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Differentials of Cox rings: Jaczewski's theorem revisited

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Dedicated to the memory of Krzysztof Jaczewski (1955–1994)

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Abstract. A generalized Euler sequence over a complete normal variety X is the unique extension of the trivial bundle $V\otimes\mathcal{O}_X$ by the sheaf of differentials Ω_X, given by the inclusion of a linear space $V\subset \mathrm{Ext}^1_X(\mathcal{O}_X,\Omega_X)$. For Λ, a lattice of Cartier divisors, let $\mathcal{R}_\Lambda$ denote the corresponding sheaf associated to V spanned by the first Chern classes of divisors in Λ. We prove that any projective, smooth variety on which the bundle $\mathcal{R}_\Lambda$ splits into a direct sum of line bundles is toric. We describe the bundle $\mathcal{R}_\Lambda$ in terms of the sheaf of differentials on the characteristic space of the Cox ring, provided it is finitely generated. Moreover, we relate the finiteness of the module of sections of $\mathcal{R}_\Lambda$ and of the Cox ring of Λ.

1. Part I, preliminaries.

1.1. Euler sequence.

In the present paper X is a normal irreducible complete complex variety of dimension n with Ω_X denoting the sheaf of Kähler differential forms on X. If $H=\mathrm{H}^1(X,\Omega_X)$ and $H_X = H\otimes\mathcal{O}_X$ then extensions of Ω_X by the trivial sheaf H_X are classified by $\mathrm{Ext}^1(H_X,\Omega_X)=\mathrm{Hom}(H,H)$. A generalized Euler sequence defined by Jaczewski [**Jac94**, Definition 2.1] is the following short exact sequence of sheaves on X which corresponds to the class of identity in $\mathrm{Hom}(H,H)$:

$$0\longrightarrow \Omega_X\longrightarrow \mathcal{R}_X\longrightarrow H_X\longrightarrow 0 \tag{1.1.1}$$

The dual of the sheaf $\mathcal{R}_X$ is called by Jaczewski *the potential sheaf* of X.

More generally, given a non-zero linear subspace $V\subseteq \mathrm{H}^1(X,\Omega_X)$, setting $V_X = V\otimes\mathcal{O}_X$, we get a sequence

$$0\longrightarrow \Omega_X\longrightarrow \mathcal{R}_V\longrightarrow V_X\longrightarrow 0 \tag{1.1.2}$$

associated to the inclusion $V\hookrightarrow \mathrm{H}^1(X,\Omega_X)$.

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1.2. Theorem of Jaczewski.
The following theorem was proved by Jaczewski, [**Jac94**, Theorem 3.1]

Theorem 1.1. *A smooth complete variety X is a toric variety if and only if there exists an effective divisor $D = \bigcup_{\alpha=1}^{r} D_\alpha$ with simple normal crossing components D_α such that*

$$\mathcal{R}_X = \bigoplus_{\alpha=1\dots r} \mathcal{O}_X(-D_\alpha).$$

The divisors D_α are then the closures of the codimension one orbits of the torus action associated to the rays of the fan defining X.

The "only if" part of the above theorem was proved earlier by Batyrev and Melnikov, [**BM86**]. The proof of "if" part given by Jaczewski involved analysing log-differentials $\Omega_X(\log D)$ and reconstructing the torus action on X. A part of the argument is recovering the Lie algebra of the torus from the sequence dual to (1.1.1).

It has been pointed to us by Yuri Prokhorov that Jaczewski's theorem is related to a conjecture of Shokurov, [**Sho00**], and results of McKernan [**McK01**] and Prokhorov [**Pro01**], [**Pro03**], about characterization of toric varieties: see e.g. [**Pro01**, Conjecture 1.1] for a formulation of the problem.

1.3. Atiyah extension and jet bundle.
A particular case of the sequence (1.1.2) is known as the Atiyah extension and it is defined in [**Ati57**]. Namely, let us consider a Cartier divisor D on X, defined on a covering $\mathcal{U} = (U_i)_{i\in I}$ of X by non-zero rational functions $f_i \in \mathbb{C}(X)^*$ satisfying $\mathrm{div}(f_i)_{|U_i} = D_{|U_i}$. In other words D is defined by a Čech cochain $(f_i) \in \mathcal{C}^0(\mathcal{U}, \mathbb{C}(X)^*)$. Its Čech boundary $g_{ij} = f_j/f_i \in \mathcal{O}_X^*(U_i \cap U_j)$, which is a Čech cocycle in $\mathcal{Z}^1(\mathcal{U}, \mathcal{O}_X^*)$, determines the associated invertible sheaf, or line bundle, $\mathcal{O}_X(D)$ (note that the order of indices in our definition of g_{ij} may differ from the one for transition functions in some standard textbooks). Recall that $\mathrm{Pic}\, X = \mathrm{H}^1(X, \mathcal{O}_X^*)$ is the Picard group of invertible sheaves (module isomorphisms).

The log-derivatives $d\log g_{ij} = dg_{ij}/g_{ij} \in \Omega_X(U_i \cap U_j)$ form a cocycle in $\mathcal{Z}^1(\mathcal{U}, \Omega_X)$ which defines (up to the constant coefficient, which we will ignore since it does not change any of our computations) the first Chern class $c_1(D) \in \mathrm{H}^1(X, \Omega_X)$ and also determines the following extension, [**Ati57**, Proposition 12, Theorem 5],

$$0 \longrightarrow \Omega_X \longrightarrow \mathcal{R}_D \longrightarrow \mathcal{O}_X \longrightarrow 0. \tag{1.3.3}$$

The twisted sheaf $\mathcal{R}_D \otimes \mathcal{O}_X(D)$ is well known to be isomorphic to the sheaf of first jets of sections of the line bundle $\mathcal{O}_X(D)$, [**Ati57**, Sections 4 and 5]. Indeed, let h be a rational function such that $s_i = hf_i$ is regular on U_i, for every $i \in I$, hence it determines a section of $\mathcal{O}_X(D)$. Then, because $s_j = g_{ij}s_i$, we have the relation $g_{ij}ds_i + s_i dg_{ij} = ds_j$. Thus the pair (ds_i, s_i) is the associated jet section of $\mathcal{R}_D \otimes \mathcal{O}_X(D)$.

By the construction, the twisted sequence (1.3.3), which now is as follows:

$$0 \longrightarrow \Omega_X \otimes \mathcal{O}(D) \longrightarrow \mathcal{R}_D \otimes \mathcal{O}(D) \longrightarrow \mathcal{O}(D) \longrightarrow 0 \tag{1.3.4}$$

splits as a sequence of sheaves of $\mathbb{C}$-modules (but not $\mathcal{O}$-modules), see [**Ati57**, Section 4], and this yields the exact sequence of global sections:

$$0 \longrightarrow \mathrm{H}^0(X, \Omega_X \otimes \mathcal{O}(D)) \longrightarrow \mathrm{H}^0(X, \mathcal{R}_D \otimes \mathcal{O}(D)) \longrightarrow \mathrm{H}^0(X, \mathcal{O}(D)) \longrightarrow 0. \tag{1.3.5}$$

Jets of sections of $\mathcal{O}_X(D)$ can be seen as differentials on the total space of the dual bundle. In fact, in [**KPSW00**, Section 2.1] it is shown that the sequence (1.3.3) arises from the sequence of sheaves of differentials associated to the projection of an associated $\mathbb{C}^*$ bundle to X. This observation will be elaborated later in the second part of the paper.

1.4. Cox rings.

From now on we assume that the variety X is projective. Let $\Lambda \subset \mathrm{CDiv}(X)$ be a finitely generated group of Cartier divisors. We will assume that Λ is free of rank r, so that $\Lambda \simeq \mathbb{Z}^r$. The elements of Λ will be denoted by λ or by D_λ, if we want to underline that they are divisors. In fact, in the course of our arguments we will fix a covering $\mathcal{U} = (U_i)_{i\in I}$ and represent in this covering the generators D_λ of Λ by rational functions $f_i^\lambda \in \mathbb{C}(X)^*$, which will imply a presentation of Λ as a subgroup of Čech cochains $\mathcal{C}^0(\mathcal{U}, \mathbb{C}(X)^*)$.

We assume that the first Chern class map $c_1 : \Lambda \longrightarrow \mathrm{H}^1(X, \Omega_X)$, defined as $\Lambda \ni D_\lambda \mapsto c_1(\mathcal{O}(D_\lambda)) \in \mathrm{H}^1(X, \Omega_X)$, is an injection so, by abuse, we will identify Λ with a lattice in $\mathrm{H}^1(X, \Omega_X)$. We define the following objects related to Λ:

- the subspace in cohomology $\Lambda_{\mathbb{C}} = \Lambda \otimes \mathbb{C} \subseteq \mathrm{H}^1(X, \Omega_X)$,
- the sheaves of $\mathcal{O}_X$-modules $\Lambda_X = \Lambda_{\mathbb{C}} \otimes \mathcal{O}_X$ and $\mathcal{R}_\Lambda$ arising as the middle term in the sequence (1.1.2) for $V = \Lambda_{\mathbb{C}}$, so that we have the sequence

$$0 \longrightarrow \Omega_X \longrightarrow \mathcal{R}_\Lambda \longrightarrow \Lambda_X \longrightarrow 0, \tag{1.4.6}$$

- the algebraic torus $\mathbb{T}_\Lambda = \mathrm{Hom}_{alg}(\Lambda, \mathbb{C}^*)$.

The torus $\mathbb{T}_\Lambda$ acts on the graded ring of rational functions $\mathcal{S}_\Lambda = \bigoplus_{\lambda\in\Lambda} S^\lambda$ where

$$S^\lambda = \Gamma(X, \mathcal{O}_X(D_\lambda)) = \{f \in \mathbb{C}(X)^* : \mathrm{div}(f) + D_\lambda \geq 0\} \cup \{0\} \tag{1.4.7}$$

and the multiplication is as in the field of rational functions $\mathbb{C}(X)$. We call $\mathcal{S}_\Lambda$ the Cox ring of the lattice Λ. We note that, if $\mathbb{Z}^r \simeq \Lambda' \subset \mathrm{CDiv}(X)$ is such that the classes of divisors from Λ and Λ' define the same subgroup of $\mathrm{Pic}\, X$ then we have the natural isomorphism $\mathcal{S}_\Lambda \simeq \mathcal{S}_{\Lambda'}$. Thus, given a $\Lambda \simeq \mathbb{Z}^r \subset \mathrm{Pic}\, X$, by $\mathcal{S}_\Lambda$ we will also denote the ring constructed by lifting Λ to some subgroup in $\mathrm{CDiv}\, X$.

The graded pieces S^λ may be zero for some $\lambda \in \Lambda$. In fact, we define a cone of effective divisors $\overline{\mathrm{Eff}}_\Lambda \subset \Lambda_{\mathbb{R}}$ which is the closure of the cone spanned by λ's with $S^\lambda \neq 0$. The cone $\overline{\mathrm{Eff}}_\Lambda$ is convex and, because X is projective, it is pointed which means that it contains no non-trivial linear space. Equivalently, there exists a linear form $\kappa : \Lambda \to \mathbb{Z}$ such that $\kappa_{|\overline{\mathrm{Eff}}_\Lambda} \geq 0$ and $\kappa(\lambda) \geq 1$ for every non-zero $\lambda \in \overline{\mathrm{Eff}}_\Lambda \cap \Lambda$ by [**BFJ09**, Proposition 1.3]. We define $\mathcal{S}_\Lambda^+ = \bigoplus_{\kappa(\lambda)>0} S^\lambda$ which is a maximal ideal in $\mathcal{S}_\Lambda$.

Given a (quasi)coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules by $\mathcal{F}(D_\lambda)$, we denote its twist $\mathcal{F} \otimes \mathcal{O}(D_\lambda)$. Next we define Λ-graded $\mathcal{S}_\Lambda$-module of sections

$$\Gamma_\Lambda(\mathcal{F}) = \bigoplus_{\lambda \in \Lambda} \mathrm{H}^0(X, \mathcal{F}(D_\lambda)). \tag{1.4.8}$$

We will skip the subscript and write $\Gamma(\mathcal{F})$ if the choice of Λ is clear from the context.

The multiplication by elements of $\mathcal{S}_\Lambda$ is defined by the standard isomorphism $\mathcal{F}(D_{\lambda_1}) \otimes \mathcal{O}(D_{\lambda_2}) \simeq \mathcal{F}(D_{\lambda_1+\lambda_2})$, see [**Har77**, Section II.5]. Clearly, Γ_Λ extends to a functor from category of (quasi) coherent sheaves on X to category of graded $\mathcal{S}_\Lambda$ modules. The functor Γ_Λ is left exact.

For the proof of Theorem 2.8 we need the following observation.

Lemma 1.2. *In the above situation, let D be a divisor on X, which is linearly equivalent to D_{λ_0} for some $\lambda_0 \in \Lambda$. Then $\Gamma_\Lambda(\mathcal{O}(D))$ is a free $\mathcal{S}_\Lambda$-module of rank* 1 *which is isomorphic to $\mathcal{S}_\Lambda[\lambda_0]$, where the square bracket denotes the shift in grading.*

1.5. Characteristic spaces of Cox rings.

The theory of Cox rings, or total coordinate rings, is pretty well understood, see for example [**Cox95b**], [**Cox95a**], [**HK00**], [**BH07**] or [**AH09**]. The reader not familiar with this subject may want to consult [**LV09**] for a concise review or [**ADHL10**] for an exhaustive overview of the subject.

We note that in the present paper the starting set-up is slightly different from that of [**ADHL10**]. In our situation the rings are associated to a choice of Λ, hence our set up is similar to this of Zariski, [**Zar62**, Section 4–5]. In [**ADHL10**] the Cox rings are built on the whole divisor class group $\mathrm{Cl}\, X$ which is assumed to be finitely generated. To distinguish the situation of [**ADHL10**] from our set-up, we will call the ring from that paper by *total coordinate rings*. We note that the construction in [**ADHL10**, Section I.4.2] allows to define the Cox ring over a torsion subgroup of $\mathrm{Cl}\, X$. We will use this construction in the proof of 2.9.

The following condition for the pair (X, Λ) is very convenient:

Assumption 1.3. Ampleness of Λ and finite generation of $\mathcal{S}_\Lambda$:

1. Λ contains an ample divisor,
2. the ring $\mathcal{S}_\Lambda$ is a finitely generated $\mathbb{C}$-algebra.

We note that the above condition is true if either Λ is generated by an ample divisor or X is a Mori Dream Space (for example, a $\mathbb{Q}$-factorial Fano variety, see [**BCHM10**, 1.3.2]) and Λ generates the Picard group of X.

Under the above assumptions $Y_\Lambda = \mathrm{Spec}\, \mathcal{S}_\Lambda$ is a well defined affine variety over $\mathbb{C}$ with the action of the torus $\mathbb{T}_\Lambda$. Moreover, the ideal $\mathcal{S}_\Lambda^+$ defines the unique $\mathbb{T}_\Lambda$ invariant closed point in Y_Λ.

If $\mathcal{S}_\Lambda$ is the total coordinate ring, then Y_Λ is normal, [**ADHL10**, Section I.5.1].

The relation of Y_Λ with X can be understood via the GIT theory, see e.g. [**HK00**, Chapter 2]. A choice of an ample divisor $D_a \in \Lambda$ determines a character of $\mathbb{T}_\Lambda$ and thus the set of semistable points of this action $\widehat{Y}_\Lambda \subseteq Y_\Lambda$, such that $Y_\Lambda \setminus \widehat{Y}_\Lambda$ is of codimension 2 at least. Moreover X is a geometric quotient of $\widehat{Y}_\Lambda$ by $\pi_\Lambda : \widehat{Y}_\Lambda \longrightarrow X$.

A slightly different view is presented in [**ADHL10**, Chapter I] and we will follow this approach. As in [**ADHL10**, I.6.1], the variety $\widehat{Y}_\Lambda$, now called the characteristic space associated to $\mathcal{S}_\Lambda$, is constructed as the relative spectrum $\operatorname{Spec}_X \widehat{\mathcal{S}}_\Lambda$ of the graded sheaf of finitely generated $\mathcal{O}_X$-algebras $\widehat{\mathcal{S}}_\Lambda = \bigoplus_{\lambda\in\Lambda} \widehat{S}^\lambda$ where $\widehat{S}^\lambda = \mathcal{O}_X(D_\lambda)$. Following [**ADHL10**], the sheaf $\widehat{\mathcal{S}}_\Lambda$ will be called the Cox sheaf of the lattice Λ and $\mathcal{S}_\Lambda$ is the algebra of its global sections. The evaluation of global sections yields a map $\iota : \widehat{Y}_\Lambda \longrightarrow Y_\Lambda$ which is an embedding onto an open subset whose complement is of codimension ≥ 2, see [**ADHL10**, I.6.3]. On the other hand the inclusion $\mathcal{O}_X = \widehat{S}^0 \hookrightarrow \widehat{\mathcal{S}}_\Lambda$ yields the map $\pi : \widehat{Y}_\Lambda \longrightarrow X$ which is a geometric quotient of the $\mathbb{T}_\Lambda$ action. In fact, since Λ consists of locally principal divisors, the map π is a locally trivial, principal $\mathbb{T}_\Lambda$-bundle, [**ADHL10**, I.3.2.7].

The ideal defining the closed set $Y_\Lambda \setminus \widehat{Y}_\Lambda$ is called the irrelevant ideal and it is the radical of an ideal generated by sections of a very ample line bundle on X, [**ADHL10**, I.6.3].

1.6. Cox rings of toric varieties.

The following theorem was partly established by Cox ("only if" part, [**Cox95b**]), Hu and Keel ("if" part, smooth case [**HK00**, Corollary 2.9]) and, eventually, by Berchtold and Hausen, [**BH07**, Corollary 4.4] who proved it in a stronger form than the one below. We recall that a variety X satisfies Włodarczyk's A_2 property if any two points of X are contained in common open affine neighbourhood, see [**Wło93**].

Theorem 1.4. *Let X be a complete normal variety which has A_2 property and has finitely generated free divisor class group. Then X is a toric variety if and only if its total coordinate ring is a polynomial ring.*

We will need a slight variation of the above result.

Theorem 1.5. *Let X be a projective normal variety with a lattice $\Lambda \subset \operatorname{Pic} X$ containing a class of an ample divisor. If $\mathcal{S}_\Lambda$ is a polynomial ring, then X is a smooth toric variety and $\Lambda = \operatorname{Pic} X = \operatorname{Cl} X$, hence $\mathcal{S}_\Lambda$ is the total coordinate ring of X.*

The proof of the first part of this theorem, that is of toricness of X, goes along the lines of the proof of [**HK00**, 2.10]. The reference regarding the linearization of the group action was suggested to us by Jürgen Hausen. The main statement is that $\mathcal{S}_\Lambda$ is the total coordinate ring of X; its short proof below, which replaced our original argument, is due to the referee.

Proof. As usually, by n we denote the dimension of X and by r the rank of the lattice Λ. Thus, in the situation of the theorem, $\mathcal{S}_\Lambda$ is the polynomial ring in $n+r$ variables. By the linearization theorem [**KP85**, 5.1] the action of $\mathbb{T}_\Lambda$ on the affine space $Y = \operatorname{Spec} \mathcal{S}_\Lambda = \mathbb{A}^{r+n}_{\mathbb{C}}$ is linear. This means that $\mathcal{S}_\Lambda = \mathbb{C}[y_1, \ldots, y_{n+r}]$ and y_i's are eigenvalues of the action of $\mathbb{T}_\Lambda$. Equivalently, if $\widehat{M}$ is the lattice of characters of the standard torus acting on $Y = \mathbb{A}^{n+r}_{\mathbb{C}}$ then we have a homomorphism $\widehat{M} \to \Lambda$ which we may assume surjective (possibly replacing the lattice Λ by the image of $\widehat{M}$; this does not change $\mathcal{S}_\Lambda$ but now the action of $\mathbb{T}_\Lambda$ is faithful). If $(\widehat{e}_1, \dots, \widehat{e}_{n+r})$ is the basis of $\widehat{M}$ associated to the coordinates $(y_1, \dots, y_{n+r})$ then $\widehat{e}_i$ is mapped to the character $\lambda_i \in \Lambda$ with which $\mathbb{T}_\Lambda$ acts on y_i.

By [**Świ99**, 2.4 and 2.5] the characteristic space $\widehat{Y}_\Lambda$ is a toric variety and the quotient of $\widehat{Y}$ by the action of $\mathbb{T}_\Lambda$, which is X, is a toric variety again. The kernel of the homomorphism $\widehat{M} \to \Lambda$ is the lattice of characters of the torus acting on the quotient X; we denote this lattice by M and the respective big torus of X by $\mathbb{T}_M$. Moreover, by [**Świ99**, Theorem 4.1], the morphism $\widehat{Y} \to X$ is surjective and toric, and the set of rays in the fan defining $\widehat{Y}$ maps surjectively onto the set of rays in the fan of X. Thus, the (torsion-free) rank of $\operatorname{Cl} X$ (which may have torsion), which is equal to the number of rays in the fan of X minus its dimension, is equal r at most. On the other hand, since $\Lambda \subseteq \operatorname{Pic} X \subseteq \operatorname{Cl} X$, it follows that $r = \operatorname{rank} \Lambda = \operatorname{rank} \operatorname{Cl} X$.

The total coordinate ring $\mathcal{S}_{tot}$ of X is the polynomial ring $\mathbb{C}[x_1, \dots, x_{n+r}]$ with variables x_i associated to the rays in the fan of X or, equivalently, to $\mathbb{T}_M$-invariant divisors D_i on X; the ring $\mathcal{S}_{tot}$ has grading in $\operatorname{Cl} X$ (see [**CLS11**, Section 5.1] and also [**ADHL10**, Section I.4.2] for a thorough explanation regarding the case when $\operatorname{Cl} X$ has torsion). By $\widetilde{M}$ let us denote the lattice of characters of the standard big torus $\mathbb{T}_{\widetilde{M}}$ acting on $\mathbb{C}[x_1, \dots, x_{n+r}]$. The elements of $\widetilde{M}$ can be interpreted as $\mathbb{T}_M$-invariant (Weil) divisors on X. We have the surjective homomorphism $\widetilde{M} \to \operatorname{Cl} X$ which gives the class group grading in $\mathcal{S}_{tot}$. Thus, we can use the inclusion $\Lambda \subseteq \operatorname{Cl} X$ to lift-up a basis of Λ to $\mathbb{T}_M$-invariant divisors in $\operatorname{CDiv} X$ and use them to define the multiplicative structure on $\mathcal{S}_\Lambda$. Therefore, the inclusion $\Lambda \subseteq \operatorname{Cl} X$ determines the natural inclusion $\mathcal{S}_\Lambda \hookrightarrow \mathcal{S}_{tot}$.

The quotient group $G = \operatorname{Cl} X / \Lambda$ is finite and it gives the grading on $\mathcal{S}_{tot}$ as well. Dually, if $\mathbb{T}_{\operatorname{Cl} X} = \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Cl} X, \mathbb{C}^*)$ is the quasi-torus associated to $\operatorname{Cl} X$ and $G^\vee = \operatorname{Hom}_{\mathbb{Z}}(G, \mathbb{C}^*) \subset \mathbb{T}_{\operatorname{Cl} X}$ the group of characters of G, then both groups act on $\mathcal{S}_{tot}$ with weights determined by the grading. The subring $\mathcal{S}_\Lambda \subseteq \mathcal{S}_{tot}$ consists of elements with trivial G-grading hence it is equal to the invariant subring $\mathcal{S}_{tot}^{G^\vee} \subseteq \mathcal{S}_{tot}$.

Geometrically, the group $G^\vee$ (which is isomorphic to G) acts on $\operatorname{Spec} \mathcal{S}_{tot} = \mathbb{A}^{r+n}_{\mathbb{C}}$ and the inclusion $\mathcal{S}_\Lambda \subseteq \mathcal{S}_{tot}$ yields the finite quotient map $\pi : \mathbb{A}^{n+r}_{\mathbb{C}} \to \mathbb{A}^{n+r}_{\mathbb{C}}$ with the Galois group $G^\vee$. By [**Hau08**, Proposition 2.2, (iii)] there exists an open subset of $\operatorname{Spec} \mathcal{S}_{tot}$ on which $\mathbb{T}_{\operatorname{Cl} X}$ acts freely and whose complement has codimension at least two. Thus the action of $G^\vee$ on $\operatorname{Spec} \mathcal{S}_{tot}$ is free in codimension 1. This implies that the map $\pi : \operatorname{Spec} \mathcal{S}_{tot} \to \operatorname{Spec} \mathcal{S}_\Lambda$ is unramified covering in codimension 1, and consequently G is trivial. This follows since $\operatorname{Spec} \mathcal{S}_\Lambda = \mathbb{A}^{n+r}_{\mathbb{C}}$ will remain simply connected after removing the set of branching points of the map π, which is of codimension ≥ 2.

Therefore $\Lambda = \operatorname{Pic} X = \operatorname{Cl} X$ which yields that X is smooth, see [**CLS11**, 4.2.6], and $\mathcal{S}_\Lambda = \mathcal{S}_{tot}$. □

Although Theorems 1.1 and 1.4 give characterisation of toricness in terms of apparently different objects, they turn out to be closely related. We will discuss this issue in the second part of the paper.

1.7. Differential forms and splitting of tangent bundle.

We will use the standard definition of the module and the sheaf of Kähler $\mathbb{C}$ differentials, as in [**Mat80**, Section 26], which we will denote by Ω_A and Ω_X, respectively. The double dual, $\Omega_A^{\vee\vee}$ or $\Omega_X^{\vee\vee}$ will be called, respectively, the module, or the sheaf of Zariski or reflexive differentials, see e.g. [**Kni73**, Section 1]. We note that, being reflexive, over normal varieties the sheaf of Zariski differentials has extension property which means that all its sections are determined uniquely on complements of codimension ≥ 2 sets, [**Har80**]. That is, if Y is normal and $\iota : \widehat{Y} \hookrightarrow Y$ is an open subset such that $Y \setminus \widehat{Y}$ is of codimension ≥ 2, then $\iota_* \Omega_{\widehat{Y}}^{\vee\vee} = \Omega_Y^{\vee\vee}$.

Let us note that the theorem of Jaczewski or, more generally, any theorem about splitting of the sheaf $\mathcal{R}_\Lambda$, see 2.8, can be considered in the context of splitting of the sheaf of Kähler differentials, Ω_X (just take $\Lambda = 0$). This question for complex Kähler manifolds has been considered by several authors: [**Bea00**], [**Dru00**], [**CP02**], [**BPT06**] and [**Hör07**], [**Hör08**]. For non-Kähler manifolds, a Hopf manifold of dimension ≥ 2, provides an example of a complex manifold whose tangent bundle (the dual of the sheaf of Kähler differentials) splits into a sum of line bundles, see [**Bea00**, Example 2.1].

2. Part II, results.

2.1. The generalized Euler extension.

In the present section we want to describe the generalized Euler sequence (1.1.2) in terms of Čech data. We fix an affine covering $\mathcal{U} = (U_i)_{i \in I}$ of X in which all divisors in Λ can be represented as principal divisors. That is, for every $D_\lambda \in \Lambda \subset \mathrm{CDiv}(X)$ and $i \in I$ we can choose a rational function $f_i^\lambda \in \mathbb{C}(X)^*$ such that $D_{\lambda|U_i} = \mathrm{div}(f_i^\lambda)$. We define also $g_{ij}^\lambda = f_j^\lambda / f_i^\lambda \in \mathcal{O}_X^*(U_i \cap U_j)$. The first Chern class $c_1(\mathcal{O}(D))$ is represented (up to the multiplicative constant) in the covering $\mathcal{U}$ by the Čech cocycle $(d \log g_{ij}^\lambda) \in \mathcal{Z}^1(\mathcal{U}, \Omega_X)$. The choice of f_i's and hence of g_{ij}'s is unique up to functions from $\mathcal{O}_X^*(U_i)$.

Thus, in order to avoid ambiguity we fix a basis of the lattice Λ and f_i's for the basis elements. This determines the choice f_i^λ for every $D_\lambda \in \Lambda$ so that we have a group homomorphism of Λ into Čech 0-cochains of rational functions

$$\Lambda \ni \lambda \mapsto (f_i^\lambda) \in \mathcal{C}^0(\mathcal{U}, \mathbb{C}(X)^*). \tag{2.1.9}$$

As the result, we get a homomorphism into Čech cocycles

$$\Lambda \ni \lambda \mapsto (g_{ij}^\lambda) \in \mathcal{Z}^1(\mathcal{U}, \mathcal{O}_X^*). \tag{2.1.10}$$

The map $c_1 : \Lambda \longrightarrow \mathrm{H}^1(X, \Omega_X)$ is represented in the covering $\mathcal{U}$ by $\psi_{ij} \in \mathrm{Hom}(\Lambda, \Omega_X(U_i \cap U_j))$ such that

$$\psi_{ij}(\lambda) = \psi_{ij}(D_\lambda) = d \log g_{ij}^\lambda = d \log f_j^\lambda - d \log f_i^\lambda.$$

Sections of the sheaf $\mathcal{R}_\Lambda$ over an open $U \subset X$ come by glueing $(\omega_i, \lambda_i) \in \Omega_X(U_i \cap U) \oplus \Lambda_X(U_i \cap U)$, with $(\omega_j, \lambda_j) \in \Omega_X(U_j \cap U) \oplus \Lambda_X(U_j \cap U)$ and $\omega_j = \omega_i + \widetilde{\psi_{ij}}(\lambda_i)$, where the restriction of ω's and λ's to $U_i \cap U_j \cap U$ is denoted by the same letter, c.f. [**Ati57**, Section

4], and $\widetilde{\psi_{ij}}$ is an extension of ψ_{ij} to a homomorphism $\Lambda_X(U\cap U_i\cap U_j) \to \Omega_X(U\cap U_i\cap U_j)$.

2.2. Differentials of the Cox sheaf.

Now, for a nonzero rational function $h \in S^\lambda = \mathrm{H}^0(X, \mathcal{O}_X(D_\lambda))$ we consider its local regular presentation in covering $\mathcal{U}$ given by

$$s_i = hf_i^\lambda \in \mathcal{O}_X(U_i). \tag{2.2.11}$$

We will write $s := (s_i)_{i\in I}$. We claim that the pairs $(ds_i, s_i \cdot D_\lambda) \in \Omega_X(U_i) \oplus \Lambda_X(U_i)$ determine a section of $\mathcal{R}_\Lambda(D_\lambda) = \mathcal{R}_\Lambda \otimes \mathcal{O}(D_\lambda)$ which we call ds. Here we use the convention that $(ds_i, s_i \cdot D_\lambda)$ is identified with $(ds_i \otimes 1, D_\lambda \otimes s_i)$ as a section of $\Omega_X(D_\lambda) \oplus \Lambda_X(D_\lambda)$ over U_i. Indeed, since over the set $U_i \cap U_j$ we have $s_j = g_{ij}^\lambda s_i$, it follows that

$$ds_j = g_{ij}^\lambda \cdot ds_i + s_i \cdot dg_{ij}^\lambda = g_{ij}^\lambda \cdot (ds_i + s_i \cdot d\log g_{ij}^\lambda) = g_{ij}^\lambda \cdot (ds_i + s_i \cdot \psi_{ij}(D_\lambda)) \tag{2.2.12}$$

hence we get the statement. The map $s \mapsto ds$ is $\mathbb{C}$ linear and it satisfies the Leibniz rule. Indeed, for $s \in S^\lambda$ and $s' \in S^{\lambda'}$, we verify that

$$d(s\cdot s') = (d(s_i\cdot s_i'), s_i\cdot s_i'\cdot(D_\lambda + D_{\lambda'})) = s_i(ds_i', s_i'\cdot D_{\lambda'}) + s_i'(ds_i, s_i\cdot D_\lambda) = s\cdot ds' + s'\cdot ds.$$

Thus the map

$$\begin{aligned} \mathcal{S}_\Lambda \supset S^\lambda = \mathrm{H}^0(X, \mathcal{O}_X(D_\lambda)) \ni s \\ \longrightarrow ds = (ds_i, s_i \cdot D_\lambda) \in \mathrm{H}^0(X, \mathcal{R}_\Lambda(D_\lambda)) \subset \Gamma(\mathcal{R}_\Lambda) \end{aligned} \tag{2.2.13}$$

is a $\mathbb{C}$-derivation of the Cox ring $\mathcal{S}_\Lambda$.

It is clear that the above construction of the map $s \to ds$ works also locally, for sections over an arbitrary open subset of X, hence it gives a map $d : \mathcal{O}_X(D_\lambda) \to \mathcal{R}(D_\lambda)$. Actually, the $\mathbb{C}$-linear map d can be described in relation to the jet bundles, which were presented in Section 1.3. The first Chern class of D_λ gives the following diagram with exact rows of left-to-right homomorphisms of sheaves of $\mathcal{O}_X$-modules

$$\begin{tikzcd} 0 \arrow[r] & \Omega_X(D_\lambda) \arrow[r] \arrow[d, equal] & \mathcal{R}_{D_\lambda}(D_\lambda) \arrow[r] \arrow[d] & \mathcal{O}_X(D_\lambda) \arrow[r] \arrow[l, bend right] \arrow[dl, "d"] \arrow[d, "c_1(D_\lambda)\otimes id"] & 0 \\ 0 \arrow[r] & \Omega_X(D_\lambda) \arrow[r] & \mathcal{R}_\Lambda(D_\lambda) \arrow[r] & \Lambda \otimes \mathcal{O}_X(D_\lambda) \arrow[r] & 0. \end{tikzcd} \tag{2.2.14}$$

As discussed in Section 1.3 the upper row of this sequence splits as a sequence of $\mathbb{C}$-modules and the map d is the composition of the splitting map with the map of extensions.

The map d is a $\mathbb{C}$-derivation of the sheaf $\widehat{S}_\Lambda$. That is, for every open $U \subseteq X$, any section $s \in \widehat{S}^\lambda(U)$ yields $ds \in (\mathcal{R}_\Lambda \otimes \widehat{S}^\lambda)(U)$ and the map $s \mapsto ds$ is a derivation. Verification of the Leibniz identity is the same as in (2.2.12). Thus, we get the homomorphism of Λ-graded quasi-coherent sheaves of $\mathcal{O}_X$-modules $\widehat{\varphi} : \pi_*\Omega_{\widehat{Y}} \longrightarrow \mathcal{R}_\Lambda \otimes_{\mathcal{O}_X} \widehat{\mathcal{S}}_\Lambda$.

Lemma 2.1. *Suppose that we are in the set-up of Section 1.5 so that, in particular,* *the pair (X,Λ) satisfies the Assumption* 1.3 *and the characteristic space $\widehat{Y}_\Lambda$ with the projection $\pi : \widehat{Y}_\Lambda \to X$ is well defined. Then the map $\widehat{\varphi}$ fits into the following commutative diagram of Λ graded sheaves of $\mathcal{O}_X$-modules whose rows are exact and vertical arrows are isomorphisms*:

$$\begin{tikzcd}
0 \arrow[r] & \pi_*\pi^*\Omega_X \arrow[r] \arrow[d] & \pi_*\Omega_{\widehat{Y}_\Lambda} \arrow[r] \arrow[d, "\widehat{\varphi}"] & \pi_*\Omega_{\widehat{Y}_\Lambda/X} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega_X \otimes_{\mathcal{O}_X} \widehat{\mathcal{S}}_\Lambda \arrow[r] & \mathcal{R}_\Lambda \otimes_{\mathcal{O}_X} \widehat{\mathcal{S}}_\Lambda \arrow[r] & \Lambda_X \otimes_{\mathcal{O}_X} \widehat{\mathcal{S}}_\Lambda \arrow[r] & 0.
\end{tikzcd} \tag{2.2.15}$$

The lower row of the above diagram comes by extending the coefficients in (1.1.2) *while the upper sequence is the relative cotangent sequence for π.*

PROOF. Let us recall that $\pi : \widehat{Y} \longrightarrow X$ is a locally trivial principal $\mathbb{T}_\Lambda$-bundle. That is, for $U_i \subseteq X$ trivializing all divisors in Λ there is an identification of $\mathcal{O}_X(U_i)$-modules $\widehat{S}^\lambda(U_i) = (f_i^\lambda)^{-1} \cdot \mathcal{O}_X(U_i)$, as submodules of $\mathbb{C}(X)(U_i)$. Thus, having in mind homomorphism (2.1.9), we get isomorphism of $\mathcal{O}_X(U_i)$-algebras $\widehat{\mathcal{S}}_\Lambda(U_i) \simeq \mathcal{O}_X(U_i)[\Lambda] = \mathcal{O}_X(U_i) \otimes \mathbb{C}[\Lambda]$. Therefore, by the formula for tensor products [**Eis95**, 16.5], we get the canonical isomorphism

$$\Omega_{\widehat{\mathcal{S}}_\Lambda(U_i)} = \Omega_X(U_i) \otimes_{\mathcal{O}_X(U_i)} \widehat{\mathcal{S}}_\Lambda(U_i) \oplus \Omega_{\mathbb{C}[\Lambda]} \otimes_{\mathbb{C}[\Lambda]} \widehat{\mathcal{S}}_\Lambda(U_i)$$

which splits the upper row over U_i. This splitting coincides with the splitting of the lower row over U_i once we note the standard isomorphism $\Omega_{\mathbb{C}[\Lambda]} = \mathbb{C}[\Lambda] \otimes_{\mathbb{Z}} \Lambda$, which sends the derivative of the monomial $t^\lambda \in \mathbb{C}[\Lambda]$ to $t^\lambda \otimes \lambda$, see [**CLS11**, Chapter 8]. Moreover, under this identification the arrows in the upper row are identical with those in the lower row, c.f. (2.2.11). Thus the diagram commutes and all vertical arrows are now defined globally, which concludes the proof of Lemma 2.1. $\square$

The above lemma is a generalization of [**KPSW00**, 2.1]. In particular we get the following corollary

COROLLARY 2.2. *Suppose that we are in the situation of Lemma* 2.1. *Then the generalized Euler sequence* (1.1.2) *associated to $\Lambda_{\mathbb{C}}$ is the $\mathbb{T}_\Lambda$-invariant part* (*zero grading with respect to* Λ) *of the exact sequence of sheaves of differentials associated to the map* π :

$$0 \longrightarrow \pi^*\Omega_X \longrightarrow \Omega_{\widehat{Y}_\Lambda} \longrightarrow \Omega_{\widehat{Y}_\Lambda/X} \longrightarrow 0. \tag{2.2.16}$$

The next corollary follows immediately by the extension property of Zariski differentials, which was explained in Section 1.7. We recall that the Cox ring $\mathcal{S}_\Lambda$ is integrally closed by [**EKW04**, Theorem 1.1].

COROLLARY 2.3. *Suppose that we are in the situation of Lemma* 2.1 *and assume that the sheaf of Kähler differentials on X is reflexive. Then the $\mathcal{S}_\Lambda$-module $\Gamma(\mathcal{R}_\Lambda)$* (*see*

1.4.8) *is isomorphic to the module of Zariski differentials of the Cox ring* $\mathcal{S}_\Lambda$:

$$\Gamma(\mathcal{R}_\Lambda) \simeq \Omega^{\vee\vee}_{\mathcal{S}_\Lambda}.$$

Clearly, the sheaf $\mathcal{R}_\Lambda$ will not change if Λ is replaced by another lattice whose $\mathbb{C}$ linear span is the same as Λ. That is, if $\Lambda' \subset \Lambda$ is a sublattice of finite index, then $\mathcal{R}_{\Lambda'} = \mathcal{R}_\Lambda$ and thus $\Gamma_{\Lambda'}(\mathcal{R}_{\Lambda'})$ is Λ' graded part of $\Gamma_\Lambda(\mathcal{R}_\Lambda)$. Thus, if we are in the situation of Corollary 2.3, this gives a clear description of Zariski differentials on $\operatorname{Spec}\mathcal{S}_{\Lambda'}$ in terms of Zariski differentials over $\operatorname{Spec}\mathcal{S}_\Lambda$. On the other hand, the behaviour of Kähler differentials is much more intricate. This problem was studied in a special situation of $X = \mathbb{P}^n$ in [**GR10**].

2.3. Generation of the Cox ring.

Let us apply the functor Γ defined in Section 1.4 to the generalized Euler sequence (1.4.6). The result is the following sequence of Λ-graded $\mathcal{S}_\Lambda$-modules

$$0 \longrightarrow \Gamma(\Omega_X) \longrightarrow \Gamma(\mathcal{R}_\Lambda) \longrightarrow \Gamma(\Lambda_X) = \Lambda \otimes_\mathbb{Z} \mathcal{S}_\Lambda.$$

We compose the right hand arrow in the above sequence with $\kappa \otimes id : \Lambda \otimes \mathcal{S}_\Lambda \to \mathcal{S}_\Lambda$ where κ is the map defined in Section 1.4. We call the resulting homomorphism $\widehat{\kappa} : \Gamma(\mathcal{R}_\Lambda) \to \mathcal{S}_\Lambda$. We note that both, $\kappa \otimes id$ and $\widehat{\kappa}$, are homomorphism of Λ-graded $\mathcal{S}$-modules. Indeed, the homomorphism $\kappa \otimes id : \Lambda \otimes \mathcal{S}_\Lambda \to \mathbb{Z} \otimes \mathcal{S}_\Lambda$ is graded with respect to the second factor, which means that $\kappa : \Lambda \to \mathbb{Z}$ is trivially Λ-graded. Equivalently, $\widehat{\kappa}$ can be defined by applying the functor Γ_Λ to the composition of homomorphisms of sheaves $\mathcal{R}_\Lambda \to \Lambda \otimes \mathcal{O}_X \to \mathcal{O}_X$, with the left-hand-side arrow coming from the Euler sequence (1.4.6) and the right-hand-side arrow being $\kappa \otimes id$.

Lemma 2.4. *The homomorphism* $\widehat{\kappa}$ *is surjective onto* $\mathcal{S}^+_\Lambda$.

Proof. First we note that the image of $\widehat{\kappa}$ is contained in $\mathcal{S}^+_\Lambda$. This is because the map $\mathrm{H}^0(X, \mathcal{R}_\Lambda) \to \mathrm{H}^0(X, \Lambda_X)$ is zero. Next we use (2.2.14) to get the following commutative diagram

$$\begin{tikzcd}
\mathrm{H}^0(X, \mathcal{R}_{D_\lambda}(D_\lambda)) \arrow[r] \arrow[d, shift right] & \mathrm{H}^0(X, \mathcal{R}_\Lambda(D_\lambda)) \arrow[d] \arrow[dr, "\widehat{\kappa}"] & \\
\mathrm{H}^0(X, \mathcal{O}(D_\lambda)) \arrow[r, "c_1(D_\lambda)\otimes id"'] \arrow[u, shift right] & \Lambda \otimes \mathrm{H}^0(X, \mathcal{O}(D_\lambda)) \arrow[r, "\kappa\otimes id"'] & \mathrm{H}^0(X, \mathcal{O}(D_\lambda)).
\end{tikzcd} \tag{2.3.17}$$

The composition of the homomorphisms in the lower row is an isomorphism because for any effective non-zero divisor $\kappa(c_1(D_\lambda)) \neq 0$. Hence $\widehat{\kappa}$ is onto $\mathcal{S}^+_\Lambda$. ☐

We recall the following observation, which is classical and probably known since Hilbert's time.

Lemma 2.5. *Let* $\mathcal{A} = \bigoplus_{m\geq 0} \mathcal{A}^m$ *be* $\mathbb{Z}_{\geq 0}$*-graded ring. If homogeneous elements* $a_1, \dots, a_t$ *generate the ideal* $\mathcal{A}^+ = \bigoplus_{m>0} \mathcal{A}^m$*, then they generate* $\mathcal{A}$ *as* $\mathcal{A}^0$*-algebra.*

Combining the two above lemmata we get

Corollary 2.6. *If $\Gamma(\mathcal{R}_\Lambda)$ is generated as $\mathcal{S}_\Lambda$-module by Λ-homogeneous elements $m_1, \ldots, m_t$, then $\mathcal{S}_\Lambda$ is a finitely generated $\mathbb{C}$-algebra with generators $\widehat{\kappa}(m_1), \ldots, \widehat{\kappa}(m_t)$.*

The above result can be put as a part of the following equivalence statement.

Theorem 2.7. *Let X be a projective, normal variety and $\Lambda \subset \operatorname{Pic} X$ a finitely generated lattice of Cartier divisors as in Section 1.4. Suppose moreover that Λ contains an ample divisor and the sheaf Ω_X is reflexive. Then the following conditions are equivalent:*

1. *The Cox ring $\mathcal{S}_\Lambda$ is a finitely generated $\mathbb{C}$-algebra.*
2. *The module $\Gamma_\Lambda(\mathcal{R}_\Lambda)$ is finitely generated over $\mathcal{S}_\Lambda$.*
3. *For every reflexive sheaf $\mathcal{F}$ over X, the graded module $\Gamma_\Lambda(\mathcal{F})$ is finitely generated over $\mathcal{S}_\Lambda$.*

Proof. We proved the implication $2 \Rightarrow 1$. The implication $3 \Rightarrow 2$ is clear since $\mathcal{R}_\Lambda$ is reflexive. To prove the implication $1 \Rightarrow 3$ suppose that $\mathcal{F}$ is reflexive and $D_{\lambda_0} \in \Lambda$ is ample. Thus $\mathcal{F}^\vee(D_{m\lambda_0})$ is generated by global section for $m \gg 0$ or, equivalently, we have a surjective morphism $\mathcal{O}^{\oplus N} \to \mathcal{F}^\vee(D_{m\lambda_0})$ for some positive integer N. Dualising and twisting we get injective $\mathcal{F} \to \mathcal{O}(D_{m\lambda_0})^{\oplus N}$. By left-exactness of Γ_Λ we can present $\Gamma_\Lambda(\mathcal{F})$ as a submodule of the free finitely generated $\mathcal{S}_\Lambda$-module $\mathcal{S}_\Lambda[m\lambda_0]^{\oplus N}$. Because $\mathcal{S}_\Lambda$ is Noetherian, $\Gamma_\Lambda(\mathcal{F})$ is finitely generated too. $\square$

2.4. The theorem of Jaczewski revisited.

Theorem 2.8. *Suppose that X is a smooth projective variety and Λ a free finitely generated group of Cartier divisors, as defined in the Section 1.4, which contains an ample divisor. If $\mathcal{R}_\Lambda$ splits into the sum of line bundles, that is $\mathcal{R}_\Lambda = \bigoplus \mathcal{L}_i$, and for every $\mathcal{L}_i$ we have $\mathcal{L}_i \simeq \mathcal{O}(-D_{\lambda_i})$ for some $\lambda_i \in \Lambda$, then X is a toric variety, $\Lambda = \operatorname{Pic} X$ and D_i's are linearly equivalent to torus invariant prime divisors.*

Proof. By Lemma 1.2 the module $\Gamma(\mathcal{R}_\Lambda)$ is free of rank $n + r$. Therefore, by Corollary 2.6 the algebra $\mathcal{S}_\Lambda$ is generated by $n + r$ elements. Since its dimension is also equal to $n + r$, it is the polynomial ring and X is a toric variety and $\Lambda = \operatorname{Pic} X$ by Theorem 1.5. On the toric variety X the Euler sheaf $R_{\operatorname{Pic} X}$ splits as a sum of line bundles associated to the negatives of all torus invariant prime divisors (c.f. [**BM86**] or Corollary 2.2). The theorem follows since such decomposition is essentially unique by [**Ati56**, Theorem 1]. $\square$

We can reformulate this result in the spirit of the original Jaczewski theorem, 1.1.

Corollary 2.9. *Let X be a smooth projective variety with $\mathrm{H}^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$. If the potential sheaf of X splits into the direct sum of line bundles, then X is a toric variety.*

Proof. We note that, by assumptions, it follows that $\operatorname{Pic} X = \mathrm{H}^2(X, \mathbb{Z})$ is a finitely generated abelian group.

First, let us assume that $\operatorname{Pic} X$ has no torsion. In this case we let $\Lambda = \operatorname{Pic} X$. Then, because of our assumptions, $\Lambda_{\mathbb{C}} = \mathrm{H}^1(X, \Omega_X)$ and therefore $\mathcal{R}_\Lambda = \mathcal{R}_X$ is the dual of the potential sheaf. Thus we are in the situation of 2.8 and the corollary follows.

Now, we consider the general case when, possibly, the torsion part of $\operatorname{Pic} X$ is non-zero, we denote it by G and we will show that it is impossible.

The group G is finite. We consider the following locally free sheaf of $\mathcal{O}_X$-modules of rank $|G|$

$$\widehat{\mathcal{S}}_G = \bigoplus_{g \in G} \mathcal{L}_g$$

where $\mathcal{L}_g$ denotes the invertible sheaf whose class in $G \subset \operatorname{Pic} X$ is g. We repeat the construction I.4.2.1 from [**ADHL10**, Section I.4.2] to define the multiplication on $\widehat{\mathcal{S}}_G$ and make it the sheaf of G-graded $\mathcal{O}_X$ algebras. As in [**ADHL10**, Section I.6, Construction I.6.1.5] we define $\widehat{Y}_G = \operatorname{Spec}_{\mathcal{O}_X} \widehat{\mathcal{S}}_G$, the characteristic space of $G \subset \operatorname{Pic} X$. The resulting morphism $\pi : \widehat{Y}_G \to X$ is étale with Galois group G. We note that this is a slight generalization of the well known construction of unramified cyclic covering, see e.g. [**BPVdV84**, Section I.17]. In fact, given a presentation of G into a product of cyclic groups we can produce an étale morphism $\pi : \widehat{Y}_G \to X$ with Galois group G by defining $\widehat{Y}_G$ as the fiber product over X of the respective cyclic coverings of X.

Now we use [**KKV89**, Lemma 2.2, Proposition 4.2] to conclude that $\operatorname{Pic} \widehat{Y}_G$ is torsion free. Since the morphism π is étale and the definition of the Euler sheaf is local (c.f. Section 2.1), we conclude that the Euler sheaf $\mathcal{R}_\Lambda$, where $\Lambda = \operatorname{Pic} \widehat{Y}_G$, is a pullback of the potential sheaf $\mathcal{R}_X$. By Theorem 2.8 the variety $\widehat{Y}_G$ is toric.

The finite group G acts freely on $\widehat{Y}_G$, however, any element of G has a fixed point on $\widehat{Y}_G$ by the holomorphic Lefschetz fixed-point formula (see [**AB68**] or [**GH94**]) which is impossible unless G is trivial. □

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