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On the theory of multilinear Littlewood–Paley g-function

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Abstract. Let $m \geq 2$ and define the multilinear Littlewood–Paley g-function by

$$g(\vec{f})(x) = \left(\int_0^\infty \left| \frac{1}{t^{mn}} \int_{(\mathbb{R}^n)^m} \psi\left(\frac{y_1}{t}, \ldots, \frac{y_m}{t}\right) \prod_{j=1}^m f_j(x-y_j)dy_j \right|^2 \frac{dt}{t} \right)^{1/2}.$$

In this paper, we establish the strong $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ to $L^p(\nu_{\vec{\omega}})$ boundedness and weak type $L^{p_1}(w_1)\times\cdots\times L^{p_m}(w_m)$ to $L^{p,\infty}(\nu_{\vec{\omega}})$ estimate for the multilinear g-function. The weighted strong and end-point estimates for the iterated commutators of g-function are also given. Here $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i^{p/p_i}$ and each w_i is a nonnegative function on $\mathbb{R}^n$.

1. Introduction.

In order to study the dyadic decomposition of Fourier series, Littlewood and Paley [**25**], [**26**], [**27**] introduced the g-function of one dimension as follows:

$$g(f)(\theta) = \left(\int_0^1 (1-\rho)|\Phi'(\rho e^{i\theta})|^2 d\rho \right)^{1/2},$$

where $\Phi(z)$ is a function which is analytic in $|z| < 1$, and whose real part has boundary value $f(\theta)$. The function g is basic in the Littlewood–Paley theory of Fourier series [**41**]. Littlewood and Paley proved that

$$A_p\|f\|_p \leq \|g(f)\|_p \leq B_p\|f\|_p, \tag{1.1}$$

where in the left side of the above inequality, it was assumed that $\int_0^{2\pi} f(\theta)d\theta = 0$. Later, Stein defined the following n-dimensional form of the Littlewood–Paley g-function and get the same norm inequality as (1.1),

$$g(f)(x) = \left(\int_0^\infty t|\nabla u(x,t)|^2 dt \right)^{1/2},$$

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where $u(x,t) = P_t * f(x)$ denotes the Poisson integral of f. From then on, efforts have been made by many mathematicians to study Littlewood–Paley g-function of higher dimensions with more general kernels. Among such achievement there are celebrated works of Taibleson [**35**], Stein and Wainger [**34**], Stein [**33**], Uchiyama [**36**], Lerner [**22**]. In particular, Stein expanded Littlewood–Paley theory by using more singular kernels in place of P_t, to make it applicable to interesting geometrical and Fourier analytical questions (see e.g. [**20**] and [**21**] and the references therein). A more general type of Littlewood–Paley g-function with a kernel much weaker than P_t was studied by Wang [**37**] in 1989.

Now we recall some background of multilinear Littlewood–Paley g-function. We begin by a quick review of the development of multilinear type operators. The multilinear Calderón–Zygmund theory was originated in the works of Coifman and Meyer in the 70s (see e.g. [**4**], [**5**]), and it was oriented towards the work on the Calderón–Zygmund commutator. This topic was retaken by several authors, including Christ and Journé [**8**], Kenig and Stein [**19**], Perez et all [**29**], [**30**], and Grafakos and Torres [**18**]. Coifman and Meyer ([**5**] and [**7**]) considered the following multilinear Calderón–Zygmund singular integral operator,

$$T(\vec{f})(x) = \text{p.v.} \int_{(\mathbb{R}^n)^m} K(y_1, \ldots, y_m) \prod_{i=1}^{m} f_i(x - y_i)\, dy_i. \tag{1.2}$$

They proved that T was bounded from $L^{p_1} \times \cdots \times L^{p_m}$ to L^p, with $1 < p,\ p_i \leq \infty$ and $1/p = 1/p_1 + \cdots + 1/p_m$. Later on, Kenig and Stein [**19**] showed that T was also bounded from $L^1 \times \cdots \times L^1$ to $L^{1/m,\infty}$. In [**6**], Coifman and Meyer introduced a class of multilinear operators as a multilinearization of Littlewood–Paley's g-function (bilinear, one dimensional) as follows:

$$B(a,f) = \int_0^\infty (f * \phi_t)(a * \Phi_t) \frac{m(t)}{t} dt, \tag{1.3}$$

where $m(t) \in L^\infty$, $a \in \text{BMO}$, $\hat{\phi}$ and $\hat{\Phi}$ have compact support $0 \notin \operatorname{supp} \hat{\Phi}$. They studied L^2 estimates of such operators, using the notion of Carleson measures. In 1982, Yabuta [**40**] obtained the $L^p (p \geq 1)$ boundedness and BMO type estimates of $B(a,f)$ by weakening the assumptions in [**6**]. In 2001, under suitable conditions assumed on each kernel φ_i, Sato and Yabuta studied the $L^{p_1} \times \cdots \times L^{p_m}$ to L^p boundedness of multilinear Littlewood–Paley g-functions with $p \geq 1/m$ for $m \geq 2$ as follows,

$$T_g(\vec{f})(x) = \int_0^\infty \prod_{i=1}^{m} ((\varphi_i)_t * f_i)(x) \frac{dt}{t}. \tag{1.4}$$

To show the importance of the multilinear Littlewood–Paley g-function and related multilinear Littlewood–Paley type estimates, we now list some related important applications. In 1982, Fabes, Jerison and Kenig [**11**] studied a collections of multilinear Littlewood–Paley estimates and then applied them to two problems in partial differential equations. In 1984, Fabes, Jerison and Kenig [**11**] obtained necessary and sufficient conditions for absolute continuity of elliptic-harmonic measure, the most important tool in the proof of sufficiency is a multilinear Littlewood–Paley estimate. Related estimates were previously obtained and applied to different problems by Coifman, McIntosh and Meyer in [**3**], Coifman, Deng and Meyer in [**2**], David and Journe in [**9**]. In 1985, Fabes, Jerison and Kenig [**13**] studied a class of multilinear square functions and applied it to Kato's problem.

In (1.3) and (1.4), the kernels are restricted to separable variable kernels, and it is only a special case of the kernel in (1.2). Therefore, we consider the multilinear Littlewood–Paley g-function with the same kind of homogeneous kernel as in (1.2). This leads to the following definition:

Definition 1.1. For any $\vec{f} = (f_1, \dots, f_m) \in \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \times \cdots \times \mathcal{S}(\mathbb{R}^n)$, and $x \notin \bigcap_{j=1}^m \operatorname{supp} f_j$, define the multilinear Littlewood–Paley g-function by

$$g(\vec{f})(x) = \left(\int_0^\infty |\psi_t * \vec{f}(x)|^2 \frac{dt}{t} \right)^{1/2}, \tag{1.5}$$

where

$$\psi_t * \vec{f}(x) = \int_{(\mathbb{R}^n)^m} \psi_t(y_1, \dots, y_m) \prod_{j=1}^m f_j(x - y_j) dy_j, \tag{1.6}$$

with $\psi_t(y_1, \dots, y_m) = (1/t^{mn}) \psi(y_1/t, \dots, y_m/t)$, here the kernel ψ is a function defined on $(\mathbb{R}^n)^m$, satisfying:

(i) Size condition: For any $(y_1, \dots, y_m) \in (\mathbb{R}^n)^m$, there is a constant $C > 0$, such that

$$|\psi(y_1, \dots, y_m)| \le \frac{C}{(1 + |y_1| + \cdots + |y_m|)^{mn+\delta}}, \qquad \text{for some} \quad \delta > 0.$$

(ii) Smoothness condition: there is a $C > 0$, such that

$$\begin{aligned} &|\psi(y_1, \dots, y_i + z, \dots, y_m) - \psi(y_1, \dots, y_i, \dots, y_m)| \\ &\quad \le \frac{C|z|^\gamma}{(1 + |y_1| + \cdots + |y_m|)^{mn+\delta+\gamma}} \end{aligned}$$

for some $\gamma > 0$, $2|z| \le \max_{i=1,\dots,m} |y_i|$ and all $(y_1, \dots, y_m) \in (\mathbb{R}^n)^m$.

Remark 1.2. The conditions assumed on the kernel in the above definition were weaker than the ones studied by Wang [**37**], even in the case for $m = 1$.

Throughout this paper, we assume that g can be extended to be a bounded operator for some $1 \le q_1, \dots, q_m \le \infty$ with $1/q = 1/q_1 + \cdots + 1/q_m$, that is

$$L^{q_1} \times \cdots \times L^{q_m} \to L^q. \tag{1.7}$$

In fact, consider the bilinear case, the bilinear Littlewood–Paley g-function can be rewritten as a 4-linear Fourier multiplier with symbol $m(\xi,\eta)=\int_0^\infty \widehat{\psi}(t\xi)\widehat{\psi}(t\eta)(dt/t)$. By the results of Grafakos–Miyachi–Tomita [**17**] and assume the kernel is sufficient smooth, we know that the bilinear Littlewood–Paley g-function does satisfy the boundedness in (1.7), which shows our assumption (1.7) is reasonable.

Our main results in this paper are as follows:

Theorem 1.1. *The operator g satisfies the endpoint estimate $L^1\times\cdots\times L^1\to L^{1/m,\infty}$, i.e., if $f_1,\dots,f_m\in L^1$, for any $\lambda>0$, there exists a constant $C>0$ such that*

$$\big|\{x\in\mathbb{R}^n : g(\vec{f})(x)>\lambda\}\big|\le \frac{C}{\lambda^{1/m}}\prod_{j=1}^m\Big(\int_{\mathbb{R}^n}|f_i|\Big)^{1/m}. \tag{1.8}$$

Let $\nu_{\vec{\omega}}$ be the multiple weights defined in Section 2. Then we obtain:

Theorem 1.2. *Suppose $\vec{\omega}\in A_{\vec{p}}$ and $1\le p_1,\dots,p_m<\infty$ with $1/p=1/p_1+\cdots+1/p_m$.*

(i) *If there is no $p_i=1$, then g is of type $L^{p_1}(\omega_1)\times\cdots\times L^{p_m}(\omega_m)\to L^p(\nu_{\vec{\omega}})$, i.e., there is a $C>0$ such that*

$$\|g(\vec{f})\|_{L^p(\nu_{\vec{\omega}})}\le C\prod_{i=1}^m\|f_i\|_{L^{p_i}(\omega_i)}. \tag{1.9}$$

(ii) *If there is a $p_i=1$, then g is of weak type $L^{p_1}(\omega_1)\times\cdots\times L^{p_m}(\omega_m)\to L^{p,\infty}(\nu_{\vec{\omega}})$, i.e., there exists a constant $C>0$ such that*

$$\|g(\vec{f})\|_{L^{p,\infty}(\nu_{\vec{\omega}})}\le C\prod_{i=1}^m\|f_i\|_{L^{p_i}(\omega_i)}. \tag{1.10}$$

Given any positive integer m, the iterated commutators of the multilinear Littlewood–Paley g-function are defined by

$$g_{\Pi\vec{b}}(\vec{f})(x)=\bigg(\int_0^\infty \big|[b,\psi_t*f](x)\big|^2\frac{dt}{t}\bigg)^{1/2}, \tag{1.11}$$

where $\vec{b}=(b_1,\dots,b_m)\in \mathrm{BMO}^m$ and

$$[b,\psi_t*f](x)=\int_{(\mathbb{R}^n)^m}\psi_t(x-y_1,\dots,x-y_m)\bigg[\prod_{j=1}^m(b_j(x)-b_j(y_j))\bigg]\prod_{i=1}^m f_i(y_i)dy_i.$$

We will prove the following strong and end-point estimates for $g_{\Pi\vec{b}}$.

Theorem 1.3. *Let $\vec{\omega}\in A_{\vec{p}}$ with $1/p=1/p_1+\cdots+1/p_m$, $1<p_j<\infty$, $j=1,\dots,m$. If $\vec{b}\in \mathrm{BMO}^m$. Then there exists a constant $C>0$ such that*

$$\|g_{\Pi\vec{b}}(\vec{f})\|_{L^p(\nu_{\vec{\omega}})} \leq C\prod_{i=1}^m \|b_i\|_{\mathrm{BMO}} \prod_{j=1}^m \|f_j\|_{L^{p_j}(\omega_j)}. \tag{1.12}$$

Theorem 1.4. *Let $\vec{\omega} \in A_{\vec{1}}$, $\vec{b} \in \mathrm{BMO}^m$. Then there exists a constant C depending on $\vec{b}$ such that*

$$\nu_{\vec{\omega}}\big(\{x \in \mathbb{R}^n : |g_{\Pi\vec{b}}(\vec{f})(x)| > t^m\}\big) \leq C\prod_{i=1}^m \left(\int_{\mathbb{R}^n} \Phi^{(m)}\left(\frac{|f_j(x)|}{t}\right)\omega_j(x)dx\right)^{1/m}, \tag{1.13}$$

where $\Phi(t) = t(1+\log^+ t)$ and $\Phi^{(m)} = \overbrace{\Phi\circ\cdots\circ\Phi}^{m}$. The estimate is sharp in the sense that $\Phi^{(m)}$ can not be replaced by $\Phi^{(k)}$ for $k < m$.

2. Definitions and notations.

Definition 2.1 ([**23**] Multiple weights). Let $1 \leq p_1, \dots, p_m < \infty$, p satisfies $1/p = 1/p_1 + \cdots + 1/p_m$. Given $\vec{\omega} = (\omega_1, \dots, \omega_m)$, set $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i^{p/p_i}$. We say that $\vec{\omega}$ satisfies the $A_{\vec{p}}$ condition if

$$\sup_B \left(\frac{1}{|B|}\int_B \prod_{i=1}^m \omega_i^{p/p_i}\right)^{1/p} \prod_{i=1}^m \left(\frac{1}{|B|}\int_B \omega_i^{1-p_i'}\right)^{1/p_i'} < \infty,$$

when

$$p_i = 1, \quad \left(\frac{1}{|B|}\int_B \omega_i^{1-p_i'}\right)^{1/p_i'}$$

is understood as $(\inf_B \omega_i)^{-1}$.

A kind of new multilinear maximal operator $\mathcal{M}$ was introduced in [**23**] in the following way

$$\mathcal{M}(\vec{f})(x) = \sup_{x\in Q}\prod_{i=1}^m \frac{1}{|Q|}\int_Q |f_i(y)|dy,$$

where the supremum is taken over all cubes Q containing x.

The authors also gave the weighted estimates for $\mathcal{M}$ as follows:

(i) If $1 < p_1, \dots, p_m < \infty$, both $\mathcal{M}$ can be extended to be a bounded operator

$$L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m) \to L^p(\nu_{\vec{\omega}}). \tag{2.1}$$

(ii) If $1 \leq p_1, \dots, p_m < \infty$, both $\mathcal{M}$ can be extended to be a bounded operator

$$L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m) \to L^{p,\infty}(\nu_{\vec{\omega}}). \tag{2.2}$$

DEFINITION 2.2 (Orlicz norms [**28**]). For $\Phi(t) = t(1+\log^+ t)$ and a cube Q in $\mathbb{R}^n$ we will consider the average $\|f\|_{\Phi,Q}$ of a function f given by the Luxemburg norm

$$\|f\|_{L\log L,Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q \Phi\left(\frac{|f(y)|}{\lambda}\right)dy \le 1\right\}.$$

We need some basic estimates of Orlicz spaces. First

$$\|f\|_{\Phi,Q} > 1 \qquad \text{if and only if} \qquad \frac{1}{|Q|}\int_Q \Phi(|f(y)|)dy > 1. \tag{2.3}$$

Then, using the generalized Hölder inequality in Orlicz spaces and the John-Nirenberg's inequality, we get

$$\frac{1}{|Q|}\int_B |(b-b_Q)|f \le C\|b\|_{\mathrm{BMO}}\|f\|_{L(\log L),Q}. \tag{2.4}$$

The maximal function (see [**38**] and [**23**])

$$M_{L(\log L)}(f)(x) = \sup_{Q\ni x}\|f\|_{L(\log L),Q},$$

will be used, which satisfies the pointwise equivalence

$$M_{L(\log L)}(f)(x) \approx M^2 f(x), \tag{2.5}$$

where M is the Hardy–Littlewood maximal operator. Multilinear maximal type operator $\mathcal{M}_{L(\log L)}$ (see [**23**]) will also be used:

$$\mathcal{M}_{L(\log L)}(\vec{f})(x) = \sup_{x\in Q}\prod_{i=1}^m \|f_i\|_{L(\log L),Q},$$

where the supremum is taken over all cubes Q containing x.

DEFINITION 2.3 (Sharp maximal functions (see [**15**] and [**23**])). For $\delta > 0$, M_δ is the maximal function

$$M_\delta f(x) = M(|f|^\delta)^{1/\delta}(x) = \left(\sup_{Q\ni x}\frac{1}{|Q|}\int_Q |f(y)|^\delta dy\right)^{1/\delta}.$$

In addition, $M^\sharp$ is the Sharp maximal function of Fefferman and Stein [**15**],

$$M^\sharp f(x) = \sup_{Q\ni x}\inf_c \frac{1}{|Q|}\int_Q |f-c|dy \approx \sup_{Q\ni x}\frac{1}{|Q|}\int_Q |f-f_Q|dy$$

and

$$M_\delta^\sharp f(x) = M^\sharp(|f|^\delta)^{1/\delta}(x).$$

The famous Fefferman–Stein type inequality is also needed.

LEMMA 2.1 ([**15**]). *Let $0 < p, \delta < \infty$ and let ω be any Mackenhoupt A_∞ weight. Then there exists a constant C independent of f such that the inequality*

$$\int_{\mathbb{R}^n} (M_\delta f(x))^p \omega(x) dx \le C \int_{\mathbb{R}^n} (M_\delta^\sharp f(x))^p \omega(x) dx, \tag{2.6}$$

holds for any function f for which the left-hand side is finite.

Moreover, if $\varphi : (0, \infty) \to (0, \infty)$ is doubling, then there is a C depending on the A_∞ constant of ω and the doubling condition of φ, such that

$$\sup_{\lambda>0} \varphi(\lambda)\omega(\{x \in \mathbb{R}^n : M_\delta f(x) < \lambda\}) \le C \sup_{\lambda>0} \varphi(\lambda)\omega(\{x \in \mathbb{R}^n : M_\delta^\sharp f(x) < \lambda\}), \tag{2.7}$$

again for any f such that the left hand side is finite.

3. Endpoint estimate for g.

First we give a pointwise estimate for $\psi_t * \vec{f}(x)$.

LEMMA 3.1 (Uniform pointwise estimate). *There exists a constant $C > 0$ independent of t such that*

$$\psi_t * \vec{f}(x) \le C\mathcal{M}(\vec{f})(x). \tag{3.1}$$

PROOF. It is easy to see that (3.1) follows by only invoking the size estimate of ψ. In fact, write simply

$$(\mathbb{R}^n)^m = \underbrace{(Q, \ldots, Q)}_{m} \cup E_2 \cup \underbrace{(Q^c, \ldots, Q^c)}_{m} = Q_1 \cup Q_2 \cup Q_3,$$

where in Q_2, Q appears ℓ times and Q^c appears $m - \ell$ times for $1 \le \ell < m$, Q is a n-dimensional cube centered at x and its measure is $(2t)^n$. Then

$$|\psi_t * \vec{f}(x)| \le C \sum_{k=1}^{3} \int_{Q_k} \frac{t^\delta}{(t + \sum_{i=1}^m |y_i|)^{mn+\delta}} \prod_{j=1}^m |f_j(x - y_j)| dy_j.$$

Therefore, (3.1) follows by analyzing each terms on the right side of the above inequality, which is a pretty much easy task. □

Proof of Theorem 1.1.

PROOF. Without loss of generalities, we assume each $\|f_i\|_{L^1} = 1$. Initially we perform a Calderón–Zygmund decomposition of each f_i. Write $f_i = d_i + b_i$, then there is a non-overlapping collection of cubes $\{Q_i^{(k_i)}\}$ such that $\mathrm{supp}\{b_i^{(k_i)}\} \subset Q_i^{(k_i)}$ where each $b_i = \sum_{k_i} b_i^{(k_i)}$. Moreover, we have

$$\int_{Q_i^{(k_i)}} b_i^{(k_i)} = 0, \qquad \Big|\bigcup_{k_i} Q_i^{(k_i)}\Big| \le \frac{C}{\lambda^{1/m}}, \tag{3.2}$$

$$\|b_i^{(k_i)}\|_{L^1} \le C\lambda^{1/m}|Q_i^{(k_i)}|, \qquad \|d_i\|_{L^\infty} \le C\lambda^{1/m}, \qquad \|d_i\|_{L^{q_i}} \le C\lambda^{1/mq_i'}. \tag{3.3}$$

For each $s = 1, \dots, 2^m$, denote $E_s = \{x \in \mathbb{R}^n : |g(\vec{h})(x)| > \lambda\}$, where $\vec{h} = (h_1, \dots, h_m)$, $h_i \in \{d_i, b_i\}$, $i = 1, \dots, m$, and every E_s is distinct from each other. In particular, denote $\vec{b} = (b_1, \dots, b_m)$, $\vec{d} = (d_1, \dots, d_m)$ and write $E_1 = \{x \in \mathbb{R}^n : |g(\vec{d})(x)| > \lambda\}$ and $E_{2^m} = \{x \in \mathbb{R}^n : |g(\vec{b})(x)| > \lambda\}$. It suffices to show that for every s there is a constant $C > 0$, such that

$$|E_s| \le \frac{C}{\lambda^{1/m}} \tag{3.4}$$

instead of (1.8).

By Chebychev's inequalities, the boundedness (1.7) and (3.3) for d_i, we have

$$|E_1| \le \frac{1}{\lambda^q}\|g(\vec{d})\|_{L^q}^q \le \frac{C}{\lambda^q}\prod_{i=1}^m \|d_i\|_{L^{q_i}}^q \le \frac{C}{\lambda^q}\prod_{i=1}^m \lambda^{q/mq_i'} = \frac{C}{\lambda^{1/m}}. \tag{3.5}$$

For usual E_s, it is clear that

$$\begin{aligned} |E_s| &= \left|\left\{x \in \mathbb{R}^n : \int_0^{+\infty} |\psi_t * \vec{h}(x)|^2 \frac{dt}{t} > \lambda^2\right\}\right| \\ &\le \left|\left\{x \in \mathbb{R}^n : \sup_{t>0} |\psi_t * \vec{h}(x)| > \lambda\right\}\right| + \left|\left\{x \in \mathbb{R}^n : \int_0^{+\infty} |\psi_t * \vec{h}(x)| \frac{dt}{t} > \lambda\right\}\right|. \end{aligned} \tag{3.6}$$

By (3.6) and (3.1), it is enough to prove

$$\left|\left\{x \in \mathbb{R}^n : \int_0^{+\infty} |\psi_t * \vec{h}(x)| \frac{dt}{t} > \lambda\right\}\right| \le \frac{C}{\lambda^{1/m}}. \tag{3.7}$$

Without lost of generalities, we suppose the bad functions appear at the entries $1, \dots, \ell$, where $1 < \ell \le m$, and the good functions appear at other entries, that is

$$h_i = \begin{cases} b_i, & i = 1, \dots, \ell \\ d_i, & i = \ell+1, \dots, m. \end{cases}$$

Let $c_i^{(k_i)}$ be the center of $Q_i^{(k_i)}$ and $B_i^{(k_i)}$ be a certain dimensional dilation of $Q_i^{(k_i)}$ with the same center $c_i^{(k_i)}$. Write $S_i = \bigcup_{k_i} B_i^{(k_i)}$, we will obtain a pointwise estimate for

$\int_0^\infty |\psi_t * \vec{h}(x)|(dt/t)$ with x outside $\bigcup_{i=1}^{\ell} S_i$ and employ the size estimate for $\bigcup_{i=1}^{\ell} S_i$ to show (3.7).

$$\begin{aligned}
&\int_0^\infty |\psi_t * \vec{h}(x)| \frac{dt}{t} \\
&\quad \le \sum_{k_1,\dots,k_\ell} \int_0^\infty \left| \int_{(\mathbb{R}^n)^m} \psi_t(x-y_1,\dots,x-y_m) \prod_{i=1}^{\ell} b_i^{k_i}(y_i) \prod_{i=\ell+1}^{m} g_i(y_i) d\vec{y} \right| \frac{dt}{t} \\
&\quad = \sum_{k_1,\dots,k_\ell} \int_0^\infty |F_{k_1,\dots,k_\ell}^{(t)}| \frac{dt}{t}, \qquad (3.8)
\end{aligned}$$

where $F_{k_1,\dots,k_\ell}^{(t)} = \int_{(\mathbb{R}^n)^m} \psi_t(x-y_1,\dots,x-y_m) \prod_{i=1}^{\ell} b_i^{k_i}(y_i) \prod_{i=\ell+1}^{m} g_i(y_i) d\vec{y}$.

For fixed $k_1,\dots,k_\ell$, we have the following claim:

Claim. *For any $x \notin \bigcup_{i=1}^{\ell} S_i$, there is a constant $C>0$ independent of $k_1,\dots,k_\ell$, such that*

$$\int_0^\infty |F_{k_1,\dots,k_\ell}^{(t)}| \frac{dt}{t} \le C \prod_{i=\ell+1}^{m} \|d_i\|_{L^\infty} \prod_{i=1}^{\ell} \frac{\|b_i^{(k_i)}\|_{L^1} (|Q_i^{(k_i)}|^{1/n})^{\gamma/\ell}}{(|x-c_i^{(k_i)}| + |Q_i^{(k_i)}|^{1/n})^{n+\gamma/\ell}}. \qquad (3.9)$$

Proof. To show the above claim is true, we first fix $x \notin \bigcup_{i=1}^{\ell} S_i$ and set $a_i = (1/2)|x-c_i^{(k_i)}|$. Without lost of generalities, we suppose that $|Q_1^{(k_1)}|$ is the smallest size of $\{|Q_1^{(k_1)}|,\dots,|Q_m^{(k_m)}|\}$ and assume $a_1 \le a_2 \le \cdots \le a_l$. Now, split the left hand side of (3.9) into three parts and consider several cases.

$$\int_0^\infty |F_{k_1,\dots,k_\ell}^{(t)}| \frac{dt}{t} = \left(\int_0^{a_1} + \sum_{j=1}^{l-1} \int_{a_j}^{a_{j+1}} + \int_{a_l}^{\infty} \right) |F_{k_1,\dots,k_\ell}^{(t)}| \frac{dt}{t} := J_1 + J_2 + J_3.$$

Case 1. When $2|y_1 - c_1^{(k_1)}| \le \max_{1\le i\le m}(|x-y_i|)$.

We split it into three subcases.

- Estimate for J_1 in the case $2|y_1 - c_1^{(k_1)}| \le \max_{1\le i\le m}(|x-y_i|)$.

When $2|y_1 - c_1^{(k_1)}| \le \max_{1\le i\le m}(|x-y_i|)$, by the cancelation condition in (3.2) and (1.1) we have

$$\left| \int_{Q_1^{(k_1)}} \psi_t(x-y_1,\dots,x-y_m) b_1^{(k_1)}(y_1) dy_1 \right| \le C \int_{Q_1^{(k_1)}} \frac{|b_1^{(k_1)}(y_1)||Q_1^{(k_1)}|^{\gamma/n} t^\delta}{(t+\sum_{i=1}^m |x-y_i|)^{mn+\delta+\gamma}} dy_1.$$

Multiplying the above derived inequalities by all the good functions and integrating over all y_i at the entries $\{\ell+1,\dots,m\}$ and taking $0<\varepsilon<\delta$, we have

$$
\begin{aligned}
H := &\Big| \int_{(\mathbb{R}^n)^{m-\ell}} \int_{\mathbb{R}^n \bigcap Q_1^{(k_1)}} \psi_t(x-y_1,\dots,x-y_m) b_1^{(k_1)}(y_1)dy_1 \prod_{i=\ell+1}^m d_i(y_i)dy_i \Big| \\
&\le C \prod_{i=\ell+1}^m \|d_i\|_{L^\infty} \int_{Q_1^{(k_1)}} \frac{|b_1^{(k_1)}(y_1)||Q_1^{(k_1)}|^{\gamma/n} t^\delta}{a_\ell^{n\ell+\delta-\varepsilon+\gamma}} dy_1 \\
&\quad \times \int_{(\mathbb{R}^n)^{m-\ell}} \frac{\prod_{i=\ell+1}^m dy_i}{(t+\sum_{i=1}^m |x-y_i|)^{(m-\ell)n+\varepsilon}}. \qquad (3.10)
\end{aligned}
$$

Observe that

$$
\int_{(\mathbb{R}^n)^{m-\ell}} \frac{\prod_{i=\ell+1}^m dy_i}{(t+\sum_{i=1}^m |x-y_i|)^{(m-\ell)n+\varepsilon}} \le C t^{-\varepsilon}. \qquad (3.11)
$$

Thus, multiplying left bad functions and integrating over all y_i at other entries, we get

$$
\begin{aligned}
&\Big| \int_{(\mathbb{R}^n)^m} \psi_t(x-y_1,\dots,x-y_m) \prod_{i=1}^\ell b_i^{(k_i)}(y_i)dy_i \prod_{i=\ell+1}^m d_i(y_i)dy_i \Big| \\
&\qquad \le C t^{\delta-\varepsilon} \prod_{i=\ell+1}^m \|d_i\|_{L^\infty} \prod_{i=1}^\ell \frac{\|b_i^{(k_i)}\|_{L^1} |Q_i^{(k_i)}|^{\gamma/n\ell}}{a_\ell^{n+(\gamma+\delta)/\ell-\varepsilon}}. \qquad (3.12)
\end{aligned}
$$

Therefore,

$$
\big|F_{k_1,\dots,k_\ell}^{(t)}\big| \le C t^{\delta-\varepsilon} \prod_{i=\ell+1}^m \|d_i\|_{L^\infty} \prod_{i=1}^\ell \frac{\|b_i^{(k_i)}\|_{L^1} |Q_i^{(k_i)}|^{\gamma/n\ell}}{a_\ell^{n+(\gamma+\delta)/\ell-\varepsilon}}.
$$

Hence, recall our definition of a_l and $|x-c_i^{(k_i)}| \sim d(x, Q_i^{(k_i)})$, we obtain

$$
\begin{aligned}
\int_0^{a_1} \big|F_{k_1,\dots,k_\ell}^{(t)}\big| \frac{dt}{t} &\le C \int_0^{a_1} \frac{t^{\delta-\varepsilon}}{t} dt \prod_{i=\ell+1}^m \|d_i\|_{L^\infty} \prod_{i=1}^\ell \frac{\|b_i^{(k_i)}\|_{L^1} |Q_i^{(k_i)}|^{\gamma/n\ell}}{a_\ell^{n+(\gamma+\delta-\varepsilon)/\ell}} \\
&\le C \prod_{i=\ell+1}^m \|d_i\|_{L^\infty} \prod_{i=1}^\ell \frac{\|b_i^{(k_i)}\|_{L^1} |Q_i^{(k_i)}|^{\gamma/n\ell}}{(|x-c_i^{(k_i)}| + |Q_i^{(k_i)}|^{1/n})^{n+\gamma/\ell}}. \qquad (3.13)
\end{aligned}
$$

- Estimate for J_2 when $2|y_1 - c_1^{(k_1)}| \le \max_{1\le i\le m}(|x-y_i|)$.

By almost the same argument as in (3.10) and (3.12) and the inequalities above, we obtain

$$\int_{a_j}^{a_{j+1}} \left|F_{k_1,\ldots,k_\ell}^{(t)}\right| \frac{dt}{t} \le C \int_{a_j}^{a_{j+1}} \frac{t^{\delta-\varepsilon}}{t} dt \prod_{i=\ell+1}^{m} \|d_i\|_{L^\infty} \prod_{i=1}^{\ell} \frac{\|b_i^{(k_i)}\|_{L^1} |Q_i^{(k_i)}|^{\gamma/n\ell}}{a_\ell^{n+(\gamma+\delta-\varepsilon)/\ell}}$$

$$\le C \prod_{i=\ell+1}^{m} \|d_i\|_{L^\infty} \prod_{i=1}^{\ell} \frac{\|b_i^{(k_i)}\|_{L^1} |Q_i^{(k_i)}|^{\gamma/n\ell}}{(|x-c_i^{(k_i)}| + |Q_i^{(k_i)}|^{1/n})^{n+\gamma/\ell}}. \tag{3.14}$$

Hence, we have shown that

$$\sum_{j=1}^{\ell} \int_{a_j}^{a_{j-1}} \left|F_{k_1,\ldots,k_\ell}^{(t)}\right| \frac{dt}{t} \le C \prod_{i=\ell+1}^{m} \|d_i\|_{L^\infty} \prod_{i=1}^{\ell} \frac{\|b_i^{(k_i)}\|_{L^1} |Q_i^{(k_i)}|^{\gamma/n\ell}}{(|x-c_i^{(k_i)}| + |Q_i^{(k_i)}|^{1/n})^{n+\gamma/\ell}}. \tag{3.15}$$

- Estimate for J_3 when $2|y_1 - c_1^{(k_1)}| \le \max_{1\le i\le m}(|x-y_i|)$.

Observe that H in (3.10) can be changed as

$$H := \left| \int_{(\mathbb{R}^n)^{m-\ell}} \int_{\mathbb{R}^n \bigcap Q_1^{(k_1)}} \psi_t(x-y_1,\ldots,x-y_m) b_1^{(k_1)}(y_1) dy_1 \prod_{i=\ell+1}^{m} d_i(y_i) dy_i \right|$$

$$\le C \prod_{i=\ell+1}^{m} \|d_i\|_{L^\infty} \int_{Q_1^{(k_1)}} \frac{|b_1^{(k_1)}(y_1)| |Q_1^{(k_1)}|^{\gamma/n} t^\delta}{a_\ell^{n\ell+\delta-\ell+\gamma}} dy_1$$

$$\times \int_{(\mathbb{R}^n)^{m-\ell}} \frac{\prod_{i=\ell+1}^{m} dy_i}{(t+\sum_{i=1}^{m} |x-y_i|)^{(m-\ell)n+\ell}}. \tag{3.16}$$

Now, repeating the step from (3.11)–(3.12) with necessary changes, we obtain

$$H \le C \prod_{i=\ell+1}^{m} \|d_i\|_{L^\infty} \int_{Q_1^{(k_1)}} \left|b_1^{(k_1)}(y_1)\right| \frac{|Q_1^{(k_1)}|^{\gamma/n} t^\delta}{a_\ell^{n\ell+\gamma+\delta-\ell}} dy_1 \cdot t^{-\ell}$$

$$\le C t^{-\ell+\delta} \prod_{i=\ell+1}^{m} \|d_i\|_{L^\infty} \|b_1^{(k_1)}\|_{L^1} \prod_{i=1}^{\ell} \frac{|Q_i^{(k_i)}|^{\gamma/n\ell}}{a_\ell^{n+(\gamma+\delta)/\ell-1}}. \tag{3.17}$$

Therefore,

$$\left|F_{k_1,\ldots,k_\ell}^{(t)}\right| \le C t^{-\ell+\delta} \prod_{i=\ell+1}^{m} \|d_i\|_{L^\infty} \prod_{i=1}^{\ell} \frac{\|b_i^{(k_i)}\|_{L^1} |Q_i^{(k_i)}|^{\gamma/n\ell}}{(|x-c_i^{(k_i)}| + |Q_i^{(k_i)}|^{1/n})^{n+(\gamma+\delta)/\ell-1}}.$$

Hence, we obtain

$$\int_{a_l}^{\infty} \left|F_{k_1,\ldots,k_\ell}^{(t)}\right| \frac{dt}{t} \le C \prod_{i=\ell+1}^{m} \|d_i\|_{L^\infty} \prod_{i=1}^{\ell} \frac{\|b_i^{(k_i)}\|_{L^1} |Q_i^{(k_i)}|^{\gamma/n\ell}}{(|x-c_i^{(k_i)}| + |Q_i^{(k_i)}|^{1/n})^{n+\gamma/\ell}}. \tag{3.18}$$

Case 2. When $2|y_1 - c_1^{(k_1)}| \geq \max_{1\leq i\leq m}\{|x-y_i|\}$.

By the size condition of ψ, we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}^n \cap Q_1^{(k_1)}} \psi_t(x-y_1,\dots,x-y_m) b_1^{(k_1)}(y_1)dy_1 \right| \\
&\leq C \int_{Q_1^{(k_1)}} |b_1^{(k_1)}(y_1)| \frac{t^\delta}{(t+\sum_{i=1}^m |x-y_i|)^{mn+\delta}} \frac{|y_1 - c_1^{(k_1)}|^\gamma}{\max_{1\leq i\leq m}\{|x-y_i|\}^\gamma} dy_1 \\
&\leq C \int_{Q_1^{(k_1)}} |b_1^{(k_1)}(y_1)| \frac{t^\delta}{(t+\sum_{i=1}^m |x-y_i|)^{mn+\delta}} \frac{|Q_1^{(k_1)}|^{\gamma/n}}{\max_{1\leq i\leq m}\{|x-y_i|\}^\gamma} dy_1.
\end{aligned}$$

Therefore, we may estimate each J_i similar as what we have done to deal (3.11)–(3.18), here we omit the detail.

Thus, we finish the proof of (3.9).

Next, by (3.3), (3.8), (3.9), we have if $x \notin \bigcup_{i=1}^\ell S_i$, then

$$\begin{aligned}
\int_0^\infty |\psi_t * \vec{h}(x)| \frac{dt}{t} &\leq C \sum_{k_1,\dots,k_\ell} \prod_{i=\ell+1}^m \|d_i\|_{L^\infty} \prod_{i=1}^\ell \frac{\|b_i^{(k_i)}\|_{L^1} |Q_i^{(k_i)}|^{\gamma/n\ell}}{(|x-c_i^{(k_i)}| + |Q_i^{(k_i)}|^{1/n})^{n+\gamma/\ell}} \\
&\leq C\lambda \prod_{i=1}^\ell \mathscr{T}_{i,\gamma/\ell}(x),
\end{aligned}$$

where

$$\mathscr{T}_{i,\gamma/\ell}(x) = \sum_{k_i} \frac{(|Q_i^{(k_i)}|^{1/n})^{n+\gamma/\ell}}{(|x-c_i^{(k_i)}| + |Q_i^{(k_i)}|^{1/n})^{n+\gamma/\ell}}$$

is the Marcinkiewicz function associated with the union of the cubes $\{Q_i^{(k_i)}\}_{k_i}$. It has been shown in [**14**] that

$$\int_{\mathbb{R}^n} \mathscr{T}_{i,\gamma/\ell} \leq C \sum_{k_i} |Q_i^{(k_i)}|. \qquad \square$$

Now, invoking the property of the Marcinkiewicz function and (3.3), we obtain

$$\begin{aligned}
\left| \left\{ x \notin \bigcup_{i=1}^\ell S_i : \int_0^\infty \frac{|\psi_t * \vec{h}(x)|}{t} dt > \lambda \right\} \right| &\leq \frac{C}{\lambda^{1/\ell}} \int_{(\bigcup_{i=1}^\ell S_i)^c} \left(\lambda \prod_{i=1}^\ell \mathscr{T}_{i,\gamma/\ell} \right)^{1/\ell} \\
&\leq C \prod_{i=1}^\ell \left(\int_{\mathbb{R}^n} \mathscr{T}_{i,\gamma/\ell} \right)^{1/\ell} \leq C \frac{1}{\lambda^{1/m}}.
\end{aligned}$$

Moreover, since

$$\left|\bigcup_{i=1}^{\ell} S_i\right| \le \frac{C}{\lambda^{1/m}}$$

which is a consequence of (3.2) and the condition that $\{Q_i^{(k_i)}\}_{k_i}$ are nonoverlapping, we proved (3.7).

Finally, we can finish the proof of Theorem 1.1 by combining (3.5), (3.6), (2.2) and (3.7). □

4. Weighted estimate for g.

To prove the weighted boundedness of g, we need the following lemma.

LEMMA 4.1 (Kolmogorov's inequality, [**10**]). *Suppose that $0<\alpha<n$ and $p,q>0$ satisfying $1/q = 1/p - \alpha/n$. Then for any measurable function f and cube Q,*

$$\left(\int_Q |f|^p\right)^{1/p} \le \left(\frac{q}{q-p}\right)^{1/p} |Q|^{\alpha/n} \|f\|_{L^{q,\infty}(Q)}. \tag{4.1}$$

LEMMA 4.2 (Pointwise sharp estimate for g). *Let $0<\delta<1/m$, then there exists a constant $C>0$ only depending on δ such that*

$$M_\delta^\sharp(g(\vec{f}))(x) \le C\mathcal{M}(\vec{f})(x), \tag{4.2}$$

where

$$M_\delta^\sharp(g(\vec{f}))(x) = \left(\sup_{Q\ni x}\inf_{c\in\mathbb{R}} \frac{1}{|Q|}\int_Q \left||g(\vec{f})|^\delta - |c|^\delta\right|\right)^{1/\delta}.$$

PROOF. Fix $x\in\mathbb{R}^n$ and a cube $Q\ni x$, the length of which is d. Denote $f_i^0 = f_i\chi_{2Q}$, then $f_i^\infty = f_i - f_i^0$, we have

$$\prod_{i=1}^m f_i = \prod_{i=1}^m (f_i^0 + f_i^\infty) = \prod_{i=1}^m f_i^0 + \sum_{r_i\in\{0,\infty\},\exists r_j=\infty} f_1^{r_1}\cdots f_m^{r_m},$$

where the latter term is abbreviated to $\vec{f^r}$.

Now, we denote $\bar{\psi}_t(x,z,\vec{y}) = \psi_t(x-y_1,\dots,x-y_m) - \psi_t(z-y_1,\dots,z-y_m)$. Then

$$\begin{aligned}
\left|g(\vec{f})(x) - (g(\vec{f^r}))_Q\right| &\le \frac{1}{|Q|}\int_Q |g(\vec{f})(x) - g(\vec{f^r})(z)|dz\\
&\le \frac{1}{|Q|}\int_Q \left|\left(\int_0^\infty \frac{1}{t^{mn}}\left|\int_{(\mathbb{R}^n)^m} \psi\left(\frac{x-y_1}{t},\dots,\frac{x-y_m}{t}\right)\prod_{i=1}^m f_i(y_i)dy_i\right|^2\frac{dt}{t}\right)^{1/2}\right.\\
&\quad \left. -\left(\int_0^\infty \frac{1}{t^{mn}}\left|\int_{(\mathbb{R}^n)^m} \psi\left(\frac{z-y_1}{t},\dots,\frac{z-y_m}{t}\right)\vec{f^r}\prod_{i=1}^m dy_i\right|^2\frac{dt}{t}\right)^{1/2}\right|dz
\end{aligned}$$

$$\leq g(\vec{f^0})(x) + \sum_{r_i\in\{0,\infty\},\exists r_j=\infty} \frac{1}{|Q|}\int_Q g_K(\vec{f})(x,z)dz, \tag{4.3}$$

where

$$g_K(\vec{f})(x,z) = \left(\int_0^\infty \left|\int_{(\mathbb{R}^n)^m} |\bar{\psi}_t(x,z,\vec{y})| \prod_{i=1}^m f_i^{r_i}(y_i)dy_i\right|^2 \frac{dt}{t}\right)^{1/2}.$$

We will estimate both of the two terms in the right side of (4.3), respectively. Using the Kolmogorov's inequality (4.1) in the case $p=\delta$, $q=1/m$ and (1.8), we deduce

$$\left(\frac{1}{|Q|}\int_Q |g(\vec{f^0})|^\delta\right)^{1/\delta} \leq C\|g(\vec{f^0})\|_{L^{1/m,\infty}(Q,dx/|Q|)} \leq C\mathcal{M}(\vec{f^0})(x). \tag{4.4}$$

For the latter term, we claim a pointwise inequality that for any $z\in Q$

$$|g_K(\vec{f})(x,z)| \leq C\mathcal{M}(\vec{f})(x). \tag{4.5}$$

We may assume that there exists j_0 such that $r_{j_0}=\infty$. Clearly, we can see that

$$\begin{aligned}|g_K(\vec{f})(x,z)| \leq &\left(\int_0^{2d}\left(\int_{(\mathbb{R}^n)^m} |\psi_t(x-y_1,\ldots,x-y_m)| \prod_{i=1}^m f_i^{r_i}(y_i)dy_i\right)^2 \frac{dt}{t}\right)^{1/2}\\ &+\left(\int_0^{2d}\left(\int_{(\mathbb{R}^n)^m} |\psi_t(z-y_1,\ldots,z-y_m)| \prod_{i=1}^m f_i^{r_i}(y_i)dy_i\right)^2 \frac{dt}{t}\right)^{1/2}\\ &+\left(\int_{2d}^{\infty}\left(\int_{(\mathbb{R}^n)^m} |\bar{\psi}_t(x,z,\vec{y})| \prod_{i=1}^m f_i^{r_i}(y_i)dy_i\right)^2 \frac{dt}{t}\right)^{1/2} := I_1+I_2+I_3.\end{aligned}$$

First, we estimate I_1. Note that there exists a j_0 such that $|x-y_{j_0}|>2d$, then

$$\begin{aligned}I_1 &\leq C\prod_{i=1}^m \int_{\mathbb{R}^n} f_i^{r_i}(y_i)\left(\int_0^{2d} \frac{t^{2\delta}t^{-1}dt}{(t+\sum|x-y_i|)^{2mn+2\delta}}\right)^{1/2} dy_i\\ &\leq C\prod_{i=1}^m\sum_{k=1}^\infty \int_{2^kQ\setminus 2^{k-1}Q} \frac{f_i^{r_i}(y_i)}{(\sum_{j=1}^m|x-y_j|)^{mn+\delta}}\left(\int_0^{2d} t^{2\delta-1}dt\right)^{1/2} dy_i\\ &\leq C\sum_{k=1}^\infty \frac{1}{(2^k)^\delta}\prod_{i=1}^m \frac{1}{|2^k|Q|^{1/n}|}\int_{2^kQ} f_i^{r_i}(y_i)dy_i \leq C\mathcal{M}(\vec{f})(x).\end{aligned}$$

Similarly, we get $I_2 \leq C\mathcal{M}(\vec{f})(x)$.

For I_3,

$$
\begin{aligned}
I_3 &\leq C\bigg(\int_{2d}^{\infty}\bigg(\int_{(\mathbb{R}^n)^m}\frac{|x-z|^{\gamma}t^{\delta}}{(t+\sum|x-y_i|)^{mn+\delta+\gamma}}\prod_{i=1}^{m}f_i^{r_i}(y_i)dy_i\bigg)^2\frac{dt}{t}\bigg)^{1/2}\\
&\leq C\int_{(\mathbb{R}^n)^m}\frac{d^{\varepsilon}\prod_{i=1}^{m}f_i^{r_i}(y_i)dy_i}{(\sum_{j=1}^{\infty}|x-y_j|)^{mn+\varepsilon}}\leq C\mathcal{M}(\vec{f})(x),
\end{aligned}
$$

where $\gamma > \varepsilon > 0$.

Until now, we have proved that

$$
\big|g(\vec{f^r})(x)-\big(g(\vec{f^r})\big)_Q\big|\leq C\mathcal{M}(\vec{f})(x), \tag{4.6}
$$

where $\big(g(\vec{f^r})\big)_Q = (1/Q)\int_Q g(\vec{f^r})$. Combining (4.4) with (4.6), we can obtain (4.2). □

In order to employ the Fefferman–Stein's inequalities, we need to introduce another lemma as follows.

Lemma 4.3. *Suppose $f_i \in C_c^{\infty}(\mathbb{R}^n)$ for any $i = 1,\ldots,m$, and* $\operatorname{supp} f_i \subset B(0,R)$. *Then there exists a constant $C_{\vec{f}}$ depending on $\vec{f}$ such that for any $x \in \mathbb{R}^n$ and $|x| > 2R$*

$$
g(\vec{f})(x)\leq C_{\vec{f}}\mathcal{M}(\vec{f})(x) \tag{4.7}
$$

holds uniformly.

Proof. When $|x| > 2R$,

$$
\begin{aligned}
g(\vec{f})(x) &\leq \bigg(\int_0^{|x|}\bigg(\int_{(B(0,R))^m}\psi_t(x-y_1,\ldots,x-y_m)\prod_{i=1}^{m}f_i^{r_i}(y_i)dy_i\bigg)^2\frac{dt}{t}\bigg)^{1/2}\\
&\quad+\bigg(\int_{|x|}^{\infty}\bigg(\int_{(B(0,R))^m}\psi_t(x-y_1,\ldots,x-y_m)\prod_{i=1}^{m}f_i^{r_i}(y_i)dy_i\bigg)^2\frac{dt}{t}\bigg)^{1/2}\\
&:= I_1 + I_2.
\end{aligned}
$$

It is easy to see that $|x-y_i| \sim |x|$, since $|x-y_i| \geq |x|/2$ and $|x-y_i| \leq 3|x|/2$. So we have $|x-y_i| \sim |x|$. For I_1

$$
\begin{aligned}
I_1 &\leq \bigg(\int_0^{|x|}\bigg(\int_{(B(0,R))^m}\frac{t^{\delta}}{(t+\sum|x-y_i|)^{mn+\delta}}\prod_{i=1}^{m}f_i^{r_i}(y_i)dy_i\bigg)^2\frac{dt}{t}\bigg)^{1/2}\\
&\leq C\int_{(B(0,R))^m}\frac{1}{(|x-y_i|)^{mn}}\prod_{i=1}^{m}f_i^{r_i}(y_i)dy_i \leq C\mathcal{M}(\vec{f})(x).
\end{aligned}
$$

For I_2

$$\begin{aligned}I_2 &\le \bigg(\int_{|x|}^{\infty}\bigg(\int_{(B(0,R))^m}\frac{t^{\delta}}{(t+\sum|x-y_i|)^{mn+\delta}}\prod_{i=1}^m f_i^{r_i}(y_i)dy_i\bigg)^2\frac{dt}{t}\bigg)^{1/2}\\ &\le C\int_{(B(0,R))^m}\prod_{i=1}^m f_i^{r_i}(y_i)\bigg(\int_{|x|}^{\infty}\frac{1}{t^{2mn+1}}dt\bigg)^{1/2}dy_i\\ &\le C\mathcal{M}(\vec{f})(x)\prod_{i=1}^m\|f_i\|_{L^1}.\end{aligned}$$

The proof of this lemma is finished. □

After having the above preparations, we are in a position to prove Theorem 1.2.

Proof of Theorem 1.2.

Proof. First, we show that Theorem 1.2 (i) holds. By (2.1), we can assume that

$$\bigg(\int_{\mathbb{R}^n}(\mathcal{M}(\vec{f}))^p\nu_{\vec{w}}\bigg)^{1/p}<\infty.$$

Without loss of generality, we may suppose $f_i>0$ and $\vec{f}\in(C_c^{\infty}(\mathbb{R}^n))^m$, we will first show that

$$\int_{\mathbb{R}^n}\big(g(\vec{f})\big)^p\upsilon_{\vec{\omega}}dx<\infty.$$

We may only consider the special case that the weights $\nu_{\vec{w}}$ are bounded functions. For the general case, denote $\nu_{\vec{w}_N}=\inf\{\nu_{\vec{w}},N\}$, then $[\nu_{\vec{w}_N}]_{A_\infty}\le C[\nu_{\vec{w}}]_{A_\infty}$. Since $|g(\vec{f})|^p\nu_{\vec{w}_N}(x)$ converges everywhere to $|g(\vec{f})|^p\nu_{\vec{w}}(x)$ as $N\to\infty$, by Fatou's lemma, we can obtain the results for the general case.

To see the proof for the special case with $\vec{f}\in(C_c^{\infty}(\mathbb{R}^n))^m$, note that

$$\int_{\mathbb{R}^n}\big(g(\vec{f})\big)^p\upsilon_{\vec{\omega}}dx\le\int_{2B}\big(g(\vec{f})\big)^p\upsilon_{\vec{\omega}}dx+\int_{(2B)^c}\big(g(\vec{f})\big)^p\upsilon_{\vec{\omega}}dx.$$

From the assumption of $\mathcal{M}$ and (4.7), we get

$$\int_{(2B)^c}\big(g(\vec{f})\big)^p\upsilon_{\vec{\omega}}dx\le C_{\vec{f}}\int_{\mathbb{R}^n}\big(\mathcal{M}(\vec{f})\big)^p\upsilon_{\vec{\omega}}<\infty.$$

Using the Holder's inequality, we have

$$\begin{aligned}\int_{2B}\big(g(\vec{f})\big)^p\upsilon_{\vec{\omega}}dx &\le\bigg(\int_{2B}\upsilon_{\vec{\omega}}^{1/(1-\rho)}dx\bigg)^{1-\rho}\bigg(\int_{2B}g(\vec{f})^{p/\rho}dx\bigg)^{\rho}\\ &\le CR^{n(1-\rho)}\prod_{i=1}^m\|\omega_i\|_\infty^{p/p_i\cdot1/(1-\rho)}\prod_{i=1}^m\|f\|_{L^{p/\rho}}^p<\infty.\end{aligned}$$

Hence, we obtain

$$\left(\int_{\mathbb{R}^n} \left(g(\vec{f})\right)^p v_{\vec{\omega}}\right)^{1/p} < \infty. \tag{4.8}$$

By Lemma 2.1, we only need to prove that

$$\left(\int_{\mathbb{R}^n} (\mathcal{M}_\rho(g_\lambda^*(\vec{f})))^p \nu_{\vec{w}}\right)^{1/p} < \infty. \tag{4.9}$$

Since $w \in A_\infty$, then there exists $p_0 > 1$, such that $w \in A_{p_0}$. We can take $\rho > 0$ small enough and $p/\rho > p_0$ such that $w \in A_{p/\rho}$. The $L^{p/\rho}$ bounds of $\mathcal{M}$ and (4.8) yield

$$\left(\int_{\mathbb{R}^n} (\mathcal{M}_\rho(g(\vec{f})))^p \nu_{\vec{w}}\right)^{1/p} \leq C\left(\int_{\mathbb{R}^n} (g(\vec{f}))^p \nu_{\vec{w}}\right)^{1/p} < \infty.$$

Thus, we obtain the desired estimates by applying Fefferman–Stein's inequality,

$$\begin{aligned}\left(\int_{\mathbb{R}^n} \left(g(\vec{f})\right)^p \nu_{\vec{\omega}}\right)^{1/p} &\leq \left(\int_{\mathbb{R}^n} \left(M_\rho\left(g(\vec{f})\right)\right)^p \nu_{\vec{\omega}}\right)^{1/p} \leq \left(\int_{\mathbb{R}^n} \left(M_\rho^\sharp\left(g(\vec{f})\right)\right)^p \nu_{\vec{\omega}}\right)^{1/p} \\ &\leq C\left(\int_{\mathbb{R}^n} \left(\mathcal{M}(\vec{f})\right)^p \nu_{\vec{\omega}}\right)^{1/p} \leq C\prod_{i=1}^m \left(\int_{\mathbb{R}^n} |f_i|^{p_i}\omega_i\right)^{1/p_i}.\end{aligned}$$

The proof of Theorem 1.2 (ii) can be treated as that of Theorem 1.2 (i) with only a slight modifications, we omit its proof here. □

5. Commutators of multilinear *g*-function.

Recently, the iterated commutators of multilinear Calderón–Zygmund operators given below are defined and studied, including the strong type and weak end-point estimates with multiple $A_{\vec{p}}$ weights [**31**].

$$T_{\Pi \mathbf{b}}(\vec{f})(x) = [b_1, [b_2, \dots [b_{m-1}, [b_m, T]_m]_{m-1} \dots]_2]_1(\vec{f})(x). \tag{5.1}$$

Inspired by the above results, we studied the iterated commutators of multilinear g-function defined by

$$\begin{aligned} g_{\Pi\vec{b}}(\vec{f})(x) &= \left\|[b_1, [b_2, \dots [b_{m-1}, [b_m, \psi_t * f(x)]_m]_{m-1} \dots]_2]_1(\vec{f})(x)\right\|_{\mathcal{H}} \\ &= \left\| \int_{(\mathbb{R}^n)^m} \psi_t(x-y_1, \dots, x-y_m)\left[\prod_{j=1}^m (b_j(x) - b_j(y_j))\right] \prod_{i=1}^m f_i(y_i) dy_i \right\|_{\mathcal{H}} \end{aligned} \tag{5.2}$$

where $\psi_t * f(x)(x)$ is the same as before, $\mathcal{H} = \{h : (\int_0^\infty |h|^2 dt/t)^{1/2} < \infty\}$ and $\vec{b} = (b_1, \dots, b_m) \in \mathrm{BMO}^m$.

5.1. Strong weighted estimate.

The proof of the Theorem 1.3 needs a pointwise estimate using sharp maximal functions.

Following [**30**], for positive integers m and j with $1 \le j \le m$, denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \dots, \sigma(j)\}$ of $\{1, \dots, m\}$ of j different elements, where we always take $\sigma(k) < \sigma(\ell)$ if $k < \ell$. For any $\sigma \in C_j^m$, we associate the complementary sequence $\sigma' \in C_{m-j}^m$ given by $\sigma' = \{1, \dots, m\} \backslash \sigma$ with the convention $C_0^m = \emptyset$. Given an m-tuple of function $\vec{b}$ and C_j^m, we also use the notation $\vec{b}_\sigma$ for the j-tuple obtained from $\vec{b}$ given by $(b_{\sigma(1)}, \dots, b_{\sigma(j)})$.

Similar to (5.2), we define for Littlewood–Paley g-function g, $\sigma \in C_j^m$, and $\vec{b}_\sigma = (b_{\sigma(1)}, \dots, b_{\sigma(j)})$ in BMO^j, the iterated commutator

$$g_{\Pi\vec{b}_\sigma}(\vec{f})(x) = \left(\int_0^\infty \left(\int_{(\mathbb{R}^n)^m} \psi_t(x - y_1, \dots, x - y_m) \right.\right.$$
$$\left.\left. \times \left[\prod_{i=1}^j (b_{\sigma_i}(x) - b_{\sigma_i}(y_{\sigma_i})) \right] \prod_{i=1}^m f_i(y_i) dy_i \right)^2 \frac{dt}{t} \right)^{1/2}. \tag{5.3}$$

Clearly, $g_{\Pi\vec{b}_\sigma} = g_{\Pi\vec{b}}$ as defined before when $\sigma = \{1, \dots, m\}$ and $g_{\Pi\vec{b}_\sigma} = g_{b_j}^j$ when $\sigma = \{j\}$.

The pointwise estimate that will serve our purposes is as follows.

Lemma 5.1. *Assume that $g_{\Pi\vec{b}}$ is a multilinear commutator with $\vec{b} \in \mathrm{BMO}^m$ and let $0 < \delta < \varepsilon$, with $0 < \delta < 1/m$. Then there exists a constant $C > 0$ depending on δ and ε such that*

$$M_\delta^\sharp(g_{\Pi\vec{b}}(\vec{f}))(x) \le C \prod_{i=1}^m \|b_i\|_{\mathrm{BMO}} \big(\mathcal{M}_{L(\log L)}(\vec{f})(x) + M_\varepsilon(g(\vec{f}))(x) \big)$$
$$+ C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \prod_{i=1}^j \|b_{\sigma(i)}\|_{\mathrm{BMO}} M_\varepsilon(g_{\Pi\vec{b}_{\sigma'}}(\vec{f}))(x) \tag{5.4}$$

for all m-tuples $\vec{f} = (f_1, \dots, m)$ of bounded measurable functions with compact support.

Proof. The way to interpret (5.4) is

$$M_\delta^\sharp(g_{\Pi\vec{b}}(\vec{f}))(x) \lesssim \prod_{i=1}^m \|b_i\|_{\mathrm{BMO}} \mathcal{M}_{L(\log L)}(\vec{f})(x) + \text{“lower order terms”},$$

as it will become apparent in its application. Here we only consider the case $m = 2$, our method still holds for general m with little modifications. Hence we will limit ourselves to establish the following version of (5.4).

For $b_1, b_2 \in \mathrm{BMO}$, we need to show

$$M_\delta^\sharp(g_{\Pi\vec{b}}(f_1,f_2))(x) \le C\|b_1\|_{\mathrm{BMO}}\|b_2\|_{\mathrm{BMO}}\big(\mathcal{M}_{L(\log L)}(f_1,f_2)(x)+M_\varepsilon(g(f_1,f_2))(x)\big) + C\big(\|b_2\|_{\mathrm{BMO}}M_\varepsilon(g^1_{b_1}(f_1,f_2))(x)+\|b_1\|_{\mathrm{BMO}}M_\varepsilon(g^2_{b_2}(f_1,f_2))(x)\big),$$

where

$$g^1_{b_1}(f_1,f_2)(x)=b_1(x)g(f_1,f_2)(x)-g(b_1f_1,f_2)(x),$$
$$g^2_{b_2}(f_1,f_2)(x)=b_2(x)g(f_1,f_2)(x)-g(f_1,b_2f_2)(x).$$

For any constant λ_1 and λ_2, fix $x\in\mathbb{R}^n$, a cube Q centered at x and a constant c, since $0<\delta<1/2$, we then have

$$\begin{aligned}
&\Big(\frac{1}{|Q|}\int_Q \|g_{\Pi\vec{b}}(\vec{f})(z)|^\delta-|c|^\delta|dz\Big)^{1/\delta}\\
&\le\Big(\frac{C}{|Q|}\int_Q|(b_1(z)-\lambda_1)(b_2(z)-\lambda_2)g(f_1,f_2)(z)|^\delta dz\Big)^{1/\delta}\\
&\quad+\Big(\frac{C}{|Q|}\int_Q|(b_1(z)-\lambda_1)g(f_1,(b_2(\cdot)-\lambda_2)f_2)(z)|^\delta dz\Big)^{1/\delta}\\
&\quad+\Big(\frac{C}{|Q|}\int_Q|(b_2(z)-\lambda_2)g((b_1(\cdot)-\lambda_1)f_1,f_2)(z)|^\delta dz\Big)^{1/\delta}\\
&\quad+\Big(\frac{C}{|Q|}\int_Q|g((b_1(\cdot)-\lambda_1)f_1,(b_2(\cdot)-\lambda_2)f_2)(z)-c|^\delta dz\Big)^{1/\delta}\\
&:=I_1+I_2+I_3+I_4.
\end{aligned}$$

We will analyze each term separately by selecting appropriate constants. Let $Q^*=3Q$ and $\lambda_j=(b_j)_{Q^*}$ be the average of b_j on Q^*, $j=1,2$. For any $1<q_1,q_2,q_3<\infty$ with $1=1/q_1+1/q_2+1/q_3$ and $q_3<\varepsilon/\delta$, we have by the Hölder's inequality,

$$\begin{aligned}
I_1&\le C\Big(\frac{1}{|Q|}\int_Q|b_1(z)-\lambda_1|^{\delta q_1}dz\Big)^{1/\delta q_1}\Big(\frac{1}{|Q|}\int_Q|b_2(z)-\lambda_2|^{\delta q_2}dz\Big)^{1/\delta q_2}\\
&\quad\times\Big(\frac{1}{|Q|}\int_Q|g(f_1,f_2)(z)|^{\delta q_3}dz\Big)^{1/\delta q_3}\\
&\le C\prod_{i=1}^2\|b_i\|_{\mathrm{BMO}}M_{\delta q_3}(g(f_1,f_2))(x)\le C\prod_{i=1}^2\|b_i\|_{\mathrm{BMO}}M_\varepsilon(g(f_1,f_2))(x).
\end{aligned}$$

Since I_2 and I_3 are symmetric, we only study I_2. Let $1<t_1,\,t_2<\infty$ with $1=1/t_1+1/t_2$ and $t_2<\varepsilon/\delta$. Then, by the Hölder's inequality and the Jensen's inequality, we obtain

$$\begin{aligned}I_2 &\le C\Big(\frac{1}{|Q|}\int_Q |b_1(z)-\lambda_1|^{\delta t_1}dz\Big)^{1/\delta t_1}\Big(\frac{1}{|Q|}\int_Q |g^2_{b_2-\lambda_2}(f_1,f_2)(z)dz|^{\delta t_2}dz\Big)^{1/\delta t_2}\\ &\le C\|b_1\|_{\mathrm{BMO}}M_{\delta t_2}(g^2_{b_2-\lambda_2}(f_1,f_2))(x) \le C\|b_1\|_{\mathrm{BMO}}M_{\varepsilon}(g^2_{b_2}(f_1,f_2))(x).\end{aligned}$$

Similarly,

$$I_3 \le C\|b_2\|_{\mathrm{BMO}}M_{\varepsilon}(g^1_{b_1-\lambda_1}(f_1,f_2))(x) = C\|b_2\|_{\mathrm{BMO}}M_{\varepsilon}(g^1_{b_1}(f_1,f_2))(x).$$

As for the the last term I_3, we split each f_i as $f_i = f_i^0 + f_i^\infty$, where $f_i^0 = f\chi_{Q^*}$ and $f_i^\infty = f_i - f_i^0$.

Let $c = \sum_{j=1}^3 c_j$, where $c_1 = g((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^\infty)$, $c_2 = g((b_1-\lambda_1)f_1^\infty, (b_2-\lambda_2)f_2^0)$, $c_3 = g((b_1-\lambda_1)f_1^\infty,(b_2-\lambda_2)f_2^\infty)$. Then, we have

$$\begin{aligned}I_4 \le{}& \Big(\frac{C}{|Q|}\int_Q |g((b_1(\cdot)-\lambda_1)f_1^0,(b_2(\cdot)-\lambda_2)f_2^0)(z)|^\delta dz\Big)^{1/\delta}\\ &+\Big(\frac{C}{|Q|}\int_Q |g((b_1(\cdot)-\lambda_1)f_1^0,(b_2(\cdot)-\lambda_2)f_2^\infty)(z)-c_1|^\delta dz\Big)^{1/\delta}\\ &+\Big(\frac{C}{|Q|}\int_Q |g((b_1(\cdot)-\lambda_1)f_1^\infty,(b_2(\cdot)-\lambda_2)f_2^0)(z)-c_2|^\delta dz\Big)^{1/\delta}\\ &+\Big(\frac{C}{|Q|}\int_Q |g((b_1(\cdot)-\lambda_1)f_1^\infty,(b_2(\cdot)-\lambda_2)f_2^\infty)(z)-c_3|^\delta dz\Big)^{1/\delta}\\ ={}& I_{41}+I_{42}+I_{43}+I_{44}.\end{aligned}$$

For I_{41}, take $1<p<1/2\delta$. Since $p\delta<1/2$, using the Hölder's inequality we get

$$\begin{aligned}I_{41} &\le \Big(\frac{C}{|Q|}\int_Q |g((b_1(\cdot)-\lambda_1)f_1^0,(b_2(\cdot)-\lambda_2)f_2^0)(z)|^{p\delta} dz\Big)^{1/p\delta}\\ &\le C\|g((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^0)\|_{L^{1/2,\infty}(Q,dx/|Q|)}\\ &\le C\prod_{i=1}^2\|b_i\|_{\mathrm{BMO}}\mathcal{M}_{L(\log L)}(f_1,f_2)(x).\end{aligned}$$

Since I_{42} and I_{43} are symmetric, we only consider I_{42}.

$$\begin{aligned}&\big|g((b_1(\cdot)-\lambda_1)f_1^0,(b_2(\cdot)-\lambda_2)f_2^\infty)(z) - g((b_1(\cdot)-\lambda_1)f_1^0,(b_2(\cdot)-\lambda_2)f_2^\infty)(x)\big|\\ &\quad\le \Big(\int_0^{2d}\Big(\int_{(\mathbb{R}^n)^2}|\psi_t(z-y_1,z-y_2)|\prod_{j=1}^2|(b_j(y_j)-\lambda_j)f_1^0(y_1)f_2^\infty(y_2)|dy_1dy_2\Big)^2\frac{dt}{t}\Big)^{1/2}\\ &\qquad+\Big(\int_0^{2d}\Big(\int_{(\mathbb{R}^n)^2}|\psi_t(x-y_1,x-y_2)|\prod_{j=1}^2|(b_j(y_j)-\lambda_j)f_1^0(y_1)f_2^\infty(y_2)|dy_1dy_2\Big)^2\frac{dt}{t}\Big)^{1/2}\end{aligned}$$

$$+\left(\int_{2d}^{\infty}\left(\int_{(\mathbb{R}^n)^2}|\psi_t(z-y_1,z-y_2)-\psi_t(x-y_1,x-y_2)|\right.\right.$$
$$\left.\left.\times\prod_{j=1}^{2}|(b_j(y_j)-\lambda_j)f_1^0(y_1)f_2^\infty(y_2)|dy_1dy_2\right)^2\frac{dt}{t}\right)^{1/2}$$
$$:=I_{421}+I_{422}+I_{423}.$$

Similar to the proof of Lemma 4.2, we have

$$\begin{aligned}
I_{421}&\le Cd^\delta\int_{\mathbb{R}^n\setminus 3Q}\frac{|(b_2(y_2)-\lambda_2)f_2(y_2)|dy_2}{|x-y_2|^{2n+\delta}}\int_{3Q}|(b_1(y_1)-\lambda_1)f_1(y_1)|dy_1\\
&\le C\sum_{k=1}^{\infty}\frac{|Q|^{\delta/n}}{(3^k|Q|^{1/n})^{2n+\delta}}\left(\int_{3^{k+1}Q}|(b_2(y_2)-\lambda_2)f_2(y_2)|dy_2\right)\\
&\quad\times\left(\int_{3^{k+1}Q}|(b_1(y_1)-\lambda_1)f_1(y_1)|dy_1\right)\\
&\le C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}}\mathcal{M}_{L(\log L)}(f_1,f_2)(x);\\
I_{422}&\le C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}}\mathcal{M}_{L(\log L)}(f_1,f_2)(x);
\end{aligned}$$

and

$$\begin{aligned}
I_{423}&\le Cd^\varepsilon\int_{\mathbb{R}^n\setminus 3Q}\frac{|(b_2(y_2)-\lambda_2)f_2(y_2)|dy_2}{|x-y_2|^{2n+\varepsilon}}\int_{3Q}|(b_1(y_1)-\lambda_1)f_1(y_1)|dy_1\\
&\le C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}}\mathcal{M}_{L(\log L)}(f_1,f_2)(x),
\end{aligned}$$

where $\gamma>\varepsilon>0$.

From I_{421}, I_{422}, and I_{423}, we get the estimate of I_{42}:

$$I_{42}\le C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}}\mathcal{M}_{L(\log L)}(f_1,f_2)(x).$$

Similarly as I_{42}, we can get the following estimate

$$I_{44}\le C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}}\mathcal{M}_{L(\log L)}(f_1,f_2)(x).$$

The proof is completed. □

We can also obtain analogous estimates to (5.4) for m-linear commutators involving $j < m$ functions in BMO. That is estimates of term

$$M_\delta^\sharp(g_{\Pi\vec{b}_\sigma}(\vec{f})(x) \lesssim \prod_{k=1}^{j} \|b_{\sigma_k}\|_{\mathrm{BMO}} \mathcal{M}_{L(\log L)_\sigma}(\vec{f})(x) + \text{“lower order terms”}, \tag{5.5}$$

where $\mathcal{M}_{L(\log L)_\sigma}$ denotes the analog of $\mathcal{M}_{L(\log L)}$ but with only log factors in the $\vec{f}_\sigma$ functions (Noting $\mathcal{M}_{L(\log L)_\sigma} = \mathcal{M}^j_{L(\log L)}$ when $\sigma = \{j\}$). The lower order terms are now of the form

$$\prod_{k=1}^{\ell} \|b_{\eta'(k)}\|_{\mathrm{BMO}} M_\varepsilon(g_{\Pi\vec{b}_\eta}(\vec{f}))(x)$$

for $\ell < j$, where η is subset of σ of cardinality ℓ, and $\eta \cup \eta' = \sigma$. Note also that

$$\mathcal{M}_{L(\log L)_\sigma}(\vec{f})(x) \le \mathcal{M}_{L(\log L)}(\vec{f})(x). \tag{5.6}$$

Lemma 5.2. *Suppose $f_i \in C_c^\infty(\mathbb{R}^n)$ and $\operatorname{supp} f_i \subset B(0,R)$ for any $i = 1,\dots,m$. Then there exists a constant $C_{\vec{f}}$ depending on $\vec{f}$ such that for any $|x| > 2R$ and bounded function $b_j(x)$ $(j = 1,\dots,m)$, the following inequality holds*

$$|g_{\Pi\vec{b}}(\vec{f})(x)| \le C_{\vec{f}} \mathcal{M}(\vec{f})(x), \tag{5.7}$$

uniformly.

Proof. Noting that $\vec{b} \in (L^\infty)^m$, the proof is almost the same as (4.7). $\square$

These pointwise estimates are the key for the strong and weak-type estimates with multiple weights. In particular, they yield an appropriate version of the following Fefferman–Stein inequalities [**1**].

Theorem 5.3. *Let ω be an A_∞ weight and let $\Phi(t) = t(1+\log^+ t)$ and $0 < p < \infty$. Suppose that $\vec{b} \in \mathrm{BMO}^m$. Then, there exists a constant C_ω (independent of $\vec{b}$) and a constant $C_\omega(\vec{b})$ such that*

$$\int_{\mathbb{R}^n} |g_{\Pi\vec{b}}(\vec{f})|^p \omega(x)dx \le C_\omega \prod_{i=1}^{m} \|b_i\|_{\mathrm{BMO}} \int_{\mathbb{R}^n} \mathcal{M}_{L(\log L)}\vec{f}(x)^p \omega(x)dx. \tag{5.8}$$

At the endpoint case, we have

$$\begin{aligned}&\sup_{t>0} \frac{1}{\Phi^m(1/t)} \omega(\{y \in \mathbb{R}^n : |g_{\Pi\vec{b}}(\vec{f})(y)| > t^m\}) \\ &\qquad \le C_\omega(\vec{b}) \sup_{t>0} \frac{1}{\Phi^m(1/t)} \omega(\{y \in \mathbb{R}^n : \mathcal{M}_{L(\log L)}\vec{f}(y) > t^m\}),\end{aligned} \tag{5.9}$$

for all bounded vector function $\vec{f} = (f_1, \dots, f_m)$ with compact support.

PROOF. The proof is routine, we refer the reader to [**23**] for details. □

Proof of Theorem 1.3.

PROOF. Since for $\vec{\omega} \in A_\infty$, the weight $\nu_{\vec{\omega}}$ is in A_∞, we can use one more result from [**23**] on strong bounds for $\mathcal{M}_{L(\log L)}$ and conclude from (5.8) that

$$\begin{aligned}\int_{\mathbb{R}^n} |g_{\Pi\vec{b}}(\vec{f})|^p \nu_{\vec{\omega}} dx &\le C_{\nu_{\vec{\omega}}} \prod_{i=1}^m \|b_i\|_\infty \int_{\mathbb{R}^n} \big(\mathcal{M}_{L(\log L)}(\vec{f}(x))\big)^p \nu_{\vec{\omega}} dx \\ &\le C_{\nu_{\vec{\omega}}} \prod_{i=1}^m \|b_i\|_\infty \prod_{j=1}^m \|f_j\|_{L^{p_j}(\omega_j)}.\end{aligned}$$
□

5.2. Endpoint estimate.

We need a weak-type end point estimate for $\mathcal{M}_{L(\log L)}$.

THEOREM 5.4 ([**31**]). *Let $\vec{\omega} \in A_{\vec{1}}$. Then there exists a constant C such that*

$$\begin{aligned}&\nu_{\vec{\omega}}\big(\{x \in \mathbb{R}^n : \mathcal{M}_{L(\log L)}(\vec{f})(x) > t^m\}\big) \\ &\quad \le C \prod_{i=1}^m \left(\int_{\mathbb{R}^n} \Phi^{(m)}\left(\frac{|f_j(x)|}{t}\right)\omega_j(x) dx\right)^{1/m}. \qquad (5.10)\end{aligned}$$

Moreover, this estimate is sharp in the sense that $\Phi^{(m)}$ can not be replaced by $\Phi^{(k)}$ for $k < m$.

Given (5.9) and (5.10) the proof of Theorem 1.4 is almost routine. We refer the reader to [**31**] for the proof of (5.10).

REMARK 5.1. To show $\Phi^{(m)}$ can not be replaced by $\Phi^{(k)}$ for $k < m$. We consider $m = 2$, $n = 1$ simply, and the functions $b_1(x) = b_2(x) = \log|1 + x|$, $f_1 = f_2 = \chi_{(0,1)}$ and $\psi(y_1, y_2) = 1/(|y_1| + |y_2|)^{2+\delta}$. Then for $x > e$, we have

$$\begin{aligned}&|g_{\Pi(b_1,b_2)}(f_1, f_2)(x)| \\ &= \left(\int_0^\infty \left|\frac{1}{t^2}\int_{(\mathbb{R})^2} \psi\left(\frac{x-y_1}{t}, \frac{x-y_2}{t}\right)\prod_{j=1}^2 (b_j(x) - b_j(y_j)) f_1(y_1) f_2(y_2) dy_1 dy_2\right|^2 \frac{dt}{t}\right)^{1/2} \\ &\ge C\left(\int_x^{2x} \left|\int_0^1\int_0^1 \frac{\log^2|1+x|}{x^{2+\delta}} t^\delta dy_1 dy_2\right|^2 \frac{dt}{t}\right)^{1/2} \\ &\ge C\left(\int_x^{2x} \frac{t^{2\delta}}{t} dt\right)^{1/2} \frac{\log^2|1+x|}{x^{2+\delta}} \ge C\frac{\log^2|1+x|}{x^2}.\end{aligned}$$

Now, by using almost the same argument as in [**31**, p. 5], we can show that $\Phi^{(m)}$ can not be replaced by $\Phi^{(k)}$ for $k < m$. Thus we finish the proof.

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