special lines of the pencil, we proceed exactly as in [**4**] and we refer to this article for details. In our present case, the monodromy relations around L_{β_2} (multiplicity 6 tangent relations) are given by

$$\xi_7 = \xi_6 = \xi_5 = \xi_4 = \xi_3 = \xi_2,$$

while the relation associated with the line L_θ (node relation) is written as $\xi_2\xi_1 = \xi_1\xi_2$. It follows immediately that the group $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian, and since the component type of C is $\{6, 1\}$, we have the isomorphism $\pi_1(\mathbb{P}^2 \setminus C) \simeq \mathbb{Z}$ (cf. Remark 1.4).

Remark 2.1. For the given curve, the commutativity of $\pi_1(\mathbb{P}^2 \setminus C)$ can also be seen by applying Nori's theorem [**8**] (in a stronger form than the one mentioned in our introduction). A similar remark also applies for the curves in Sections 2.2 and 2.3.

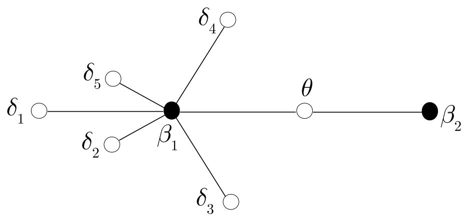

Figure 3.

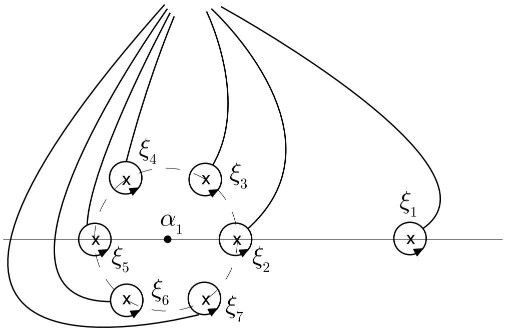

Figure 4.

2.2. Exponents $(6,1;5,2)$.

For this set of exponents, we can assume that the polynomials f and g are of the form $f(y) = a\,(y-\beta_1)^6\,(y-\beta_2)$ and $g(x) = b\,(x-\alpha_1)^5\,(x-\alpha_2)^2$ and that their real graphs are as in Figure 5, so that there exist real numbers $\theta \in (\beta_1, \beta_2)$ and $\gamma \in (\alpha_1, \alpha_2)$ satisfying $f'(\theta) = g'(\gamma) = 0$ and $f(\theta) = g(\gamma)$. (This assertion can be proved using an argument similar to that described in Section 2.1. Since the proof does not involve any new idea, in order to avoid repetitions this kind of argument will be systematically omitted.) The set of singularities of the corresponding curve C is $\boldsymbol{B}_{6,5} \oplus \boldsymbol{A}_5 \oplus \boldsymbol{A}_1$, and its component type is $\{7\}$ — actually, we can take

$$f(y) = -\frac{12500}{823543}\,y^6(y-1) \quad \text{and} \quad g(x) = \frac{46656}{823543}\,x^5(x-1)^2,$$

so that

$$f(y) - g(x) = -\frac{12500}{823543}\,y^7 + \frac{12500}{823543}\,y^6 - \frac{46656}{823543}\,x^7 + \frac{93312}{823543}\,x^6 - \frac{46656}{823543}\,x^5.$$

The special lines of the pencil with respect to C are given by the vertices of the dessin d'enfants in Figure 3.

Let us take generators $\xi_1, \ldots, \xi_7$ of $\pi_1(L_{\beta_2-\varepsilon} \setminus C)$ as in Figure 6. Then, the monodromy relations around L_{β_2} are given by:

$L_{\beta_2}(\alpha_1)$: $\xi_7 = \xi_6 = \xi_5 = \xi_4 = \xi_3$ (multiplicity 5 tangent relations);

$L_{\beta_2}(\alpha_2)$: $\xi_2 = \xi_1$ (multiplicity 2 tangent relation);

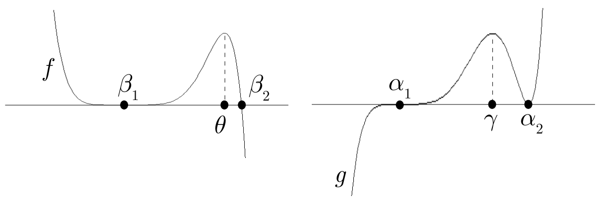

Figure 5. Real graphs of f and g (exponents $(6,1;5,2)$).

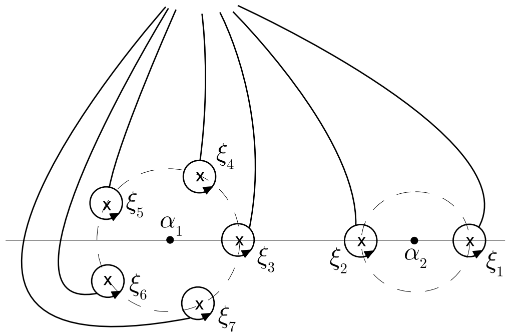

Figure 6.

and the relation associated with the line L_θ (node relation) is written as $\xi_3\xi_2 = \xi_2\xi_3$. Hence, $\pi_1(\mathbb{P}^2 \setminus C) \simeq \mathbb{Z}_7$.

2.3. Exponents $(6,1;4,3)$.

For these exponents, we can assume that f and g are of the form $f(y) = a\,(y-\beta_1)^6\,(y-\beta_2)$ and $g(x) = b\,(x-\alpha_1)^4\,(x-\alpha_2)^3$ and that their real graphs are as in Figure 7, so that there exist real numbers $\theta \in (\beta_1,\beta_2)$ and $\gamma \in (\alpha_1,\alpha_2)$ satisfying $f'(\theta) = g'(\gamma) = 0$ and $f(\theta) = g(\gamma)$. The set of singularities of the corresponding curve C is $\boldsymbol{B}_{6,4} \oplus \boldsymbol{B}_{6,3} \oplus \boldsymbol{A}_1$, and its component type is $\{7\}$ — actually, we can take

$$f(y) = -\frac{6912}{823543}\,y^6(y-1) \quad \text{and} \quad g(x) = -\frac{46656}{823543}\,x^4(x-1)^3,$$

so that $f(y) - g(x)$ is given by

$$-\frac{6912}{823543}\,y^7 + \frac{6912}{823543}\,y^6 + \frac{46656}{823543}\,x^7 - \frac{139968}{823543}\,x^6 + \frac{139968}{823543}\,x^5 - \frac{46656}{823543}\,x^4.$$

The special lines of the pencil with respect to this curve are given by the vertices of the dessin d'enfants in Figure 3.

Take generators $\xi_1, \dots, \xi_7$ of the group $\pi_1(L_{\beta_2-\varepsilon} \setminus C)$ as in Figure 8. Then, the monodromy relations associated with the line L_{β_2} are given by:

$L_{\beta_2}(\alpha_1)$: $\xi_7 = \xi_6 = \xi_5 = \xi_4$ (multiplicity 4 tangent relations);

$L_{\beta_2}(\alpha_2)$: $\xi_3 = \xi_2 = \xi_1$ (multiplicity 3 tangent relations);

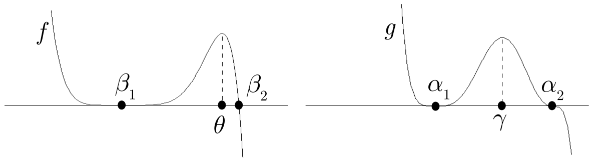

Figure 7. Real graphs of f and g (exponents $(6,1;4,3)$).

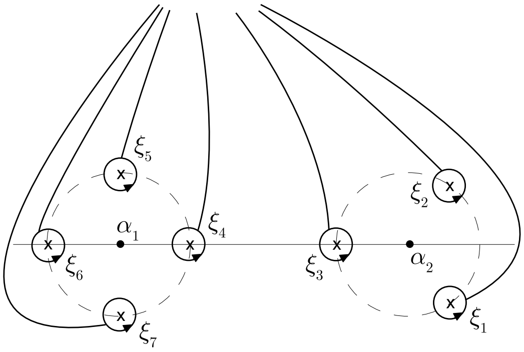

Figure 8.

and the relation around L_θ (node relation) is written as $\xi_4\xi_3 = \xi_3\xi_4$. Hence, $\pi_1(\mathbb{P}^2 \setminus C) \simeq \mathbb{Z}_7$.

2.4. Exponents $(6, 1; 4, 2, 1)$.

For this set of exponents, we can suppose that the polynomials f and g are:

(1) either of the form $f(y) = a\,(y-\beta_1)^6\,(y-\beta_2)$ and $g(x) = b\,(x-\alpha_1)^4\,(x-\alpha_2)^2\,(x-\alpha_3)$, with real graphs as in Figure 9, so that the following condition is satisfied:

$$\begin{cases} \exists\ \theta \in (\beta_1, \beta_2),\ \gamma_1 \in (\alpha_1, \alpha_2),\ \gamma_2 \in (\alpha_2, \alpha_3) \\ \text{such that } f'(\theta) = g'(\gamma_1) = g'(\gamma_2) = 0 \\ \text{and } f(\theta) = g(\gamma_1) = g(\gamma_2); \end{cases} \tag{2.7}$$

(2) or of the form $f(y) = a\,(y-\beta_1)^6\,(y-\beta_2)$ and $g(x) = b\,(x-\alpha_1)^2\,(x-\alpha_2)^4\,(x-\alpha_3)$, with real graphs as in Figure 10, so that Condition (2.7) is satisfied.

For the given exponents, the proof of this assertion is slightly more sophisticated than for

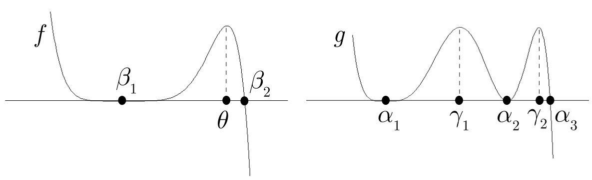

Figure 9. Real graphs of f and g (exponents $(6, 1; 4, 2, 1)$ — first case).

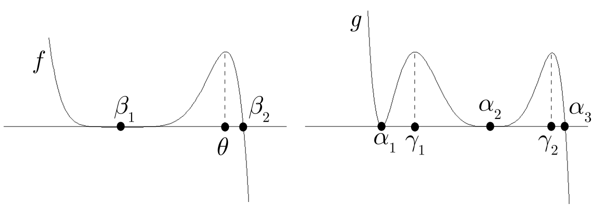

Figure 10. Real graphs of f and g (exponents $(6, 1; 4, 2, 1)$ — second case).

the previous ones. The argument is as follows. First, observe that an $\mathbb{R}$-join-type septic with exponents $(6,1;4,2,1)$ necessarily satisfies one of the following two conditions:

(a) it is topologically equivalent — or it degenerates to a curve topologically equivalent — to the curve defined by a pair of polynomials f and g as above;
(b) it degenerates to an $\mathbb{R}$-join-type septic with exponents $(6,1;5,2)$ or $(6,1;4,3)$.

Indeed, for the same reason as explained in Section 2.1, it is enough to consider the polynomials f_1, f_2, g_1, g_2 and g_3 given in Figure 11. (The numbers refer to the exponents.) Clearly, the curves $f_2(y) = g_1(x)$ and $f_2(y) = g_3(x)$ do not have any exceptional singularity. The curve $f_1(y) = g_2(x)$ degenerates to the curve $f_1(y) = \tilde{g}_2(x)$, with exponents $(6,1;4,3)$, where $\tilde{g}_2$ is as in Figure 12 — push the lower critical point of g_2 up to the horizontal axis. Similarly, the curve $f_2(y) = g_2(x)$ degenerates to the curve $f_2(y) = \hat{g}_2(x)$, where $\hat{g}_2$ is as in Figure 12 — push the upper critical point of g_2 up to the horizontal axis. In this case, the exponents of the degenerated curve are $(6,1;5,2)$. Now, by the degeneration principle and since the fundamental group of any $\mathbb{R}$-join-type septic with exponents $(6,1;4,3)$ or $(6,1;5,2)$ is abelian (cf. Sections 2.3 and 2.2), the fundamental groups associated with the curves $f_1(y) = g_2(x)$ and $f_2(y) = g_2(x)$ are also abelian. Therefore, to show that the fundamental group of an $\mathbb{R}$-join-type curve with exponents $(6,1;4,2,1)$ is abelian, it suffices to consider the curve defined by the polynomials $f := f_1$ and $g := g_1$ given in Figure 9 and the curve defined by the polynomials $f := f_1$ and $g := g_3$ given in Figure 10.

In both cases, (1) and (2), the set of singularities of the corresponding curve C is $\boldsymbol{B}_{6,4} \oplus \boldsymbol{A}_5 \oplus 2\,\boldsymbol{A}_1$. Moreover, C is irreducible (see below). In both cases, the special lines of the pencil are given by the vertices of the dessin d'enfants in Figure 3. To compute the fundamental group, in the case (1), take generators $\xi_1, \ldots, \xi_7$ of $\pi_1(L_{\beta_2-\varepsilon} \setminus C)$ as in Figure 13. Then, the monodromy relations around L_{β_2} are given by:

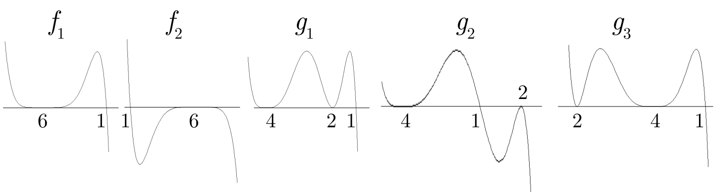

Figure 11.

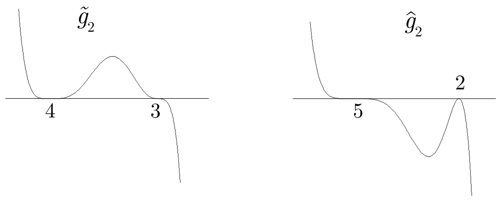

Figure 12.

$L_{\beta_2}(\alpha_1)$: $\xi_7 = \xi_6 = \xi_5 = \xi_4$ (multiplicity 4 tangent relations);

$L_{\beta_2}(\alpha_2)$: $\xi_3 = \xi_2$ (multiplicity 2 tangent relation).

After simplification, the relations associated with the line L_θ (node relation) are written as $\xi_4\xi_2 = \xi_2\xi_4$ and $\xi_2\xi_1 = \xi_1\xi_2$, while the relations around L_{δ_4} (multiplicity 2 tangent relations) give $\xi_4 = \xi_1 = \xi_2$. By the vanishing relation at infinity, $\xi_1^7 = e$, and therefore,

$$\pi_1(\mathbb{P}^2 \setminus C) \simeq \langle \xi_1 \mid \xi_1^7 = e \rangle \simeq \mathbb{Z}_7.$$

In particular, C is irreducible (cf. Remark 1.4).

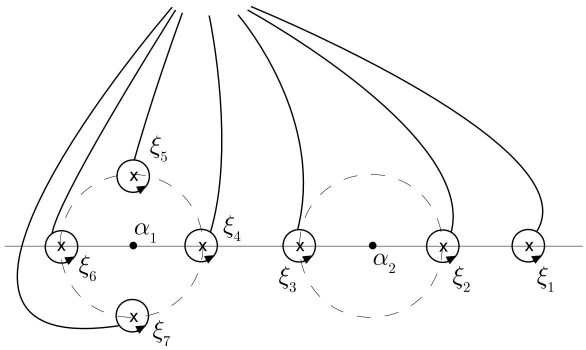

Figure 13.

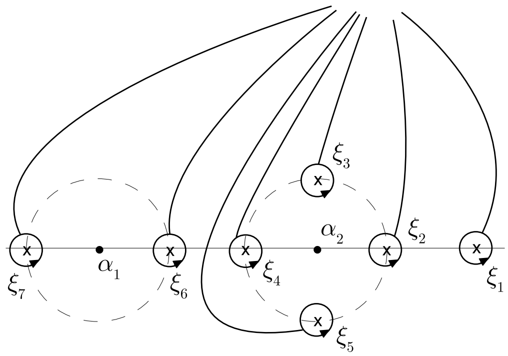

Figure 14.

In the case (2), take generators $\xi_1, \ldots, \xi_7$ of the group $\pi_1(L_{\beta_2-\varepsilon} \setminus C)$ as in Figure 14. Then, the monodromy relations around L_{β_2} are given by:

$L_{\beta_2}(\alpha_1)$: $\xi_7 = \xi_6$ (multiplicity 2 tangent relation);

$L_{\beta_2}(\alpha_2)$: $\xi_5 = \xi_4 = \xi_3 = \xi_2$ (multiplicity 4 tangent relations).

After simplification, the relations associated with the line L_θ (node relation) are written as $\xi_6\xi_2 = \xi_2\xi_6$ and $\xi_2\xi_1 = \xi_1\xi_2$, while the relations around L_{δ_4} and L_{δ_5} (multiplicity 2 tangent relations) reduce to $\xi_6 = \xi_2$ and $\xi_2 = \xi_1$ respectively. Again, since $\xi_1^7 = e$ (vanishing relation at infinity), we have $\pi_1(\mathbb{P}^2 \setminus C) \simeq \mathbb{Z}_7$.

Remark 2.2. By contrast with Remark 2.1, in the present case, we cannot substitute Nori's theorem to the Zariski–van Kampen theorem. Indeed, here, the component type of C (needed to apply Nori's theorem) is deduced from the structure of the group $\pi_1(\mathbb{P}^2 \setminus C)$, and not the opposite. For the same reason, Nori's theorem cannot be used in the next sections either.

2.5. Exponents $(6,1;3,2,2)$.

For this set of exponents, we can suppose that the polynomials f and g are of the form $f(y) = a\,(y-\beta_1)^6\,(y-\beta_2)$ and $g(x) = b\,(x-\alpha_1)^2\,(x-\alpha_2)^2\,(x-\alpha_3)^3$ and that their real graphs are as in Figure 15, so that there exist real numbers $\theta \in (\beta_1, \beta_2)$, $\gamma_1 \in (\alpha_1, \alpha_2)$ and $\gamma_2 \in (\alpha_2, \alpha_3)$ satisfying $f'(\theta) = g'(\gamma_1) = g'(\gamma_2) = 0$ and $f(\theta) = g(\gamma_1) = g(\gamma_2)$. (This assertion can be proved using a degeneration argument similar to that described in Section 2.4. Again, since the proof does not involve any new idea, in order to avoid repetitions this kind of argument will be systematically omitted.) The set of singularities of the corresponding curve C is $\boldsymbol{B}_{6,3} \oplus 2\,\boldsymbol{A}_5 \oplus 2\,\boldsymbol{A}_1$, its component type is $\{7\}$ (see below), and the special lines of the pencil $\mathscr{P}$ with respect to this curve are given by the vertices of the dessin d'enfants in Figure 3.

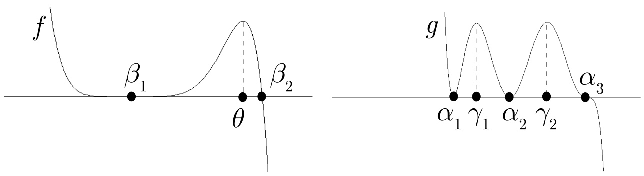

Figure 15. Real graphs of f and g (exponents $(6,1;3,2,2)$).

Let us choose generators $\xi_1, \ldots, \xi_7$ of $\pi_1(L_{\beta_2-\varepsilon} \setminus C)$ as in Figure 16. Then, the monodromy relations around the line L_{β_2} are given by:

$L_{\beta_2}(\alpha_1)$: $\xi_7 = \xi_6$ (multiplicity 2 tangent relation);

$L_{\beta_2}(\alpha_2)$: $\xi_5 = \xi_4$ (multiplicity 2 tangent relation);

$L_{\beta_2}(\alpha_3)$: $\xi_3 = \xi_2 = \xi_1$ (multiplicity 3 tangent relations).

After simplification, the relations around L_θ (node relations) are written as $\xi_6\xi_4 = \xi_4\xi_6$ and $\xi_4\xi_1 = \xi_1\xi_4$, while the relations associated with the lines L_{δ_4} and L_{δ_5} (multiplicity 2 tangent relations) give $\xi_6 = \xi_1$ and $\xi_4 = \xi_1$ respectively. Therefore, the vanishing relation at infinity is written as $\xi_1^7 = e$ and $\pi_1(\mathbb{P}^2 \setminus C) \simeq \mathbb{Z}_7$. (As above, this immediately implies that C is irreducible.)

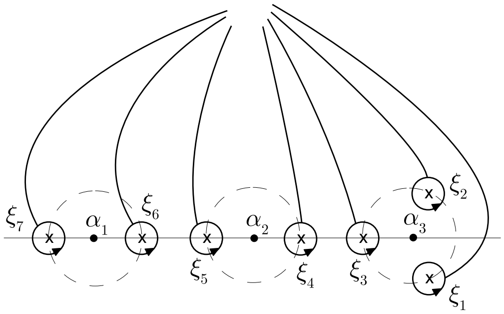

Figure 16.

2.6. Exponents $(6, 1; 2, 2, 2, 1)$.

For these exponents, we can assume that the polynomials f and g are of the form $f(y) = a\,(y-\beta_1)^6\,(y-\beta_2)$ and $g(x) = b\,(x-\alpha_1)^2\,(x-\alpha_2)^2\,(x-\alpha_3)^2\,(x-\alpha_4)$ and that their real graphs are as in Figure 17, so that there exist real numbers $\theta \in (\beta_1, \beta_2)$, $\gamma_1 \in (\alpha_1, \alpha_2)$, $\gamma_2 \in (\alpha_2, \alpha_3)$ and $\gamma_3 \in (\alpha_3, \alpha_4)$ satisfying $f'(\theta) = g'(\gamma_1) = g'(\gamma_2) = g'(\gamma_3) = 0$ and $f(\theta) = g(\gamma_1) = g(\gamma_2) = g(\gamma_3)$. The set of singularities of the corresponding curve C is $3\,\boldsymbol{A}_5 \oplus 3\,\boldsymbol{A}_1$, its component type is $\{7\}$ (see below), and the special lines of the pencil are given, once again, by the vertices of the dessin d'enfants in Figure 3.

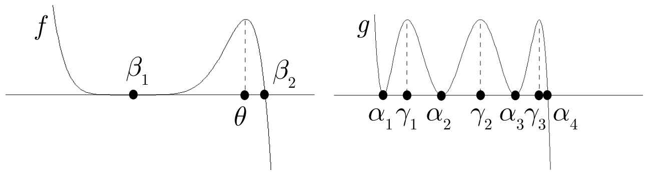

Figure 17. Real graphs of f and g (exponents $(6, 1; 2, 2, 2, 1)$).

Take generators $\xi_1, \ldots, \xi_7$ of the group $\pi_1(L_{\beta_2-\varepsilon} \setminus C)$ as in Figure 18. Then, the monodromy relations associated with the line L_{β_2} (multiplicity 2 tangent relations) are given by $\xi_7 = \xi_6$, $\xi_5 = \xi_4$ and $\xi_3 = \xi_2$. After simplification, the relations around L_θ (node relations) and L_{δ_4} (multiplicity 2 tangent relations) are written as:

$$L_\theta\colon\ \xi_6\xi_4 = \xi_4\xi_6,\ \xi_4\xi_2 = \xi_2\xi_4,\ \xi_2\xi_1 = \xi_1\xi_2;$$

$$L_{\delta_4}\colon\ \xi_6 = \xi_2,\ \xi_2^{-1}\xi_6\xi_2 = \xi_1,\ \xi_1^{-1}\xi_4\xi_1 = \xi_2.$$

Then, the vanishing relation at infinity is written as $\xi_1^7 = e$. Hence, $\pi_1(\mathbb{P}^2 \setminus C) \simeq \mathbb{Z}_7$.

2.7. Exponents $(5, 2; 5, 2)$.

For this set of exponents, we can assume that the polynomials f and g are of the form $f(y) = a\,(y-\beta_1)^5\,(y-\beta_2)^2$ and $g(x) = b\,(x-\alpha_1)^5\,(x-\alpha_2)^2$ and that their real graphs are as in Figure 19, so that there exist real numbers $\theta \in (\beta_1, \beta_2)$ and $\gamma \in (\alpha_1, \alpha_2)$ satisfying $f'(\theta) = g'(\gamma) = 0$ and $f(\theta) = g(\gamma)$. The set of singularities of the corresponding curve C is $\boldsymbol{B}_{5,5} \oplus 2\,\boldsymbol{A}_4 \oplus 2\,\boldsymbol{A}_1$, and its component type is $\{6, 1\}$ — actually, we can take

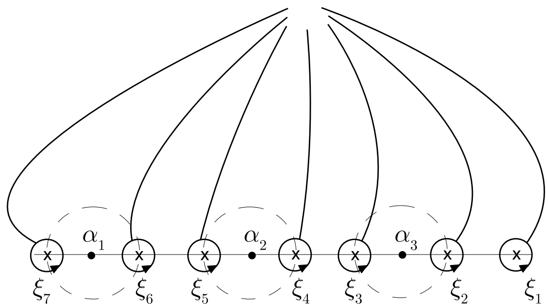

Figure 18.

$f(y) = -y^5\,(y-1)^2$ and $g(x) = -x^5\,(x-1)^2$, so that $f(y) - g(x)$ is given by

$$(x-y)(x^6 - 2\,x^5 + yx^5 + y^2x^4 - 2\,yx^4 + x^4 + x^3y^3 - 2\,y^2x^3 \\ + yx^3 + x^2y^4 - 2\,x^2y^3 + y^2x^2 + xy^5 - 2\,xy^4 + xy^3 + y^6 - 2\,y^5 + y^4).$$

The special lines of the pencil $\mathscr{P}$ with respect to this curve are given by the vertices of the dessin d'enfants in Figure 20.

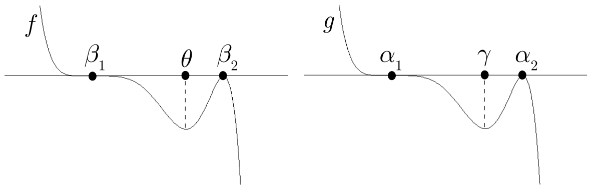

Figure 19. Real graphs of f and g (exponents $(5,2;5,2)$).

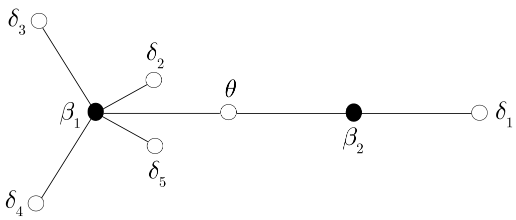

Figure 20.

We choose generators $\xi_1, \ldots, \xi_7$ of the group $\pi_1(L_{\beta_2+\varepsilon} \setminus C)$ as in Figure 6. Then, after simplification, the monodromy relations associated with the line L_{β_2} are given by:

$L_{\beta_2}(\alpha_1)$: $\xi_7 = \xi_5 = \xi_3$, $\xi_6 = \xi_4$ and $\xi_4\xi_3\xi_4\xi_3\xi_4 = \xi_3\xi_4\xi_3\xi_4\xi_3$ ($(2,5)$-type relations[3]);

$L_{\beta_2}(\alpha_2)$: $\xi_2\xi_1 = \xi_1\xi_2$ (node relation);

[3] A (β_j, α_i)-type relation with respect to the pencil $\mathscr{P}$ is a Brieskorn–Pham relation which is described by the local model $y^{\beta_j} = x^{\alpha_i}$.

and the relations around L_{δ_1}, L_θ and L_{δ_5} are written as:

$$\begin{aligned} L_{\delta_1}&: \ \xi_3 = \xi_2 \ \text{(tangent relation)};\\ L_\theta&: \ \xi_4\xi_1 = \xi_1\xi_4 \ \text{(node relation)};\\ L_{\delta_5}&: \ \xi_2\xi_4 = \xi_4\xi_2 \ \text{(tangent relation)}. \end{aligned}$$

This already shows that $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian, and since the component type of C is $\{6,1\}$, it follows that $\pi_1(\mathbb{P}^2 \setminus C) \simeq \mathbb{Z}$.

2.8. Exponents $(5,2;4,3)$.

For these exponents, we can suppose that the polynomials f and g are of the form $f(y) = a\,(y-\beta_1)^5\,(y-\beta_2)^2$ and $g(x) = b\,(x-\alpha_1)^3\,(x-\alpha_2)^4$ and that their real graphs are as in Figure 21, so that there exist real numbers $\theta \in (\beta_1,\beta_2)$ and $\gamma \in (\alpha_1,\alpha_2)$ satisfying $f'(\theta) = g'(\gamma) = 0$ and $f(\theta) = g(\gamma)$. The set of singularities of the corresponding curve C is $\boldsymbol{B}_{5,4} \oplus \boldsymbol{E}_8 \oplus \boldsymbol{A}_3 \oplus \boldsymbol{A}_2 \oplus \boldsymbol{A}_1$, and its component type is $\{7\}$ — actually, we can take

$$f(y) = -\frac{6912}{823543}\,y^5(y-1)^2 \quad \text{and} \quad g(x) = -\frac{12500}{823543}\,x^3(x-1)^4,$$

so that

$$\begin{aligned} f(y) - g(x) = \; & -\frac{6912}{823543}\,y^7 + \frac{13824}{823543}\,y^6 - \frac{6912}{823543}\,y^5 + \frac{12500}{823543}\,x^7\\ & -\frac{50000}{823543}\,x^6 + \frac{75000}{823543}\,x^5 - \frac{50000}{823543}\,x^4 + \frac{12500}{823543}\,x^3. \end{aligned}$$

The special lines of the pencil with respect to this curve are given by the vertices of the dessin d'enfants in Figure 20.

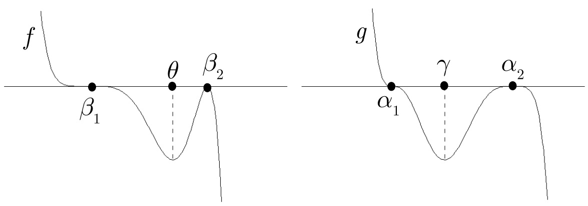

Figure 21. Real graphs of f and g (exponents $(5,2;4,3)$).

Let us choose generators $\xi_1,\dots,\xi_7$ of the group $\pi_1(L_{\beta_2+\varepsilon}\setminus C)$ as in Figure 22. Then, the monodromy relations associated with the line L_{δ_1} is given by $\xi_5 = \xi_4$ (multiplicity 2 tangent relation). After simplification, the relations around L_{β_2} are written as:

$$L_{\beta_2}(\alpha_1)\colon \ \xi_7 = \xi_4 \ \text{and} \ \xi_6\xi_4\xi_6 = \xi_4\xi_6\xi_4 \ \text{((2,3)-type relations)};$$

$$L_{\beta_2}(\alpha_2)\colon \ \xi_3 = \xi_1, \ \xi_4 = \xi_2, \ \text{and} \ \xi_1\xi_2\xi_1\xi_2 = \xi_2\xi_1\xi_2\xi_1 \ \text{((2,4)-type relations)};$$

and the relations associated with the lines L_θ, L_{δ_2}, L_{δ_3} are given by:

$$L_\theta:\ \xi_6\xi_1 = \xi_1\xi_6 \text{ (node relation);}$$
$$L_{\delta_2}:\ \xi_6 = \xi_2 \text{ (multiplicity 2 tangent relation);}$$
$$L_{\delta_3}:\ \xi_2 = \xi_1 \text{ (multiplicity 2 tangent relation).}$$

Hence, $\pi_1(\mathbb{P}^2 \setminus C) \simeq \mathbb{Z}_7$.

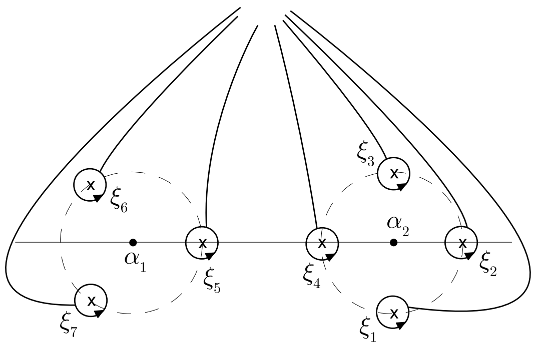

Figure 22.

2.9. Exponents $(5, 2; 4, 2, 1)$.

For this set of exponents, we can suppose that the polynomials f and g are:

(1) either of the form $f(y) = a\,(y-\beta_1)^5\,(y-\beta_2)^2$ and $g(x) = b\,(x-\alpha_1)^4\,(x-\alpha_2)^2\,(x-\alpha_3)$, with real graphs as in Figure 23, so that the following condition is satisfied:

$$\begin{cases} \exists\ \theta \in (\beta_1, \beta_2),\ \gamma_1 \in (\alpha_1, \alpha_2),\ \gamma_2 \in (\alpha_2, \alpha_3) \\ \text{such that } f'(\theta) = g'(\gamma_1) = g'(\gamma_2) = 0 \\ \text{and } f(\theta) = g(\gamma_1) = g(\gamma_2); \end{cases} \tag{2.8}$$

(2) or of the form $f(y) = a\,(y-\beta_1)^5\,(y-\beta_2)^2$ and $g(x) = b\,(x-\alpha_1)^2\,(x-\alpha_2)^4\,(x-\alpha_3)$, with real graphs as in Figure 24, so that Condition (2.8) is satisfied.

In both cases, (1) and (2), the set of singularities of the corresponding curve C is $\boldsymbol{B}_{5,4} \oplus \boldsymbol{A}_4 \oplus \boldsymbol{A}_3 \oplus 3\,\boldsymbol{A}_1$, its component type is $\{7\}$ (see below), and the special lines of the pencil with respect to C are given by the vertices of the dessin d'enfants in Figure 20.

To compute the fundamental group, in the case (1), we choose generators $\xi_1, \ldots, \xi_7$ of the group $\pi_1(L_{\beta_2+\varepsilon} \setminus C)$ as in Figure 13. Then, the monodromy relations associated with the line L_{δ_1} are given by $\xi_4 = \xi_3$ and $\xi_2 = \xi_1$ (multiplicity 2 tangent relations). After simplification, the relations around L_{β_2} are written as:

$$L_{\beta_2}(\alpha_1):\ \xi_7 = \xi_5,\ \xi_6 = \xi_3,\ \xi_3\xi_5\xi_3\xi_5 = \xi_5\xi_3\xi_5\xi_3 \text{ ((2, 4)-type relations);}$$
$$L_{\beta_2}(\alpha_2):\ \xi_3\xi_1 = \xi_1\xi_3 \text{ (node relation).}$$

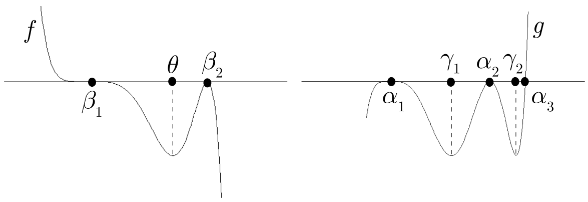

Figure 23. Real graphs of f and g (exponents $(5,2;4,2,1)$ — first case).

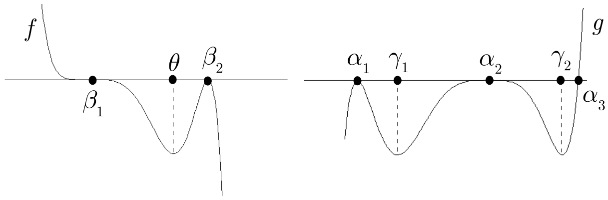

Figure 24. Real graphs of f and g (exponents $(5,2;4,2,1)$ — second case).

The relations around L_θ give $\xi_5\xi_1 = \xi_1\xi_5$ (node relation), and the relations associated with L_{δ_2} are written as $\xi_3 = \xi_1$ and $\xi_5 = \xi_1$ (multiplicity 2 tangent relations). By the vanishing relation at infinity, we have $\xi_1^7 = e$. Therefore, $\pi_1(\mathbb{P}^2 \setminus C) \simeq \mathbb{Z}_7$.

In the case (2), we choose generators $\xi_1, \ldots, \xi_7$ of $\pi_1(L_{\beta_2+\varepsilon} \setminus C)$ as in Figure 14. Then, the monodromy relations associated with the line L_{δ_1} are given by $\xi_6 = \xi_4$ and $\xi_2 = \xi_1$ (multiplicity 2 tangent relations). After simplification, the relations around L_{β_2} are written as:

$$L_{\beta_2}(\alpha_1)\colon\ \xi_4\xi_7 = \xi_7\xi_4 \text{ (node relation)},$$

$$L_{\beta_2}(\alpha_2)\colon\ \xi_1 = \xi_4,\ \xi_3 = \xi_5,\ \xi_3\xi_1\xi_3\xi_1 = \xi_1\xi_3\xi_1\xi_3 \text{ ((2,4)-type relations)}.$$

The relations around L_θ (node relations) are written as $\xi_7\xi_3 = \xi_3\xi_7$ and $\xi_1\xi_3 = \xi_3\xi_1$, and the relations associated with the lines L_{δ_2} and L_{δ_3} (multiplicity 2 tangent relations) give $\xi_3 = \xi_1$ and $\xi_7 = \xi_1$ respectively. Again, as $\xi_1^7 = e$ (vanishing relation at infinity), we have $\pi_1(\mathbb{P}^2 \setminus C) \simeq \mathbb{Z}_7$.

2.10. Exponents $(5, 2; 3, 2, 2)$.

Here, we can assume that f and g are of the form $f(y) = a\,(y-\beta_1)^5\,(y-\beta_2)^2$ and $g(x) = b\,(x-\alpha_1)^3\,(x-\alpha_2)^2\,(x-\alpha_3)^2$ and that their real graphs are as in Figure 25, so that there exist real numbers $\theta \in (\beta_1, \beta_2)$, $\gamma_1 \in (\alpha_1, \alpha_2)$ and $\gamma_2 \in (\alpha_2, \alpha_3)$ satisfying $f'(\theta) = g'(\gamma_1) = g'(\gamma_2) = 0$ and $f(\theta) = g(\gamma_1) = g(\gamma_2)$. The set of singularities of the corresponding curve C is $\boldsymbol{E}_8 \oplus 2\,\boldsymbol{A}_4 \oplus \boldsymbol{A}_2 \oplus 4\,\boldsymbol{A}_1$, its component type is $\{7\}$ (see below), and the special lines of the pencil, once again, are given by the vertices of the dessin d'enfants in Figure 20.

Let us choose generators $\xi_1, \ldots, \xi_7$ of the group $\pi_1(L_{\beta_2+\varepsilon} \setminus C)$ as in Figure 26. Then, the monodromy relations associated with the line L_{δ_1} are given by $\xi_5 = \xi_4$ and $\xi_3 = \xi_2$ (multiplicity 2 tangent relations). After simplification, the relations corresponding to L_{β_2} are:

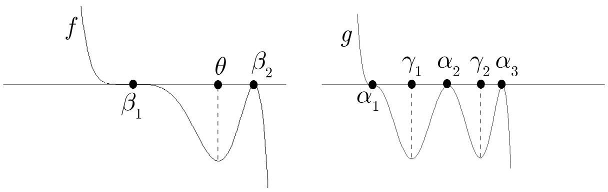

Figure 25. Real graphs of f and g (exponents $(5,2;3,2,2)$).

$$L_{\beta_2}(\alpha_1)\colon\ \xi_7=\xi_4 \text{ and } \xi_6\xi_4\xi_6=\xi_4\xi_6\xi_4 \ \ ((2,3)\text{-type relations});$$
$$L_{\beta_2}(\alpha_2)\colon\ \xi_2\xi_4=\xi_4\xi_2 \text{ (node relation)};$$
$$L_{\beta_2}(\alpha_3)\colon\ \xi_2\xi_1=\xi_1\xi_2 \text{ (node relation)};$$

and the relations associated with the lines L_θ and L_{δ_2} are given by:

$$L_\theta\colon\ \xi_6\xi_2=\xi_2\xi_6 \text{ and } \xi_4\xi_1=\xi_1\xi_4 \text{ (node relations)};$$
$$L_{\delta_2}\colon\ \xi_6=\xi_2 \text{ and } \xi_2=\xi_1 \text{ (multiplicity 2 tangent relations)}.$$

Since $\xi_1^7=e$ (vanishing relation at infinity), it follows that $\pi_1(\mathbb{P}^2\setminus C)\simeq\mathbb{Z}_7$.

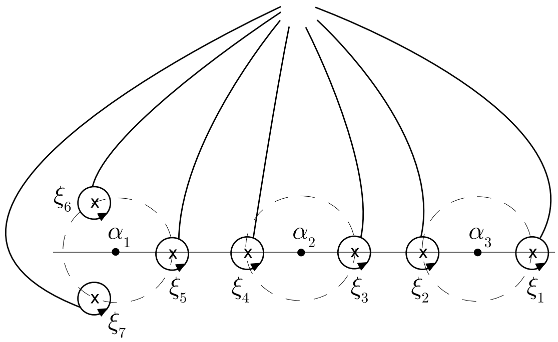

Figure 26.

2.11. Exponents $(5,2;2,2,2,1)$.

For this set of exponents, we can assume that f and g are of the form $f(y)=a\,(y-\beta_1)^5\,(y-\beta_2)^2$ and $g(x)=b\,(x-\alpha_1)^2\,(x-\alpha_2)^2\,(x-\alpha_3)^2\,(x-\alpha_4)$ and that their real graphs are as in Figure 27, so that there exist real numbers $\theta\in(\beta_1,\beta_2)$, $\gamma_1\in(\alpha_1,\alpha_2)$, $\gamma_2\in(\alpha_2,\alpha_3)$ and $\gamma_3\in(\alpha_3,\alpha_4)$ satisfying $f'(\theta)=g'(\gamma_1)=g'(\gamma_2)=g'(\gamma_3)=0$ and $f(\theta)=g(\gamma_1)=g(\gamma_2)=g(\gamma_3)$. The set of singularities of the corresponding curve C is $3\,\boldsymbol{A}_4\oplus 6\,\boldsymbol{A}_1$, its component type is $\{7\}$ (see below), and the special lines of the pencil with respect to this curve are given by the vertices of the dessin d'enfants in Figure 20.

Let us choose generators $\xi_1,\dots,\xi_7$ of the group $\pi_1(L_{\beta_2+\varepsilon}\setminus C)$ as in Figure 18. Then, the monodromy relations around L_{δ_1} (multiplicity 2 tangent relations) are given by $\xi_6=\xi_5$, $\xi_4=\xi_3$ and $\xi_2=\xi_1$. After simplification, the relations associated with the

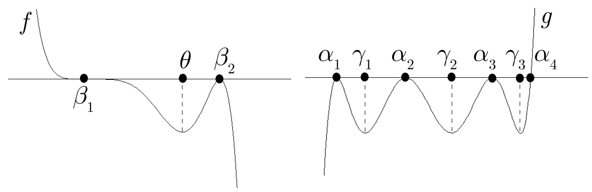

Figure 27. Real graphs of f and g (exponents $(5,2;2,2,2,1)$).

lines L_{β_2} and L_θ (node relations) give:

$$\begin{aligned} L_{\beta_2}\colon\ & \xi_7\xi_5=\xi_5\xi_7,\ \xi_5\xi_3=\xi_3\xi_5,\ \xi_3\xi_1=\xi_1\xi_3;\\ L_\theta\colon\ & \xi_7\xi_3=\xi_3\xi_7,\ \xi_5\xi_1=\xi_1\xi_5; \end{aligned}$$

and the relations corresponding to L_{δ_2} are written as $\xi_5=\xi_1$, $\xi_7=\xi_1$ and $\xi_5=\xi_3$. The vanishing relation at infinity says $\xi_1^7=e$. Hence, $\pi_1(\mathbb{P}^2\setminus C)\simeq\mathbb{Z}_7$.

2.12. Exponents $(4,3;4,3)$.

Here, we can assume that f and g are of the form $f(y)=a\,(y-\beta_1)^4\,(y-\beta_2)^3$ and $g(x)=b\,(x-\alpha_1)^4\,(x-\alpha_2)^3$ and that their real graphs are as in Figure 28, so that there exist real numbers $\theta\in(\beta_1,\beta_2)$ and $\gamma\in(\alpha_1,\alpha_2)$ satisfying $f'(\theta)=g'(\gamma)=0$ and $f(\theta)=g(\gamma)$. The set of singularities of the corresponding curve C is $\boldsymbol{B}_{4,4}\oplus 2\,\boldsymbol{E}_6\oplus\boldsymbol{D}_4\oplus\boldsymbol{A}_1$, and its component type is $\{6,1\}$ — actually, we can take $f(y)=-y^4(y-1)^3$ and $g(x)=-x^4(x-1)^3$, so that

$$\begin{aligned} f(y)-g(x)=(x-y)(&x^6-3\,x^5+yx^5+3\,x^4-3\,yx^4+y^2x^4\\ &+y^3x^3-3\,y^2x^3+3\,yx^3-x^3+x^2y^4-3\,x^2y^3+3\,y^2x^2-yx^2\\ &+xy^5-3\,xy^4+3\,xy^3-y^2x+y^6-3\,y^5+3\,y^4-y^3). \end{aligned}$$

The special lines of the pencil $\mathscr{P}$ with respect to this curve are given by the vertices of the dessin d'enfants in Figure 29.

Take generators $\xi_1,\dots,\xi_7$ of the group $\pi_1(L_{\beta_1+\varepsilon}\setminus C)$ as in Figure 8. Then, the monodromy relations are given as follows:

$L_{\delta_3}\colon\ \xi_7\xi_6\xi_5\xi_4=\xi_6\xi_5\xi_4\xi_2$ (multiplicity 2 tangent relation);

$L_{\delta_2}\colon\ \xi_5\xi_3\xi_2=\xi_3\xi_2\xi_1$ (multiplicity 2 tangent relation);

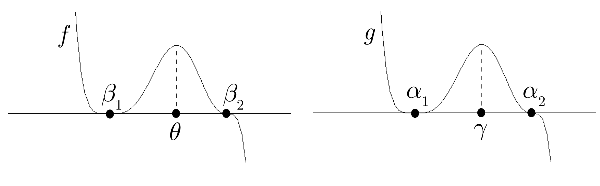

Figure 28. Real graphs of f and g (exponents $(4,3;4,3)$).

$L_{\delta_1}:$ $\xi_6\xi_7\xi_6\xi_5\xi_4 = \xi_7\xi_6\xi_5\xi_4\xi_1$ (multiplicity 2 tangent relation);

$L_{\beta_1}(\alpha_1):$ $\xi_4\xi_7\xi_6\xi_5 = \xi_7\xi_6\xi_5\xi_4,\ \xi_5\xi_7\xi_6\xi_5\xi_4 = \xi_7\xi_6\xi_5\xi_4\xi_5,\ \xi_6\xi_7\xi_6\xi_5\xi_4 = \xi_7\xi_6\xi_5\xi_4\xi_6$ ((4,4)-type relations);

$L_{\beta_1}(\alpha_2):$ $\xi_1\xi_3\xi_2\xi_1 = \xi_3\xi_2\xi_1\xi_2,\ \xi_2\xi_3\xi_2\xi_1 = \xi_3\xi_2\xi_1\xi_3$ ((4,3)-type relations);

$L_\theta:$ $\xi_4\xi_3 = \xi_3\xi_4$ (node relation);

$L_{\beta_2}(\alpha_1):$ $\xi_7 = \xi_3,\ \xi_5\xi_7\xi_6\xi_5 = \xi_7\xi_6\xi_5\xi_3,\ \xi_6\xi_7\xi_6\xi_5\xi_3 = \xi_7\xi_6\xi_5\xi_3\xi_5$ ((3,4)-type relations);

$L_{\beta_2}(\alpha_2):$ $\xi_1\xi_4\xi_2 = \xi_4\xi_2\xi_1,\ \xi_2\xi_4\xi_2\xi_1 = \xi_4\xi_2\xi_1\xi_2$ ((3,3)-type relations);

$L_{\delta_5}:$ $\xi_7\xi_6\xi_5\xi_3 = \xi_6\xi_5\xi_3\xi_2$ (multiplicity 2 tangent relation);

$L_{\delta_4}:$ $\xi_5\xi_4\xi_2 = \xi_4\xi_2\xi_1$ (multiplicity 2 tangent relation);

$R_\infty:$ $\xi_7\xi_6\xi_5\xi_4\xi_3\xi_2\xi_1 = e$ (vanishing relation at infinity).

After simplification, we find $\pi_1(\mathbb{P}^2 \setminus C) \simeq \langle\, \xi_1 \mid - \,\rangle \simeq \mathbb{Z}$.

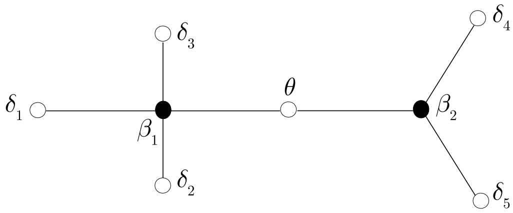

Figure 29.

2.13. Exponents $(4,3;4,2,1)$.

For this set of exponents, we can assume that the polynomials f and g are:

(1) either of the form $f(y) = a\,(y-\beta_1)^4\,(y-\beta_2)^2\,(y-\beta_3)$ and $g(x) = b\,(x-\alpha_1)^4\,(x-\alpha_2)^3$, with real graphs as in Figure 30, so that the following consdition is satisfied:

$$\begin{cases} \exists\ \theta_1 \in (\beta_1,\beta_2),\ \theta_2 \in (\beta_2,\beta_3),\ \gamma \in (\alpha_1,\alpha_2) \\ \text{such that } f'(\theta_1) = f'(\theta_2) = g'(\gamma) = 0 \\ \text{and } f(\theta_1) = f(\theta_2) = g(\gamma); \end{cases} \tag{2.9}$$

(2) or of the form $f(y) = a\,(y-\beta_1)^2\,(y-\beta_2)^4\,(y-\beta_3)$ and $g(x) = b\,(x-\alpha_1)^4\,(x-\alpha_2)^3$, with real graphs as in Figure 31, so that Condition (2.9) is satisfied.

In both cases, (1) and (2), the set of singularities of the corresponding curve C is $\boldsymbol{B}_{4,4} \oplus \boldsymbol{E}_6 \oplus \boldsymbol{A}_3 \oplus \boldsymbol{A}_2 \oplus 2\,\boldsymbol{A}_1$, and the curve is irreducible (see below).

To compute the fundamental group, note that, in the case (1), the special lines of the pencil $\mathscr{P}$ with respect to C are given by the vertices of the dessin d'enfants in Figure 32. Let us take generators $\xi_1,\dots,\xi_7$ of the group $\pi_1(L_{\beta_3-\varepsilon} \setminus C)$ as in Figure 8. Then, the monodromy relations around the line L_{β_3} are given by:

$$L_{\beta_3}(\alpha_1)\colon\ \xi_7=\xi_6=\xi_5=\xi_4 \text{ (multiplicity 4 tangent relations);}$$

$$L_{\beta_3}(\alpha_2)\colon\ \xi_3=\xi_2=\xi_1 \text{ (multiplicity 3 tangent relations).}$$

After simplification, the relation associated with the line L_{θ_2} (node relation) is written as $\xi_4\xi_1=\xi_1\xi_4$, while the relations around the line L_{β_2} ($(2,4)$ and $(2,3)$-type relations) give $\xi_1=\xi_4$. The vanishing relation at infinity says that $\xi_1^7=e$, and consequently $\pi_1(\mathbb{P}^2\setminus C)\simeq\mathbb{Z}_7$.

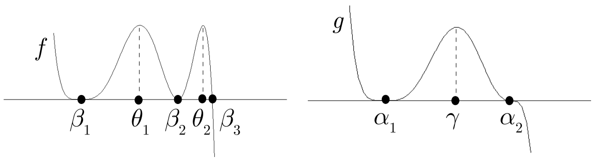

Figure 30. Real graphs of f and g (exponents $(4,3;4,2,1)$ — first case).

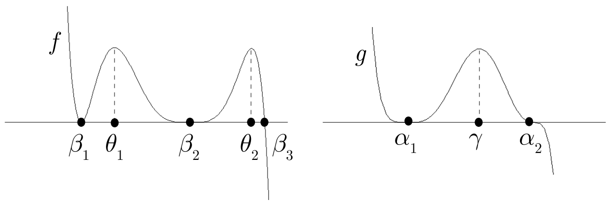

Figure 31. Real graphs of f and g (exponents $(4,3;4,2,1)$ — second case).

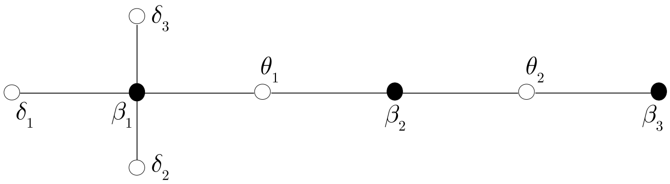

Figure 32.

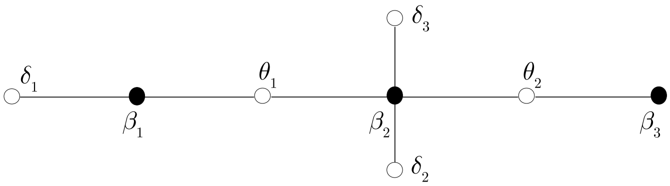

Figure 33.

In the case (2), the special lines of the pencil with respect to C are given by the vertices of the dessin d'enfants in Figure 33. Let us also take generators $\xi_1,\dots,\xi_7$ of the group $\pi_1(L_{\beta_3-\varepsilon}\setminus C)$ as in Figure 8. Then, as above and after simplification, the monodromy relations around the special lines L_{β_3}, L_{θ_2} and L_{β_2} are given by:

$$L_{\beta_3}(\alpha_1)\colon\ \xi_7 = \xi_6 = \xi_5 = \xi_4 \text{ (multiplicity 4 tangent relations);}$$

$$L_{\beta_3}(\alpha_2)\colon\ \xi_3 = \xi_2 = \xi_1 \text{ (multiplicity 3 tangent relations);}$$

$$L_{\theta_2}\colon\ \xi_4\xi_1 = \xi_1\xi_4 \text{ (node relation);}$$

$$L_{\beta_2}(\alpha_2)\colon\ \xi_4 = \xi_1 \text{ ((4,3)-type relation).}$$

Again, since $\xi_1^7 = e$ (vanishing relation at infinity), we have $\pi_1(\mathbb{P}^2 \setminus C) \simeq \mathbb{Z}_7$.

2.14. Exponents $(4,3;3,2,2)$.

Here, we can suppose that f and g are of the form $f(y) = a\,(y-\beta_1)^3\,(y-\beta_2)^4$ and $g(x) = b\,(x-\alpha_1)^3\,(x-\alpha_2)^2\,(x-\alpha_3)^2$ and that their real graphs are as in Figure 34, so that there exist real numbers $\theta \in (\beta_1, \beta_2)$, $\gamma_1 \in (\alpha_1, \alpha_2)$ and $\gamma_2 \in (\alpha_2, \alpha_3)$ satisfying $f'(\theta) = g'(\gamma_1) = g'(\gamma_2) = 0$ and $f(\theta) = g(\gamma_1) = g(\gamma_2)$. The set of singularities of the corresponding curve C is $\boldsymbol{D}_4 \oplus \boldsymbol{E}_6 \oplus 2\,\boldsymbol{A}_3 \oplus 2\,\boldsymbol{A}_2 \oplus 2\,\boldsymbol{A}_1$, and the curve is irreducible (see below). Here, the special lines of the pencil $\mathscr{P}$ with respect to C are given by the vertices of the dessin d'enfants in Figure 35.

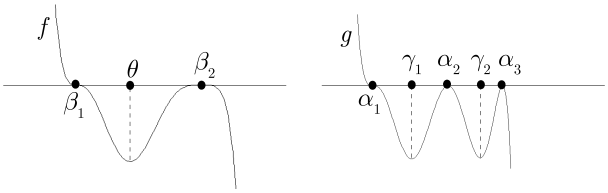

Figure 34. Real graphs of f and g (exponents $(4,3;3,2,2)$).

Take generators $\xi_1, \dots, \xi_7$ of the group $\pi_1(L_{\beta_2+\varepsilon} \setminus C)$ as in Figure 26. Then, the monodromy relations around L_{δ_1}, L_{δ_2} and L_{δ_3} (multiplicity 2 tangent relations) give:

$$L_{\delta_1}\colon\ \xi_5 = \xi_4 \text{ and } \xi_3 = \xi_2;$$

$$L_{\delta_2}\colon\ \xi_7\xi_6\xi_4 = \xi_6\xi_4\xi_2 \text{ and } \xi_4\xi_2 = \xi_2\xi_1;$$

$$L_{\delta_3}\colon\ \xi_6\xi_4 = \xi_4\xi_2.$$

In particular, this implies $\xi_7 = \xi_6$. Then, the relations around L_{β_2} give $\xi_6 = \xi_4$. It follows immediately that $\xi_4 = \xi_2 = \xi_1$, and since $\xi_1^7 = e$ (vanishing relation at infinity), we have $\pi_1(\mathbb{P}^2 \setminus C) \simeq \mathbb{Z}_7$.

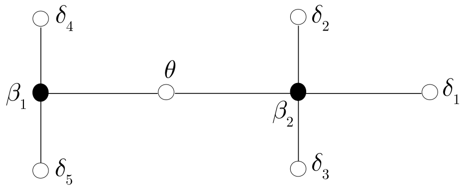

Figure 35.

2.15. Exponents $(4,3;2,2,2,1)$.

For these exponents, we can assume that the polynomials f and g are of the form $f(y)=a\,(y-\beta_1)^4\,(y-\beta_2)^3$ and $g(x)=b\,(x-\alpha_1)^2\,(x-\alpha_2)^2\,(x-\alpha_3)^2(x-\alpha_4)$ and that their real graphs are as in Figure 36, so that there exist real numbers $\theta\in(\beta_1,\beta_2)$, $\gamma_1\in(\alpha_1,\alpha_2)$, $\gamma_2\in(\alpha_2,\alpha_3)$ and $\gamma_3\in(\alpha_3,\alpha_4)$ satisfying $f'(\theta)=g'(\gamma_1)=g'(\gamma_2)=g'(\gamma_3)=0$ and $f(\theta)=g(\gamma_1)=g(\gamma_2)=g(\gamma_3)$. The set of singularities of the corresponding curve C is $3\,\boldsymbol{A}_3\oplus 3\,\boldsymbol{A}_2\oplus 3\,\boldsymbol{A}_1$, its component type is $\{7\}$ (see below), and the special lines of the pencil with respect to this curve are given by the vertices of the dessin d'enfants in Figure 29.

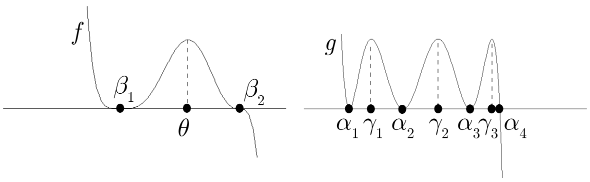

Figure 36. Real graphs of f and g (exponents $(4,3;2,2,2,1)$).

Take generators $\xi_1,\ldots,\xi_7$ of the group $\pi_1(L_{\beta_1-\varepsilon}\setminus C)$ as in Figure 18. Then, the monodromy relations around the line L_{δ_1} are given by $\xi_6=\xi_5$, $\xi_4=\xi_3$ and $\xi_2=\xi_1$ (multiplicity 2 tangent relations). After simplification, the relations associated with the line L_{δ_2} (which are also multiplicity 2 tangent relations) are written as $\xi_7=\xi_5=\xi_3=\xi_1$. Then, the vanishing relation at infinity says that $\xi_1^7=e$, and therefore $\pi_1(\mathbb{P}^2\setminus C)\simeq\mathbb{Z}_7$.

2.16. Exponents $(4,2,1;4,2,1)$.[4]

For this set of exponents, we can assume that the polynomials f and g are:

(1) either of the form $f(y)=a\,(y-\beta_1)^4\,(y-\beta_2)^2\,(y-\beta_3)$ and $g(x)=b\,(x-\alpha_1)^4\,(x-\alpha_2)^2\,(x-\alpha_3)$, with real graphs as in Figure 37, so that the following condition is satisfied:

$$\begin{cases}\exists\ \theta_1\in(\beta_1,\beta_2),\ \theta_2\in(\beta_2,\beta_3),\ \gamma_1\in(\alpha_1,\alpha_2),\ \gamma_2\in(\alpha_2,\alpha_3)\\ \text{such that } f'(\theta_1)=f'(\theta_2)=g'(\gamma_1)=g'(\gamma_2)=0\\ \text{and } f(\theta_1)=f(\theta_2)=g(\gamma_1)=g(\gamma_2);\end{cases}\tag{2.10}$$

(2) or of the form $f(y)=a\,(y-\beta_1)^2\,(y-\beta_2)^4\,(y-\beta_3)$ and $g(x)=b\,(x-\alpha_1)^4\,(x-\alpha_2)^2\,(x-\alpha_3)$, with real graphs as in Figure 38, so that Condition (2.10) is satisfied;

(3) or, finally, of the form $f(y)=a\,(y-\beta_1)^2\,(y-\beta_2)^4\,(y-\beta_3)$ and $g(x)=b\,(x-\alpha_1)^2\,(x-\alpha_2)^4\,(x-\alpha_3)$, with real graphs as in Figure 39, so that Condition (2.10) is satisfied.

In all the cases, the set of singularities of the corresponding curve C is $\boldsymbol{B}_{4,4}\oplus 2\,\boldsymbol{A}_3\oplus 5\,\boldsymbol{A}_1$. As for the fundamental group, we proceed case-by-case. In the first case, the special lines of the pencil $\mathscr{P}$ are given by the vertices of the dessin d'enfants in Figure 32. Let us choose generators $\xi_1,\ldots,\xi_7$ of the group $\pi_1(L_{\beta_3-\varepsilon}\setminus C)$ as in Figure 13. Then, the monodromy relations are as follows:

[4]Compare with [**1**].

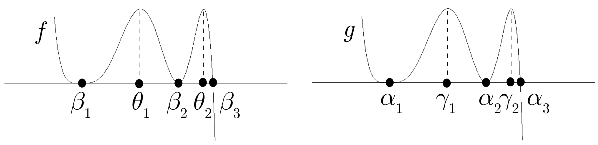

Figure 37. Real graphs of f and g (exponents $(4,2,1;4,2,1)$ — first case).

$$L_{\beta_3}(\alpha_1)\colon\ \xi_7=\xi_6=\xi_5=\xi_4 \text{ (multiplicity 4 tangent relations);}$$
$$L_{\beta_3}(\alpha_2)\colon\ \xi_3=\xi_2 \text{ (multiplicity 2 tangent relation);}$$
$$L_{\theta_2}\colon\ \xi_4\xi_2=\xi_2\xi_4 \text{ and } \xi_2\xi_1=\xi_1\xi_2 \text{ (node relations);}$$
$$L_{\beta_2}(\alpha_1)\colon\ \xi_4=\xi_2 \text{ ((2,4)-type relation).}$$

The other monodromy relations do not give any new relation. Hence,

$$\pi_1(\mathbb{P}^2\setminus C)\simeq\langle\xi_1,\xi_2\mid\xi_2\xi_1=\xi_1\xi_2,\ \xi_2^6\xi_1=e\rangle\simeq\langle\xi_2\mid -\rangle\simeq\mathbb{Z}.$$

REMARK 2.3. Here, the component type of C is $\{6,1\}$. Indeed, the fundamental group $\pi_1(\mathbb{C}\setminus\Sigma)$ acts on the generic fibre $L_{\beta_3-\varepsilon}\cap C$ (which consists of seven points) by so-called *braid action*. (Here, Σ is the set of parameters corresponding to the special lines of the pencil.) It is well known that the component type of C is $\{d_1,\dots,d_r\}$ if the seven points of $L_{\beta_3-\varepsilon}\cap C$ split into r orbits of $d_1,\dots,d_r$ elements, respectively, under this action. See also [**11**, Proposition (6.1)].

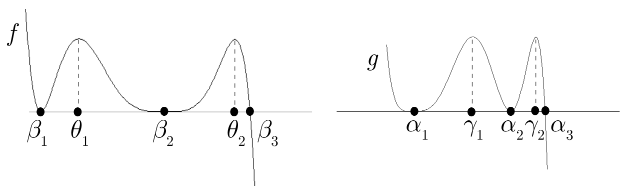

Figure 38. Real graphs of f and g (exponents $(4,2,1;4,2,1)$ — second case).

In the second case, the special lines of the pencil are given by the vertices of the dessin d'enfants in Figure 33. Let us also take generators $\xi_1,\dots,\xi_7$ of the group $\pi_1(L_{\beta_3-\varepsilon}\setminus C)$ as in Figure 13. Then, the monodromy relations associated with the line L_{β_3} are given by:

$$L_{\beta_3}(\alpha_1)\colon\ \xi_7=\xi_6=\xi_5=\xi_4 \text{ (multiplicity 4 tangent relations);}$$
$$L_{\beta_3}(\alpha_2)\colon\ \xi_3=\xi_2 \text{ (multiplicity 2 tangent relation).}$$

After simplification, the relations corresponding to L_{θ_2} (node relations) are written as $\xi_4\xi_2=\xi_2\xi_4$ and $\xi_2\xi_1=\xi_1\xi_2$, while the relations associated with the line L_{δ_2} (multiplicity 2 tangent relations) give $\xi_4=\xi_2=\xi_1$. Finally, by the vanishing relation at infinity, we have $\xi_1^7=e$. Hence, $\pi_1(\mathbb{P}^2\setminus C)\simeq\mathbb{Z}_7$.

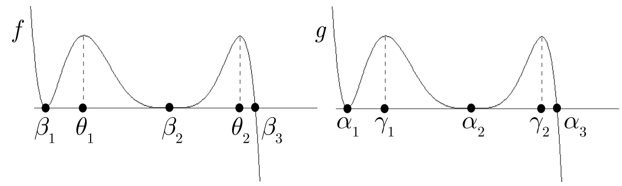

Figure 39. Real graphs of f and g (exponents $(4,2,1;4,2,1)$ — third case).

Finally, in the third case, the special lines of the pencil are given by the vertices of the dessin d'enfants in Figure 33. Here, we choose generators $\xi_1, \dots, \xi_7$ of the group $\pi_1(L_{\beta_3-\varepsilon} \setminus C)$ as in Figure 14. Then, the monodromy relations are as follows:

$$L_{\beta_3}(\alpha_1)\colon\ \xi_7 = \xi_6 \text{ (multiplicity 2 tangent relation);}$$

$$L_{\beta_3}(\alpha_2)\colon\ \xi_5 = \xi_4 = \xi_3 = \xi_2 \text{ (multiplicity 4 tangent relations);}$$

$$L_{\theta_2}\colon\ \xi_6\xi_2 = \xi_2\xi_6 \text{ and } \xi_2\xi_1 = \xi_1\xi_2 \text{ (node relations);}$$

$$L_{\delta_2}\colon\ \xi_6 = \xi_2 \text{ (multiplicity 2 tangent relation).}$$

The other monodromy relations do not give any new relation. Therefore, as in the first case, $\pi_1(\mathbb{P}^2 \setminus C) \simeq \mathbb{Z}$.

Remark 2.4. For the same reason as in Remark 2.3, here the component type of C is $\{6,1\}$.

2.17. Exponents $(4,2,1;3,2,2)$.

For these exponents, we can suppose that f and g are:

(1) either of the form $f(y) = a\,(y-\beta_1)^2\,(y-\beta_2)^4\,(y-\beta_3)$ and $g(x) = b\,(x-\alpha_1)^2\,(x-\alpha_2)^2\,(x-\alpha_3)^3$, with real graphs as in Figure 40, so that the following condition is satisfied:

$$\begin{cases} \exists\ \theta_1 \in (\beta_1,\beta_2),\ \theta_2 \in (\beta_2,\beta_3),\ \gamma_1 \in (\alpha_1,\alpha_2),\ \gamma_2 \in (\alpha_2,\alpha_3) \\ \text{such that } f'(\theta_1) = f'(\theta_2) = g'(\gamma_1) = g'(\gamma_2) = 0 \\ \text{and } f(\theta_1) = f(\theta_2) = g(\gamma_1) = g(\gamma_2); \end{cases} \tag{2.11}$$

(2) or of the form $f(y) = a\,(y-\beta_1)^4\,(y-\beta_2)^2\,(y-\beta_3)$ and $g(x) = b\,(x-\alpha_1)^2\,(x-\alpha_2)^2\,(x-\alpha_3)^3$, with real graphs as in Figure 41, so that Condition (2.11) is satisfied.

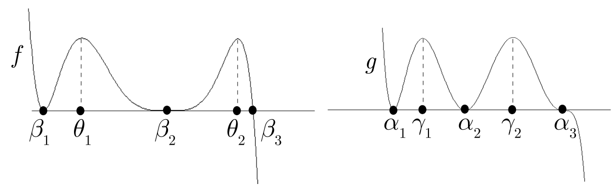

Figure 40. Real graphs of f and g (exponents $(4,2,1;3,2,2)$ — first case).

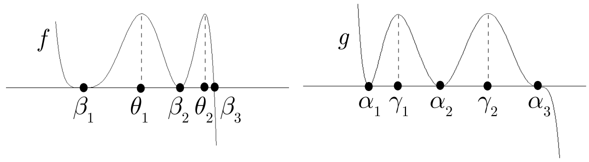

Figure 41. Real graphs of f and g (exponents $(4,2,1;3,2,2)$ — second case).

In both cases, the set of singularities of the corresponding curve C is $\boldsymbol{E}_6 \oplus 2\,\boldsymbol{A}_3 \oplus \boldsymbol{A}_2 \oplus 6\,\boldsymbol{A}_1$, and its component type is $\{7\}$ (see below). As for the fundamental group, in the first case, the special lines of the pencil are given by the vertices of the dessin d'enfants in Figure 33. Take generators $\xi_1, \ldots, \xi_7$ of the group $\pi_1(L_{\beta_3-\varepsilon} \setminus C)$ as in Figure 16. Then, the monodromy relations associated with the line L_{β_3} are given by $\xi_7 = \xi_6$, $\xi_5 = \xi_4$ (multiplicity 2 tangent relations) and $\xi_3 = \xi_2 = \xi_1$ (multiplicity 3 tangent relation). After simplification, the relations around the line L_{θ_2} (node relations) are written as $\xi_6\xi_4 = \xi_4\xi_6$ and $\xi_4\xi_1 = \xi_1\xi_4$, while the relations corresponding to $L_{\beta_2}(\alpha_3)$ ($(4,3)$-type relation) and L_{δ_3} (multiplicity 2 tangent relation) give $\xi_4 = \xi_1$ and $\xi_6 = \xi_1$ respectively. Since $\xi_1^7 = e$ (vanishing relation at infinity), it follows immediately that $\pi_1(\mathbb{P}^2 \setminus C) \simeq \mathbb{Z}_7$.

In the second case, the special lines of the pencil are given by the vertices of the dessin d'enfants in Figure 32. Let us also take generators $\xi_1, \ldots, \xi_7$ of the group $\pi_1(L_{\beta_3-\varepsilon} \setminus C)$ as in Figure 16. Then, after simplification, the monodromy relations are given as follows:

$$
\begin{aligned}
L_{\beta_3}(\alpha_1)&: \ \xi_7 = \xi_6 \text{ (multiplicity 2 tangent relation)};\\
L_{\beta_3}(\alpha_2)&: \ \xi_5 = \xi_4 \text{ (multiplicity 2 tangent relation)};\\
L_{\beta_3}(\alpha_3)&: \ \xi_3 = \xi_2 = \xi_1 \text{ (multiplicity 3 tangent relations)};\\
L_{\theta_2}&: \ \xi_6\xi_4 = \xi_4\xi_6 \text{ and } \xi_4\xi_1 = \xi_1\xi_4 \text{ (node relations)};\\
L_{\beta_2}(\alpha_2)&: \ \xi_6\xi_1 = \xi_1\xi_6 \text{ (node relation)};\\
L_{\beta_2}(\alpha_3)&: \ \xi_4 = \xi_1 \text{ ((2,3)-type relation)};\\
L_{\delta_3}&: \ \xi_6 = \xi_1 \text{ (multiplicity 2 tangent relation)}.
\end{aligned}
$$

Then, the vanishing relation at infinity is written as $\xi_1^7 = e$. Again, $\pi_1(\mathbb{P}^2 \setminus C) \simeq \mathbb{Z}_7$.

2.18. Exponents $(4,2,1;2,2,2,1)$.

For this set of exponents, we can assume that the polynomials f and g are:

(1) either of the form $f(y) = a\,(y-\beta_1)^4\,(y-\beta_2)^2\,(y-\beta_3)$ and $g(x) = b\,(x-\alpha_1)^2\,(x-\alpha_2)^2\,(x-\alpha_3)^2\,(x-\alpha_4)$, with real graphs as in Figure 42, so that the following condition is satisfied:

$$
\begin{cases}
\exists\ \theta_1 \in (\beta_1,\beta_2),\ \theta_2 \in (\beta_2,\beta_3),\\
\gamma_1 \in (\alpha_1,\alpha_2),\ \gamma_2 \in (\alpha_2,\alpha_3),\ \gamma_3 \in (\alpha_3,\alpha_4) \text{ such that}\\
f'(\theta_1) = f'(\theta_2) = g'(\gamma_1) = g'(\gamma_2) = g'(\gamma_3) = 0 \text{ and}\\
f(\theta_1) = f(\theta_2) = g(\gamma_1) = g(\gamma_2) = g(\gamma_3);
\end{cases}
\tag{2.12}
$$

(2) or of the form $f(y) = a\,(y-\beta_1)^2\,(y-\beta_2)^4\,(y-\beta_3)$ and $g(x) = b\,(x-\alpha_1)^2\,(x-\alpha_2)^2\,(x-\alpha_3)^2\,(x-\alpha_4)$, with real graphs as in Figure 43, so that Condition (2.12) is satisfied.

In both cases, the set of singularities of the corresponding curve C is $3\,\boldsymbol{A}_3 \oplus 9\,\boldsymbol{A}_1$, and the curve is irreducible (see below). As for the fundamental group, in the first case, the special lines of the pencil $\mathscr{P}$ are given by the vertices of the dessin d'enfants in Figure 32. Take generators $\xi_1, \ldots, \xi_7$ of the group $\pi_1(L_{\beta_3-\varepsilon} \setminus C)$ as in Figure 18. Then, after simplification, the monodromy relations are given as follows:

$L_{\beta_3}:$ $\xi_7 = \xi_6,\ \xi_5 = \xi_4$ and $\xi_3 = \xi_2$ (multiplicity 2 tangent relations);

$L_{\theta_2}:$ $\xi_6\xi_4 = \xi_4\xi_6,\ \xi_4\xi_2 = \xi_2\xi_4$ and $\xi_1\xi_2 = \xi_2\xi_1$ (node relations);

$L_{\beta_2}:$ $\xi_6\xi_2 = \xi_2\xi_6$ and $\xi_4\xi_1 = \xi_1\xi_4$ (node relations);

$L_{\theta_1}:$ $\xi_6\xi_1 = \xi_1\xi_6$ (node relation);

$L_{\delta_3}:$ $\xi_4 = \xi_1,\ \xi_6 = \xi_2$ and $\xi_6 = \xi_4$ (multiplicity 2 tangent relations).

Then, the vanishing relation at infinity is written as $\xi_1^7 = e$, and it follows immediately that $\pi_1(\mathbb{P}^2 \setminus C) \simeq \mathbb{Z}_7$.

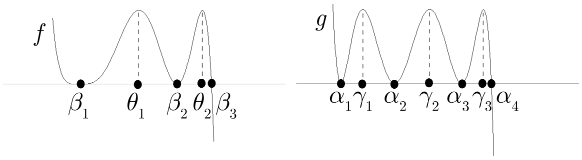

Figure 42. Real graphs of f and g (exponents $(4,2,1;2,2,2,1)$ — first case).

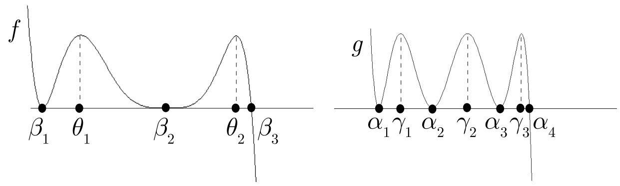

Figure 43. Real graphs of f and g (exponents $(4,2,1;2,2,2,1)$ — second case).

In the second case, the special lines of the pencil are given by the vertices of the dessin d'enfants in Figure 33. Again, we take generators $\xi_1, \ldots, \xi_7$ of the group $\pi_1(L_{\beta_3-\varepsilon} \setminus C)$ as in Figure 18. Then, after simplification, the monodromy relations are given as follows:

$L_{\beta_3}:$ $\xi_7 = \xi_6,\ \xi_5 = \xi_4$ and $\xi_3 = \xi_2$ (multiplicity 2 tangent relations);

$L_{\theta_2}:$ $\xi_6\xi_4 = \xi_4\xi_6,\ \xi_4\xi_2 = \xi_2\xi_4$ and $\xi_1\xi_2 = \xi_2\xi_1$ (node relations);

$L_{\delta_3}:$ $\xi_6 = \xi_2,\ \xi_2^{-1}\xi_6\xi_2 = \xi_1$ and $\xi_1^{-1}\xi_4\xi_1 = \xi_2$ (multiplicity 2 tangent relations).

Again, the vanishing relation at infinity is written as $\xi_1^7 = e$, and $\pi_1(\mathbb{P}^2 \setminus C) \simeq \mathbb{Z}_7$.

2.19. Exponents $(3, 2, 2; 3, 2, 2)$.

Here, we can assume that the polynomials f and g are of the form $f(y) = a\,(y-\beta_1)^3\,(y-\beta_2)^2\,(y-\beta_3)^2$ and $g(x) = b\,(x-\alpha_1)^3\,(x-\alpha_2)^2\,(x-\alpha_3)^2$ and that their real graphs are as in Figure 44, so that there exist real numbers $\theta_1 \in (\beta_1, \beta_2)$, $\theta_2 \in (\beta_2, \beta_3)$, $\gamma_1 \in (\alpha_1, \alpha_2)$ and $\gamma_2 \in (\alpha_2, \alpha_3)$ satisfying $f'(\theta_1) = f'(\theta_2) = g'(\gamma_1) = g'(\gamma_2) = 0$ and $f(\theta_1) = f(\theta_2) = g(\gamma_1) = g(\gamma_2)$. The set of singularities of the corresponding curve C is $\boldsymbol{D}_4 \oplus 4\,\boldsymbol{A}_2 \oplus 8\,\boldsymbol{A}_1$. The special lines of the pencil with respect to this curve are given by the vertices of the dessin d'enfants in Figure 45.

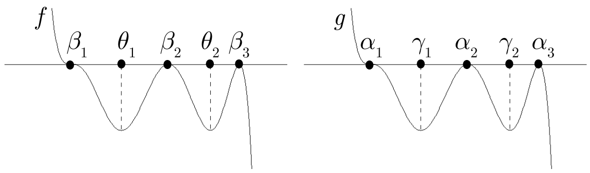

Figure 44. Real graphs of f and g (exponents $(3, 2, 2; 3, 2, 2)$).

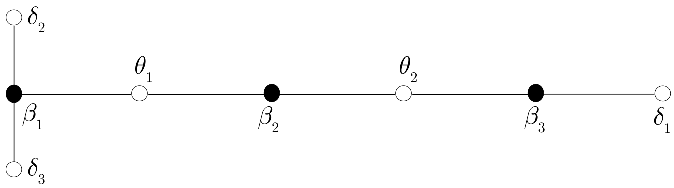

Figure 45.

Take generators $\xi_1, \dots, \xi_7$ of $\pi_1(L_{\beta_3+\varepsilon} \setminus C)$ as in Figure 26. Then, the monodromy relations around the line L_{δ_1} are given by $\xi_5 = \xi_4$ and $\xi_3 = \xi_2$ (multiplicity 2 tangent relations). After simplification, the relations associated with the line L_{β_3} are written as $\xi_7 = \xi_4$, $\xi_6\xi_4\xi_6 = \xi_4\xi_6\xi_4$ ((2, 3)-type relations) and $\xi_4\xi_2 = \xi_2\xi_4$, $\xi_2\xi_1 = \xi_1\xi_2$ (node relations), while the relations around the line L_{θ_2} (which are also node relations) give $\xi_6\xi_2 = \xi_2\xi_6$ and $\xi_4\xi_1 = \xi_1\xi_4$. Finally, the relations associated with the line L_{β_2} simplify to $\xi_6 = \xi_2 = \xi_4$. The other monodromy relations do not give any new relation. It follows that

$$\pi_1(\mathbb{P}^2 \setminus C) \simeq \langle\, \xi_1, \xi_2 \mid \xi_2\xi_1 = \xi_1\xi_2,\ \xi_2^6\xi_1 = e\,\rangle \simeq \mathbb{Z}.$$

Remark 2.5. For the same reason as in Remark 2.3, here the component type of C is $\{6, 1\}$.

2.20. Exponents $(3, 2, 2; 2, 2, 2, 1)$.

Here, we can assume that the polynomials f and g are of the form $f(y) = a\,(y-\beta_1)^2\,(y-\beta_2)^2\,(y-\beta_3)^3$ and $g(x) = b\,(x-\alpha_1)^2\,(x-\alpha_2)^2\,(x-\alpha_3)^2\,(x-\alpha_4)$ and that their real graphs are as in Figure 46, so that there exist real numbers $\theta_1 \in (\beta_1, \beta_2)$, $\theta_2 \in (\beta_2, \beta_3)$, $\gamma_1 \in (\alpha_1, \alpha_2)$, $\gamma_2 \in (\alpha_2, \alpha_3)$ and $\gamma_3 \in (\alpha_3, \alpha_4)$ satisfying $f'(\theta_1) = f'(\theta_2) = g'(\gamma_1) = g'(\gamma_2) = g'(\gamma_3) = 0$ and $f(\theta_1) = f(\theta_2) = g(\gamma_1) = g(\gamma_2) = g(\gamma_3)$. The set of singularities of the corresponding curve C is $3\,\boldsymbol{A}_2 \oplus 12\,\boldsymbol{A}_1$, and the curve is irreducible (see below). The special lines of the pencil $\mathscr{P}$ are given by the vertices of the dessin d'enfants in Figure 47.

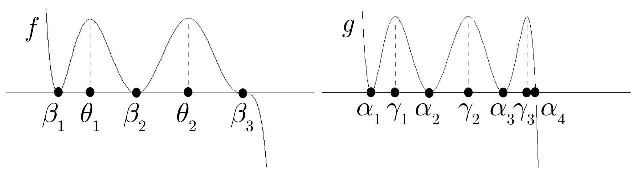

Figure 46. Real graphs of f and g (exponents $(3,2,2;2,2,2,1)$).

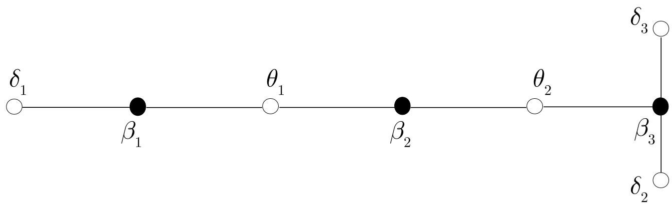

Figure 47.

Take generators $\xi_1,\dots,\xi_7$ of $\pi_1(L_{\beta_1-\varepsilon}\setminus C)$ as in Figure 18. Then, the monodromy relations around the line L_{δ_1} are given by $\xi_6=\xi_5$, $\xi_4=\xi_3$ and $\xi_2=\xi_1$ (multiplicity 2 tangent relations). After simplification, the relations associated with the line L_{β_1}, L_{θ_1}, L_{β_2} (node relations) and L_{δ_2} (multiplicity 2 tangent relations) are written as:

$$L_{\beta_1}:\ \xi_7\xi_5=\xi_5\xi_7,\ \xi_5\xi_3=\xi_3\xi_5 \text{ and } \xi_3\xi_1=\xi_1\xi_3;$$
$$L_{\theta_1}:\ \xi_7\xi_3=\xi_3\xi_7 \text{ and } \xi_5\xi_1=\xi_1\xi_5;$$
$$L_{\beta_2}:\ \xi_7\xi_1=\xi_1\xi_7;$$
$$L_{\delta_2}:\ \xi_7=\xi_5=\xi_3=\xi_1.$$

Therefore, the vanishing relation at infinity is written as $\xi_1^7=e$, and we have $\pi_1(\mathbb{P}^2\setminus C)\simeq\mathbb{Z}_7$.

2.21. Remark on the pseudo-maximality.

There is no general rule to detect the pseudo-maximality property. It should be checked case-by-case. It is not difficult to see that, in each section $2.i$ ($i=1,\dots,20$), the curve C that we considered is pseudo-maximal. Indeed, in each case, it is clear that there is no further degeneration within the real numbers. There is no further degeneration within the complex numbers either, because, in each case, the critical values (for both f and g) are all positive or all negative.

2.22. Exponents in Table 2.

If the set of exponents of C is in Table 2, then one checks easily that C degenerates either to a curve that we have already encountered in Sections 2.1–2.20 or to a curve whose exponents appear previously in the list. (Again, the proof is similar to that described in Section 2.4, and details are left to the reader.) Therefore, by the degeneration principle, its fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$ is necessarily isomorphic to $\mathbb{Z}_7$ or $\mathbb{Z}$.

2.23. Exponents in Table 3.

If the set of exponents of C is in Table 3, then C has only singularities of type $\boldsymbol{A}_1$, and therefore its fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian by the Zariski–Fulton–Deligne theorem (cf. [**14**], [**16**], [**5**], [**3**]). For the sets of exponents numbered 86, 87 and 88, the maximal number of nodes is 3. Then, by Bezout's theorem, the corresponding curves are necessarily irreducible, and therefore their fundamental groups are isomorphic to $\mathbb{Z}_7$. For the sets of exponents numbered 89, 90, 91 and 92, we have at most 6 nodes, so the component type can be only $\{7\}$ or $\{6,1\}$, and therefore the fundamental group is $\mathbb{Z}_7$ or $\mathbb{Z}$. For the sets numbered 93, 94 and 98, we have at most 9 nodes. In this case too, the component type can be only $\{7\}$ or $\{6,1\}$, and the fundamental group is $\mathbb{Z}_7$ or $\mathbb{Z}$. If the set of exponents of C is the set № 96, 97, 99, 100, 101, 102, 103 or 104, then one checks easily that C degenerates to a curve whose exponents are in Tables 1 or 2. Therefore, by the degeneration principle, the group $\pi_1(\mathbb{P}^2 \setminus C)$ is necessarily isomorphic to $\mathbb{Z}_7$ or $\mathbb{Z}$. The sets $(2,2,2,1;2,2,2,1)$ (№ 95) and $(1,\dots,1;1,\dots,1)$ (№ 105) are 'special' and should be discussed separately.

For the set $(2,2,2,1;2,2,2,1)$, the curve C has at most $15\,\boldsymbol{A}_1$, except when f and g have the form $f(y) = (y-\beta_1)^2(y-\beta_2)^2(y-\beta_3)^2(y-\beta_4)$ and $g(x) = (x-\alpha_1)^2(x-\alpha_2)^2(x-\alpha_3)^2(x-\alpha_4)$ with the real graphs given in Figure 48. In the latter case, there exist real numbers $\theta_1 \in (\beta_1,\beta_2)$, $\theta_2 \in (\beta_2,\beta_3)$, $\theta_3 \in (\beta_3,\beta_4)$, $\gamma_1 \in (\alpha_1,\alpha_2)$, $\gamma_2 \in (\alpha_2,\alpha_3)$ and $\gamma_3 \in (\alpha_3,\alpha_4)$ satisfying

$$\begin{cases} f'(\theta_1) = f'(\theta_2) = f'(\theta_3) = g'(\gamma_1) = g'(\gamma_2) = g'(\gamma_3) = 0, \\ f(\theta_1) = f(\theta_2) = f(\theta_3) = g(\gamma_1) = g(\gamma_2) = g(\gamma_3), \end{cases}$$

and the curve C has $18\,\boldsymbol{A}_1$.

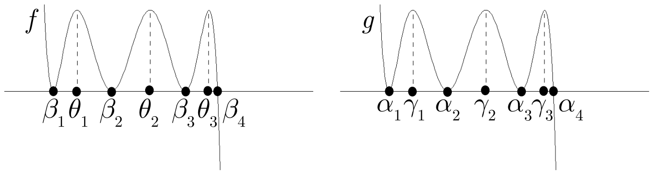

Figure 48. Real graphs of f and g (exponents $(2,2,2,1;2,2,2,1)$ — case where $\mathrm{Sing}(C) = 18\,\boldsymbol{A}_1$).

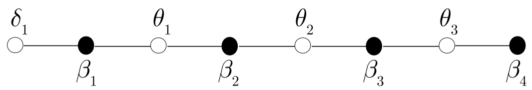

Figure 49.

When the curve has at most $15\,\boldsymbol{A}_1$, it degenerates to a curve whose exponents are in Tables 1 or 2, and therefore its fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$ is $\mathbb{Z}_7$ or $\mathbb{Z}$. When the curve has 18 nodes, its fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$ is isomorphic to $\mathbb{Z}^3$. Indeed, in this case, the special lines of the pencil $\mathscr{P}$ are given by the vertices of the dessin d'enfants in

Figure 49. Let us choose generators $\xi_1, \dots, \xi_7$ of the group $\pi_1(L_{\beta_1 - \varepsilon} \setminus C)$ as in Figure 18. Then, the monodromy relations are given as follows:

$$L_{\delta_1}\colon\ \xi_6 = \xi_5,\ \xi_4 = \xi_3,\ \xi_2 = \xi_1;$$
$$L_{\beta_1}\colon\ \xi_7\xi_5 = \xi_5\xi_7,\ \xi_5\xi_3 = \xi_3\xi_5,\ \xi_3\xi_1 = \xi_1\xi_3;$$
$$L_{\theta_1}\colon\ \xi_7\xi_3 = \xi_3\xi_7,\ \xi_5\xi_1 = \xi_1\xi_5;$$
$$L_{\beta_2}\colon\ \xi_7\xi_1 = \xi_1\xi_7.$$

The relations associated with the lines L_{θ_2}, L_{β_3}, L_{θ_3} and L_{β_4} do not give any new relation. By the vanishing relation at infinity, $\xi_7 = (\xi_5^2\xi_3^2\xi_1^2)^{-1}$. Hence,

$$\pi_1(\mathbb{P}^2 \setminus C) \simeq \langle \xi_1, \xi_3, \xi_5 \mid \xi_1\xi_3 = \xi_3\xi_1,\ \xi_1\xi_5 = \xi_5\xi_1,\ \xi_3\xi_5 = \xi_5\xi_3 \rangle \simeq \mathbb{Z}^3.$$

Remark 2.6. For the same reason as in Remark 2.3, here the component type of C is $\{2, 2, 2, 1\}$.

For the set $(1, \dots, 1; 1, \dots, 1)$, the curve C has at most $15\,\boldsymbol{A}_1$, except when f and g are as in Figure 50. In the latter case, there exist real numbers $\theta_j \in (\beta_j, \beta_{j+1})$ and $\gamma_i \in (\alpha_i, \alpha_{i+1})$ for $1 \leq i, j \leq 6$ satisfying

$$\begin{cases} f'(\theta_1) = \cdots = f'(\theta_6) = g'(\gamma_1) = \cdots = g'(\gamma_6) = 0, \\ f(\theta_1) = f(\theta_3) = f(\theta_5) = g(\gamma_1) = g(\gamma_3) = g(\gamma_5) > 0, \\ f(\theta_2) = f(\theta_4) = f(\theta_6) = g(\gamma_2) = g(\gamma_4) = g(\gamma_6) < 0, \end{cases}$$

and the curve C has $18\,\boldsymbol{A}_1$.

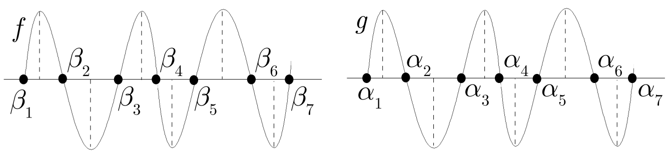

Figure 50. Real graphs of f and g (exponents $(1, \dots, 1; 1, \dots, 1)$ — case where $\mathrm{Sing}(C) = 18\,\boldsymbol{A}_1$).

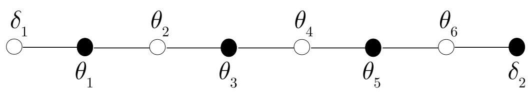

Figure 51.

When C has at most $15\,\boldsymbol{A}_1$, it degenerates to a curve whose exponents appear in Tables 1 or 2, and therefore its fundamental group is $\mathbb{Z}_7$ or $\mathbb{Z}$. When the curve has 18 nodes, the group $\pi_1(\mathbb{P}^2 \setminus C)$ is isomorphic to $\mathbb{Z}^3$. Indeed, in this case, the special lines of the pencil are given by the vertices of the dessin d'enfants in Figure 51. (The black vertices correspond to the roots of the equation $f(y) = f(\theta_1) = g(\gamma_1)$, and the white ones to the roots of $f(y) = f(\theta_2) = g(\gamma_2)$.) Take generators $\xi_1, \dots, \xi_7$ of the group $\pi_1(L_{\theta_1-\varepsilon} \setminus C)$ as in Figure 18. Then, the monodromy relations are given as follows:

$$\begin{aligned} L_{\delta_1}&: \ \xi_6 = \xi_5, \ \xi_4 = \xi_3, \ \xi_2 = \xi_1; \\ L_{\theta_1}&: \ \xi_5\xi_7 = \xi_7\xi_5, \ \xi_3\xi_5 = \xi_5\xi_3, \ \xi_3\xi_1 = \xi_1\xi_3; \\ L_{\theta_2}&: \ \xi_7\xi_3 = \xi_3\xi_7, \ \xi_5\xi_1 = \xi_1\xi_5; \\ L_{\theta_3}&: \ \xi_7\xi_1 = \xi_1\xi_7. \end{aligned}$$

The relations associated with the lines $L_{\theta_4}, L_{\theta_5}, L_{\theta_6}$ and L_{δ_2} do not give any new relation. By the vanishing relation at infinity, $\xi_7 = (\xi_5^2\xi_3^2\xi_1^2)^{-1}$. Hence, as above, $\pi_1(\mathbb{P}^2 \setminus C) \simeq \mathbb{Z}^3$.

Remark 2.7. For the same reason as in Remark 2.3, here the component type of C is $\{2, 2, 2, 1\}$.

This completes the proof of Theorem 1.3.

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