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A note on maximal commutators and commutators of maximal functions

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Abstract. In this paper maximal commutators and commutators of maximal functions with functions of bounded mean oscillation are investigated. New pointwise estimates for these operators are proved.

1. Introduction.

Given a locally integrable function f on $\mathbb{R}^n$, the Hardy-Littlewood maximal function Mf of f is defined by

$$Mf(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy, \qquad (x \in \mathbb{R}^n),$$

where the supremum is taken over all cubes Q containing x. The operator $M: f \to Mf$ is called the Hardy-Littlewood maximal operator.

For any $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let $M^{\#} f$ be the sharp maximal function of Fefferman-Stein defined by

$$M^{\#} f(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where the supremum extends over all cubes containing x, and f_Q is the mean value of f on Q. For a fixed $\delta \in (0,1)$, any suitable function g and $x \in \mathbb{R}^n$, let $M_\delta^{\#} g(x) := [M^{\#}(|g|^\delta)(x)]^{1/\delta}$ and $M_\delta g(x) := [M(|g|^\delta)(x)]^{1/\delta}$.

Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$. Then f is said to be in $\mathrm{BMO}(\mathbb{R}^n)$ if the seminorm given by

$$\|f\|_* := \sup_Q \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy$$

is finite.

Let T be the Calderón-Zygmund singular integral operator

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$$Tf(x) := \text{p.v.} \int_{\mathbb{R}^n} K(x-y) f(y)\, dy$$

with kernel $K(x) = \Omega(x)/|x|^n$, where Ω is homogeneous of degree zero, infinitely differentiable on the unit sphere $\mathbb{S}^{n-1}$, and $\int_{\mathbb{S}^{n-1}} \Omega = 0$.

The well-known result of Coifman, Rochberg, and Weiss [**6**] states that if $b \in \mathrm{BMO}(\mathbb{R}^n)$, then $[T,b]$ defined initially for $f \in L_c^\infty(\mathbb{R}^n)$, by

$$[T,b](f) := T(bf) - bT(f), \tag{1.1}$$

is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$; conversely, if $[R_i, b]$ is bounded on $L^p(\mathbb{R}^n)$ for every Riesz transform R_i, then $b \in \mathrm{BMO}(\mathbb{R}^n)$. Janson [**12**] observed that actually for any singular integral T (with kernel satisfying the above-mentioned conditions) the boundedness of $[T,b]$ on $L^p(\mathbb{R}^n)$ implies $b \in \mathrm{BMO}(\mathbb{R}^n)$.

Unlike the classical theory of singular integral operators, a simple example shows that $[T,b]$ fails to be of weak-type (1,1) when $b \in \mathrm{BMO}(\mathbb{R}^n)$, and satisfies weak-type $L(\log L)$ inequality (see [**16**]).

We consider the commutator of the Hardy-Littlewood maximal operator M and a measurable function b.

Definition 1.1. Given a measurable function b the commutator of the Hardy-Littlewood maximal operator M and b is defined by

$$[M,b]f(x) := M(bf)(x) - b(x)Mf(x)$$

for all $x \in \mathbb{R}^n$.

The operator $[M,b]$ was studied by Milman et al. in [**15**] and [**2**]. This operator arises, for example, when one tries to give a meaning to the product of a function in H^1 and a function in BMO (which may not be a locally integrable function, see, for instance, [**4**]). Using real interpolation techniques, in [**15**], Milman and Schonbek proved the L_p-boundedness of the operator $[M,b]$. Bastero, Milman and Ruiz [**2**] proved the next theorem.

Theorem 1.2. *Let $1 < p < \infty$. Then the following assertions are equivalent*:

(i) $[M,b]$ *is bounded on* $L_p(\mathbb{R}^n)$.
(ii) $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and* $b^- \in L_\infty(\mathbb{R}^n)$.[1]

As we know only these two papers are devoted to the problem of boundedness of the commutator of maximal function in Lebesgue spaces.

In order to investigate $[M,b]$ we start with the consideration of the maximal commutator, which is an easier one.

Definition 1.3. Given a measurable function b the maximal commutator is de-

[1]Denote by $b^+(x) = \max\{b(x), 0\}$ and $b^-(x) = -\min\{b(x), 0\}$, consequently $b = b^+ - b^-$ and $|b| = b^+ + b^-$.

fined by

$$C_b(f)(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |b(x) - b(y)||f(y)|dy$$

for all $x \in \mathbb{R}^n$.

This operator plays an important role in the study of commutators of singular integral operators with BMO symbols (see, for instance, [**8**], [**14**], [**18**], [**19**]). Garcia-Cuerva et al. [**8**] proved the following.

Theorem 1.4. *Let $1 < p < \infty$. C_b is bounded on $L_p(\mathbb{R}^n)$ if and only if $b \in$ BMO$(\mathbb{R}^n)$.*

C_b enjoys weak-type $L(\log L)$ estimate.

Theorem 1.5 (see, for instance, [**1**] and [**10**]). *If $b \in$ BMO$(\mathbb{R}^n)$, then*

$$|\{x \in \mathbb{R}^n : C_b(f)(x) > \lambda\}| \le C \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \left(1 + \log^+ \left(\frac{|f(x)|}{\lambda}\right)\right) dx. \tag{1.2}$$

The maximal operator C_b has been studied intensively and there exist plenty of results about it.

Our results are the following.

Theorem 1.6. *Let $b \in$ BMO$(\mathbb{R}^n)$ such that $b^- \in L_\infty(\mathbb{R}^n)$. Then there exists a positive constant C such that*

$$\begin{aligned} &|\{x \in \mathbb{R}^n : |[M,b]f(x)| > \lambda\}| \\ &\quad \le CC_0(1 + \log^+ C_0) \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \left(1 + \log^+ \left(\frac{|f(x)|}{\lambda}\right)\right) dx, \end{aligned} \tag{1.3}$$

for all $f \in L(1 + \log^+ L)$ and $\lambda > 0$, where $C_0 = \|b^+\|_* + \|b^-\|_\infty$.

Remark 1.7. Unfortunately, in Theorem 1.6 we have only sufficient part, and we are not able to prove that the condition $b \in$ BMO$(\mathbb{R}^n)$ is also necessary for inequality (1.3) to hold.

Theorem 1.8. *The following assertions are equivalent*:

(i) *There exists a positive constant C such that for each $\lambda > 0$, inequality* (1.2) *holds for all $f \in L(1 + \log^+ L)(\mathbb{R}^n)$.*

(ii) *$b \in$ BMO$(\mathbb{R}^n)$.*

Remark 1.9. The fact that (ii) implies (i) is the statement of Theorem 1.5. We give another proof for this.

For the proofs of our theorems we prove the following estimate.

THEOREM 1.10. *Let* $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and let* $0 < \delta < 1$. *Then, there exists a positive constant* $C = C_\delta$ *such that*

$$M_\delta(C_b(f))(x) \le C\|b\|_* M^2 f(x) \qquad (x \in \mathbb{R}^n) \tag{1.4}$$

for all $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

This theorem improves the known inequality

$$M_\delta^\#(C_b(f))(x) \lesssim \|b\|_* M^2 f(x),$$

(see, for instance, [**11**, Lemma 1]). Indeed, since $M_\delta^\# \lesssim M_\delta$,

$$M_\delta^\#(C_b(f))(x) \lesssim M_\delta(C_b(f))(x) \le C\|b\|_* M^2 f(x) \qquad (x \in \mathbb{R}^n).$$

By Theorem 1.10 we can prove all the theorems in a unified style. In particular we can give easier proof for Theorems 1.2, 1.4 and 1.5 (see Theorems 1.8 and 1.13).

As corollaries of Theorem 1.10 we obtain the following.

COROLLARY 1.11. *Let* $b \in \mathrm{BMO}(\mathbb{R}^n)$. *Then, there exists a positive constant* C *such that*

$$C_b(f)(x) \le C\|b\|_* M^2 f(x) \qquad (x \in \mathbb{R}^n) \tag{1.5}$$

for all $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

COROLLARY 1.12. *Let* $b \in \mathrm{BMO}(\mathbb{R}^n)$ *such that* $b^- \in L_\infty(\mathbb{R}^n)$. *Then, there exists a positive constant* C *such that*

$$|[M,b]f(x)| \le C(\|b^+\|_* + \|b^-\|_\infty) M^2 f(x) \tag{1.6}$$

for all $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

Inequalities (1.5) and (1.6) allow us to state the boundedness of both operators on any Banach spaces of measurable functions on which the Hardy-Littlewood maximal operator is bounded.

THEOREM 1.13. *Let* $b \in \mathrm{BMO}(\mathbb{R}^n)$. *Suppose that* X *is a Banach space of measurable functions defined on* $\mathbb{R}^n$. *Assume that* M *is bounded on* X. *Then the operator* C_b *is bounded on* X, *and the inequality*

$$\|C_b f\|_X \le C\|b\|_* \|f\|_X$$

holds with constant C *independent of* f.

Moreover, if $b^- \in L_\infty(\mathbb{R}^n)$, *then the operator* $[M,b]$ *is bounded on* X, *and the inequality*

$$\|[M,b]f\|_X \le C(\|b^+\|_* + \|b^-\|_\infty)\|f\|_X$$

holds with constant C independent of f.

The paper is organized as follows. Section 2 contains some preliminaries along with the standard ingredients used in the proofs. In Section 3 we prove the pointwise estimates. In Section 4 we give the proof of Theorem 1.8. Finally, in Section 5 we prove Theorem 1.6 and show that $[M,b]$ fails to be of weak type $(1,1)$ in general.

2. Notations and Preliminaries.

Now we make some conventions. Throughout the paper, we always denote by C a positive constant, which is independent of main parameters, but it may vary from line to line. However a constant with subscript such as C_1 does not change in different occurrences. By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent. For a measurable set E, χ_E denotes the characteristic function of E. Throughout this paper cubes will be assumed to have their sides parallel to the coordinate axes. Given $\lambda > 0$ and a cube Q, λQ denotes the cube with the same center as Q and whose side is λ times that of Q. For a fixed p with $p \in [1,\infty)$, p' denotes the dual exponent of p, namely, $p' = p/(p-1)$. For any measurable set E and any integrable function f on E, we denote by f_E the mean value of f over E, that is, $f_E = (1/|E|)\int_E f(x)dx$.

For the sake of completeness we recall the definitions and some properties of the spaces we are going to use.

The non-increasing rearrangement (see, e.g., [**5**, p. 39]) of a measurable function f on $\mathbb{R}^n$ is defined by

$$f^*(t) = \inf\{\lambda > 0 : |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}| \le t\} \quad (0 < t < \infty).$$

Let $p \in [1,\infty)$. The Lorentz space $L^{p,\infty}$ is defined by

$$L^{p,\infty}(\mathbb{R}^n) := \Big\{ f : \|f\|_{L^{p,\infty}(\mathbb{R}^n)} := \sup_{0<t<\infty} t^{1/p} f^*(t) < \infty \Big\}.$$

The most important result regarding BMO is the following theorem of F. John and L. Nirenberg [**13**] (see also [**7**, p. 164]).

THEOREM 2.1. *There exists constants C_1 and C_2 depending only on the dimension n, such that*

$$|\{x \in Q : |f(x) - f_Q| > t\}| \le C_1 |Q| \exp\Big\{ -\frac{C_2}{\|f\|_*} t \Big\} \tag{2.1}$$

for every $f \in \mathrm{BMO}(\mathbb{R}^n)$, every cube Q and every $t > 0$.

LEMMA 2.2 ([**7**, p. 166]). *Let $f \in \mathrm{BMO}(\mathbb{R}^n)$ and $p \in (0,\infty)$. Then for every λ such that $0 < \lambda < C_2/\|f\|_*$, we have*

$$\sup_Q \frac{1}{|Q|} \int_Q \exp\{\lambda |f(x) - f_Q|\} dx < \infty,$$

where C_2 is the same constant appearing in (2.1).

Lemma 2.3 ([**13**] and [**3**]). *For $p \in (0,\infty)$, $\mathrm{BMO}(p)(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$, with equivalent norms, where*

$$\|f\|_{\mathrm{BMO}(p)(\mathbb{R}^n)} := \sup_Q \left(\frac{1}{|Q|} \int_Q |f(y) - f_Q|^p dy \right)^{1/p}.$$

A function $\Psi : [0,\infty] \to [0,\infty]$ is a Young function if it is continuous, convex and increasing satisfying $\Psi(0) = 0$ and $\Psi(t) \to \infty$ as $t \to \infty$. Let us define the Ψ-average of a function f over a cube Q by means of the following Luxemburg norm

$$\|f\|_{\Psi,Q} := \inf \left\{ \alpha > 0 : \frac{1}{|Q|} \int_Q \Psi\left(\frac{|f(y)|}{\alpha} \right) dy \le 1 \right\}$$

(see, for instance, [**17**]). The following generalized Hölder's inequality holds:

$$\frac{1}{|Q|} \int_Q |f(y) g(y)| dy \le \|f\|_{\Phi,Q} \|g\|_{\Psi,Q}, \tag{2.2}$$

where Φ is the complementary Young function associated to Ψ.

The maximal function of f with respect to Ψ is defined by

$$M_\Psi f(x) := \sup_{Q \ni x} \|f\|_{\Psi,Q}.$$

The main example that we are going to use is $\Phi(t) = t(1 + \log^+ t)$ with maximal function defined by $M_{L(\log L)}$. The complementary Young function is given by $\Psi(t) \approx e^t$ with the corresponding maximal function denoted by $M_{\exp L}$.

Recall the definition of quasinorm of Zygmund space:

$$\|f\|_{L(1+\log^+ L)} := \int_{\mathbb{R}^n} |f(x)| (1 + \log^+ |f(x)|) dx$$

The size of M^2 is estimated as follows.

Lemma 2.4 ([**16**, Lemma 1.6]). *There exists a positive constant C such that for any function f and for all $\lambda > 0$,*

$$|\{x \in \mathbb{R}^n : M^2 f(x) > \lambda\}| \le C \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \left(1 + \log^+ \left(\frac{|f(x)|}{\lambda} \right) \right) dx. \tag{2.3}$$

3. Pointwise estimates.

Operators C_b and $[M, b]$ essentially differ from each other. For example, C_b is a positive and sublinear operator, but $[M, b]$ is neither positive nor sublinear. However, if b satisfies some additional conditions, then operator C_b controls $[M, b]$.

LEMMA 3.1. *Let b be any non-negative locally integrable function. Then*

$$|[M, b]f(x)| \leq C_b(f)(x) \tag{3.1}$$

for all $f \in L_1^{\text{loc}}(\mathbb{R}^n)$.

PROOF. It is easy to see that for any $f,\, g \in L_1^{\text{loc}}(\mathbb{R}^n)$ the following pointwise estimate holds:

$$|Mf(x) - Mg(x)| \leq M(f-g)(x). \tag{3.2}$$

Since b is non-negative, we can write, by (3.2),

$$\begin{aligned} |[M, b]f(x)| &= |M(bf)(x) - b(x)Mf(x)| = |M(bf)(x) - M(b(x)f)(x)| \\ &\leq M(bf - b(x)f)(x) = M((b - b(x))f)(x) = C_b(f)(x). \end{aligned}$$

□

LEMMA 3.2. *Let b be any locally integrable function on $\mathbb{R}^n$. Then*

$$|[M, b]f(x)| \leq C_b(f)(x) + 2b^-(x)Mf(x). \tag{3.3}$$

holds for all $f \in L_1^{\text{loc}}(\mathbb{R}^n)$.

PROOF. Since

$$|[M, b]f(x) - [M, |b|]f(x)| \leq 2b^-(x)Mf(x)$$

(see [**2**, p. 3330], for instance),

$$|[M, b]f(x)| \leq |[M, |b|]f(x)| + 2b^-(x)Mf(x), \tag{3.4}$$

and by Lemma 3.1 we have

$$|[M, b]f(x)| \leq C_{|b|}f(x) + 2b^-(x)Mf(x).$$

Since $||a| - |b|| \leq |a - b|$ holds for any $a, b \in \mathbb{R}$, we get $C_{|b|}f(x) \leq C_b f(x)$ for all $x \in \mathbb{R}^n$. □

PROOF OF THEOREM 1.10. Let $x \in \mathbb{R}^n$ and fix a cube $Q \ni x$. Let $f = f_1 + f_2$, where $f_1 = f\chi_{3Q}$. Since for any $y \in \mathbb{R}^n$

$$\begin{aligned} C_b(f)(y) &= M((b-b(y))f)(y) = M((b-b_{3Q}+b_{3Q}-b(y))f)(y) \\ &\le M((b-b_{3Q})f_1)(y) + M((b-b_{3Q})f_2)(y) + |b(y)-b_{3Q}|Mf(y), \end{aligned}$$

we have

$$\begin{aligned} \left(\frac{1}{|Q|}\int_Q (C_b(f)(y))^\delta dy\right)^{1/\delta} &\lesssim \left(\frac{1}{|Q|}\int_Q |M((b-b_{3Q})f_1)(y)|^\delta dy\right)^{1/\delta} \\ &\quad + \left(\frac{1}{|Q|}\int_Q |M((b-b_{3Q})f_2)(y)|^\delta dy\right)^{1/\delta} \\ &\quad + \left(\frac{1}{|Q|}\int_Q |b(y)-b_{3Q}|^\delta (Mf(y))^\delta dy\right)^{1/\delta} \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned} \tag{3.5}$$

Since

$$\begin{aligned} \int_Q |M((b-b_{3Q})f_1)(y)|^\delta dy &\le \int_0^{|Q|} [(M((b-b_{3Q})f_1))^*(t)]^\delta dt \\ &\le \left[\sup_{0<t<|Q|} t(M((b-b_{3Q})f_1))^*(t)\right]^\delta \int_0^{|Q|} t^{-\delta} dt, \end{aligned}$$

using the boundedness of M from $L_1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$ we have

$$\begin{aligned} \int_Q |M((b-b_{3Q})f_1)(y)|^\delta dy &\lesssim \|(b-b_{3Q})f_1\|_{L_1(\mathbb{R}^n)}^\delta |Q|^{-\delta+1} \\ &= \|(b-b_{3Q})f\|_{L_1(3Q)}^\delta |Q|^{-\delta+1}. \end{aligned}$$

Thus

$$\mathrm{I} \lesssim \frac{1}{|Q|}\int_{3Q} |b(y)-b_{3Q}||f(y)|dy.$$

By (2.2), we get

$$\mathrm{I} \lesssim \|b-b_{3Q}\|_{\exp L,3Q}\|f\|_{L\log L,3Q}.$$

Lemma 2.2 shows that there exists a constant $C>0$ such that for any cube Q,

$$\|b-b_Q\|_{\exp L,Q} \le C\|b\|_*.$$

Then we have

$$\mathrm{I} \lesssim \|b\|_* M_{L\log L} f(x). \tag{3.6}$$

Let us estimate II. Since II is comparable to $\inf_{y\in Q} M((b-b_{3Q})f)(y)$ (see [**7**, p. 160], for instance), we have

$$\mathrm{II} \lesssim M((b-b_{3Q})f)(x).$$

Again by (2.2) and Lemma 2.2, we get

$$\mathrm{II} \lesssim \sup_{x\in Q} \|b-b_{3Q}\|_{\exp L,3Q}\|f\|_{L\log L,3Q} \lesssim \|b\|_* M_{L\log L} f(x). \tag{3.7}$$

Let $\delta<\varepsilon<1$. To estimate III we use Hölder's inequality with exponents r and r', where $r=\varepsilon/\delta>1$:

$$\mathrm{III} \le \left(\frac{1}{|Q|}\int_Q |b(y)-b_{3Q}|^{\delta r'}\,dy\right)^{1/\delta r'} \left(\frac{1}{|Q|}\int_Q (Mf(y))^{\delta r}\,dy\right)^{1/\delta r}.$$

By Lemma 2.3 we get

$$\mathrm{III} \lesssim \|b\|_*\left(\frac{1}{|Q|}\int_Q (Mf(y))^{\varepsilon}\,dy\right)^{1/\varepsilon} \le \|b\|_* M_\varepsilon(Mf)(x). \tag{3.8}$$

Finally, since $M^2 \approx M_{L\log L}$ (see [**16**, p. 174] and [**9**, p. 159], for instance), we get, by (3.5)–(3.8),

$$M_\delta(C_b(f))(x) \le C\|b\|_*\big(M_\varepsilon(Mf)(x)+M^2f(x)\big). \tag{3.9}$$

Since

$$M_\varepsilon(Mf)(x)\le M^2f(x), \quad \text{when} \quad 0<\varepsilon<1,$$

we have (1.4). □

Proof of Corollary 1.11. Since, by the Lebesgue differentiation theorem

$$C_b(f)(x)\le M_\delta(C_b(f))(x),$$

the statement follows from Theorem 1.10. □

Now we are in a position to prove Corollary 1.12.

Proof of Corollary 1.12. By Lemma 3.2 and Corollary 1.11, we have

$$|[M,b]f(x)| \le C\big(\|b\|_* M^2 f(x) + b^-(x)Mf(x)\big). \tag{3.10}$$

Since $f\le Mf$ and $\|b\|_* \le \|b^+\|_* + \|b^-\|_* \lesssim \|b^+\|_* + \|b^-\|_{L_\infty}$, we obtain Corollary 1.12. □

4. Proof of Theorem 1.8.

(i) $\Rightarrow$ (ii). Let Q_0 be any fixed cube and let $f = \chi_{Q_0}$. For any $\lambda > 0$ we have

$$\begin{aligned}|\{x \in \mathbb{R}^n : C_b(f)(x) > \lambda\}| &= \left|\left\{x \in \mathbb{R}^n : \sup_{x\in Q} \frac{1}{|Q|}\int_{Q\cap Q_0} |b(x)-b(y)|dy > \lambda\right\}\right| \\ &\geq \left|\left\{x \in Q_0 : \sup_{x\in Q} \frac{1}{|Q|}\int_{Q\cap Q_0} |b(x)-b(y)|dy > \lambda\right\}\right| \\ &\geq \left|\left\{x \in Q_0 : \frac{1}{|Q_0|}\int_{Q_0} |b(x)-b(y)|dy > \lambda\right\}\right| \\ &\geq |\{x \in Q_0 : |b(x)-b_{Q_0}| > \lambda\}|,\end{aligned}$$

since

$$|b(x)-b_{Q_0}| \leq \frac{1}{|Q_0|}\int_{Q_0} |b(x)-b(y)|dy.$$

By assumption we have

$$|\{x \in Q_0 : |b(x)-b_{Q_0}| > \lambda\}| \leq C|Q_0|\frac{1}{\lambda}\left(1+\log^+\frac{1}{\lambda}\right).$$

For $0<\delta<1$ we have

$$\begin{aligned}\int_{Q_0} |b-b_{Q_0}|^\delta &= \delta\int_0^\infty \lambda^{\delta-1}|\{x\in Q_0 : |b(x)-b_{Q_0}|>\lambda\}|d\lambda \\ &= \delta\left\{\int_0^1 + \int_1^\infty\right\}\lambda^{\delta-1}|\{x\in Q_0 : |b(x)-b_{Q_0}|>\lambda\}|d\lambda \\ &\leq \delta|Q_0|\int_0^1 \lambda^{\delta-1}d\lambda + C\delta|Q_0|\int_1^\infty \lambda^{\delta-1}\frac{1}{\lambda}\left(1+\log^+\frac{1}{\lambda}\right)d\lambda \\ &= |Q_0| + C\delta|Q_0|\int_1^\infty \lambda^{\delta-2}d\lambda = \left(1+C\frac{\delta}{1-\delta}\right)|Q_0|.\end{aligned}$$

Thus $b \in \mathrm{BMO}_\delta(\mathbb{R}^n)$. By Lemma 2.3 we get $b \in \mathrm{BMO}(\mathbb{R}^n)$.

(ii) $\Rightarrow$ (i). By Theorem 1.11 and Lemma 2.4, we have

$$\begin{aligned}|\{x \in \mathbb{R}^n : C_b(f)(x) > \lambda\}| &\leq \left|\left\{x \in \mathbb{R}^n : M^2 f(x) > \frac{\lambda}{C\|b\|_*}\right\}\right| \\ &\leq C\int_{\mathbb{R}^n} \frac{C\|b\|_*|f(x)|}{\lambda}\left(1+\log^+\left(\frac{C\|b\|_*|f(x)|}{\lambda}\right)\right)dx.\end{aligned}$$

Since the inequality

$$1+\log^+(ab) \le (1+\log^+ a)(1+\log^+ b) \tag{4.1}$$

holds for any a, $b>0$, we get

$$\begin{aligned}&|\{x\in\mathbb{R}^n : C_b(f)(x)>\lambda\}|\\ &\quad\le C\|b\|_*(1+\log^+\|b\|_*)\int_{\mathbb{R}^n}\frac{|f(x)|}{\lambda}\left(1+\log^+\left(\frac{|f(x)|}{\lambda}\right)\right)dx.\end{aligned}$$

□

5. Proof of Theorem 1.6.

By Lemma 3.2, we have

$$\begin{aligned}&|\{x\in\mathbb{R}^n : |[M,b]f(x)|>\lambda\}|\\ &\le\left|\left\{x\in\mathbb{R}^n : C_b(f)(x)>\frac{\lambda}{2}\right\}\right|+\left|\left\{x\in\mathbb{R}^n : |2b^-|Mf(x)>\frac{\lambda}{2}\right\}\right|\\ &\le\left|\left\{x\in\mathbb{R}^n : C_b(f)(x)>\frac{\lambda}{2}\right\}\right|+\left|\left\{x\in\mathbb{R}^n : 2\|b^-\|_\infty Mf(x)>\frac{\lambda}{2}\right\}\right|.\end{aligned}$$

By (4.1), we have

$$\begin{aligned}&\left|\left\{x\in\mathbb{R}^n : C_b(f)(x)>\frac{\lambda}{2}\right\}\right|\\ &\quad\le CC_0(1+\log^+ C_0)\int_{\mathbb{R}^n}\frac{|f(x)|}{\lambda}\left(1+\log^+\left(\frac{|f(x)|}{\lambda}\right)\right)dx.\end{aligned}\tag{5.1}$$

On the other hand, since the maximal operator M is a weak type (1,1), we get

$$\left|\left\{x\in\mathbb{R}^n : 2\|b^-\|_\infty\, Mf(x)>\frac{\lambda}{2}\right\}\right|\le C\|b^-\|_\infty\int_{\mathbb{R}^n}\frac{|f(x)|}{\lambda}dx. \tag{5.2}$$

Combining (5.1) and (5.2), we get (1.3). □

Remark 5.1. We show that $[M,b]$ fails to be of weak type $(1,1)$ in general. We use the idea due to C. Perez (see [**16**, p. 175]). Let $b(x)=\log|1+x|\in \mathrm{BMO}(\mathbb{R}^n)$ and let $f(x)=\chi_{(0,1)}(x)$. It is easy to see that for any $x<0$

$$Mf(x)=\sup_{0<t<1}\frac{t}{t-x}=\frac{1}{1-x}.$$

On the other hand, for any $x<0$

$$M(bf)(x)=\sup_{0<t<1}\frac{\int_0^t\log|1+y|dy}{t-x}=\sup_{0<t<1}\frac{(1+t)\log(1+t)-t}{t-x}=\frac{2\log 2-1}{1-x}.$$

Thus

$$[M,b]f(x) = \frac{2\log 2 - 1}{1-x} - \frac{\log|1+x|}{1-x}.$$

If $x < -100$, then

$$\log|1+x| - (2\log 2 - 1) > \frac{1}{2}\log|x|.$$

Therefore, for any $\lambda > 0$,

$$\begin{aligned}
\lambda|\{x \in \mathbb{R} : |[M,b]f(x)| > \lambda\}| &\geq \lambda\left|\left\{x < 0 : \left|\frac{2\log 2 - 1}{1-x} - \frac{\log|1+x|}{1-x}\right| > \lambda\right\}\right| \\
&\geq \lambda\left|\left\{x < -100 : \frac{1}{2}\frac{\log|x|}{1-x} > \lambda\right\}\right| \\
&\geq \lambda\left|\left\{x < -100 : \frac{1}{4}\frac{\log|x|}{|x|} > \lambda\right\}\right| \\
&= \lambda(\varphi^{-1}(-100) - \varphi^{-1}(4\lambda)),
\end{aligned}$$

where φ is the increasing function $\varphi : (-\infty, -e) \to (0, e^{-1})$, given by $\varphi(x) = \log|x|/|x|$. Observe that the right hand side of the estimate is unbounded as $\lambda \to 0$:

$$\lim_{\lambda\to 0} \lambda\varphi^{-1}(\lambda) = \lim_{\lambda\to\infty} \lambda\varphi(\lambda) = \infty.$$

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