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Twisting the q-deformations of compact semisimple Lie groups

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Abstract. Given a compact semisimple Lie group G of rank r, and a parameter $q > 0$, we can define new associativity morphisms in $\mathrm{Rep}(G_q)$ using a 3-cocycle Φ on the dual of the center of G, thus getting a new tensor category $\mathrm{Rep}(G_q)^\Phi$. For a class of cocycles Φ we construct compact quantum groups G_q^τ with representation categories $\mathrm{Rep}(G_q)^\Phi$. The construction depends on the choice of an r-tuple τ of elements in the center of G. In the simplest case of $G = SU(2)$ and $\tau = -1$, our construction produces Woronowicz's quantum group $SU_{-q}(2)$ out of $SU_q(2)$. More generally, for $G = SU(n)$, we get quantum group realizations of the Kazhdan–Wenzl categories.

Introduction.

A known problem in the theory of quantum groups is classification of quantum groups with fusion rules of a given Lie group G, see e.g. [**Wor88**], [**WZ94**], [**Ban96**], [**Ohn99**], [**Bic03**], [**Ohn05**], [**Mro15**]. Although this problem has been completely solved in a few cases, most notably for $G = SL(2,\mathbb{C})$ [**Ban96**], [**Bic03**], as the rank of G grows the situation quickly becomes complicated. Already for $G = SL(3,\mathbb{C})$, even when requiring the dimensions of the representations to remain classical, one gets a large list of quantum groups that is not easy to grasp [**Ohn99**], [**Ohn05**]. A categorical version of the same problem turns out to be more manageable. Namely, the problem is to classify semisimple rigid monoidal $\mathbb{C}$-linear categories with fusions rules of G. As was shown by Kazhdan and Wenzl [**KW93**], for $G = SL(n,\mathbb{C})$ such categories $\mathcal{C}$ are parametrized by pairs $(q_\mathcal{C}, \tau_\mathcal{C})$ of nonzero complex numbers, defined up to replacing $(q_\mathcal{C}, \tau_\mathcal{C})$ by $(q_\mathcal{C}^{-1}, \tau_\mathcal{C}^{-1})$, such that $q_\mathcal{C}^{n(n-1)/2} = \tau_\mathcal{C}^n$ and $q_\mathcal{C}$ is not a nontrivial root of unity.[1] Concretely, these are twisted representation categories $\mathcal{C} = \mathrm{Rep}(SL_q(n))^\zeta$, where q is not a nontrivial root of unity and ζ is a root of unity of order n; the corresponding parameters are $q_\mathcal{C} = q^2$ and $\tau_\mathcal{C} = \zeta^{-1}q^{n-1}$. The twists are defined by choosing a $\mathbb{T}$-valued 3-cocycle on the dual of the center of $SL(n,\mathbb{C})$ and by using this cocycle to define new associativity morphisms in $\mathrm{Rep}(SL_q(n))$. The third cohomology group of the dual of the center is cyclic of order n, and this explains the parametrization of twists of $\mathrm{Rep}(SL_q(n))$ by roots of unity. A partial extension of the result of Kazhdan and Wenzl to types BCD was obtained by Tuba and Wenzl [**TW05**].

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[1] This is not how the result is formulated in [**KW93**]. There is a known mistake in [**KW93**, Proposition 5.1], see [**PR11**, Section 7] for a discussion.

Although two problems are clearly related, a solution of the latter does not immediately say much about the former. The present work is motivated by the natural question whether there exist quantum groups with representation categories $\mathrm{Rep}(SL_q(n))^\zeta$ for all ζ such that $\zeta^n = 1$. Equivalently, do the categories $\mathrm{Rep}(SL_q(n))^\zeta$ always admit fiber functors? For $n = 2$ there is essentially nothing to solve, since for $q \neq 1$ the category $\mathrm{Rep}(SL_q(2))^{-1}$ is equivalent to $\mathrm{Rep}(SL_{-q}(2))$. For $q = 1$ the answer is also known: the quantum group $SU_{-1}(2)$ defined by Woronowicz (which has nothing to do with the quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{sl}_2)$ at $q = -1$) has representation category $\mathrm{Rep}(SL(2,\mathbb{C}))^{-1}$. For $n \geq 2$, quantum groups with fusion rules of $SL(n,\mathbb{C})$ have been studied by many authors, see e.g. [**Hai00**] and the references therein. Usually, one starts by finding a solution of the quantum Yang–Baxter equation satisfying certain conditions, and from this derives a presentation of the algebra of functions on the quantum group [**RTF89**]. This approach cannot work in our case, since the category $\mathrm{Rep}(SL_q(n))^\zeta$ does not have a braiding unless $\zeta^2 = 1$.

The approach we take works, to some extent, for any compact semisimple simply connected Lie group G. Assume that Φ is a $\mathbb{T}$-valued 3-cocycle on the dual of the center of G. To construct a fiber functor φ from the category $\mathrm{Rep}(G_q)^\Phi$ with associativity morphisms defined by Φ, such that $\dim\varphi(U) = \dim U$, is the same as to find an invertible element F in a completion $\mathcal{U}(G_q \times G_q)$ of $\mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{U}_q(\mathfrak{g})$ satisfying

$$\Phi = (\iota \otimes \hat{\Delta}_q)(F^{-1})(1 \otimes F^{-1})(F \otimes 1)(\hat{\Delta}_q \otimes \iota)(F).$$

Then, using the twist (or a pseudo-2-cocycle in the terminology of [**EV96**]) F, we can define a new comultiplication on $\mathcal{U}(G_q)$, thus getting a new quantum group with representation category $\mathrm{Rep}(G_q)^\Phi$.

Our starting point is the simple remark that to solve the above cohomological equation we do not have to go all the way to G_q, it might suffice to pass from the center $Z(G)$ to a (quantum) subgroup of G_q, for example, to the maximal torus T. For simple G this is indeed enough: any 3-cocycle on $\widehat{Z(G)}$ becomes a coboundary when lifted to the dual $P = \hat{T}$ of T. The reason is that, for simple G, the center is contained in a torus of dimension at most 2. However, a 2-cochain f on P such that $\partial f = \Phi$ is unique only up to a 2-cocycle on P. Already for trivial Φ this leads to deformations of G_q by 2-cocycles on P that are not very well studied [**AST91**], [**LS91**], with associated C^*-algebras of functions (for $q > 0$) that are typically not of type I.

Our next observation is that, for arbitrary G, if Φ lifts to a coboundary on P, then the cochain f can be chosen to be of a particular form. This leads to a very special class of quantum groups G^τ_q, whose construction depends on the choice of elements $\tau_1, \dots, \tau_r \in Z(G)$, where r is the rank of G. We show that the quantum groups G^τ_q are as close to G_q as one could hope. For example, they can be defined in terms of finite central extensions of $\mathcal{U}_q(\mathfrak{g})$.

Since we are, first of all, interested in compact quantum groups in the sense of Woronowicz, we will concentrate on the case $q > 0$, when the categories $\mathrm{Rep}(G_q)^\Phi$ have a C^*-structure and, correspondingly, G^τ_q become compact quantum groups. We then show that the C^*-algebras $C(G^\tau_q)$ are KK-isomorphic to $C(G)$, they are of type I, and their primitive spectra are only slightly more complicated than that of $C(G_q)$. For $G = SU(n)$ we also find explicit generators and relations of the algebras $\mathbb{C}[SU_q^\tau(n)]$ of regular functions on $SU_q^\tau(n)$.

To summarize, our construction produces quantum groups with nice properties and with representation category $\mathrm{Rep}(G_q)^\Phi$ for any 3-cocycle Φ on $\widehat{Z(G)}$ that lifts to a coboundary on $\hat{T}$. This covers the cases when G is simple, but in the general semisimple case there exist cocycles that do not have this property. For such cocycles the existence of fiber functors for $\mathrm{Rep}(G_q)^\Phi$ remains an open problem.

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1. Preliminaries.

1.1. Compact quantum groups.

A *compact quantum group* $\mathbb{G}$ is given by a unital C^*-algebra $C(\mathbb{G})$ together with a coassociative unital $*$-homomorphism $\Delta\colon C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$ satisfying the cancellation condition

$$[\Delta(C(\mathbb{G}))(C(\mathbb{G}) \otimes 1)] = C(\mathbb{G}) \otimes C(\mathbb{G}) = [\Delta(C(\mathbb{G}))(1 \otimes C(\mathbb{G}))],$$

where brackets denote the closed linear span. Here we only introduce the relevant terminology and summarize the essential results, see e.g. [**NT13**] for details.

A theorem of Woronowicz gives a distinguished state h, the Haar state, which is an analogue of the normalized Haar measure over compact groups. Denote by $C_r(\mathbb{G})$ the quotient of $C(\mathbb{G})$ by the kernel of the GNS-representation defined by h. We will be interested in the case where h is faithful, so that $C_r(\mathbb{G}) = C(\mathbb{G})$. This condition is automatically satisfied for coamenable compact quantum groups. The quantum groups studied in this paper will be coamenable thanks to Banica's theorem [**Ban99**, Proposition 6.1] and [**NT13**, Theorem 2.7.14].

A finite dimensional unitary representation of $\mathbb{G}$ is given by a unitary element $U \in B(\mathcal{H}_U) \otimes C(\mathbb{G})$ satisfying the condition $U_{13}U_{23} = (\iota \otimes \Delta)(U)$. The tensor product of two representations is defined by $U \mathbin{\bigcirc\!\!\!\!\top} V = U_{13}V_{23}$. The category $\mathrm{Rep}(\mathbb{G})$ of finite dimensional unitary representations of $\mathbb{G}$ has the structure of a rigid C^*-tensor category with a unitary fiber functor ('forgetful functor') $U \mapsto \mathcal{H}_U$ to the category Hilb_f of finite dimensional Hilbert spaces. Woronowicz's Tannaka–Krein duality theorem states that the reduced quantum group $(C_r(\mathbb{G}), \Delta)$ can be axiomatized in terms of $\mathrm{Rep}(\mathbb{G})$ and the fiber functor.

We denote by $\mathbb{C}[\mathbb{G}] \subset C(\mathbb{G})$ the Hopf $*$-algebra of matrix coefficients of finite dimensional representations of $\mathbb{G}$. Denote by $\mathcal{U}(\mathbb{G})$ the dual $*$-algebra of $\mathbb{C}[\mathbb{G}]$, so $\mathcal{U}(\mathbb{G}) = \prod_{U \in \mathrm{Irrep}(\mathbb{G})} B(\mathcal{H}_U)$. It can be considered from many different angles: as the algebra of functions on the dual discrete quantum group $\hat{\mathbb{G}}$, as the algebra of endomorphisms of the forgetful functor, as the multiplier algebra of the convolution algebra $\widehat{\mathbb{C}[\mathbb{G}]}$ of $\mathbb{G}$. We also write $\mathcal{U}(\mathbb{G}^n)$ for $n \geq 2$ to denote the 'tensor product' multipliers, such as

$$\mathcal{U}(\mathbb{G}^2) = \prod_{U,V \in \mathrm{Irrep}(\mathbb{G})} B(\mathcal{H}_U) \otimes B(\mathcal{H}_V).$$

By duality, the multiplication map $m\colon \mathbb{C}[\mathbb{G}] \otimes \mathbb{C}[\mathbb{G}] \to \mathbb{C}[\mathbb{G}]$ defines a 'coproduct' $\hat{\Delta}\colon \mathcal{U}(\mathbb{G}) \to \mathcal{U}(\mathbb{G}^2)$.

1.2. Twisting of quantum groups.

Let $\mathbb{G}$ be a compact quantum group, and Φ be an invariant unitary 3-cocycle over the discrete dual of $\mathbb{G}$ [**NT13**, Chapter 3]. Thus, Φ is a unitary element in $\mathcal{U}(\mathbb{G}^3)$ satisfying the cocycle condition

$$(1 \otimes \Phi)(\iota \otimes \hat{\Delta} \otimes \iota)(\Phi)(\Phi \otimes 1) = (\iota \otimes \iota \otimes \hat{\Delta})(\Phi)(\hat{\Delta} \otimes \iota \otimes \iota)(\Phi) \tag{1.1}$$

and the invariance condition $[\Phi, (\hat{\Delta} \otimes \iota)\hat{\Delta}(x)] = 0$ for $x \in \mathcal{U}(\mathbb{G})$.

Then, the representation category $\mathrm{Rep}(\mathbb{G})$ can be twisted into a new C^*-tensor category $\mathrm{Rep}(\mathbb{G})^\Phi$, by using the action by Φ on $\mathcal{H}_U \otimes \mathcal{H}_V \otimes \mathcal{H}_W$ as the new associativity morphism $(U \mathbin{\bigcirc\!\!\!\!\top} V) \mathbin{\bigcirc\!\!\!\!\top} W \to U \mathbin{\bigcirc\!\!\!\!\top} (V \mathbin{\bigcirc\!\!\!\!\top} W)$ for $U, V, W \in \mathrm{Rep}(\mathbb{G})$. The category $\mathrm{Rep}(\mathbb{G})^\Phi$ can be considered as the module category of the discrete quasi-bialgebra $(\widehat{\mathbb{C}[\mathbb{G}]}, \hat{\Delta}, \Phi)$ [**Dri89**].

Suppose the category $\mathrm{Rep}(\mathbb{G})^\Phi$ is rigid. This is equivalent to the condition that the central element

$$\Phi_1 \hat{S}(\Phi_2)\Phi_3 = m(m \otimes \iota)(\iota \otimes \hat{S} \otimes \iota)(\Phi)$$

in $\mathcal{U}(\mathbb{G})$ is invertible. Suppose also that there exists a unitary $F \in \mathcal{U}(\mathbb{G}^2)$ such that

$$\Phi = (\iota \otimes \hat{\Delta})(F^*)(1 \otimes F^*)(F \otimes 1)(\hat{\Delta} \otimes \iota)(F). \tag{1.2}$$

Then the discrete quantum group $\mathcal{U}(\mathbb{G})$ can be deformed into another one, with the new coproduct $\hat{\Delta}_F(x) = F\hat{\Delta}(x)F^*$. By duality, the function algebra $\mathbb{C}[\mathbb{G}]$ can be endowed with the new product

$$x \cdot_F y = m(F^* \triangleright (x \otimes y) \triangleleft F).$$

Here, $\triangleright$ and $\triangleleft$ are the natural actions of $\mathcal{U}(\mathbb{G})$ on $\mathbb{C}[\mathbb{G}]$ given by $X \triangleright a = \langle X, a_{[2]} \rangle a_{[1]}$ and $a \triangleleft X = \langle X, a_{[1]} \rangle a_{[2]}$. We denote the corresponding compact quantum group by $\mathbb{G}_F$. Note that in general the involution on $\mathbb{C}[\mathbb{G}_F]$ differs from the original one, see [**NT13**, Example 2.3.9].

We have a unitary monoidal equivalence of the C^*-tensor categories $\mathrm{Rep}(\mathbb{G})^\Phi$ and $\mathrm{Rep}(\mathbb{G}_F)$. The tensor functor $\varphi\colon \mathrm{Rep}(\mathbb{G})^\Phi \to \mathrm{Rep}(\mathbb{G}_F)$ is given by the identity map on objects and morphisms, but with the nontrivial tensor transformation $\varphi(U) \mathbin{\bigcirc\!\!\!\!\top} \varphi(V) \to \varphi(U \mathbin{\bigcirc\!\!\!\!\top} V)$ defined by

$$\mathcal{H}_U \otimes \mathcal{H}_V \to \mathcal{H}_U \otimes \mathcal{H}_V, \quad \xi \otimes \eta \mapsto F^*(\xi \otimes \eta).$$

In terms of fiber functors, F gives a tensor functor $\mathrm{Rep}(\mathbb{G})^\Phi \to \mathrm{Hilb}_\mathrm{f}$ which is the same as that of $\mathrm{Rep}(\mathbb{G})$ on objects and morphisms, but with the modified tensor transformation $\mathcal{H}_U \otimes \mathcal{H}_V \to \mathcal{H}_{U \oplus V}$ given by $\xi \otimes \eta \mapsto F^*(\xi \otimes \eta)$.

Examples of invariant 3-cocycles can be obtained as follows. Assume $\mathbb{H}$ is a closed central subgroup of $\mathbb{G}$, so $\mathbb{H}$ is a compact abelian group and we are given a surjective homomorphism $\pi\colon \mathbb{C}[\mathbb{G}] \to \mathbb{C}[\mathbb{H}]$ of Hopf $*$-algebras such that the image of $\mathcal{U}(\mathbb{H})$ under the dual homomorphism $\mathcal{U}(\mathbb{H}) \to \mathcal{U}(\mathbb{G})$ is a central subalgebra of $\mathcal{U}(\mathbb{G})$, or equivalently, for any irreducible unitary representation U of $\mathbb{G}$ the element $(\iota \otimes \pi)(U)$ has the form $1 \otimes \chi_U$ for a character χ_U of $\mathbb{H}$. Unitary 3-cocycles in $\mathcal{U}(\mathbb{H}^3)$ are nothing else than $\mathbb{T}$-valued 3-cocycles on the Pontryagin dual $\hat{\mathbb{H}}$. Any such cocycle defines an invariant cocycle Φ in $\mathcal{U}(\mathbb{G}^3)$; when $\mathbb{G}$ is itself compact abelian, this is just the usual pullback homomorphism $Z^3(\hat{\mathbb{H}}; \mathbb{T}) \to Z^3(\hat{\mathbb{G}}; \mathbb{T})$. Explicitly, the action of Φ on $\mathcal{H}_U \otimes \mathcal{H}_V \otimes \mathcal{H}_W$ is by multiplication by $\Phi(\chi_U, \chi_V, \chi_W)$. For such cocycles Φ the C^*-tensor category $\mathrm{Rep}(\mathbb{G})^\Phi$ is always rigid.

1.3. Quantized universal enveloping algebra.

Throughout the whole paper G denotes a semisimple simply connected compact Lie group, and $\mathfrak{g}$ denotes its complexified Lie algebra. We fix a maximal torus T in G, and denote the corresponding Cartan subalgebra by $\mathfrak{h}$. The root lattice is denoted by Q, and the weight lattice by P. We fix a choice of positive roots, and denote the corresponding positive simple roots by $\{\alpha_1, \dots, \alpha_r\}$. We also fix an ad-invariant symmetric form on $\mathfrak{g}$ such that it is negative definite on the real Lie algebra of G. If G is simple, we assume that this form is standardly normalized, meaning that $(\alpha, \alpha) = 2$ for every short root α. The Cartan matrix is denoted by $(a_{ij})_{1 \le i,j \le r}$, and the Weyl group is denoted by W. The center $Z(G)$ of G is contained in T and can be identified with the dual of P/Q.

In what follows the variable q ranges over the strictly positive real numbers, although many results remain true for all $q \neq 0$ such that the numbers $q_i = q^{(\alpha_i, \alpha_i)/2}$ are not nontrivial roots of unity. For $q \neq 1$, the *quantized universal enveloping algebra* $\mathcal{U}_q(\mathfrak{g})$ is the universal algebra over $\mathbb{C}$ generated by the elements E_i, F_i, and $K_i^{\pm 1}$ for $1 \le i \le r$ satisfying the relations

$$[K_i, K_j] = 0, \quad K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j, \quad K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j,$$

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} E_i^k E_j E_i^{1-a_{ij}-k} = 0,$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} F_i^k F_j F_i^{1-a_{ij}-k} = 0.$$

It has the structure of a Hopf $*$-algebra defined by the operations

$$\hat{\Delta}_q(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \hat{\Delta}_q(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \hat{\Delta}_q(K_i) = K_i \otimes K_i,$$

$$\hat{S}_q(E_i) = -K_i^{-1} E_i, \quad \hat{S}_q(F_i) = -F_i K_i^{-1}, \quad \hat{S}_q(K_i) = K_i^{-1},$$

$$\hat{\epsilon}_q(E_i) = \hat{\epsilon}_q(F_i) = 0, \quad \hat{\epsilon}_q(K_i) = 1,$$

$$E_i^* = F_i K_i, \quad F_i^* = K_i^{-1} E_i, \quad K_i^* = K_i.$$

A representation (π, V) of $\mathcal{U}_q(\mathfrak{g})$ is said to be *admissible* when V admits a decomposition $\bigoplus_{\chi \in P} V_\chi$ such that $\pi(K_i)|_{V_\chi}$ is equal to the scalar $q^{(\alpha_i, \chi)}$. The category of finite dimensional admissible $*$-representations of $\mathcal{U}_q(\mathfrak{g})$ is a C^*-tensor category with the forgetful functor. We denote the associated compact quantum group by G_q. There is a natural inclusion of T into $\mathcal{U}(G_q)$. Then the set $Z(G_q)$ of group-like central elements in $\mathcal{U}(G_q)$ coincides with $Z(G)$. The class of representations of G_q on which $Z(G)$ acts trivially corresponds to a quotient quantum group denoted by $G_q/Z(G)$.

2. Twisted q-deformations.

2.1. Extension of the QUE-algebra.

For $q > 0$, we let $\tilde{\mathcal{U}}_q(\mathfrak{g})$ denote the universal $*$-algebra generated by $\mathcal{U}_q(\mathfrak{g})$ and unitary central elements $C_1, \dots, C_r$. It is not difficult to check that for $q \neq 1$ the following formulas define a Hopf $*$-algebra structure on $\tilde{\mathcal{U}}_q(\mathfrak{g})$:

$$\hat{\Delta}(E_i) = E_i \otimes C_i + K_i \otimes E_i, \quad \hat{\Delta}(K_i) = K_i \otimes K_i, \quad \hat{\Delta}(C_i) = C_i \otimes C_i.$$

Similarly, for $q = 1$, we define

$$\hat{\Delta}(E_i) = E_i \otimes C_i + 1 \otimes E_i, \quad \hat{\Delta}(H_i) = H_i \otimes 1 + 1 \otimes H_i, \quad \hat{\Delta}(C_i) = C_i \otimes C_i.$$

There is a Hopf $*$-algebra homomorphism from $\tilde{\mathcal{U}}_q(\mathfrak{g})$ onto $\mathcal{U}_q(\mathfrak{g})$, defined by $C_i \mapsto 1$ and by the identity map on the copy of $\mathcal{U}_q(\mathfrak{g})$. There is also a Hopf $*$-algebra homomorphism onto $\mathbb{C}[(C_i)_{i=1}^r]$, given by $E_i \mapsto 0$, $F_i \mapsto 0$, $K_i \mapsto 1$, and by the identity map on the C_i's. We regard representations of $\mathcal{U}_q(\mathfrak{g})$ and of $\mathbb{C}[(C_i)_{i=1}^r]$ as the ones of $\tilde{\mathcal{U}}_q(\mathfrak{g})$ via these homomorphisms.

Remark 2.1. The Hopf algebra $\tilde{\mathcal{U}}_q(\mathfrak{g})$ is closely related to the Drinfeld double $\mathcal{D}(\mathcal{U}_q(\mathfrak{b}_+))$ of $\mathcal{U}_q(\mathfrak{b}_+) = \langle E_i, K_i \mid 1 \le i \le r \rangle$. Namely, put

$$X_i^+ = E_i C_i^{-1}, \quad K_i^+ = K_i C_i^{-1}, \quad X_i^- = F_i, \quad K_i^- = K_i C_i.$$

Then we see that the elements X_i^+ and K_i^+ generate a copy of $\mathcal{U}_q(\mathfrak{b}_+)$, while the X_i^- and K_i^- generate a copy of $\mathcal{U}_q(\mathfrak{b}_-)$, and taking together these subalgebras give a copy of $\mathcal{D}(\mathcal{U}_q(\mathfrak{b}_+))$ in $\tilde{\mathcal{U}}_q(\mathfrak{g})$. The homomorphism $\tilde{\mathcal{U}}_q(\mathfrak{g}) \to \mathcal{U}_q(\mathfrak{g})$ is an extension of the standard projection $\mathcal{D}(\mathcal{U}_q(\mathfrak{b}_+)) \to \mathcal{U}_q(\mathfrak{g})$. If we add square roots of $K_i^\pm$ to $\mathcal{D}(\mathcal{U}_q(\mathfrak{b}_+))$, thus getting a Hopf algebra $\mathcal{D}(\widetilde{\mathcal{U}_q(\mathfrak{b}_+)})$, we can recover $\tilde{\mathcal{U}}_q(\mathfrak{g})$ by letting $C_i = (K_i^-)^{1/2} (K_i^+)^{-1/2}$. Therefore we have inclusions of Hopf algebras $\mathcal{D}(\mathcal{U}_q(\mathfrak{b}_+)) \subset \tilde{\mathcal{U}}_q(\mathfrak{g}) \subset \mathcal{D}(\widetilde{\mathcal{U}_q(\mathfrak{b}_+)})$.

Let $\tau = (\tau_1, \dots, \tau_r)$ be an r-tuple of elements in $Z(G)$. We say that a representation (π, V) of $\tilde{\mathcal{U}}_q(\mathfrak{g})$ is *τ-admissible* if its restriction to $\mathcal{U}_q(\mathfrak{g})$ is admissible and the elements C_i act on the weight spaces V_χ as scalars $\langle \tau_i, \chi \rangle$. The category of τ-admissible representations is a rigid C^*-tensor category with forgetful functor. Moreover, the $G_q/Z(G)$-representations are naturally included in the τ-admissible representations as a C^*-tensor subcategory.

Definition 2.2. We let G_q^τ denote the compact quantum group realizing the category of finite dimensional τ-admissible $*$-representations of $\tilde{\mathcal{U}}_q(\mathfrak{g})$ together with its canonical fiber functor.

In other words, $\mathbb{C}[G_q^\tau] \subset \tilde{\mathcal{U}}_q(\mathfrak{g})^*$ is spanned by matrix coefficients of finite dimensional τ-admissible representations, and the Hopf $*$-algebra structure on $\mathbb{C}[G_q^\tau]$ is defined by duality using that of $\tilde{\mathcal{U}}_q(\mathfrak{g})$.

Since every admissible representation of $\mathcal{U}_q(\mathfrak{g})$ extends uniquely to a τ-admissible representation of $\tilde{\mathcal{U}}_q(\mathfrak{g})$, and every τ-admissible representation is obtained this way, we can identify the $*$-algebra $\mathcal{U}(G_q^\tau)$ with $\mathcal{U}(G_q)$. The image $\mathcal{U}_q^\tau(\mathfrak{g})$ of $\tilde{\mathcal{U}}_q(\mathfrak{g})$ in $\mathcal{U}(G_q^\tau) = \mathcal{U}(G_q)$ plays the role of a quantized universal enveloping algebra for G_q^τ. As an algebra it is generated by E_i, F_i, $K_i^{\pm 1}$ and τ_i (which is the image of C_i), but is endowed with a modified coproduct

$$\hat{\Delta}(E_i) = E_i \otimes \tau_i + K_i \otimes E_i, \quad \hat{\Delta}(K_i) = K_i \otimes K_i, \quad \hat{\Delta}(\tau_i) = \tau_i \otimes \tau_i. \tag{2.1}$$

To put it differently, as a $*$-algebra, $\mathcal{U}_q^\tau(\mathfrak{g})$ is the tensor product of $\mathcal{U}_q(\mathfrak{g})$ and the group algebra of the group $T_\tau \subset Z(G)$ generated by $\tau_1, \dots, \tau_r$, while the coproduct is defined by (2.1). As a quotient of $\tilde{\mathcal{U}}_q(\mathfrak{g})$, the Hopf $*$-algebra $\mathcal{U}_q^\tau(\mathfrak{g})$ is obtained by requiring that the unitaries $C_1, \dots, C_r$ satisfy the same relations as $\tau_1, \dots, \tau_r \in Z(G)$.

2.2. Twisting and associator.

Given $\tau = (\tau_1, \dots, \tau_r) \in Z(G)^r$, we obtain a 3-cocycle on $\widehat{Z(G)} = P/Q$ as follows.

First, let $f(\lambda, \mu)$ be a $\mathbb{T}$-valued function on $P \times P$ satisfying

$$f(\lambda, \mu + Q) = f(\lambda, \mu), \quad f(\lambda + \alpha_i, \mu) = \langle \tau_i, \mu \rangle f(\lambda, \mu). \tag{2.2}$$

These conditions imply that f can be determined by its restriction to the image of a set-theoretic section $(P/Q)^2 \to P^2$. For example, if $\lambda_1, \dots, \lambda_n$ is a system of representatives of P/Q, then we can put

$$f\Big(\lambda_i + \sum_{j=1}^r m_j \alpha_j, \mu\Big) = \prod_{j=1}^r \langle \tau_j, \mu \rangle^{m_j}$$

for all $1 \le i \le n$ and $(m_1, \dots, m_r) \in \mathbb{Z}^r$.

Using (2.2), the coboundary of f,

$$(\partial f)(\lambda, \mu, \nu) = f(\mu, \nu) f(\lambda + \mu, \nu)^{-1} f(\lambda, \mu + \nu) f(\lambda, \mu)^{-1},$$

is seen to be invariant under the translation by Q in each variable. Thus, ∂f can be considered as a 3-cochain on P/Q with values in $\mathbb{T}$. By construction, it is a cocycle. If

f' satisfies the same condition as f above, the difference $f'f^{-1}$ is Q^2-invariant, that is, it defines a function on $(P/Q)^2$. Thus, the cohomology class of ∂f in $H^3(P/Q;\mathbb{T})$ depends only on τ. It also follows that the twisted coproduct $\hat\Delta_f(x)=f\hat\Delta_q(x)f^*$ does not depend on the choice of f.

Since $(\partial f)^*$ belongs to $\mathcal{U}(Z(G)^3)$, as we discussed in Section 1.2, it can be regarded as an invariant 3-cocycle in $\mathcal{U}(G_q^3)$ which is denoted by Φ^τ. Similarly, f can be considered as a unitary in $\mathcal{U}(G_q^2)$, and we have

$$\Phi^\tau=(\iota\otimes\hat\Delta_q)(f^*)(1\otimes f^*)(f\otimes 1)(\hat\Delta_q\otimes\iota)(f).$$

Proposition 2.3. *The coproduct $\hat\Delta_f$ on $\mathcal{U}(G_q)$ coincides with the coproduct $\hat\Delta$ defined by* (2.1).

Proof. Since f is contained in $\mathcal{U}(T^2)\subset\mathcal{U}(G_q^2)$, $\hat\Delta_f=\hat\Delta_q$ on the elements K_i. For E_i, since the action of E_i on an admissible module increases the weight of a vector by α_i, identities (2.2) imply that $f(K_i\otimes E_i)f^*=K_i\otimes E_i$ and $f(E_i\otimes 1)f^*=E_i\otimes\tau_i$. Comparing these identities with (2.1), we obtain the assertion. □

Corollary 2.4. *The representation category of G_q^τ is unitarily monoidally equivalent to* $\operatorname{Rep}(G_q)^{\Phi^\tau}$*, the representation category of G_q with associativity morphisms defined by Φ^τ.*

This result can also be interpreted as follows. Let $\Phi_{\mathrm{KZ},q}\in\mathcal{U}(G^3)$ be the Drinfeld associator coming from the Knizhnik–Zamolodchikov equations associated with the parameter $\hbar=\log(q)/\pi i$. The representation category of G_q is equivalent to that of G with associativity morphisms defined by $\Phi_{\mathrm{KZ},q}$. The equivalence is given by a unitary Drinfeld twist $F_D\in\mathcal{U}(G^2)$ satisfying (1.2) for $\Phi_{\mathrm{KZ},q}$ [**NT13**, Chapter 4]. It follows that $\operatorname{Rep}(G_q^\tau)$ is unitarily monoidally equivalent to the category $\operatorname{Rep}(G)$ with associativity morphisms defined by

$$\Phi^\tau_{\mathrm{KZ},q}=(\iota\otimes\hat\Delta)(F_D^*)(1\otimes F_D^*)\Phi^\tau(F_D\otimes 1)(\hat\Delta\otimes\iota)(F_D)=\Phi^\tau\Phi_{\mathrm{KZ},q},$$

where we now consider Φ^τ as an element of $\mathcal{U}(G^3)$. Correspondingly, the unitary $F_D^\tau=fF_D\in\mathcal{U}(G^2)$ plays the role of a Drinfeld twist for G_q^τ.

Remark 2.5. The construction of [**NT10**] can be carried out for G_q^τ to obtain a spectral triple over $\mathbb{C}[G_q^\tau]$ as an isospectral deformation of the spin Dirac operator on G. Indeed, it is enough to verify the boundedness of $[1\otimes(\iota\otimes\gamma)(t),(\pi\otimes\iota\otimes\widetilde{\mathrm{ad}})(\Phi^\tau_{\mathrm{KZ},q})]$ for any irreducible representation π, where t is the standard symmetric tensor $\sum_i x_i\otimes x_i$ [**NT10**, Corollary 3.2]. Since $(\pi\otimes\iota\otimes\widetilde{\mathrm{ad}})(\Phi^\tau)\in\mathbb{C}\otimes\mathcal{U}(Z(G))\otimes\mathbb{C}$ commutes with $1\otimes(\iota\otimes\gamma)(t)$, we can reduce the proof to the case of trivial τ.

A natural question is how large the class of cocycles of the form Φ^τ is. These cocycles are analyzed in detail in Appendix. Using that analysis we point out the following.

Proposition 2.6. *A $\mathbb{T}$-valued 3-cocycle Φ on P/Q is cohomologous to Φ^τ for some*

$\tau_1, \ldots, \tau_r \in Z(G)$ *if and only if* Φ *lifts to a coboundary on* P. *This is always the case if* P/Q *can be generated by not more than two elements. For example, this is the case if* G *is simple.*

Proof. The first statement is proved in Corollary A.4. It is also shown there that another equivalent condition on Φ is that it vanishes on $\bigwedge^3(P/Q) \subset H_3(P/Q;\mathbb{Z})$. This condition is obviously satisfied if P/Q can be generated by two elements. Finally, if G is simple, then it is known that P/Q is cyclic in all cases except for $G = \mathrm{Spin}(4n)$, in which case $P/Q \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. □

Therefore for simple G the quantum groups G_q^τ realize all possible associativity morphisms on $\mathrm{Rep}(G_q)$ defined by 3-cocycles on the dual of the center. In the semisimple case this is not true as soon as the center becomes slightly more complicated, namely, as soon as $\bigwedge^3(P/Q) \neq 0$. We conjecture that in this case, if we take a cocycle Φ on P/Q that does not lift to a coboundary on P, then there are no unitary fiber functors on $\mathrm{Rep}(G)^\Phi$, that is, there are no compact quantum groups with this representation category. Note that by Corollary A.5 any such cocycle Φ is cohomologous to product of a cocycle Φ^τ and a 3-character on P/Q that is nontrivial on $\bigwedge^3(P/Q) \subset (P/Q)^{\otimes 3}$.

2.3. Isomorphisms of twisted quantum groups.

Denote the cohomology class of the cocycle Φ^τ in $H^3(P/Q;\mathbb{T})$ by $\Theta(\tau)$. This way we obtain a homomorphism

$$\Theta\colon Z(G)^r \to H^3(P/Q;\mathbb{T}).$$

Assume $\tau \in \ker\Theta$. Let f be a function satisfying (2.2). Then there exists a 2-cochain $g\colon (P/Q)^2 \to \mathbb{T}$ such that $\partial f = \partial g$, so that fg^{-1} is a 2-cocycle on P. Another choice of f and g would give us a cocycle that differs from fg^{-1} by a 2-cocycle on P/Q. Therefore taking the cohomology class of fg^{-1} we get a well-defined homomorphism

$$\Upsilon\colon \ker\Theta \to H^2(P;\mathbb{T})/H^2(P/Q;\mathbb{T}).$$

Proposition 2.7. *Assume* $\tau', \tau \in Z(G)^r$ *are such that*

$$\tau'\tau^{-1} \in \ker\Theta \quad \text{and} \quad \tau'\tau^{-1} \in \ker\Upsilon.$$

Then the quantum groups $G_q^{\tau'}$ *and* G_q^τ *are isomorphic.*

Proof. Denote by $\hat\Delta'$ and $\hat\Delta$ the coproducts on $\mathcal{U}(G_q)$ defined by τ' and τ, see (2.1). Let f' and f be functions satisfying (2.2) for τ' and τ, respectively, so that $\hat\Delta' = \hat\Delta_{f'}$ and $\hat\Delta = \hat\Delta_f$. The assumptions of the proposition mean that there exists a 2-cochain g on P/Q such that $f'f^{-1}g$ is a coboundary on P. In other words, there exists a unitary $u \in \mathcal{U}(T^2) \subset \mathcal{U}(G_q^2)$ such that

$$f'g = (u \otimes u) f \hat\Delta_q(u)^*.$$

Then $\mathrm{Ad} u$ is an isomorphism of $(\mathcal{U}(G_q), \hat{\Delta})$ onto $(\mathcal{U}(G_q), \hat{\Delta}')$, hence $G_q^\tau \cong G_q^{\tau'}$. □

Apart from the isomorphisms given by this proposition, we have $G_q^\tau \cong G_{q^{-1}}^{\tau^{-1}}$. There also are isomorphisms induced by symmetries of the based root datum of G. Finally, for $q = 1$ there can be additional isomorphisms defined by conjugation by elements in $\mathcal{U}(G)$ that lie in the normalizer of the maximal torus.

3. Function algebras of twisted quantum groups.

3.1. Crossed product description.

As before, assume $\tau = (\tau_1, \dots, \tau_r) \in Z(G)^r$. Recall that we denote by T_τ the subgroup of $Z(G)$ generated by the elements $\tau_1, \dots, \tau_r$. There is a homomorphism

$$\psi\colon \hat{T}_\tau \to T/Z(G)$$

defined as follows. Given $\chi \in \hat{T}_\tau$, we define a character on the root lattice Q by $\alpha_i \mapsto \chi(\tau_i)$. It can be extended to P, and we obtain an element $\tilde{\psi}(\chi) \in \hat{P} = T$. The ambiguity of this extension is in $Q^\perp \cap T = Z(G)$. Thus, the image $\psi(\chi)$ of $\tilde{\psi}(\chi)$ in $T/Z(G)$ is well-defined.

The homomorphism ψ allows us to define an action of $\hat{T}_\tau$ by conjugation on G_q, that is, we have an action $\mathrm{Ad}\psi$ of $\hat{T}_\tau$ on $C(G_q)$ defined by

$$(\mathrm{Ad}\psi(\chi))(a) = \langle \tilde{\psi}(\chi^{-1}), a_{[1]} \rangle \langle \tilde{\psi}(\chi), a_{[3]} \rangle a_{[2]};$$

recall that the elements of T define characters of $C(G_q)$, that is, they are group-like unitary elements in $\mathcal{U}(G_q)$.

Theorem 3.1. *There is a canonical isomorphism*

$$C(G_q^\tau) \cong (C(G_q) \rtimes_{\mathrm{Ad}\psi} \hat{T}_\tau)^{T_\tau},$$

where the group T_τ acts on $C(G_q) \rtimes_{\mathrm{Ad}\psi} \hat{T}_\tau$ by right translations ρ on $C(G_q)$ and by the dual action on $C^*(\hat{T}_\tau)$.

Proof. Let us first identify the compact quantum group $\tilde{G}_q^\tau$ defined by the category of finite dimensional representations of $\mathcal{U}_q^\tau(\mathfrak{g})$ such that their restrictions to $\mathcal{U}_q(\mathfrak{g})$ are admissible. Any such irreducible representation is tensor product of an irreducible admissible representation of $\mathcal{U}_q(\mathfrak{g})$ and a character of T_τ; recall that these can be regarded as representations of $\mathcal{U}_q^\tau(\mathfrak{g})$. It follows that the Hopf $*$-algebra $\mathbb{C}[\tilde{G}_q^\tau]$ contains copies of $\mathbb{C}[G_q]$ and $C^*(\hat{T}_\tau)$, and as a space it is tensor product of these Hopf $*$-subalgebras. It remains to find relations between elements of $\mathbb{C}[G_q]$ and $C^*(\hat{T}_\tau)$ inside $\mathbb{C}[\tilde{G}_q^\tau]$.

Let (π, V) be a finite dimensional admissible representation of $\mathcal{U}_q(\mathfrak{g})$, and χ be a character of T_τ. Then, on the one hand, $\pi \otimes \chi$ is a representation on V with E_i acting by $\chi(\tau_i)\pi(E_i)$. On the other hand, $\chi \otimes \pi$ is also a representation on the same space V with E_i acting by $\pi(E_i)$. From this we see that the operator $\pi(\tilde{\psi}(\chi))$, where we consider the standard extension of π to $\mathcal{U}(G_q)$, intertwines $\chi\otimes\pi$ with $\pi\otimes\chi$. In other words, if $U_\pi\in B(V)\otimes\mathbb{C}[G_q]$ is the representation of G_q defined by π, then in $B(V)\otimes\mathbb{C}[\tilde G^\tau_q]$ we have

$$(\pi(\tilde\psi(\chi))\otimes u_\chi)U_\pi = U_\pi(\pi(\tilde\psi(\chi))\otimes u_\chi).$$

Since

$$(\pi(\tilde\psi(\chi)^{-1})\otimes 1)U_\pi(\pi(\tilde\psi(\chi))\otimes 1) = (\iota\otimes \mathrm{Ad}\psi(\chi))(U_\pi),$$

this exactly means that if $a\in\mathbb{C}[G_q]$ is a matrix coefficient of π, then $u_\chi a = (\mathrm{Ad}\psi(\chi))(a)u_\chi$. Therefore $\mathbb{C}[\tilde G^\tau_q] = \mathbb{C}[G_q]\rtimes_{\mathrm{Ad}\psi}\hat T_\tau$.

Now, the quantum group G^τ_q is the quotient of $\tilde G^\tau_q$ defined by the category of τ-admissible representations. By definition, a representation $\pi\otimes\chi$ of $\mathcal{U}^\tau_q(\mathfrak{g})$ is τ-admissible if $\pi(\tau_i)=\chi(\tau_i)$. Therefore $\mathbb{C}[G^\tau_q]\subset\mathbb{C}[\tilde G^\tau_q]=\mathbb{C}[G_q]\rtimes_{\mathrm{Ad}\psi}\hat T_\tau$ is spanned by elements of the form au_χ, where a is a matrix coefficient of an admissible representation π such that $\pi(\tau_i)=\chi(\tau_i)$. If π is irreducible, then $\pi(\tau_i)$ is scalar, and we have $\rho(\tau_i)(a)=\pi(\tau_i)a$. Hence $\mathbb{C}[G^\tau_q] = (\mathbb{C}[G_q]\rtimes_{\mathrm{Ad}\psi}\hat T_\tau)^{T_\tau}$. □

COROLLARY 3.2. *The C^*-algebra $C(G^\tau_q)$ is of type* I.

PROOF. Since $C(G^\tau_q)\subset C(G_q)\rtimes_{\mathrm{Ad}\psi}\hat T_\tau$, this follows from the known fact that the C^*-algebra $C(G_q)$ is of type I. □

Recall that the family $(C(G_q))_{0<q<\infty}$ has canonical structure of a continuous field of C^*-algebras [**NT11**].

COROLLARY 3.3. *The C^*-algebras $(C(G^\tau_q))_{0<q<\infty}$ form a continuous field of C^*-algebras.*

3.2. Primitive spectrum.

Let us turn to a description of the primitive spectrum of $C(G^\tau_q)$. We will concentrate on the case $q\neq 1$, the case $q=1$ can be treated similarly. First of all observe that the action of T_τ on $C(G_q)\rtimes_{\mathrm{Ad}\psi}\hat T_\tau$ is saturated, since every spectral subspace contains a unitary. We thus obtain a strong Morita equivalence

$$C(G^\tau_q)\sim_M C(G_q)\rtimes_{\mathrm{Ad}\psi}\hat T_\tau\rtimes_{\rho,\widehat{\mathrm{Ad}\psi}} T_\tau\cong C(G_q)\rtimes_\rho T_\tau\rtimes_{\mathrm{Ad}\psi,\hat\rho}\hat T_\tau. \tag{3.1}$$

Recall how to describe primitive spectra of crossed products, see e.g. [**Wil07**]. Let Γ be a finite group acting on a separable C^*-algebra A. Then any primitive ideal J of $A\rtimes\Gamma$ is determined by the Γ-orbit of an ideal $I\in\mathrm{Prim}(A)$ and an ideal $J_0\in\mathrm{Prim}(A\rtimes\mathrm{Stab}_\Gamma(I))$ by the condition $J_0\cap A=I$ and $J=\mathrm{Ind}\,J_0$.

If A is of type I, the ideals J_0 can, in turn, be described as follows. Put $\Gamma_0=\mathrm{Stab}_\Gamma(I)$. We want to describe irreducible representations of $A\rtimes\Gamma_0$ whose restrictions to A have kernel I. Let H be the space of an irreducible representation π of A with kernel I. Then the action of Γ_0 on A/I is implemented by a projective unitary representation

$\gamma \mapsto u_\gamma$ of Γ_0 on H. Let ω be the corresponding 2-cocycle. Consider the regular $\bar{\omega}$-representation $\gamma \mapsto \lambda^{\bar{\omega}}_\gamma$ of Γ_0 on $\ell^2(\Gamma_0)$. Then $A \rtimes \Gamma_0$ has a representation on $H \otimes \ell^2(\Gamma_0)$ defined by $a \mapsto \pi(a) \otimes 1$, $\gamma \mapsto u_\gamma \otimes \lambda^{\bar{\omega}}_\gamma$. Any irreducible representation of $A \rtimes \Gamma_0$ whose restriction to A has kernel I is a subrepresentation of this representation. So it remains to decompose the representation of $A \rtimes \Gamma_0$ on $H \otimes \ell^2(\Gamma_0)$ into irreducible subrepresentations. The von Neumann algebra generated by the image of $A \rtimes \Gamma_0$ is $B(H) \otimes C^*(\Gamma_0; \bar{\omega})$. Therefore the representations we are interested in are in a one-to-one correspondence with irreducible representations of $C^*(\Gamma_0; \bar{\omega})$.

To summarize, if A is a separable C^*-algebra of type I and Γ is a finite group acting on A, then the primitive spectrum $\mathrm{Prim}(A \rtimes \Gamma)$ can be identified with the set of pairs $([I], J)$, where $[I]$ is the Γ-orbit of an ideal $I \in \mathrm{Prim}(A)$, $J \in \mathrm{Prim}(C^*(\Gamma_I; \bar{\omega}_I))$, and ω_I is the 2-cocycle on $\Gamma_I = \mathrm{Stab}_\Gamma(I)$ defined by a projective representation of Γ_I implementing the action of Γ_I on the image of A under an irreducible representation with kernel I.

Returning to $C(G^\tau_q)$, for an element $w \in W$ of the Weyl group and a character $\chi \in \hat{T}_\tau$, put $\theta_w(\chi) = w^{-1}(\tilde{\psi}(\chi))\tilde{\psi}(\chi)^{-1}$. This defines a homomorphism from $\hat{T}_\tau$ to T.

PROPOSITION 3.4. *For $q > 0$, $q \neq 1$, the primitive spectrum of $C(G^\tau_q)$ can be identified with*

$$\coprod_{w \in W} (\theta_w(\hat{T}_\tau)\backslash T/T_\tau) \times \widehat{\theta_w^{-1}(T_\tau)}.$$

PROOF. In view of the strong Morita equivalence (3.1) it suffices to describe the primitive spectrum of

$$C(G_q) \rtimes_\rho T_\tau \rtimes_{\mathrm{Ad}\psi, \hat{\rho}} \hat{T}_\tau.$$

Recall that the spectrum of $C(G_q)$ is $W \times T$. The right translation action of T_τ on $C(G_q)$ defines an action on $W \times T$ that is simply the action by translations on T. Therefore $\mathrm{Prim}(C(G_q) \rtimes_\rho T_\tau)$ can be identified with $W \times T/T_\tau$, and every irreducible representation of $C(G_q) \rtimes_\rho T_\tau$ is induced from an irreducible representation of $C(G_q)$.

Next, we have to understand the action of $\hat{T}_\tau$ on $\mathrm{Prim}(C(G_q) \rtimes_\rho T_\tau)$. Since the dual action preserves the equivalence class of any induced representation, we just have to look at the action $\mathrm{Ad}\psi$. Given a representation $\pi_w \otimes \pi_t$ of $C(G_q)$ corresponding to $(w, t) \in W \times T$, we have

$$(\pi_w \otimes \pi_t)(\mathrm{Ad}\psi(\chi^{-1})) \sim \pi_w \otimes \pi_{\theta_w(\chi)t}$$

by [**NT12**, Lemma 3.4] and [**Yam13**, Lemma 8]. It follows that the action of $\hat{T}_\tau$ on $\mathrm{Prim}(C(G_q) \rtimes_\rho T_\tau) = W \times T/T_\tau$ is by translations on T/T_τ via the homomorphisms $\theta_w \colon \hat{T}_\tau \to T$. Hence the space of $\hat{T}_\tau$-orbits is $\coprod_{w \in W} \theta_w(\hat{T}_\tau)\backslash T/T_\tau$, and the stabilizer of a point (w, tT_τ) is $\theta_w^{-1}(T_\tau) \subset \hat{T}_\tau$.

To finish the proof of the proposition it remains to show that the action $(\mathrm{Ad}\psi, \hat{\rho})$ of $\theta_w^{-1}(T_\tau)$ on $C(G_q) \rtimes_\rho T_\tau$ can be implemented in the space of the induced representation

$\mathrm{Ind}(\pi_w \otimes \pi_t)$ by a unitary representation of $\theta_w^{-1}(T_\tau)$. For this, in turn, it suffices to show that the equivalences

$$(\pi_w \otimes \pi_{t'})(\mathrm{Ad}t^{-1}) \sim \pi_w \otimes \pi_{w^{-1}(t)t^{-1}t'}$$

from [**NT12**, Lemma 3.4] and [**Yam13**, Lemma 8] can be implemented by a unitary representation $t \mapsto v_t$ of $T/Z(G)$ on the space of representation π_w. But this is easy to see. Specifically, using the notation of [**NT12**] and [**Yam13**], if $w = s_i$ is the reflection corresponding to a simple root α_i, then the required representation $t \mapsto v_t$ on $\ell^2(\mathbb{Z}_+)$ can be defined by $v_t e_n = \langle t, \alpha_i \rangle^n e_n$. For arbitrary w we just have to take tensor products of such representations. □

Remark 3.5. A description of the topology on $\mathrm{Prim}(C(G_q))$ is given in [**NT12**]. The above argument is, however, not quite enough to understand the topology on $\mathrm{Prim}(C(G_q^\tau))$.

3.3. *K*-theory.

The maximal torus T is embedded in $\mathcal{U}(G_q^\tau)$, so it can be considered as a subgroup of G_q^τ. Let us consider the right translation action ρ of T on $C(G_q^\tau)$. The crossed product $C(G_q^\tau) \rtimes_\rho T$ is a $\hat{T}$-C^*-algebra with respect to the dual action.

Proposition 3.6. *The dual action of $\hat{T}$ on $C(G_q^\tau) \rtimes_\rho T$ is equivariantly strongly Morita equivalent to an action on $C(G_q) \rtimes_\rho T$ that is homotopic to the dual action.*

Proof. If we identify $C(G_q^\tau)$ with $(C(G_q) \rtimes_{\mathrm{Ad}\psi} \hat{T}_\tau)^{T_\tau}$, then the action of T by right translations on $C(G_q^\tau)$ extends to an action on $C(G_q) \rtimes_{\mathrm{Ad}\psi} \hat{T}_\tau$ that is trivial on $C^*(T_\tau)$ and coincides with the action by right translations on $C(G_q)$. This action of T on $C(G_q) \rtimes_{\mathrm{Ad}\psi} \hat{T}_\tau$ commutes with the action of T_τ. Hence the strong Morita equivalence (3.1) is T-equivariant, and taking crossed products we get a $\hat{T}$-equivariant strong Morita equivalence

$$C(G_q^\tau) \rtimes_\rho T \sim_M C(G_q) \rtimes_{\mathrm{Ad}\psi} \hat{T}_\tau \rtimes_{\rho, \widehat{Ad\psi}} T_\tau \rtimes_\rho T. \tag{3.2}$$

Denote the C^*-algebra on the right hand side by A. We claim that A is isomorphic to

$$B = C(G_q) \rtimes_{\mathrm{Ad}\psi} \hat{T}_\tau \rtimes_{\widehat{\mathrm{Ad}\psi}} T_\tau \rtimes_\rho T.$$

Indeed, the map $au_\chi u_t u_{t'} \mapsto au_\chi u_t u_{tt'}$ for $a \in C(G_q)$, $\chi \in \hat{T}_\tau$, $t \in T_\tau$ and $t' \in T$ is the required isomorphism. The dual action of $\hat{T}$ on A corresponds to an action β on B which is given by the dual action on the copy of $C^*(T)$ and by the dual action on the copy of $C^*(T_\tau)$ via the canonical homomorphism $r\colon \hat{T} \to \hat{T}_\tau$.

The map $\hat{T} \ni \chi \mapsto u_{r(\chi)} \in C^*(\hat{T}_\tau) \subset M(B)$ is a 1-cocycle for the action β. Therefore β is strongly Morita equivalent to the action γ defined by $\gamma_\chi = (\mathrm{Ad}u_{r(\chi)})\beta_\chi$. This action is already trivial on $C^*(T_\tau)$, while on $C(G_q)$ it is given by $\mathrm{Ad}\psi(r(\chi))$, and on $C^*(T)$ it coincides with the dual action.

Denote by δ the restriction of γ to $C(G_q) \rtimes_\rho T \subset M(B)$. Then, similarly to (3.2), the actions δ and γ are strongly Morita equivalent.

Combining the Morita equivalences that we have obtained, we conclude that the dual action of $\hat{T}$ on $C(G_q^\tau) \rtimes_\rho T$ is strongly Morita equivalent to the action $\delta = (\mathrm{Ad}\psi(r(\cdot)), \hat{\rho})$ on $C(G_q) \rtimes_\rho T$. Choosing a basis in $\hat{T} = P$ and paths from $\tilde{\psi}(r(\chi))$ to the neutral element in T for every basis element χ, we see that δ is homotopic to the dual action on $C(G_q) \rtimes_\rho T$. □

Theorem 3.7. *The C^*-algebra $C(G_q^\tau)$ is KK-isomorphic to $C(G_q)$, hence to $C(G)$.*

Proof. Since the torsion-free commutative group $\hat{T}$ satisfies the strong Baum–Connes conjecture, the functor $A \mapsto A \rtimes \hat{T}$ maps homotopic actions into KK-isomorphisms of the corresponding crossed products. By the previous proposition, this, together with the Takesaki–Takai duality, implies that $C(G_q^\tau)$ and $C(G_q)$ are KK-isomorphic. By [**NT12**] we also know that $C(G_q)$ is KK-isomorphic to $C(G)$. □

Remark 3.8.

(i) The above proof shows that the continuous field of Corollary 3.3 is a KK-fibration in the sense of [**ENOO09**]. The argument of [**NT11**] applies to the Dirac operator D given by Remark 2.5, and we obtain that the K-homology class of D is independent of q. The bi-equivariance of D and the construction in the proof of Proposition 3.6 imply that the K-homology class of D is also independent of τ up to the isomorphism of Theorem 3.7.

(ii) For the group $\hat{T}$ the strong Baum–Connes conjecture is a consequence of the Pimsner–Voiculescu sequence in KK-theory. Therefore the proof of Theorem 3.7 can be written such that it relies only on this sequence, see e.g. [**San11**, Section 5.1] for a related argument.

4. Twisted $SU_q(n)$.

4.1. Special unitary group.

Let us review the structure of $SU(n)$, see e.g. [**FH91**, Chapter 15]. For the sake of presentation, it is convenient to consider also the unitary group $U(n)$. We take the subgroup of the diagonal matrices $\tilde{T}$ as a maximal torus of $U(n)$, and take $T = \tilde{T} \cap SU(n)$ as a maximal torus of $SU(n)$. We will often identify $\tilde{T}$ with $\mathbb{T}^n$. We write the corresponding Cartan subalgebras as $\tilde{\mathfrak{h}} \subset \mathfrak{gl}_n$ and $\mathfrak{h} \subset \mathfrak{sl}_n$.

Let $\{e_{ij}\}_{i,j=1}^n$ be the matrix units in $M_n(\mathbb{C}) = \mathfrak{gl}_n$, and $\{\tilde{L}_i\}_{i=1}^n$ be the basis in $\tilde{\mathfrak{h}}^*$ dual to the basis $\{e_{ii}\}_{i=1}^n$ in $\tilde{\mathfrak{h}}$. Denote by L_i the image of $\tilde{L}_i$ in $\mathfrak{h}^*$. Therefore any $n-1$ elements among $L_1, \dots, L_n$ form a basis in $\mathfrak{h}^*$, and we have $\sum_i L_i = 0$.

The weight lattice $P \subset \mathfrak{h}^*$ is generated by the elements L_i. The pairing between T and P is given by $\langle t, L_i \rangle = t_i$ for $t \in T \subset \mathbb{T}^n$. As simple roots we take

$$\alpha_i = L_i - L_{i+1}, \quad 1 \le i \le n-1.$$

The fundamental weights are then given by

$$\varpi_i = L_1 + \cdots + L_i, \quad 1 \le i \le n-1.$$

Consider the homomorphism $|\cdot|\colon P \to \mathbb{Z}$ such that $L_1 \mapsto n-1$ and $L_i \mapsto -1$ for $1 < i \le n$. In other words,

$$|a_1\varpi_1 + \cdots + a_{n-1}\varpi_{n-1}| = \lambda_1 + \cdots + \lambda_{n-1},$$

where λ_{n-i} is given by $a_1 + \cdots + a_i$. The image of Q under $|\cdot|$ is $n\mathbb{Z}$, and therefore we can use this homomorphism to identify P/Q with $\mathbb{Z}/n\mathbb{Z}$.

4.2. Twisted quantum special unitary groups.

By Proposition A.3, the cohomology group $H^3(\mathbb{Z}/n\mathbb{Z};\mathbb{T})$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$, and a cocycle generating this group can be defined by

$$\phi(a,b,c) = \zeta_n^{\omega_n(a,b)c}, \quad \text{where} \quad \zeta_n = e^{2\pi i/n} \quad \text{and} \quad \omega_n(a,b) = \left\lfloor \frac{a+b}{n} \right\rfloor - \left\lfloor \frac{a}{n} \right\rfloor - \left\lfloor \frac{b}{n} \right\rfloor.$$

Using this generator we identify $H^3(\mathbb{Z}/n\mathbb{Z};\mathbb{T})$ with the group $\mu_n \subset \mathbb{T}$ of units of order n. Therefore, given $\zeta \in \mu_n$, we have a category $\mathrm{Rep}(SU_q(n))^\zeta$ with associativity morphisms defined by multiplication by $\zeta^{\omega_n(|\lambda|,|\eta|)|\nu|}$ on the tensor product $V_\lambda \otimes V_\eta \otimes V_\nu$ of irreducible $\mathcal{U}_q(\mathfrak{g})$-modules with highest weights λ, η, ν. This agrees with the conventions of Kazhdan and Wenzl [**KW93**].

It is also convenient to identify $Z(SU(n))$ with the group μ_n. Thus, for $\tau = (\tau_1, \dots, \tau_{n-1}) \in \mu_n^{n-1}$, we can define a twisting $SU_q^\tau(n)$ of $SU_q(n)$. Its representation category is one of $\mathrm{Rep}(SU_q(n))^\zeta$, and to find ζ we have to compute the homomorphism $\Theta\colon Z(SU(n))^{n-1} \to H^3(P/Q;\mathbb{T})$ introduced in Section 2.3. Under our identifications this becomes a homomorphism $\mu_n^{n-1} \to \mu_n$.

Proposition 4.1. *We have* $\Theta(\tau) = \prod_{i=1}^{n-1} \tau_i^{-i}$.

Proof. Recall the construction of Θ. We choose a function $f\colon P \times P \to \mathbb{T}$ such that it factors through $P \times (P/Q)$ and $f(\lambda + \alpha_i, \mu) = \overline{\langle \tau_i, \mu \rangle} f(\lambda, \mu)$. Then $\Theta(\tau)$ is the cohomology class of ∂f in $H^3(P/Q;\mathbb{T})$.

Note that $\langle \tau_i, \mu \rangle = \tau_i^{-|\mu|}$, which is immediate for $\mu = L_j$, and define a character χ of $Q \otimes (P/Q) = Q \otimes (\mathbb{Z}/n\mathbb{Z})$ by

$$\chi(\alpha_i \otimes k) = \tau_i^k \quad \text{for} \quad 1 \le i \le n-1 \quad \text{and} \quad k \in \mathbb{Z}/n\mathbb{Z},$$

so that $f(\lambda+\alpha, \mu) = \chi(\alpha \otimes |\mu|) f(\lambda, \mu)$ for all $\alpha \in Q$. By Proposition A.6, the cohomology class of ∂f depends only on the restriction of χ to

$$\ker(Q \otimes (\mathbb{Z}/n\mathbb{Z}) \to P \otimes (\mathbb{Z}/n\mathbb{Z})) \cong \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z},$$

and by varying τ we get this way an isomorphism $\mathrm{Hom}(\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}), \mathbb{T}) \cong H^3(\mathbb{Z}/n\mathbb{Z};\mathbb{T})$. In order to compute this isomorphism we can use the resolution $n\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ instead of $Q \to P \xrightarrow{|\cdot|} \mathbb{Z}/n\mathbb{Z}$. Define a morphism between these resolutions by

$\mathbb{Z} \to P$, $1 \mapsto \varpi_{n-1} = -L_n$. By pulling back χ under this morphism, we get a character $\tilde{\chi}$ of $(n\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z})$ such that

$$\tilde{\chi}(n \otimes k) = \chi(n\varpi_{n-1} \otimes k).$$

We have $n\varpi_{n-1} = \sum_{i=1}^{n-1} i\alpha_i$. Therefore

$$\tilde{\chi}(n \otimes k) = \zeta^k, \quad \text{where} \quad \zeta = \prod_{i=1}^{n-1} \tau_i^i.$$

Then the function $\tilde{f}\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{T}$ defined by

$$\tilde{f}(a, b) = \zeta^{\lfloor a/n \rfloor b},$$

factors through $\mathbb{Z} \times (\mathbb{Z}/n\mathbb{Z})$, $\tilde{f}(a + n, b) = \tilde{\chi}(n \otimes b)\tilde{f}(a, b)$ and $(\partial\tilde{f})(a, b, c) = \zeta^{-\omega_n(a,b)c}$. Therefore the class of $\partial\tilde{f}$ in $H^3(\mathbb{Z}/n\mathbb{Z}; \mathbb{T}) = \mu_n$ is ζ^{-1}. □

In Section 2.3 we also introduced a homomorphism Υ. In the present case we have $H^2(P/Q; \mathbb{T}) = 0$, so Υ is a homomorphism $\ker\Theta \to H^2(P; \mathbb{T})$.

LEMMA 4.2. *The homomorphism* $\Upsilon\colon \ker\Theta \to H^2(P; \mathbb{T})$ *is injective.*

PROOF. Assume $\tau \in \ker\Theta$, so $\prod_{i=1}^{n-1} \tau_i^i = 1$. In this case the character χ of $Q \otimes (P/Q)$ from the proof of the previous proposition extends to $P \otimes (P/Q)$ by

$$\chi(L_i \otimes \mu) = (\tau_1 \cdots \tau_{i-1})^{-|\mu|} \quad \text{for} \quad 1 \le i \le n \quad \text{and} \quad \mu \in P.$$

Therefore if we consider χ as a function on $P \times P$, we can take it as a function f in that proof. Then f is a 2-cocycle, and by definition, the image of τ under Υ is the cohomology class of $\bar{f}$. It is well-known, and also follows from Proposition A.1, that f is a coboundary if and only if f is symmetric. For $1 < i < j \le n$ we have

$$f(L_i, L_j)\overline{f(L_j, L_i)} = (\tau_i \cdots \tau_{j-1})^{-1}.$$

So if f is symmetric, then $\tau_2 = \cdots = \tau_{n-1} = 1$, but then also $\tau_1 = 1$. □

Therefore Proposition 2.7 does not give us any nontrivial isomorphisms between the quantum groups $SU_q^\tau(n)$. On the other hand, the flip map on the Dynkin diagram induces an automorphism of $\mathcal{U}(SU_q(n))$ such that $K_i \mapsto K_{n-i}$ and $E_i \mapsto E_{n-i}$ for $1 \le i \le n-1$. On $Z(SU(n)) \subset \mathcal{U}(SU_q(n))$ this automorphism is $t \mapsto t^{-1}$. It follows that it induces isomorphisms

$$SU_q^{(\tau_1,\dots,\tau_{n-1})}(n) \cong SU_q^{(\tau_{n-1}^{-1},\dots,\tau_1^{-1})}(n).$$

For $0 < q < 1$, these seem to be the only obvious isomorphisms between the quantum groups $SU_q^\tau(n)$.

4.3. Generators and relations.

The C^*-algebra $C(SU_q(n))$ is generated by the matrix coefficients $(u_{ij})_{1\le i,j\le n}$ of the natural representation of $SU_q(n)$ on $\mathbb{C}^n$, the fundamental representation with highest weight ϖ_1. They satisfy the relations [**Dri87**] and [**Wor88**]

$$u_{ij}u_{il} = qu_{il}u_{ij} \quad (j<l), \quad u_{ij}u_{kj} = qu_{kj}u_{ij} \quad (i<k), \tag{4.1}$$

$$u_{ij}u_{kl} = u_{kl}u_{ij} \quad (i>k, j<l), \quad u_{ij}u_{kl} - u_{kl}u_{ij} = (q-q^{-1})u_{il}u_{kj} \quad (i<k, j<l), \tag{4.2}$$

$$\operatorname{qdet}((u_{ij})_{i,j}) = \sum_{\sigma\in S_n} (-q)^{|\sigma|} u_{1\sigma(1)}\cdots u_{n\sigma(n)} = 1. \tag{4.3}$$

Here, $|\sigma|$ is the inversion number of the permutation σ. The involution is defined by

$$u_{ij}^* = (-q)^{j-i}\operatorname{qdet}\big(U^{\hat{i}}_{\hat{j}}\big),$$

where $U^{\hat{i}}_{\hat{j}}$ is the matrix obtained from $U = (u_{kl})_{k,l}$ by deleting the i-th row and j-th column.

In order to find generators and relations of $\mathbb{C}[SU_q^\tau(n)]$, we will use the embedding of the algebra $\mathbb{C}[SU_q^\tau(n)]$ into $\mathbb{C}[SU_q(n)] \rtimes_{\mathrm{Ad}\psi} \hat{T}_\tau$ described in Theorem 3.1. Recall that $\psi\colon \hat{T}_\tau \to T/Z(SU(n)) = T/\mu_n$ is the homomorphism such that $\langle\tilde{\psi}(\chi),\alpha_i\rangle = \chi(\tau_i)$, where $\tilde{\psi}(\chi)$ is a lift of $\psi(\chi)$ to T. Hence

$$\tilde{\psi}(\chi) = (z, z\chi(\tau_1)^{-1}, \dots, z\chi(\tau_1\cdots\tau_{n-1})^{-1}) \in T \subset \mathbb{T}^n,$$

where $z\in\mathbb{T}$ is a number such that $z^n = \prod_{i=1}^{n-1}\chi(\tau_i)^{-i}$. It follows that

$$(\mathrm{Ad}\psi(\chi))(u_{ij}) = \Big(\prod_{1\le p<i}\chi(\tau_p)\Big)\Big(\prod_{1\le p<j}\chi(\tau_p)^{-1}\Big)u_{ij}. \tag{4.4}$$

Now, the algebra $\mathbb{C}[SU_q^\tau(n)]$ is generated by matrix coefficients of the fundamental representation of $SU_q^\tau(n)$ with highest weight ϖ_1. Under the embedding $\mathbb{C}[SU_q^\tau(n)] \hookrightarrow \mathbb{C}[SU_q(n)] \rtimes_{\mathrm{Ad}\psi} \hat{T}_\tau$, these matrix coefficients correspond to $v_{ij} = u_{ij}u_{\chi_{\mathrm{nat}}}$, where $\chi_{\mathrm{nat}} \in \hat{T}_\tau$ is the character determined by the natural representation of $SU_q(n)$ on $\mathbb{C}^n$, so $\chi_{\mathrm{nat}}(\tau_i) = \tau_i$. From (4.1)–(4.3) we then get the following relations:

$$v_{ij}v_{il} = \Big(\prod_{j\le p<l}\tau_p^{-1}\Big)qv_{il}v_{ij} \quad (j<l), \quad v_{ij}v_{kj} = \Big(\prod_{i\le p<k}\tau_p\Big)qv_{kj}v_{ij} \quad (i<k), \tag{4.5}$$

$$v_{ij}v_{kl} = \Big(\prod_{k\le p<i}\tau_p^{-1}\Big)\Big(\prod_{j\le p<l}\tau_p^{-1}\Big)v_{kl}v_{ij} \quad (i>k, j<l), \tag{4.6}$$

$$\Big(\prod_{j\le p<l}\tau_p\Big)v_{ij}v_{kl} - \Big(\prod_{i\le p<k}\tau_p\Big)v_{kl}v_{ij} = (q-q^{-1})v_{il}v_{kj} \quad (i<k, j<l), \tag{4.7}$$

$$\sum_{\sigma\in S_n} \tau^{m(\sigma)}(-q)^{|\sigma|} v_{1\sigma(1)}\cdots v_{n\sigma(n)} = 1, \tag{4.8}$$

where $m(\sigma) = (m(\sigma)_1, \dots, m(\sigma)_{n-1})$ is the multi-index given by $m(\sigma)_i = \sum_{k=2}^n (k-1) m_i^{(k,\sigma(k))}$, and

$$m_i^{(k,j)} = \begin{cases} 1, & \text{if } k \le i < j, \\ -1, & \text{if } j \le i < k, \\ 0, & \text{otherwise.} \end{cases}$$

PROPOSITION 4.3. *For any $\tau \in \mu_n^{n-1}$, the algebra $\mathbb{C}[SU_q^\tau(n)]$ is a universal algebra generated by elements v_{ij} satisfying relations* (4.5)–(4.8).

PROOF. We already know that relations (4.5)–(4.8) are satisfied in $\mathbb{C}[SU_q^\tau(n)]$, so we just have to show that there are no other relations. Let $\mathcal{A}$ be a universal algebra generated by elements w_{ij} satisfying relations (4.5)–(4.8). We can define an action of $\hat{T}_\tau$ on $\mathcal{A}$ by (4.4). Then in $\mathcal{A} \rtimes \hat{T}_\tau$ the elements $w_{ij} u_{\chi_{\text{nat}}}^{-1}$ satisfy the defining relations of $\mathbb{C}[SU_q(n)]$, so we have a homomorphism $\mathbb{C}[SU_q(n)] \to \mathcal{A} \rtimes \hat{T}_\tau$ mapping u_{ij} into $w_{ij} u_{\chi_{\text{nat}}}^{-1}$. It extends to a homomorphism $\mathbb{C}[SU_q(n)] \rtimes \hat{T}_\tau \to \mathcal{A} \rtimes \hat{T}_\tau$ that is identity on the group algebra of $\hat{T}_\tau$. Restricting to $\mathbb{C}[SU_q^\tau(n)] \subset \mathbb{C}[SU_q(n)] \rtimes \hat{T}_\tau$, we get a homomorphism $\mathbb{C}[SU_q^\tau(n)] \to \mathcal{A}$ mapping v_{ij} into w_{ij}. □

The involution on $\mathbb{C}[SU_q^\tau(n)]$ is determined by requiring the invertible matrix $(v_{ij})_{i,j}$ to be unitary. An explicit formula can be easily found using that for $\mathbb{C}[SU_q(n)]$.

REMARK 4.4. The relations in $\mathbb{C}[SU_q^\tau(n)]$ cannot be obtained using the FRT-approach, since the categories $\operatorname{Rep}(SU_q(n))^\zeta$ are typically not braided. More precisely, $\operatorname{Rep}(SU_q(n))^\zeta$ has a braiding if and only if either $\zeta = 1$ or n is even and $\zeta = -1$. This statement is already implicit in [**KW93**], and it can be proved as follows. If $\zeta = 1$ or n is even and $\zeta = -1$, then a braiding indeed exists, see e.g. [**Pin07**]. Conversely, suppose we have a braiding. In other words, there exists an R-matrix $\mathcal{R}$ for $(\mathcal{U}(SU_q(n)), \hat{\Delta}_q, \Phi)$, where $\Phi = \zeta^{\omega_n(|\lambda|,|\eta|)|\nu|}$. Recall that this means that $\mathcal{R}$ is an invertible element in $\mathcal{U}(SU_q(n) \times SU_q(n))$ such that $\hat{\Delta}_q^{\text{op}} = \mathcal{R}\hat{\Delta}_q(\cdot)\mathcal{R}^{-1}$ and

$$(\hat{\Delta}_q \otimes \iota)(\mathcal{R}) = \Phi_{312}\mathcal{R}_{13}\Phi_{132}^{-1}\mathcal{R}_{23}\Phi, \quad (\iota \otimes \hat{\Delta}_q)(\mathcal{R}) = \Phi_{231}^{-1}\mathcal{R}_{13}\Phi_{213}\mathcal{R}_{12}\Phi^{-1}.$$

Since Φ is central and symmetric in the first two variables, the last two identities can be written as

$$(\hat{\Delta}_q^{\text{op}} \otimes \iota)(\mathcal{R}) = \mathcal{R}_{23}\mathcal{R}_{13}\Phi, \quad (\iota \otimes \hat{\Delta}_q)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}\Phi_{321}^{-1}.$$

On the other hand, we know that $\operatorname{Rep}(SU_q(n))$ is braided, so there exists an element $\mathcal{R}_q$ satisfying the above properties with Φ replaced by 1. Consider the element $F = \mathcal{R}_q^{-1}\mathcal{R}$. Then F is invariant, meaning that it commutes with the image of $\hat{\Delta}_q$. Furthermore, we have

$$(F\otimes 1)(\hat{\Delta}_q\otimes\iota)(F)=(\mathcal{R}_q^{-1}\otimes 1)(\hat{\Delta}_q^{\mathrm{op}}\otimes\iota)(\mathcal{R}_q^{-1})(\hat{\Delta}_q^{\mathrm{op}}\otimes\iota)(\mathcal{R})(\mathcal{R}\otimes 1)$$
$$=((\mathcal{R}_q)_{23}(\mathcal{R}_q)_{13}(\mathcal{R}_q)_{12})^{-1}\mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}\Phi,$$

and similarly

$$(1\otimes F)(\iota\otimes\hat{\Delta}_q)(F)=(\iota\otimes\hat{\Delta}_q)(\mathcal{R}_q^{-1})(1\otimes\mathcal{R}_q^{-1})(1\otimes\mathcal{R})(\iota\otimes\hat{\Delta}_q)(\mathcal{R})$$
$$=((\mathcal{R}_q)_{23}(\mathcal{R}_q)_{13}(\mathcal{R}_q)_{12})^{-1}\mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}\Phi_{321}^{-1}.$$

Therefore

$$(\iota\otimes\hat{\Delta}_q)(F^{-1})(1\otimes F^{-1})(F\otimes 1)(\hat{\Delta}_q\otimes\iota)(F)=\Phi_{321}\Phi.$$

This implies that $\mathrm{Rep}(SU_q(n))$ is monoidally equivalent to $\mathrm{Rep}(SU_q(n))^{\Phi_{321}\Phi}$. Since the cocycle $\Phi_{321}\Phi$ on the dual of the center is cohomologous to the cocycle $\zeta^{2\omega_n(|\lambda|,|\eta|)|\nu|}$, this means that $\mathrm{Rep}(SU_q(n))$ is monoidally equivalent to $\mathrm{Rep}(SU_q(n))^{\zeta^2}$. By the Kazhdan–Wenzl classification this is the case only if $\zeta^2=1$.

Appendix A. Cocycles on abelian groups.

Let Γ be a discrete abelian group. As is common in operator algebra, we denote the generators of the group algebra $\mathbb{Z}[\Gamma]$ by λ_γ ($\gamma\in\Gamma$). Let $(C_*(\Gamma),d)$ be the nonnormalized bar-resolution of the $\mathbb{Z}[\Gamma]$-module $\mathbb{Z}$, so $C_n(\Gamma)$ ($n\ge 0$) is the free $\mathbb{Z}[\Gamma]$-module with basis consisting of n-tuples of elements in Γ, written as $[\gamma_1|\cdots|\gamma_n]$, and the differential $d\colon C_n(\Gamma)\to C_{n-1}(\Gamma)$ is defined by

$$d[\gamma_1|\cdots|\gamma_n]=\lambda_{\gamma_1}[\gamma_2|\cdots|\gamma_n]+\sum_{i=1}^{n-1}(-1)^i[\gamma_1|\cdots|\gamma_i+\gamma_{i+1}|\cdots|\gamma_n]+(-1)^n[\gamma_1|\cdots|\gamma_{n-1}].$$

Let M be a commutative group endowed with the trivial Γ-module structure. The group cohomology $H^*(\Gamma;M)$ can be computed from the standard complex induced by the bar-resolution. Concretely, we have a cochain complex

$$C^*(\Gamma;M)=\mathrm{Hom}_{\mathbb{Z}[\Gamma]}(C_*(\Gamma),M)=\mathrm{Map}(\Gamma^*,M),$$

endowed with the boundary map $\partial\colon C^n(\Gamma;M)\to C^{n+1}(\Gamma;M)$ defined by

$$(\partial\phi)(\gamma_1,\dots,\gamma_{n+1})=\phi(\gamma_2,\dots,\gamma_{n+1})-\phi(\gamma_1+\gamma_2,\gamma_3,\dots,\gamma_{n+1})+\cdots$$
$$+(-1)^n\phi(\gamma_1,\dots,\gamma_{n-1},\gamma_n+\gamma_{n+1})+(-1)^{n+1}\phi(\gamma_1,\dots,\gamma_n).$$

By M-valued cocycles on Γ we mean cocycles in $(C^*(\Gamma;M),\partial)$. We will consider only $\mathbb{T}$-valued cocycles, but with minor modifications everything what we say remains true for cocycles with values in any divisible group M.

For the sake of computation, it is also convenient to introduce the integer homology

$H_*(\Gamma) = H_*(\Gamma;\mathbb{Z})$, which is given as the homology of the complex $C_*(\Gamma;\mathbb{Z}) = \mathbb{Z} \otimes_{\mathbb{Z}[\Gamma]} C_*(\Gamma)$. Since the action of Γ on $\mathbb{T}$ is trivial, we have $C^*(\Gamma;\mathbb{T}) = \mathrm{Hom}_{\mathbb{Z}[\Gamma]}(C_*(\Gamma),\mathbb{T}) = \mathrm{Hom}(C_*(\Gamma;\mathbb{Z}),\mathbb{T})$. Moreover, the injectivity of $\mathbb{T}$ as a $\mathbb{Z}$-module implies that any character of $H_n(\Gamma;\mathbb{Z})$ can be lifted to a character of $C_n(\Gamma;\mathbb{Z})$. It follows that the groups $H^n(\Gamma;\mathbb{T})$ and $H_n(\Gamma)$ are Pontryagin dual to each other. This is a particular case of the Universal Coefficient Theorem.

A map $\phi\colon \Gamma^n \to \mathbb{T}$ $(n \geq 1)$ is called an n-character on Γ if it is a character in every variable, so it is defined by a character on $\Gamma^{\otimes n}$ (unless specified otherwise, all tensor products in this appendix are over $\mathbb{Z}$). It is easy to see that every n-character is a $\mathbb{T}$-valued cocycle. An n-character ϕ is called alternating if $\phi(\gamma_1,\dots,\gamma_n) = 1$ as long as $\gamma_i = \gamma_{i+1}$ for some i; then $\phi(\gamma_{\sigma(1)},\dots,\gamma_{\sigma(n)}) = \phi(\gamma_1,\dots,\gamma_n)^{\mathrm{sgn}(\sigma)}$ for any $\sigma \in S_n$. In other words, an n-character is alternating if it factors through the exterior power $\bigwedge^n \Gamma$, which is the quotient of $\Gamma^{\otimes n}$ by the subgroup generated by elements $\gamma_1 \otimes \cdots \otimes \gamma_n$ such that $\gamma_i = \gamma_{i+1}$ for some i. It will sometimes be convenient to view $\bigwedge^n \Gamma$ as a subgroup of $\Gamma^{\otimes n}$ via the embedding

$$\gamma_1 \wedge \cdots \wedge \gamma_n \mapsto \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\gamma_{\sigma(1)} \otimes \cdots \otimes \gamma_{\sigma(n)}.$$

We will also consider $\bigwedge^n \Gamma$ as a subgroup of $H_n(\Gamma)$. The embedding $\bigwedge^* \Gamma \hookrightarrow H_*(\Gamma)$ is constructed using the canonical isomorphism $\Gamma \cong H_1(\Gamma)$ and the Pontryagin product on $H_*(\Gamma)$, see [**Bro94**, Theorem V.6.4]. On the chain level the latter product can be defined using the shuffle product, so that $\gamma_1 \wedge \cdots \wedge \gamma_n$ is identified with the homology class of the cycle

$$\sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)(1 \otimes [\gamma_{\sigma(1)}|\cdots|\gamma_{\sigma(n)}]) \in C_n(\Gamma;\mathbb{Z}).$$

For free abelian groups we have $\bigwedge^* \Gamma = H_*(\Gamma)$. By duality we get the following description of cocycles.

Proposition A.1. *If Γ is free abelian, then for every $n \geq 1$ we have*:

(i) *any $\mathbb{T}$-valued n-cocycle on Γ is cohomologous to an alternating n-character*;
(ii) *an n-character is a coboundary if and only if it vanishes on $\bigwedge^n \Gamma \subset \Gamma^{\otimes n}$; in particular, an alternating n-character is a coboundary if and only its order divides $n!$.*

Proof. The value of an n-cocycle ϕ on $\gamma_1 \wedge \cdots \wedge \gamma_n \in H_n(\Gamma)$ is

$$\langle \phi, \gamma_1 \wedge \cdots \wedge \gamma_n \rangle = \prod_{\sigma \in S_n} \phi(\gamma_{\sigma(1)},\dots,\gamma_{\sigma(n)})^{\mathrm{sgn}(\sigma)}.$$

This immediately implies (ii), since if ϕ is an n-character, then the above product is exactly the value of ϕ on $\gamma_1 \wedge \cdots \wedge \gamma_n$ considered as an element of $\Gamma^{\otimes n}$.

Turning to (i), assume ψ is an n-cocycle. It defines a character χ of $H_n(\Gamma) = \bigwedge^n \Gamma$.

Let ϕ be a character of $\bigwedge^n \Gamma$ such that $\phi^{n!} = \chi$. Then ϕ is an alternating n-character, and ϕ is cohomologous to ψ, since both cocycles ϕ and ψ define the same character χ of $H_n(\Gamma) = \bigwedge^n \Gamma$. $\square$

We now turn to the more complicated case of finite abelian groups and concentrate on 3-cocycles. In this case $\bigwedge^3 \Gamma$ is a proper subgroup of $H_3(\Gamma)$: as follows from Proposition A.3 below, the quotient $H_3(\Gamma)/\bigwedge^3 \Gamma$ is (noncanonically) isomorphic to $\Gamma \oplus (\Gamma \bigwedge \Gamma)$. Correspondingly, not every third cohomology class can be represented by a 3-character. Additional 3-cocycles can be obtained by the following construction.

Lemma A.2. *Assume $\Gamma = \Gamma_1/\Gamma_0$ for some abelian groups Γ_1 and Γ_0. Suppose $f\colon \Gamma_1 \times \Gamma_1 \to \mathbb{T}$ is a function such that*

$$f(\alpha, \beta + \gamma) = f(\alpha, \beta) \quad \text{and} \quad f(\alpha + \gamma, \beta) = \chi(\gamma \otimes \beta) f(\alpha, \beta)$$

for all $\alpha, \beta \in \Gamma_1$ and $\gamma \in \Gamma_0$, where χ is a character of $\Gamma_0 \otimes \Gamma$. Then the function

$$(\partial f)(\alpha, \beta, \gamma) = f(\beta, \gamma) f(\alpha + \beta, \gamma)^{-1} f(\alpha, \beta + \gamma) f(\alpha, \beta)^{-1}$$

on Γ_1^3 is Γ_0^3-invariant, hence it defines a $\mathbb{T}$-valued 3-cocycle on Γ.

Proof. This is a straightforward computation. $\square$

In order to describe explicitly generators of $H^3(\Gamma; \mathbb{T})$, let us introduce some notation. For natural numbers $n_1, \dots, n_k$, denote by $(n_1, \dots, n_k)$ their greatest common divisor. For $n \in \mathbb{N}$, denote by χ_n the character of $\mathbb{Z}/n\mathbb{Z}$ defined by $\chi_n(1) = e^{2\pi i/n}$. Finally, for integers a and b and a natural number n, put

$$\omega_n(a, b) = \left\lfloor \frac{a+b}{n} \right\rfloor - \left\lfloor \frac{a}{n} \right\rfloor - \left\lfloor \frac{b}{n} \right\rfloor.$$

Note that ω_n is a well-defined function on $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ with values 0 or 1.

Proposition A.3. *Assume $\Gamma = \bigoplus_{i=1}^m \mathbb{Z}/n_i\mathbb{Z}$ for some $n_i \ge 1$. Then*

$$H^3(\Gamma; \mathbb{T}) \cong \bigoplus_i \mathbb{Z}/n_i\mathbb{Z} \oplus \bigoplus_{i<j} \mathbb{Z}/(n_i, n_j)\mathbb{Z} \oplus \bigoplus_{i<j<k} \mathbb{Z}/(n_i, n_j, n_k)\mathbb{Z}.$$

Explicitly, generators ϕ_i of $\mathbb{Z}/n_i\mathbb{Z}$, ϕ_{ij} of $\mathbb{Z}/(n_i, n_j)\mathbb{Z}$ and ϕ_{ijk} of $\mathbb{Z}/(n_i, n_j, n_k)\mathbb{Z}$ can be defined by

$$\phi_i(a, b, c) = \chi_{n_i}(\omega_{n_i}(a_i, b_i) c_i), \quad \phi_{ij}(a, b, c) = \chi_{n_j}(\omega_{n_i}(a_i, b_i) c_j),$$
$$\phi_{ijk}(a, b, c) = \chi_{(n_i, n_j, n_k)}(a_i b_j c_k).$$

Proof. Recall first how to compute the homology of finite cyclic groups. Consider the group $\mathbb{Z}/n\mathbb{Z}$. Then there is a free resolution (P_*, d) of the $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$-module $\mathbb{Z}$ such that P_k is generated by one basis element e_k, and

$$de_{2k+1} = \lambda_1 e_{2k} - e_{2k} \quad \text{and} \quad de_{2k+2} = \sum_{a\in\mathbb{Z}/n\mathbb{Z}} \lambda_a e_{2k+1} \quad \text{for} \quad k \geq 0.$$

The morphism $P_0 \to \mathbb{Z}$ is given by $e_0 \mapsto 1$. Using this resolution we get

$$H_{2k+1}(\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z} \quad \text{and} \quad H_{2k+2}(\mathbb{Z}/n\mathbb{Z}) = 0 \quad \text{for} \quad k \geq 0.$$

Turning to the proof of the proposition, the first statement is equivalent to

$$H_3(\Gamma) \cong \bigoplus_i \mathbb{Z}/n_i\mathbb{Z} \oplus \bigoplus_{i<j} \mathbb{Z}/(n_i, n_j)\mathbb{Z} \oplus \bigoplus_{i<j<k} \mathbb{Z}/(n_i, n_j, n_k)\mathbb{Z}.$$

This, in turn, is proved by induction on m using the isomorphisms

$$H_1(\Gamma) \cong \Gamma, \quad H_2(\Gamma) \cong \Gamma \bigwedge \Gamma,$$

which are valid for any abelian group Γ, and the Künneth formula, which gives that $H_3(\Gamma \oplus \mathbb{Z}/n\mathbb{Z})$ is isomorphic to

$$H_3(\Gamma) \oplus (H_2(\Gamma) \otimes H_1(\mathbb{Z}/n\mathbb{Z})) \oplus H_3(\mathbb{Z}/n\mathbb{Z}) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_1(\Gamma), H_1(\mathbb{Z}/n\mathbb{Z})).$$

Note only that

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/k\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/(k,n)\mathbb{Z} \cong \mathbb{Z}/k\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}.$$

Let us check next that the functions ϕ_i, ϕ_{ij} and ϕ_{ijk} are indeed 3-cocycles. For ϕ_{ijk} this is clear, since it is a 3-character. Concerning ϕ_i, consider the function

$$f_i(a,b) = \chi_{n_i}\bigg(- \bigg\lfloor \frac{a_i}{n_i} \bigg\rfloor b_i \bigg)$$

on $\mathbb{Z}^m \times \mathbb{Z}^m$. It is of the type described in Lemma A.2 for $\Gamma_1 = \mathbb{Z}^m$ and $\Gamma_0 = \bigoplus_{i=1}^m n_i\mathbb{Z}$, so $\phi_i(a,b,c) = (\partial f_i)(a,b,c)$ is a 3-cocycle on Γ. Similarly, consider the function

$$f_{ij}(a,b) = \chi_{n_j}\bigg(- \bigg\lfloor \frac{a_i}{n_i} \bigg\rfloor b_j \bigg).$$

It is again of the type described in Lemma A.2, so $\phi_{ij} = \partial f_{ij}$ is a 3-cocycle.

Our next goal is to construct a 'dual basis' in $H_3(\Gamma)$. Let u_i be the generator $1 \in \mathbb{Z}/n_i\mathbb{Z} \subset \Gamma$. Denote by θ_{ijk} the cycle representing $u_i \wedge u_j \wedge u_k \in \bigwedge^3 \Gamma \subset H_3(\Gamma)$ obtained by the shuffle product, so

$$\theta_{ijk} = \sum_{\sigma\in S_3} \mathrm{sgn}(\sigma)(1 \otimes [u_{\sigma(i)}|u_{\sigma(j)}|u_{\sigma(k)}]),$$

where we consider S_3 as the group of permutations of $\{i,j,k\}$.

Consider the $\mathbb{Z}[\mathbb{Z}/n_i\mathbb{Z}]$-resolution (P^i_*, d) of $\mathbb{Z}$ described at the beginning of the proof. Let e^i_n be the basis element of P^i_n. We have a chain map $P^i_* \to C_*(\mathbb{Z}/n_i\mathbb{Z})$ of resolutions of $\mathbb{Z}$ defined by

$$e^i_0 \mapsto [\emptyset], \quad e^i_1 \mapsto [1], \quad e^i_2 \mapsto \sum_{a\in\mathbb{Z}/n_i\mathbb{Z}} [a|1], \quad e^i_3 \mapsto \sum_{a\in\mathbb{Z}/n_i\mathbb{Z}} [1|a|1], \dots . \tag{A.1}$$

It follows that we have a 3-cycle $\theta_i \in C_3(\Gamma;\mathbb{Z})$ defined by

$$\theta_i = \sum_{a=0}^{n_i-1} 1 \otimes [u_i|au_i|u_i].$$

Finally, consider the $\mathbb{Z}[\mathbb{Z}/n_i\mathbb{Z}\oplus\mathbb{Z}/n_j\mathbb{Z}]$-resolution $P^i_*\otimes P^j_*$ of $\mathbb{Z}$. Using this resolution we get a third homology class represented by

$$\frac{n_j}{(n_i,n_j)} 1\otimes e^i_2 \otimes e^j_1 + \frac{n_i}{(n_i,n_j)} 1 \otimes e^i_1 \otimes e^j_2.$$

A chain map between the resolutions $P^i_* \otimes P^j_*$ and $C_*(\mathbb{Z}/n_i\mathbb{Z} \oplus \mathbb{Z}/n_j\mathbb{Z})$ can be defined by the tensor product of the chain maps (A.1) and the shuffle product. This gives us a 3-cycle $\theta_{ij} \in C_3(\Gamma;\mathbb{Z})$. Explicitly,

$$\begin{aligned}\theta_{ij} = {} & \frac{n_j}{(n_i,n_j)} \sum_{a=0}^{n_i-1} 1 \otimes ([au_i|u_i|u_j] - [au_i|u_j|u_i] + [u_j|au_i|u_i]) \\ & + \frac{n_i}{(n_i,n_j)} \sum_{b=0}^{n_j-1} 1 \otimes ([u_i|bu_j|u_j] - [bu_j|u_i|u_j] + [bu_j|u_j|u_i]).\end{aligned}$$

The only nontrivial pairings between the cocycles ϕ_i, ϕ_{ij}, ϕ_{ijk} and the cycles θ_i, θ_{ij}, θ_{ijk} are

$$\langle \phi_i, \theta_i \rangle = \zeta_{n_i}, \quad \langle \phi_{ij}, \theta_{ij} \rangle = \zeta_{n_j}^{n_j/(n_i,n_j)} = \zeta_{(n_i,n_j)}, \quad \langle \phi_{ijk}, \theta_{ijk} \rangle = \zeta_{(n_i,n_j,n_k)},$$

where $\zeta_n = e^{2\pi i/n}$. This implies that these cocycles and cycles are the required generators of the Pontryagin dual groups $H^3(\Gamma;\mathbb{T})$ and $H_3(\Gamma)$. □

COROLLARY A.4. *Assume Γ is a finite abelian group. Write Γ as Γ_1/Γ_0 for a finite rank free abelian group Γ_1. Then for any $\mathbb{T}$-valued 3-cocycle ϕ on Γ the following conditions are equivalent*:

(i) *ϕ vanishes on $\bigwedge^3 \Gamma \subset H_3(\Gamma)$;*
(ii) *ϕ lifts to a coboundary on Γ_1;*
(iii) *$\phi = \partial f$ for a function $f\colon \Gamma_1 \times \Gamma_1 \to \mathbb{T}$ as in Lemma* A.2.

Proof. The equivalence of (i) and (ii) is clear, since a cocycle on Γ_1 is a coboundary if and only if it vanishes on $H_3(\Gamma_1) = \bigwedge^3 \Gamma_1$. Also, obviously (iii) implies (ii). Therefore the only nontrivial statement is that (i), or (ii), implies (iii). Assume ϕ is a cocycle that vanishes on $\bigwedge^3 \Gamma \subset H_3(\Gamma)$. We can identify Γ_1 with $\mathbb{Z}^m$ in such a way that $\Gamma_0 = \bigoplus_{i=1}^m n_i\mathbb{Z}$ for some $n_i \geq 1$. Then in the notation of the proof of the above proposition the assumption on ϕ means that ϕ vanishes on the cycles θ_{ijk}, whose homology classes are exactly $u_i \wedge u_j \wedge u_k \in \bigwedge^3 \Gamma \subset H_3(\Gamma)$. It follows that ϕ is cohomologous to product of powers of cocycles ϕ_i and ϕ_{ij}. But the cocycles ϕ_i and ϕ_{ij} are of the form ∂f with $f\colon \Gamma_1 \times \Gamma_1 \to \mathbb{T}$ as in Lemma A.2. Therefore ϕ is cohomologous to a cocycle of the form ∂f, hence ϕ itself is of the same form. □

Since every character of $\bigwedge^3 \Gamma \subset \Gamma^{\otimes 3}$ extends to a 3-character on Γ, this corollary can also be formulated as follows.

Corollary A.5. *With $\Gamma = \Gamma_1/\Gamma_0$ as in the previous corollary, any $\mathbb{T}$-valued* 3*-cocycle ϕ on Γ can be written as product of a* 3*-character χ on Γ and a cocycle ∂f with $f\colon \Gamma_1 \times \Gamma_1 \to \mathbb{T}$ as in Lemma* A.2*. Such a cocycle ϕ lifts to a coboundary on Γ_1 if and only if χ vanishes on $\bigwedge^3 \Gamma \subset \Gamma^{\otimes 3}$, and in this case $\phi = \partial g$ with $g\colon \Gamma_1 \times \Gamma_1 \to \mathbb{T}$ as in Lemma* A.2.

Let us now look more carefully at the construction of cocycles described in Lemma A.2. As Corollary A.4 shows, the class of 3-cocycles obtained by this construction does not depend on the presentation of Γ as quotient of a finite rank free abelian group. It is also clear that there is a lot of redundancy in this construction, since the group $H_3(\Gamma)$ can be much smaller than $\Gamma_0 \otimes \Gamma$. The following proposition makes these observations a bit more precise.

Proposition A.6. *Assume Γ is a finite abelian group, and write Γ as Γ_1/Γ_0 for a finite rank free abelian group Γ_1. Let $f\colon \Gamma_1 \times \Gamma_1 \to \mathbb{T}$ be a function as in Lemma* A.2*, and χ be the associated character of $\Gamma_0 \otimes \Gamma$. Then the cohomology class of ∂f in $H^3(\Gamma; \mathbb{T})$ depends only on the restriction of χ to*

$$\ker(\Gamma_0 \otimes \Gamma \to \Gamma_1 \otimes \Gamma) \cong \mathrm{Tor}_1^{\mathbb{Z}}(\Gamma, \Gamma) \cong \Gamma \otimes \Gamma.$$

Therefore by varying χ we get a natural in Γ homomorphism

$$\mathrm{Hom}(\mathrm{Tor}_1^{\mathbb{Z}}(\Gamma, \Gamma), \mathbb{T}) \to H^3(\Gamma; \mathbb{T}),$$

whose image is the annihilator of $\bigwedge^3 \Gamma \subset H_3(\Gamma)$.

Proof. It is easy to see that the cohomology class of ∂f depends only on χ, so we have a homomorphism $\mathrm{Hom}(\Gamma_0 \otimes \Gamma, \mathbb{T}) \to H^3(\Gamma; \mathbb{T})$. We have to check that if a character χ of $\Gamma_0 \otimes \Gamma$ vanishes on $\ker(\Gamma_0 \otimes \Gamma \to \Gamma_1 \otimes \Gamma)$, then the image of χ in $H^3(\Gamma; \mathbb{T})$ is zero. But this is clear, since we can extend χ to a character f of $\Gamma_1 \otimes \Gamma$, and then f, considered as a function on $\Gamma_1 \times \Gamma_1$, is of the type described in Lemma A.2, with associated character χ, and f is a 2-character, so $\partial f = 0$.

Naturality of the homomorphism $\mathrm{Hom}(\mathrm{Tor}_1^{\mathbb{Z}}(\Gamma,\Gamma),\mathbb{T}) \to H^3(\Gamma;\mathbb{T})$ in Γ is straightforward to check. The statement that its image coincides with the annihilator of $\bigwedge^3\Gamma \subset H_3(\Gamma)$ follows from Corollary A.4. □

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