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Nontrivial attractor-repellor maps of S^2 and rotation numbers

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Abstract. We consider an orientation preserving homeomorphism h of S^2 which admits a repellor denoted ∞ and an attractor $-\infty$ such that h is not a North-South map and that the basins of ∞ and $-\infty$ intersect. We study various aspects of the rotation number of $h : S^2 \setminus \{\pm\infty\} \to S^2 \setminus \{\pm\infty\}$, especially its relationship with the existence of periodic orbits.

1. Introduction.

Let h be a homeomorphism of the 2-sphere S^2. A fixed point a of h is called an *attractor* if there is an open disk V containing a such that $h(\mathrm{Cl}(V)) \subset V$ and $\bigcap_{i\in\mathbb{N}} h^i(V) = \{a\}$. For such V, the set $W_a = \bigcup_{i\in\mathbb{N}} h^{-i}(V)$ is called the *basin* of a. A point x of W_a is characterised by the property: $\lim_{i\to\infty} h^i(x) = a$. An attractor b of the inverse h^{-1} is called a *repellor* of h, and its basin W_b is defined likewise. The basins are invariant by h and homeomorphic to open disks.

Let ∞ and $-\infty$ be distinct points of S^2.

DEFINITION 1.1. A homeomorphism h of S^2 which satisfy the following conditions is called a *nontrivial attractor-repellor map*.

(1) h is orientation preserving.
(2) $-\infty$ is an attractor of h with basin $W_{-\infty}$ and ∞ a repellor with basin W_∞.
(3) $Z = S^2 \setminus (W_{-\infty} \cup W_\infty)$ is nonempty.
(4) $W_{-\infty} \cap W_\infty \neq \emptyset$.

Condition (3) is equivalent to saying that h is *not* a North-South map. Condition (4) is equivalent to saying that there is *no* h-invariant continuum separating $-\infty$ and ∞. Denote by $\mathcal{H}$ the set of nontrivial attractor-repellor maps.

Let $W^*_{\pm\infty} = W_{\pm\infty} \cup \partial W^*_{\pm\infty}$ be the prime end compactification of $W_{\pm\infty}$, where $\partial W^*_{\pm\infty}$ is the set of prime ends of $W_{\pm\infty}$. The homeomorphism $h \in \mathcal{H}$ induces a homeomorphism $h^*_{\pm\infty}$ of $W^*_{\pm\infty}$. See Section 2 for more details.

The open annulus $\mathbb{A} = S^1 \times \mathbb{R}$ is identified with $S^2 \setminus \{\pm\infty\}$ in such a way that the end $S^1 \times \{\pm\infty\}$ is identified with the deleted point $\pm\infty$. Then h induces an orientation and end preserving homeomorphism of $\mathbb{A}$, which we still denoted by h. The set $U_{\pm\infty} = W_{\pm\infty} \setminus \{\pm\infty\}$ is considered to be a subset of $\mathbb{A}$. Denote $U^*_{\pm\infty} = W^*_{\pm\infty} \setminus \{\pm\infty\}$.

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The universal covering space of $U_{\pm\infty}$ is defined as the set of the homotopy classes of paths from the base point, and is considerd to be simultaneously a subspace of $\tilde{\mathbb{A}}$, the universal covering space of $\mathbb{A}$ and of $\tilde{U}^*_{\pm\infty}$, the universal covering space of $U^*_{\pm\infty}$.

Denote the both covering maps by $\pi : \tilde{\mathbb{A}} \to \mathbb{A}$ and $\pi : \tilde{U}^*_{\pm\infty} \to U^*_{\pm\infty}$. The inverse image $\pi^{-1}(U_{\pm\infty})$ is simultaneously considered to be a subspace of $\tilde{\mathbb{A}}$ and of $\tilde{U}^*_{\pm\infty}$.

Fix once for all a lift $\tilde{h} : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ of h. Corresponding to $\tilde{h}$, a lift $\tilde{h}^*_{\pm\infty} : \tilde{U}^*_{\pm\infty} \to \tilde{U}^*_{\pm\infty}$ of $h^*_{\pm\infty}$ is specified in such a way that they coincide on $\pi^{-1}(U_{\pm\infty})$ under the above identification.

The rotation number (taking value in $\mathbb{R}$) of the restriction of $\tilde{h}^*_{\pm\infty}$ to the boundary $\partial\tilde{U}^*_{\pm\infty} = \pi^{-1}(\partial U^*_{\pm\infty})$ is called the *prime end rotation number of* $\tilde{h}$ *at* $\pm\infty$ and is denoted by $\mathrm{rot}(\tilde{h}, \pm\infty)$.

In [**13**] it is shown that if one of the prime end rotation numbers, say $\mathrm{rot}(\tilde{h}, \infty)$ of $h \in \mathcal{H}$ is rational, then there are periodic points in Z. In [**9**] a partial converse is shown: if $\mathrm{rot}(\tilde{h}, \infty)$ is irrational and if the point $-\infty$ is *accessible* from W_∞, then there is no periodic points in Z. The second condition means that there is a path $\gamma : [0, 1] \to S^2$ such that $\gamma([0, 1)) \subset W_\infty$ and $\gamma(1) = -\infty$. Our first result shows that the accessibility condition is actually necessary, contrary to a conjecture therein.

Theorem 1.2. *Given any real numbers α and β, there is a homeomorphism $h \in \mathcal{H}$ with its lift $\tilde{h}$ such that* $\mathrm{rot}(\tilde{h}, \infty) = \alpha$ *and* $(\tilde{h}, -\infty) = \beta$.

The accessibility condition is necessary since if we choose α to be rational and β irrational, there is a periodic point ([**13**]) and thus the irrationality of one prime end rotation number does dot imply the nonexistence of periodic point.

A nontrivial attractor repellor map h has a structure similar to a gradient flow. Most relevant to this structure is the chain recurrent set C of h. Except $\pm\infty$, C is contained in Z, and partitioned into the union of chain transitive classes. Each chain transitive class is closed and h-invariant. See Section 3 for a review of these concepts.

The example in Theorem 1.2 constructed in Section 2 shows that Poincaré-Birkhoff type theorem does not hold for $h \in \mathcal{H}$. But when restricted to a single chain transitive class, we get a variant of it.

We consider h to be a homeomorphism of the annulus $\mathbb{A}$, and fix a lift $\tilde{h} : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ of h. Then for any h-invariant probability measure μ of Z, the rotation number $\mathrm{rot}(\tilde{h}, \mu)$ is defined as follows. Denote by $\Pi_1 : \tilde{\mathbb{A}} \to \mathbb{R}$ the projection onto the first factor. Then the function $\Pi_1 \circ \tilde{h} - \Pi_1$ is invariant under the covering transformations, and hence defines a function on $\mathbb{A}$. We set

$$\mathrm{rot}(\tilde{h}, \mu) = \langle \mu, \Pi_1 \circ \tilde{h} - \Pi_1 \rangle.$$

For a periodic point x of h, we denote by $\mathrm{rot}(\tilde{h}, x)$ the rotation number $\mathrm{rot}(\tilde{h}, \mu)$ for μ the average of the point masses along the orbit of x.

Theorem 1.3. *Suppose x_1 and x_2 are periodic points of h belonging to the same chain transitive class C_0 such that* $\mathrm{rot}(\tilde{h}, x_\nu) = \alpha_\nu$ $(\nu = 1, 2)$. *Then for any rational number $\alpha \in [\alpha_1, \alpha_2]$ there is a periodic point x in C_0 such that* $\mathrm{rot}(\tilde{h}, x) = \alpha$.

Let us define the *rotation set* $\mathrm{rot}(\tilde{h}, C_0)$ of a chain transitive class C_0 as the set of the values $\mathrm{rot}(\tilde{h}, \mu)$, where μ runs over the space of h-invariant probability measures supported on C_0. The rotation set is a closed interval or a singleton.

COROLLARY 1.4. *Suppose C_0 is a chain transitive class with $\mathrm{rot}(\tilde{h}, C_0) = [\alpha_1, \alpha_2]$, where α_ν are distinct rational numbers. Then for any rational number $\alpha \in [\alpha_1, \alpha_2]$ there is a periodic point x in C_0 such that $\mathrm{rot}(\tilde{h}, x) = \alpha$.*

The proof of Theorem 1.3 and Corollary 1.4, as well as an example of $h \in \mathcal{H}$ which shows that Theorem 1.3 is nonvoid is given in Section 3. The author cannot improve Corollary 1.4 so as to include the case where α_ν is irrational.

Next we study an influence of the prime end rotation number $\mathrm{rot}(\tilde{h}, \infty)$ on the dynamics of h on Z.

THEOREM 1.5. *If $\mathrm{rot}(\tilde{h}, \infty) = p/q$ $((p, q) = 1)$, there is a periodic point $x \in \mathbb{A}$ of period q such that $\mathrm{rot}(\tilde{h}, x) = p/q$.*

This is already known ([**13**]) except for the last statement about the rotation number. Section 4 is devoted to the proof of Theorem 1.5.

Our last theorem is concerned about the case where $-\infty$ is accessible from U_∞. Then the dynamics of h on Z is shown to be quite simple in the view point of rotation numbers. This is a refinement of a result in [**9**] cited above. The proof is given in Section 5.

THEOREM 1.6. *Assume that $-\infty$ is accessible from U_∞ and let $\alpha = \mathrm{rot}(\tilde{h}, \infty)$. Then*

(1) $\mathrm{rot}(\tilde{h}, \mu) = \alpha$ *for any h-invariant probability measure supported on Z.*
(2) $\mathrm{rot}(\tilde{h}, -\infty) = \alpha$.

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2. Prime end rotation numbers.

2.1. First of all, we recall fundamental facts about the prime end compactification of $W_{\pm\infty}$. See [**3**], [**11**], [**14**], [**12**] for an detailed exposition.

A properly embedded copy of the real line c in $W_{\pm\infty}$ which does not pass through $\pm\infty$ is called a *cross cut* of $W_{\pm\infty}$. The word "proper" means that the inverse image of any compact set is compact. The connected component of the complement of a cross cut c which does not contain the point $\pm\infty$ is denoted by $V(c)$. A sequence $\{c_i\}_{i\in\mathbb{N}}$ of cross cuts is called a *topological chain* ([**11**]) if the following conditions are satisfied.

(1) $c_{i+1} \subset V(c_i)$, $\forall i \in \mathbb{N}$.
(2) $\mathrm{Cl}(c_i) \cap \mathrm{Cl}(c_j) = \emptyset$ if $i \neq j$, where $\mathrm{Cl}(\cdot)$ denotes the closure in S^2.
(3) $\mathrm{diam}(c_i) \to 0$ as $i \to \infty$, where the diameter is taken with respect to the spherical metric of S^2.

Two topological chains $\{c_i\}$ and $\{c'_i\}$ are said to be *equivalent* if for any i, there is j such that $c_j \subset V(c'_i)$ and $c'_j \subset V(c_i)$.

An equivalence class of topological chains is called a *prime end* of $W_{\pm\infty}$. The set of prime ends is denoted by $\partial W^*_{\pm\infty}$. The set $W^*_{\pm\infty} = W_{\pm\infty} \cup \partial W^*_{\pm\infty}$ is called the *prime end compactification of* $W_{\pm\infty}$. It is topologized as follows. A neighbourhood system in $W^*_{\pm\infty}$ of a point in $W_{\pm\infty}$ is the same as a given system for $W_{\pm\infty}$. Choose a point ξ in $\partial W^*_{\pm\infty}$ represented by a topological chain $\{c_i\}$. The set of points in $V(c_i)$, together with the prime ends represented by topological chains contained in $V(c_i)$, for each i, forms a fundamental neighbourhood system of ξ. It is a classical fact due to Carathéodory that $W^*_{\pm\infty}$ is homeomorphic to a closed disk.

It is clear by the topological nature of the definition that the homeomorphism h of S^2 induces a homeomorphism $h^*_{\pm\infty}$ of $W^*_{\pm\infty}$.

2.2. Now let us embark upon the costruction of the homeomorphism $h \in \mathcal{H}$ in Theorem 1.2. We shall construct it as a homeomorphism of the annulus $\mathbb{A}$. Roughly speaking, on the subannulus $S^1 \times [5, \infty)$, h is of the form

$$h(\theta, t) = (f_\alpha(\theta), t - g(\theta, t)),$$

where f_α is a rigid rotation of S^1 if α is rational, and a Denjoy homeomorphism if irrational. By choosing the $[0, 1]$-valued function g appropriately, one can form the homeomorphism h which has a unique minimal set on the level $t = 10$. Also h satisfies

$$h(S^1 \times [5, \infty)) = S^1 \times [4, \infty).$$

Likewise we define h on $(-\infty, -5]$ using a homeomorphism f_β of S^1 of rotation number β. It has a unique minimal set on the level $t = -10$. Finally on $S^1 \times [-5, 5]$, we define h as

$$h(\theta, t) = (\varphi_t(\theta), t - 1),$$

by using an isotopy φ_t ($t \in [-5, 5]$) joining f_β and f_α.

Let us start a concrete construction. Given $\alpha \in \mathbb{R}$, let us define a homeomorphism f_α of S^1 and its lift $\tilde{f}_\alpha : \mathbb{R} \to \mathbb{R}$ with rotation number $\mathrm{rot}(\tilde{f}_\alpha) = \alpha$ as follows. For α rational, let $\tilde{f}_\alpha$ be the translation by α. Thus f_α is the rigid rotation of S^1. For α irrational, let f_α be a Denjoy homeomorphism and $\tilde{f}_\alpha$ the lift of f_α such that $\mathrm{rot}(\tilde{f}) = \alpha$. Let $C_\alpha \subset S^1$ be a minimal set of f_α. Thus C_α is a single periodic orbit if α is rational, and a Cantor set if α is irrational. For α irrational, we assume furthermore that the complement of C_α consists of the orbit of a single wandering interval. That is, there is a connected component I_α of $S^1 \setminus C_\alpha$ such that $\bigcup_{i\in\mathbb{Z}} f^i_\alpha(I_\alpha) = S^1 \setminus C_\alpha$.

Define a continuous function $g_\alpha : S^1 \to [0, 1]$ such that

(a) $g_\alpha^{-1}(0) = C_\alpha$, and
(b) for any $\theta \in S^1 \setminus C_\alpha$, $\sum_{i\geq 0} g_\alpha(f^i_\alpha(\theta)) = \infty$ and $\sum_{i\leq 0} g_\alpha(f^i_\alpha(\theta)) = \infty$.

The existence of such g_α is clear for α rational. For α irrational, first define g_α on the interval $\mathrm{Cl}(I_\alpha)$ so that $g_\alpha^{-1}(0) = \partial I_\alpha$. For any $i \in \mathbb{Z} \setminus \{0\}$, define g_α on $f^i_\alpha(I_\alpha)$ by $g_\alpha(f^i_\alpha(\theta)) = |i|^{-1} g_\alpha(\theta)$. Finally set $g_\alpha = 0$ on C_α. Then g_α is continuous and satisfies

(a) and (b).

For $\beta \in \mathbb{R}$, we define f_β, $\tilde{f}_\beta$, C_β and g_β likewise.

Define a continuous function $g : S^1 \times \mathbb{R} \to [0,1]$, differentiable along the $\mathbb{R}$-direction, such that

(c) $g^{-1}(0) = C_\alpha \times \{10\} \cup C_\beta \times \{-10\}$,
(d) for $t \in [9,11]$, $g(\theta,t) \geq g_\alpha(\theta)$ and for $t \in [-11,-9]$, $g(\theta,t) \geq g_\beta(\theta)$, with the equality only for $t = \pm 10$,
(e) $g = 1$ on $S^1 \times ((-\infty, 15] \cup [-5,5] \cup [15,\infty))$ and
(f) $\partial g/\partial t < 1$.

Choose a continuous family φ_t ($t \in \mathbb{R}$) of homeomorphisms of S^1 and its continuous lift $\tilde{\varphi}_t$ such that

(g) $\tilde{\varphi}_t = \tilde{f}_\alpha$ for $t \in [5,\infty)$ and $\tilde{\varphi}_t = \tilde{f}_\beta$ for $t \in (-\infty,-5]$.

Finally define a homeomorphism $h : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ by

$$h(\theta,t) = (\varphi_t(\theta), t - g(\theta,t)).$$

2.3. We shall show that h satisfies the conditions of Theorem 1.2. First let us verify that h is a homeomorphism of $\mathbb{A}$. Clearly h is continuous and by (e) maps the circle $S^1 \times \{15\}$ (resp. $S^1 \times \{-15\}$) onto $S^1 \times \{14\}$ (resp. $S^1 \times \{-16\}$). This shows that h is surjective. To show that h is injective, assume $h(\theta_1,t_1) = h(\theta_2,t_2)$. Then by (e) h maps $S^1 \times [5,\infty)$, $S^1 \times (-\infty,-5]$ and $S^1 \times [-5,5]$ respectively onto $S^1 \times [4,\infty)$, $S^1 \times (-\infty,-6]$ and $S^1 \times [-6,4]$. Therefore the two points (θ_1,t_1) and (θ_2,t_2) must simultaneously belong to either one of the subannuli $S^1 \times [5,\infty)$, $S^1 \times (-\infty,-5]$ and $S^1 \times [-5,5]$. In the first case we have

$$h(\theta_i,t_i) = (f_\alpha(\theta_i), t_i - g(\theta_i,t_i)),$$

and thus $\theta_1 = \theta_2$. On the other hand by (f), $h|_{\{\theta\}\times\mathbb{R}}$ is injective, showing that $t_1 = t_2$. The second case can be dealt with similarly.

In the last case, we have

$$h(\theta_i,t_i) = (\varphi_{t_i}(\theta_i), t_i - 1).$$

Thus $t_1 = t_2$, which implies $\theta_1 = \theta_2$.

Next let us show that $h \in \mathcal{H}$. Conditions (1) $\sim$ (3) of Definition 1.1 are clear. Let us show (4). Consider the basin W_∞ of the repellor ∞ (corresponding to the end $S^1 \times \{\infty\}$ of the cylinder $\mathbb{A}$). Recall the notation $U_\infty = W_\infty \setminus \{\infty\} \subset \mathbb{A}$. We shall show

$$U_\infty \cap (S^1 \times [5,\infty)) = (S^1 \times [5,\infty)) \setminus (C_\alpha \times [5,10]). \tag{2.1}$$

To show this, first notice that the minimum value of g on $S^1 \times [10+\varepsilon,\infty)$ is positive for any $\varepsilon > 0$. This shows that $S^1 \times (10,\infty) \subset U_\infty$. Next by (d) and (b), any point in $(S^1 \setminus C_\alpha) \times (10,11)$ can be moved below the level $t = 10$ by an iterate of h. Since

$$U_\infty = \bigcup_{i \in \mathbb{N}} h^i(S^1 \times (10, \infty)),$$

we have

$$(S^1 \times [5, \infty)) \setminus (C_\alpha \times [5, 10]) \subset U_\infty.$$

On the other hand, the opposite inclusion of (2.1) is easy to show.

The basin U_∞ is obtained as the increasing union of the images of the set in (2.1) by the positive iterates of h. Therefore it is clear that $U_\infty \cap (S^1 \times (-5, 5))$ is open and dense in $S^1 \times (-5, 5)$. Likewise we can prove that $U_{-\infty} \cap (S^1 \times (-5, 5))$ is open and dense in $S^1 \times (-5, 5)$. This shows that $U_\infty \cap U_{-\infty} \neq \emptyset$, as is required.

What is left is to show that $\mathrm{rot}(\tilde{h}, \infty) = \alpha$, the other assertion $\mathrm{rot}(\tilde{h}, -\infty) = \beta$ being proven similarly.

Now for any point $\theta \in S^1$, define the ray $r_\theta : (0, \infty) \to U_\infty$ by

$$r_\theta(t) = (\theta, t^{-1} + 10).$$

For $\theta \notin C_\alpha$, the end point $r_\theta(\infty) = (\theta, 10)$ is a point in U_∞. For $\theta \in C_\alpha$, the end point $r_\theta(\infty)$ is defined as a prime end, i.e. a point of $\partial U_\infty^* (= \partial W_\infty^*)$ as follows.

For any $i \in \mathbb{N}$, let S_i be the circle centered at $(\theta, 10)$ and of radius i^{-1}, and c_i the cross cut of U_∞ obtained as the connected component of $S_i \cap U_\infty$ that intersects the ray r_θ. Clearly $\{c_i\}$ is a topological chain. Denote the prime end it determines by $r_\theta(\infty)$.

Define a map $\gamma : S^1 \to U_\infty^*$ by $\gamma(\theta) = r_\theta(\infty)$. The map γ is clearly injective. It is also continuous according to the definition of the topology of U_∞^*. The intersection C^* of the curve γ with the set of prime ends ∂U_∞^* is either a finite set or a Cantor set, and γ maps C_α homeomorphically onto C^* in a way to preserve the cyclic order and conjugates $f_\alpha|_{C_\alpha}$ to $h_\infty^*|_{C^*}$. Moreover there is a lift of γ defined on $\mathbb{R}$ taking values on $\tilde{U}_\infty^*$ which maps $\pi^{-1}(C_\alpha)$ homeomorphically onto $\pi^{-1}(C^*)$ in an order preserving way and conjugates $\tilde{f}_\alpha|\pi^{-1}(C_\alpha)$ to $\tilde{h}_\infty^*|\pi^{-1}(C^*)$. Since the lift $\tilde{h}$ of h is determined by (g), we have $\mathrm{rot}(\tilde{h}, \infty) = \alpha$, completing the proof of Theorem 1.2.

3. The rotation set of a chain transitive class.

3.1. Fix $h \in \mathcal{H}$. For $\varepsilon > 0$ and $x, y \in S^2$, a sequence $\{x = x_0, x_2, \dots, x_r = y\}$ of points of S^2 is called an *ε-chain of h of length r from x to y* if for any $0 \le i \le n-1$, $d(h(x_i), x_{i+1}) < \varepsilon$, and an *$\varepsilon$-cycle at x* if furthermore $x = y$. A point $x \in S^2$ is called *chain recurrent* if for any $\varepsilon > 0$, there is an ε-cycle at x. The set C of the chain recurrent points is called the *chain recurrent set*. It is a closed set invariant by h.

Two points x and y of C are said to be *chain transitive*, denoted $x \sim y$, if for any $\varepsilon > 0$, there are an ε-chain from x to y and another from y to x. An equivalence class of $\sim$ is called a *chain transitive class*. Again it is closed and invariant by h.

Definition 3.1. A continuous function $H : S^2 \to \mathbb{R}$ is called a *complete Lyapunov function* of h if it satisfies the following conditions.

(1) If $x \notin C$, then $H(h(x)) < H(x)$.
(2) If $x, y \in C$, $H(x) = H(y)$ if and only if $x \sim y$.
(3) The set of values $H(C)$ is closed and Lebesgue null in $\mathbb{R}$.

A value in $\mathbb{R} \setminus H(C)$ is called a *dynamically regular value* of H. If a is dynamically regular, then $H^{-1}(a)$ is mapped by h into $H^{-1}((-\infty, a))$

The existence of a complete Lyapunov function for any homeomorphism of a compact metric space is shown in [**4**]. For our purpose, the following proposition is more convenient. The proof can be found in Appendix.

PROPOSITION 3.2. *For any $h \in \mathcal{H}$, there is a C^∞ complete Lyapunov function H of h.*

Condition (3) above and the Sard theorem say that dynamically regular and regular (in the usual sense) values are Lebesgue full. For such a value a, $H^{-1}(a)$ is a 1 dimensional C^∞ submanifold and h maps $H^{-1}(a)$ into $H^{-1}((-\infty, a))$.

3.2. Let us construct an example of C^∞ diffeomorphism $h \in \mathcal{H}$ which admits a chain transitive class C_0 such that the rotation set $\mathrm{rot}(\tilde{h}, C_0)$ is a nontrivial interval. We construct h as an area preserving diffeomorphism of the annulus $\mathbb{A}$, which is so to call a "winding horseshoe map". See Figure 1. Let us denote by m the (infinite) measure on $\mathbb{A}$ given by the area form $d\theta \wedge dt$.

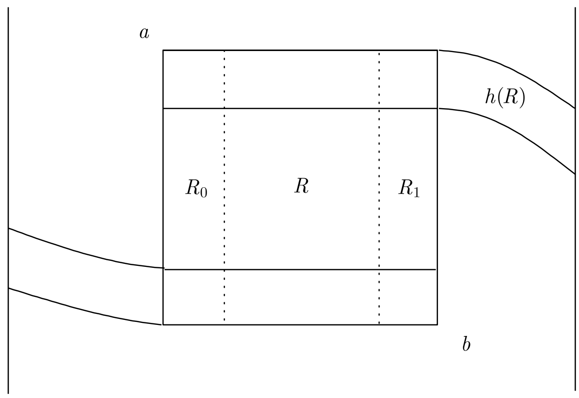

Figure 1.

Choose a rectangle $R = [0, 4^{-1}] \times [-4^{-1}, 0]$ in $\mathbb{A} = (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$. Stretch R horizontally by 5 and contract vertically by 5^{-1}, and embed the resultant long and thin rectangle into $\mathbb{A}$ in a way to wind the annulus $\mathbb{A}$. The precise conditions for a map $h : R \to \mathbb{A}$ is the following.

(1) h is an m-preserving C^∞ embedding.
(2) Restricted to the subrectangle $R_0 = [0, 20^{-1}] \times [-4^{-1}, 0]$,

$$h(\theta, t) = (5\theta, 5^{-1}t).$$

(3) Restricted to the subrectangle $R_1 = [5^{-1}, 4^{-1}] \times [-4^{-1}, 0]$,

$$h(\theta, t) = (5\theta - 1, 5^{-1}t - 5^{-1}).$$

(4) $h^{-1}(R) = R_0 \cup R_1$.
(5) $R \cup h(R)$ separates both ends of $\mathbb{A}$.

Notice that the points $a = (0, 0)$ and $b = (4^{-1}, -4^{-1})$ are the (only) fixed points of h. Extend h to a C^∞ diffeomorphism h_0 of $\mathbb{A}$ so as to satisfy the following conditions.

(6) On $S^1 \times ((-\infty, -10] \cup [10, \infty))$, $h_0(\theta, t) = (\theta, t - 1)$.

The measure $(h_0)_* m$ coincides with m on $h_0(R)$, since h_0 is m-preserving on R, and likewise on $h_0(S^1 \times ((-\infty, -10] \cup [10, \infty)))$. Now by Moser's lemma ([**10**, p. 16]), there is a C^∞ diffeomorphsim h_1 on $\mathbb{A}$ such that $(h_1)_*((h_0)_*(m)) = m$ which is the identity on $h_0(R \cup (S^1 \times (-\infty, -10] \cup [10, \infty)))$.

Now the composite $h = h_1 \circ h_0$ is m-preserving. Let us show that h satisfies the condition raised in the beginning of **3.2**. First of all clearly h satisfies condition (1) $\sim$ (3) of Definition 1.1. Moreover since h is m-preserving, it cannot admit a invariant continuum separating both ends of $\mathbb{A}$. Therefore it satisfies (4) also.

Choose a lift $\tilde{h}$ of h so that each point of $\pi^{-1}(a)$ is fixed by $\tilde{h}$. Then we have $\mathrm{rot}(\tilde{h}, a) = 0$ and $\mathrm{rot}(\tilde{h}, b) = 1$.

Finally we have $a \sim b$, since the stable manifold of a intersects the unstable manifold of b, and the unstable manifold of a intersects the stable manifold of b. Therefore the chain transitive class C_0 of a and b satisfies $[0, 1] \subset \mathrm{rot}(\tilde{h}, C_0)$.

3.3. Here we shall show Theorem 1.3 by a rather lengthy argument. We consider $h \in \mathcal{H}$ to be a homeomorphism of the annulus $\mathbb{A}$. Denote the generator of the covering transformations by $T : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$: $T(\theta, t) = (\theta + 1, t)$. First of all we have the following fundamental lemma.

Lemma 3.3. *Suppose $h^q(z) = z$ ($q \in \mathbb{N}$, $z \in \mathbb{A}$) and let $p \in \mathbb{Z}$. (We do not assume $(p, q) = 1$.) Then the following conditions are equivalent.*

(1) $\mathrm{rot}(\tilde{h}, z) = p/q$.
(2) *$\tilde{h}^q(\tilde{z}) = T^p(\tilde{z})$ for a lift $\tilde{z}$ of z.*

Notice that condition (2) is independent of the choice of the lift $\tilde{z}$. This is because $\tilde{h}$ commutes with T.

Proof. Recall that $\pi : \tilde{\mathbb{A}} \to \mathbb{A}$ is the universal covering map and $\Pi_1 : \tilde{\mathbb{A}} \to \mathbb{R}$ is the canonical projection onto the first factor. Define a function $\varphi : \mathbb{A} \to \mathbb{R}$ by

$$\pi \circ \varphi = \Pi_1 \circ \tilde{h} - \Pi_1.$$

Denote the average of the Dirac masses along the orbit of z by μ, that is,

$$\mu = q^{-1}(\delta_z + \delta_{h(z)} + \cdots + \delta_{h^{q-1}(z)}).$$

Notice that for any $i \in \mathbb{N}$,

$$\begin{aligned}\langle \delta_{h^i(z)}, \varphi\rangle &= \langle \pi_* \delta_{\tilde{h}^i(\tilde{z})}, \varphi\rangle = \langle \delta_{\tilde{h}^i(\tilde{z})}, \varphi \circ \pi\rangle \\ &= \langle \delta_{\tilde{h}^i(\tilde{z})}, \Pi_1 \circ \tilde{h} - \Pi_1\rangle = \Pi_1(\tilde{h}^{i+1}(\tilde{z})) - \Pi_1(\tilde{h}^i(\tilde{z})). \end{aligned} \tag{3.1}$$

Now assume (1): $\mathrm{rot}(\tilde{h}, z) = \langle \mu, \varphi\rangle = p/q$. Then we have by (3.1)

$$\Pi_1(h^q(\tilde{z})) - \Pi_1(\tilde{z}) = p.$$

That is,

$$\Pi_1(h^q(\tilde{z})) = \Pi_1(T^p(\tilde{z})). \tag{3.2}$$

On the other hand, since $h^q(z) = z$, we have

$$\tilde{h}^q(\tilde{z}) = T^j(\tilde{z}) \tag{3.3}$$

for some $j \in \mathbb{Z}$. Now (3.2) and (3.3) imply that $j = p$. We obtain condition (2). The converse can be shown by a reversed argument. □

Let us begin the proof of Theorem 1.3. Let $h \in \mathcal{H}$ and C_0 a chain transitive class of h. Assume $x_\nu \in C_0$ are periodic points such that $\mathrm{rot}(\tilde{h}, x_\nu) = \alpha_\nu$ ($\nu = 1, 2$) and let α be a rational number in $[\alpha_1, \alpha_2]$. If $\alpha_1 = \alpha_2$, there is nothing to prove. So assume $\alpha_1 < \alpha < \alpha_2$. Then it is possible to choose a number $q \in \mathbb{N}$ such that

(a) the rational numbers α_ν and α are written as

$$\alpha_\nu = p_\nu/q \quad (\nu = 1, 2), \quad \alpha = p/q, \quad p_1 < p < p_2, \quad \text{and}$$

(b) the periodic points x_ν satisfies $h^q(x_\nu) = x_\nu$.

By Lemma 3.3, lifts $\tilde{x}_\nu$ of x_ν satisfy $\tilde{h}^q(\tilde{x}_\nu) = T^{p_\nu}(\tilde{x}_\nu)$. Our purpose is to show the existence of a periodic point $x \in C_0$ of period q such that $\mathrm{rot}(\tilde{h}, x) = p/q$, that is, whose lifts $\tilde{x}$ satisfy $\tilde{h}^q(\tilde{x}) = T^p(\tilde{x})$.

However a simultaneous proof for all $p \in (p_1, p_2)$ has an elementary number theoretic difficulty. We shall avoid it by employing an induction on $p - p_1$. Namely we first show only for $p = p_1 + 1$. Then the newly obtained periodic points can serve as an assumption for the next step $p = p_1 + 2$. This way, Theorem 1.3 reduces to the following.

Proposition 3.4. *Let $q > 0$ and $p_1 + 1 < p_2$ and let C_0 be a chain transitive class of $h \in \mathcal{H}$. Assume there are points $x_\nu \in C_0$ with a lift $\tilde{x}_\nu$ such that $\tilde{h}^q(\tilde{x}_\nu) = T^{p_\nu}(\tilde{x}_\nu)$ ($\nu = 1, 2$). Then there is a point $x \in C_0$ with a lift $\tilde{x}$ such that $\tilde{h}^q(\tilde{x}) = T^{p_1+1}(\tilde{x})$.*

Now Proposition 3.4 itself reduces to the following.

Proposition 3.5. *Let $q > 0$ and $p_1 + 1 < p_2$ and let C_0 be a chain transitive class of $h \in \mathcal{H}$. Assume there are points $x_\nu \in C_0$ with a lift $\tilde{x}_\nu$ such that $\tilde{h}^q(\tilde{x}_\nu) = T^{p_\nu}(\tilde{x}_\nu)$*

($\nu = 1, 2$). *Let H be a C^∞ complete Lyapunov function such that $H(C_0) = 0$, and let $-a' < 0$ and $a > 0$ be dynamically regular and regular values of H. Then there is a point x in the subsurface $H^{-1}([-a', a])$ with a lift $\tilde{x}$ such that $\tilde{h}^q(\tilde{x}) = T^{p_1+1}(\tilde{x})$.*

Postponing the proof, we shall show the reduction. Thanks to the Sard theorem and condition (3) of Definition 3.1, one can find dynamically regular and regular values $-a' < 0 < a$ of H as close to 0 as we want. Let

$$F_0 = \pi(\mathrm{Fix}(\tilde{h}^q \circ T^{-p_1-1})).$$

Then Proposition 3.5 says that $F_0 \cap H^{-1}([-a', a]) \neq \emptyset$ for any such values. By the compactness of F_0, this implis that $F_0 \cap H^{-1}(0) \neq \emptyset$. On the other hand, since H is a complete Lyapunov function, $C \cap H^{-1}(0) = C_0$, showing $F_0 \cap C_0 \neq \emptyset$, as is required in Proposition 3.4.

The rest of this paragraph is devoted to the proof of Proposition 3.5. The subsurface $H^{-1}([-a', a])$ admits a single distinguished connected component X which is homotopically nontrivial in $\mathbb{A}$. In fact, if there were more than one such components, then in the complement, one could find a forward invariant compact subannulus. The intersection of its forward images would be a h-invariant continuum separating U_∞ and $U_{-\infty}$, contradicting condition (4) of Definition 1.1.

Let us consider the upper boundary $H^{-1}(a) \cap X$ of X. It has a unique homotopically nontrivial component ∂A^+. The curve ∂A^+ bounds an infinite annulus A^+ on the opposite side of X. The intersection of $\mathrm{Int}(A^+)$ with the level $H^{-1}(a)$ consists of finitely many circles ∂D_i^+. They bound discs D_i^+ in A^+. See Figure 2.

The components of $H^{-1}(a) \cap X$ other than ∂A^+ are denoted by ∂E_k^+. They are finite in number and bound discs E_k^+ in $\mathbb{A}$.

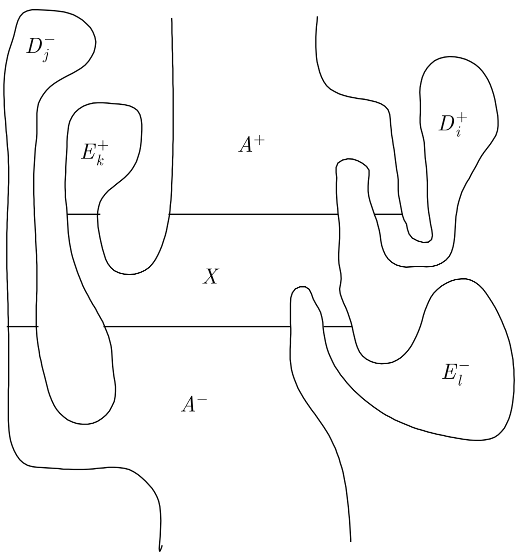

Figure 2.

Likewise we define an annulus A^-, discs D_j^- and E_l^- by considering the lower boundary $H^{-1}(-a') \cap X$ of X. Then we have

$$\mathbb{A} = X \cup A^- \cup A^+ \cup \bigcup_k E_k^+ \cup \bigcup_l E_l^-.$$

Let us study how family of the discs $\mathcal{D}^+ = \{D_i^+\}$ are mapped by h, and show the following. Denote $|\mathcal{D}^+| = \bigcup_i D_i^+$.

PROPOSITION 3.6. *The chain transitive class C_0 is disjoint from $A^+ \cup A^-$.*

This is not obvious since discs $D_i^+ \subset A^+$ may intersect $H^{-1}(0)$. Our overall strategy after having shown Proposition 3.6 is to replace h by a homeomorphism which has no periodic points in $A^+ \cup A^-$, and seek for periodic points in the rest of $\mathbb{A}$. The argument will be divided into two cases. In the first case we employ a topological method, while in the second a dynamical.

To establish Proposition 3.6, we must prepare some lemmas.

LEMMA 3.7. *For any small $\varepsilon > 0$, an ε-chain joining two points in C_0 is contained in $H^{-1}((-a', a))$.*

PROOF. Notice that $-a'$ is a dynamically regular value (Definiton 3.1) and therefore h maps the level $H^{-1}(-a')$ below itself. Therefore there is $\varepsilon_0 > 0$ such that the ε_0-neighbourhood of any point in $H^{-1}((-\infty, -a'])$ is mapped by h into $H^{-1}((-\infty, -a'))$. If we choose $\varepsilon < \varepsilon_0$ and if the ε-chain joining two points of C_0 falls into $H^{-1}((-\infty, -a'])$, then the rest of the chain cannot escape $H^{-1}((-\infty, -a'))$ forever. A contradiction. The opposite case of falling into $H^{-1}([a, \infty))$ can be dealt with similarly by considering h^{-1} and the reversed chain. $\square$

Let $B^+ = A^+ \setminus |\mathcal{D}^+|$. Then we have $h^{-1}(B^+) \subset B^+$. In fact, a point $z \in B^+$ is characterized by the existence of a path in $H^{-1}([a, \infty))$ starting at z and ending at a point in ∂A^+ without passing $H^{-1}(a)$ in the middle. This property is inherited to $h^{-1}(z)$ since there is a path from $h^{-1}(\partial A^+)$ to ∂A^+ which does not pass $H^{-1}(a)$ in the middle.

The above inclusion implies that any disc $D_i^+ \in \mathcal{D}^+$ is mapped by h to the complement of B^+, either into $\mathrm{Int}(D_{i'}^+)$ for some $D_{i'}^+ \in \mathcal{D}^+$ or into $\mathbb{A} \setminus A^+$. Notice that $\mathbb{A} \setminus A^+$ is forward invariant by h.

Let us call a sequence in $\mathcal{D}^+$

$$\mathcal{P} = \{D_{i_0}^+, D_{i_1}^+, \ldots, D_{i_n}^+\}$$

a *cycle of discs*, if $h(D_{i_{j-1}}^+) \subset \mathrm{Int}(D_{i_j}^+)$ $(0 < j \leq n)$ and $D_{i_n}^+ = D_{i_0}^+$. Denote $|\mathcal{P}| = \bigcup_{j=0}^{n-1} D_{i_j}^+$.

LEMMA 3.8. *If $C_0 \cap A^+ \neq \emptyset$, then there is a cycle of discs $\mathcal{P}$ in $\mathcal{D}^+$ such that $C_0 \cap |\mathcal{P}| \neq \emptyset$.*

PROOF. One can show as in the proof of Lemma 3.7 that for any small $\varepsilon > 0$, there is no ε-chain from a point in $\mathbb{A} \setminus A^+$ to a point in A^+, since $h(\mathrm{Cl}(\mathbb{A} \setminus A^+)) \subset \mathbb{A} \setminus A^+$.

Notice that C_0 is h-invariant. Now if $D_i^+ \cap C_0 \neq \emptyset$ for some $D_i^+ \in \mathcal{D}^+$, D_i^+ cannot be mapped into $\mathbb{A} \setminus A^+$ by a positive iterate of h. Then the iterated images of $\mathcal{D}_i^+$ are eventually periodic. That is, there are $m > 0$ and a cycle of discs $\mathcal{P}$ such that $h^m(D_i^+) \subset |\mathcal{P}|$. Since C_0 is h-invariant, this shows the lemma. □

Lemma 3.9. *If $C_0 \cap |\mathcal{P}| \neq \emptyset$ for some cycle of discs $\mathcal{P}$ of $\mathcal{D}^+$, then $C_0 \subset |\mathcal{P}|$.*

Proof. The set $H^{-1}([-a', a]) \setminus |\mathcal{P}|$ is compact, as well as $|\mathcal{P}|$. Thus there is $\varepsilon_0 > 0$ such that any $z \in H^{-1}([-a', a]) \setminus |\mathcal{P}|$ and $w \in |\mathcal{P}|$ satisfy $d(z, w) > \varepsilon_0$.

Let $x \in C_0 \cap |\mathcal{P}|$ and let y be an arbitrary point in C_0. Choose ε small enough so that $\varepsilon < \varepsilon_0$ and that any ε-chain $x_0, x_1, \ldots, x_r$ from x to y is contained in $H^{-1}((-a', a))$ (Lemma 3.7). We shall show inductively that $x_i \in |\mathcal{P}|$. This is true for $i = 0$. Assume $x_{i-1} \in |\mathcal{P}|$. Then $h(x_{i-1}) \in |\mathcal{P}|$ since $|\mathcal{P}|$ is forward invariant. On the other hand, $d(x_i, h(x_{i-1})) < \varepsilon_0$ and $x_i \in H^{-1}([-a', a])$. By the definition of ε_0, this implies $x_i \in |\mathcal{P}|$. Inductively we have $y \in |\mathcal{P}|$, as is required. □

Lemma 3.10. *If two periodic points z_ν $(\nu = 1, 2)$ are contained in $|\mathcal{P}|$, where $\mathcal{P}$ is a cycle of discs in $\mathcal{D}^+$, then we have $\mathrm{rot}(\tilde{h}, z_1) = \mathrm{rot}(\tilde{h}, z_2)$.*

Proof. By replacing z_ν by their iterate, one may assume both z_ν belong to the disc $D_{i_0}^+$. Choose the lift $\tilde{z}_\nu$ of z_ν from the same lift of $D_{i_0}^+$. Then for any $j \in \mathbb{N}$, their images $\tilde{h}^j(\tilde{z}_\nu)$ must belong to the same lift of the same disc $D_{i_j}^+$, showing the lemma. □

Proof of Proposition 3.6. Assume on the contrary that $C_0 \cap A^+ \neq \emptyset$. Then by Lemmas 3.8 and 3.9, $C_0 \subset |\mathcal{P}|$ for a cycle of discs $\mathcal{P}$ in $\mathcal{D}^+$. But then Lemma 3.10 contradicts the assumption of C_0 (the existence of two periodic points of different rotation number). The case $C_0 \cap A^- \neq \emptyset$ can be dealt with similarly. □

Now let us deform the homeomorphism h in $A^- \cup A^+$ so that it has no periodic points in $A^- \cup A^+$. Namely we replace h with a map in $\mathcal{H}$ with very simple dynamics in $A^- \cup A^+$. Notice that for any small ε, any ε-chain starting and ending at C_0 never falls into $A^+ \cup A^-$. Proposition 3.6, together with this fact, shows that the chain transitive class C_0 of the old h is unchanged for the new h. Especially the points $x_\nu \in C_0$ in the assumption of Proposition 3.5 are still the periodic points of the new h. Moreover if we find a periodic point of the new h in C_0, it is a periodic point of the old h in C_0 of the same rotation number. Therefore in the proof of Proposition 3.5, it is no loss of generality to assume the following.

Assumption 3.11. There is $\beta > 0$ such that for any $z \in A^- \cup A^+$, we have $d(z, h^q(z)) > \beta$.

The rest of the proof is divided into two cases according to whether $F_0 \cap (\bigcup_k E_k^+ \cup \bigcup_l E_l^-) = \emptyset$ or not, where $F_0 = \pi(\mathrm{Fix}(\tilde{h}^q \circ T^{-p_1-1}))$.

Case 1. $F_0 \cap (\bigcup_k E_k^+ \cup \bigcup_l E_l^-) \neq \emptyset$.

The argument in this case is based upon the Nielsen fixed point theory ([**8**]), which is a refinement of the Lefschetz index theorem. Let us give a brief summary of the theory for the special case of a continuous map f of the closed annulus $\mathbb{B}$. Let us denote by $\pi : \tilde{\mathbb{B}} \to \mathbb{B}$ the universal covering map. Let $\{\tilde{f}_i\}_{i\in I}$ be the family of the lifts of f to $\tilde{\mathbb{B}}$. Then $F_i = \pi(\mathrm{Fix}(\tilde{f}_i))$ is a closed subset of $\mathrm{Fix}(f)$, called a *Nielsen class* of $\mathrm{Fix}(f)$. It is empty but for finitely many lifts $\tilde{f}_i$, and $\mathrm{Fix}(f)$ is partitioned into a finite disjoint union of nonempty Nielsen classes. To each Nielsen class F_i, an integer $\mathrm{Index}(f, F_i)$, called the *index of* F_i, is assigned so that the sum of indices is equal to the Lefschetz number of f.

The most important feature of the index is the following. Suppose that two maps f and f' are homotopic, and a lift $\tilde{f}_i$ of f is joined with a lift $\tilde{f}'_i$ of f' by a lift of the homotopy. Then the corresponding Nielsen classes $F_i = \pi(\mathrm{Fix}(\tilde{f}_i))$ and $F'_i = \pi(\mathrm{Fix}(\tilde{f}'_i))$ have the same index: $\mathrm{Index}(f, F_i) = \mathrm{Index}(f', F'_i)$.

In particular if $f : \mathbb{B} \to \mathbb{B}$ is homotopic to the identity, then for any Nielsen class F_i, we have $\mathrm{Index}(f, F_i) = 0$, since f is homotopic to a fixed point free homeomorphism.

The index $\mathrm{Index}(f, F_i)$ is computed as follows. If a Nielsen class F_i is partitioned into a finite disjoint union of closed subsets: $F_i = \bigcup_j G_j$, then

$$\mathrm{Index}(f, F_i) = \sum_j \mathrm{Index}(f, G_j).$$

Assume there is a closed disc D such that

$$G_j = \mathrm{Fix}(f) \cap D \subset \mathrm{Int}(D). \tag{3.4}$$

Let $\tilde{f}_i$ be the lift correspoinding to the Nielsen class F_i that contains G_j and $\tilde{D}$ any lift of D. Consider an inclusion $\tilde{\mathbb{B}} \subset \mathbb{R}^2$. Then $\mathrm{Index}(f, G_j)$ is the mapping degree of the map

$$\mathrm{Id} - \tilde{f}_i : \partial\tilde{D} \to \mathbb{R}^2 \setminus \{0\}.$$

This is independent of the choice of the disc D satisfying (3.4). In particular if G_j is nonempty and if $f^{-1}(D) \subset \mathrm{Int}(D)$, then $\mathrm{Index}(f, G_j) = 1$.

Now let us start the proof of Proposition 3.5 in Case 1. We apply the Nielsen fixed point theory to the map h^q. For this purpose, the homeomophism $h^q : \mathbb{A} \to \mathbb{A}$ must be deformed in the exterior of a compact subannulus and extended to a homeomorphism of $\mathbb{B}$ in such a way that the fixed point of the new extended h^q is the same as the original h^q. But this can easily be done. In the sequal, we forget about this change, and just consider the original h^q.

We are interested in the particular lift $\tilde{h}^q \circ T^{-p_1-1}$ and the corresponding Nielsen class $F_0 = \pi(\mathrm{Fix}(\tilde{h}^q \circ T^{-p_1-1}))$. Our purpose is to show that $F_0 \cap H^{-1}([-a', a]) \neq \emptyset$. We have

$$\mathrm{Index}(h^q, F_0) = 0. \tag{3.5}$$

Assume $F_0 \cap E_k^+ \neq \emptyset$ for some k. Then we have $h^{-q}(E_k^+) \subset \mathrm{Int}(E_k^+)$. Condition (3.4) above is satisfied for $f = h^q$, $D = E_k^+$ and $G_j = F_0 \cap E_k^+$. Thus we have $\mathrm{Index}(h^q, F_0 \cap$

$E_k^+) = 1$. Likewise if $F_0 \cap E_l^- \neq \emptyset$, then $\mathrm{Index}(h^q, F_0 \cap E_l^-) = 1$.

On the other hand by Assumption 3.11, $F_0 \cap (A^- \cup A^+) = \emptyset$. By (3.5), this implies that $\mathrm{Index}(h^q, F_0 \cap X) < 0$, showing that $F_0 \cap X \neq \emptyset$, and hence $F_0 \cap H^{-1}([-a', a]) \neq \emptyset$, as is required.

Case 2. $F_0 \cap (\bigcup_k E_k^+ \cup \bigcup_l E_l^-) = \emptyset$.

In this case, F_0, if nonempty, must be contained in $X \subset H^{-1}([-a', a])$. Therefore we only need to show that F_0 is nonempty in $\mathbb{A}$. The proof is by absurdity. Assume throughout Case 2 that the map $\tilde{h}^q \circ T^{-p_1-1}$ is fixed point free. This, together with Assumption 3.11, implies that there is $\alpha > 0$ with the following property.

(1) For any $\tilde{z} \in \tilde{\mathbb{A}}$, $d(\tilde{h}^q(\tilde{z}), T^{p_1+1}(\tilde{z})) > 2\alpha$.

Here d denotes the distance function given by the lift of the standard Riemannian metric $d\theta^2 + dt^2$ of $\mathbb{A}$. Thus the covering transformation T is an isometry for d.

There is $\delta > 0$ such that for a lift $\tilde{\varphi}$ of a homeomorphism φ of $\mathbb{A}$, the following holds. We denote by $\|\cdot\|_0$ the supremum norm.

(2) If $\|\tilde{\varphi} - \mathrm{Id}\|_0 < 2\delta$, then $\|(\tilde{\varphi} \circ \tilde{h})^q - \tilde{h}^q\|_0 < \alpha$.

Conditions (1) and (2) implies in particular that for any $\tilde{z} \in \tilde{\mathbb{A}}$, we have

$$d((\tilde{\varphi} \circ \tilde{h})^q(\tilde{z}), T^{p_1+1}(\tilde{z})) > \alpha,$$

and therefore we have the following.

(3) There is no fixed point of $(\tilde{\varphi} \circ \tilde{h})^q \circ T^{-p_1-1}$.

Fix once and for all the number $\delta > 0$ that satisfies (2).

Recall the periodic points x_ν and their lift $\tilde{x}_\nu$ in the assumption of Proposition 3.5. Consider a δ-chain $\gamma = (z_0, z_1, \ldots, z_i)$ of length i from x_ν to $x_{\nu'}$ (ν, $\nu' = 1, 2$). Let $\tilde{\gamma} = (\tilde{z}_0, \tilde{z}_1, \ldots, \tilde{z}_i)$ be a lift of γ starting at $\tilde{x}_\nu$ which is a δ-chain for $\tilde{h}$. Assume that $\tilde{\gamma}$ ends at $T^j(\tilde{x}_{\nu'})$ for some $j \in \mathbb{Z}$. Then the pair (i, j) is called the *dynamical index* of γ. We have the following lemma, which is a variant of the method for finding periodic points invented in [**5**].

Lemma 3.12. *There is no δ-cycle at x_1 of dynamical index $(\xi q, \xi(p_1 + 1))$ for any $\xi \in \mathbb{N}$.*

Proof. Assume for contradiction that there is a δ-cycle $\gamma = (z_0, z_1 \ldots, z_r)$ at x_1 of dynamical index $\xi(q, p_1 + 1)$ for some $\xi > 0$, Thus $r = \xi q$ and $z_0 = z_r = x_1$. Then there is a homeomorphism φ of $\mathbb{A}$ such that $\varphi(h(z_i)) = z_{i+1}$ $(0 \leq i < \xi q)$ and that $\|\varphi - \mathrm{Id}\|_0 < 2\delta$.

To show this, consider the product $\mathbb{A} \times [0, 1]$ and the line segments joining $(h(z_i), 0)$ to $(z_{i+1}, 1)$. A general position argument shows that the line segments can be moved slightly so that they are mutually disjoint. Define a vector field X pointing upwards, tangent to the segments. With an appropriate choice of X, the holonomy map of X from $\mathbb{A} \times \{0\}$ to $\mathbb{A} \times \{1\}$ yields a desired homeomorphism φ. See Figure 3.

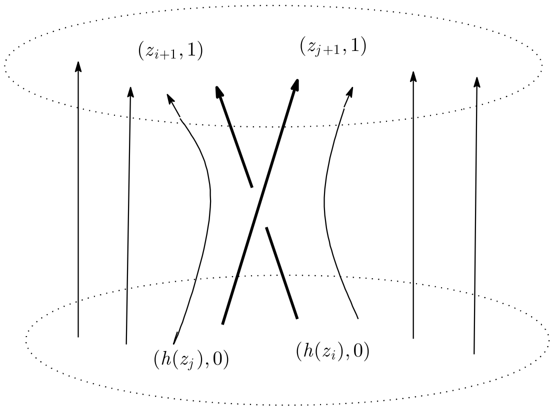

Figure 3.

Let $\tilde{\varphi}$ be the lift of φ such that $\|\tilde{\varphi} - \mathrm{Id}\|_0 < 2\delta$. Now the sequence $(z_0, z_q, z_{2q}, \dots, z_{\xi q})$ is a periodic orbit of $(\varphi \circ h)^q$. It has a lift $\tilde{z}_0, \tilde{z}_q, \dots, \tilde{z}_{\xi q}$ that is a periodic orbit of $(\tilde{\varphi} \circ \tilde{h})^q \circ T^{-p_1-1}$, since the dynamical index of γ is $\xi(q, p_1+1)$. Hence by the Brouwer plane fixed point theorem, there is a fixed point of $(\tilde{\varphi} \circ \tilde{h})^q \circ T^{-p_1-1}$. This is contrary to condition (3). The proof is complete now. □

In the rest we shall construct a δ-chain prohibited in Lemma 3.12, by using the condition $x_1 \sim x_2$. The absurdity will show that $F_0 \neq \emptyset$, as is required.

Let γ_1 be a δ-chain from x_1 to x_2 of dynamical index (i_1, j_1), and γ_3 another from x_2 to x_1 of dynamical index (i_2, j_2). One can assume that $i_1 + i_2$ is a multiple of q. In fact, if it is not the case, consider the concatenation $(\gamma_1 \cdot \gamma_3)^{q-1} \cdot \gamma_1$ instead of γ_1, leaving γ_3 unchanged. Thus we can set

(4) $i_1 + i_2 = aq$ for some $a \in \mathbb{N}$ and $j_1 + j_2 = b$ ($b \in \mathbb{Z}$).

Let

$$\gamma_2 = (x_2, h(x_2), \dots, h^{q-1}(x_2), x_2), \quad \text{and}$$
$$\gamma_4 = (x_1, h(x_1), \dots, h^{q-1}(x_1), x_1).$$

They are periodic orbits, and hence δ-cycles, of dynamical indices (q, p_2) and (q, p_1) respectively. Consider the concatenation $\gamma_1 \cdot \gamma_2^\eta \cdot \gamma_3 \cdot \gamma_4^\zeta$ for some η, $\zeta \in \mathbb{N}$. It is a δ-cycle at x_1 of dynamical index $(qa + q\zeta + q\eta, b + \zeta p_1 + \eta p_2)$.

We shall show there are ξ, ζ and η such that the above concatenation becomes a δ-cycle of dynamical index $\xi(q, p_1+1)$ forbidden in Lemma 3.12. The equation for it is the following.

(5) $a + \zeta + \eta = \xi$.
(6) $b + \zeta p_1 + \eta p_2 = \xi(p_1 + 1)$.

Now for any large $\eta > 0$, define ξ and ζ by

$$\xi = \eta(p_2 - p_1) + (b - p_1 a) \quad \text{and} \quad \zeta = \eta(p_2 - p_1 - 1) + (b - p_1 a - a).$$

Then η, ξ and ζ are positive integeres satisfying (5) and (6). A contradiction shows Proposition 3.5. We are done with the proof of Theorem 1.3.

3.4. Finally let us show Corollary 1.4. In view of Theorem 1.3, we only need to show the existence of a periodic point $x_\nu \in C_0$ such that $\mathrm{rot}(\tilde{h}, x_\nu) = \alpha_\nu$. For this we proceed just as in **3.3**. The assumption that $[\alpha_1, \alpha_2]$ is a nondegenerate interval is necessary for Proposition 3.6, which uses Lemma 3.10. The proof in the present case is exactly the same except at the last step, CASE 2. At that point, we need the following proposition.

PROPOSITION 3.13. *Suppose C_0 is a chain transitive class with* $\mathrm{rot}(\tilde{h}, C_0) = [\alpha_1, \alpha_2]$ *with $\alpha_1 = p/q$, $(p, q) = 1$. Then the homeomorphism $\tilde{h}^q \circ T^{-p}$ admits a fixed point in $\tilde{\mathbb{A}}$.*

We emphasize that we have only to show the existence of the fixed point *in the whole* $\tilde{\mathbb{A}}$, since we have followed the argument in **3.3**. The rest of this paragraph is devoted to the proof of Proposition 3.13. The assumption $\mathrm{rot}(\tilde{h}, C_0) = [p/q, \alpha_2]$ implies the following.

LEMMA 3.14. *We have* $\mathrm{rot}(\tilde{h}^q, C_0) = [p, q\alpha_2]$.

Here C_0 may not be a single chain transitive class for h^q. But the rotation set $\mathrm{rot}(\tilde{h}^q, C_0)$ is defined, in the same way, as the set of the values $\mathrm{rot}(\tilde{h}, \mu)$, where μ runs over the space of the h^q-invariant probability measures supported on C_0.

PROOF. Clearly a h-invariant probability measure μ is h^q-invariant and $\mathrm{rot}(\tilde{h}^q, \mu) = q \cdot \mathrm{rot}(\tilde{h}, \mu)$. To show this, notice that

$$\mathrm{rot}(\tilde{h}^q, \mu) = \langle \mu, \Pi_1 \circ \tilde{h}^q - \Pi_1 \rangle = \sum_{i=0}^{q} \langle \mu,\, \Pi_1 \circ \tilde{h}^{i+1} - \Pi_1 \circ \tilde{h}^i \rangle,$$

$$\begin{aligned} \text{where } \ &\langle \mu,\, \Pi_1 \circ \tilde{h}^{i+1} - \Pi_1 \circ \tilde{h}^i \rangle = \langle \mu,\, (\Pi_1 \circ \tilde{h} - \Pi_1) \circ h^i \rangle \\ &= \langle h^i_* \mu,\, \Pi_1 \circ \tilde{h} - \Pi_1 \rangle = \langle \mu,\, \Pi_1 \circ \tilde{h} - \Pi_1 \rangle = \mathrm{rot}(\tilde{h}, \mu). \end{aligned}$$

Thus we get

$$q \cdot \mathrm{rot}(\tilde{h}, C_0) \subset \mathrm{rot}(\tilde{h}^q, C_0).$$

On the other hand, given a h^q-invariant probability measure ν, the average $\hat{\nu} = q^{-1} \sum_{i=0}^{q-1} h^i_* \nu$ is h-invariant, and we have

$$\langle \hat{\nu},\, \Pi_1 \circ \tilde{h} - \Pi_1 \rangle = q^{-1} \sum_{i=0}^{q-1} \langle \tilde{h}^i_* \nu,\, \Pi_1 \circ \tilde{h} - \Pi_1 \rangle = q^{-1} \langle \nu,\, \Pi_1 \circ \tilde{h}^q - \Pi \rangle,$$

showing $\mathrm{rot}(\tilde{h}, \hat{\nu}) = q^{-1} \cdot \mathrm{rot}(\tilde{h}^q, \nu)$. This implies the converse inclusion

$$q \cdot \mathrm{rot}(\tilde{h}, C_0) \supset \mathrm{rot}(\tilde{h}^q, C_0). \qquad \square$$

Since p is an extremal point of the rotation set $[p, q\alpha_2]$, there is an ergodic h^q-invariant probability measure μ supported on C_0 such that $\mathrm{rot}(\tilde{h}^q, \mu) = p$. To see this, any h^q-invariant measure is a convex integral of the ergodic components, and since p is extremal, almost any ergodic component has rotation number p.

We use the following version of the Atkinson theorem ([**1**]), whose proof is found at Proposition 12.1 of [**7**].

PROPOSITION 3.15. *Suppose $T : X \to X$ is an ergodic automorphism of a probability space (X, μ) and let $\varphi : X \to \mathbb{R}$ be an integrable function with $\langle \mu, \varphi \rangle = 0$. Let $S(n, x) = \sum_{i=0}^{n-1} \varphi(T^i(x))$. Then for any $\varepsilon > 0$ the set of x such that $|S(n, x)| < \varepsilon$ for infinitely many n is a full measure subset of X.*

Notice that Proposition 3.15 holds only for $\mathbb{R}$-valued functions, and fails e.g. for $\mathbb{C}$-valued functions. We apply Proposition 3.15 for the transformation $h^q : C_0 \to C_0$, an ergodic measure μ with $\mathrm{rot}(\tilde{h}^q, \mu) = p$, the function $\varphi : C_0 \to \mathbb{R}$ defined by

$$\varphi \circ \pi = \Pi_1 \circ \tilde{h}^q \circ T^{-p} - \Pi_1 = \Pi_1 \circ \tilde{h}^q - \Pi_1 - p,$$

and $\varepsilon = 1$. Notice that the condition $\mathrm{rot}(\tilde{h}^q, \mu) = p$ is equivalent to $\langle \mu, \varphi \rangle = 0$.

Since for any $i \in \mathbb{N}$,

$$\varphi \circ h^{qi} \circ \pi = \varphi \circ \pi \circ (\tilde{h}^q \circ T^{-p})^i = \Pi_1 \circ (\tilde{h}^q \circ T^{-p})^{i+1} - \Pi_1 \circ (\tilde{h}^q \circ T^{-p})^i,$$

we have

$$S(n, \cdot) \circ \pi = \Pi_1 \circ (\tilde{h}^q \circ T^{-p})^n - \Pi_1.$$

By Proposition 3.15, there is a point $x \in C_0$ such that $|S(n, x)| < 1$ for infinitely many $n \in \mathbb{N}$.

Then a lift $\tilde{x}$ of x satisfies

$$|\Pi_1((\tilde{h}^q \circ T^{-p})^n(\tilde{x})) - \Pi_1(\tilde{x})| < 1 \tag{3.6}$$

for infinitely many $n \in \mathbb{N}$. Since the orbit of $\tilde{x}$ is contained in $\pi^{-1}(C_0)$, a subset in $\tilde{\mathbb{A}}$ bounded from above and below, (3.6) implies that the ω-limit set of $\tilde{x}$ for the homeomorphism $\tilde{h}^q \circ T^{-p}$ is nonempty. Especially the nonwandering set of $\tilde{h}^q \circ T^{-p}$ is nonempty. This implies the existence of a fixed poit of $\tilde{h}^q \circ T^{-p}$ by virtue of (a variant of) the Brouwer plane fixed point theorem ([**6**]). This completes the proof of Proposition 3.13.

4. Realization of a rational prime end rotation number.

The purpose of this section is to give a proof of Theorem 1.5. We assume throughout that $\mathrm{rot}(\tilde{h}, \infty) = p/q$ for $h \in \mathcal{H}$, and that H is a C^∞ complete Lyapunov function defined on $\mathbb{A}$. Let $F_1 = \pi(\mathrm{Fix}(\tilde{h}^q \circ T^{-p}))$, the Nielsen class associated to the lift $\tilde{h}^q \circ T^{-q}$ of h^q. Our purpose is to show that F_1 is nonvoid.

Let a be a regular and dynamically regular value of H which satisfies the following

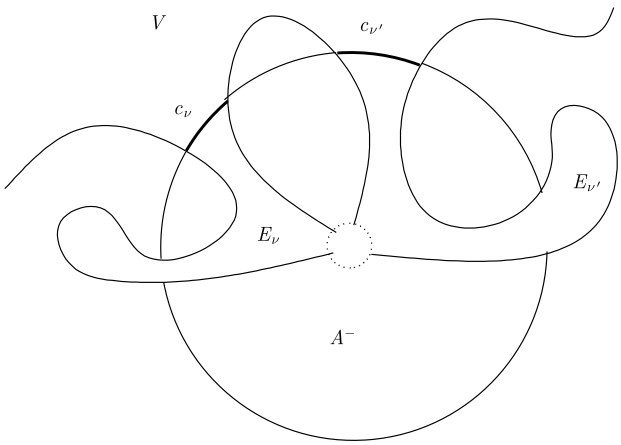

Figure 4.

condition:

$$A^+ \cap \mathrm{Fr}(U_\infty) \neq \emptyset, \tag{4.1}$$

where A^+ is the upper subannulus bounded by the unique homotopically nontrivial simple closed curve in $H^{-1}(a)$. The lower subannulus is denoted by A^-.

Let V be the unique unbounded component of $U_\infty \cap A^+$. See Figure 4. Let

$$\mathrm{Cl}_{U_\infty}(V) \cap \partial A^- = \coprod_{\nu \in I} c_\nu,$$

where c_ν are cross cuts of U_∞. The cross cuts c_ν are at most countable and oriented according to the orientation of V. Let E_ν be the connected component of $U_\infty \setminus c_\nu$ disjoint from V. Since A^- is forward invariant, $\mathrm{Cl}(E_\nu)$ is mapped by h^q into some $E_{\nu'}$.

Let p_ν (resp. q_ν) be the innitial point (resp. terminal point) of c_ν. As in **2.3**, the cross cut c_ν with endpoint p_ν (resp. q_ν) defines a prime end denoted by $\hat{p}_\nu$ (resp. $\hat{q}_\nu$). Denote by $\hat{c}_\nu$ the closed interval in the set of prime ends ∂U_∞^* bounded by $\hat{p}_\nu$ and $\hat{q}_\nu$. In other words,

$$\hat{c}_\nu = \mathrm{Cl}_{U_\infty^*}(E_\nu) \cap \partial U_\infty^*.$$

Of course they are mutually disjoint, and $\hat{c}_\nu$ is mapped by $(h_\infty^*)^q$ into the interior of some $\hat{c}_{\nu'}$. If $\nu \neq \nu'$, then there is no fixed point of $(h_\infty^*)^q$ in $\hat{c}_\nu$. If $\nu = \nu'$, $\hat{c}_\nu$ is mapped into the interior of itself by $\hat{c}_\nu$. On the other hand, there is a fixed point of $(h_\infty^*)^q$, since $\mathrm{rot}(\tilde{h}, \infty) = p/q$. Therefore there must be a fixed point ξ of $(h_\infty^*)^q$ in the set

$$\Xi = \partial U_\infty^* \setminus \left(\bigcup_\nu \hat{c}_\nu \right).$$

More precisely, any lift $\tilde{\xi}$ of ξ to $\partial \tilde{U}_\infty^*$, the universal cover of ∂U_∞^*, is fixed by $(\tilde{h}_\infty^*)^q \circ T^{-p}$.

The *principal point set* $\Pi(\xi)$ of the prime end ξ is defined to be the set of all the limit points of topological chains which represent the prime end ξ. As is well known ([**14**]), the principal point set $\Pi(\xi)$ is closed, connected and invariant by h^q. Clearly it is contained in $\mathrm{Fr}(U_\infty)$. Also since ξ is contained in Ξ, any cross cut c_i of any topological chain $\{c_i\}$ representing ξ must intersect $\mathrm{Cl}(V)$. The set $\Pi(\xi)$ is contained in $\mathrm{Cl}(V)$, since $\mathrm{diam}(c_i) \to 0$. This implies that $\Pi(\xi)$ is compact. Let $\hat{\Pi}(\xi)$ be the union of $\Pi(\xi)$ with all the bounded connected components of the complement. The set $\hat{\Pi}(\xi)$ is also a h^q-invariant continuum, and therefore it does not separate two ends of $\mathbb{A}$ by the assumption on h. It is also nonseparating, in the sense that its complement is connected.

The Cartwright-Littlewood theorem ([**2**]) asserts that any planar homeomorphism leaving a nonseparating continuum invariant has a fixed point in it. Thus there is a fixed point y of h^q in $\hat{\Pi}(\xi)$. In the rest of this section, we shall show $y \in F_1$, i.e. a lift $\tilde{y}$ of y is a fixed point of $\tilde{h}^q \circ T^{-p}$. But in fact, we shall find such a point $\tilde{y}$ at the very end of the proof.

Recall that for a bounded cross cut c of U_∞, $V(c)$ denotes the component of $U_\infty \setminus c$ which is homeomorphic to an open disc. Likewise we define the component $V(\tilde{c})$ for a lift $\tilde{c}$ of c to be the lift of $V(c)$ bounded by $\tilde{c}$.

Given a topological chain $\{c_i\}$ of U_∞, a *lift* $\{\tilde{c}_i\}$ of $\{c_i\}$ is defined as follows. For $i = 1$, let $\tilde{c}_1$ be an arbitrary lift of c_1. For $i > 1$, let $\tilde{c}_i$ be the unique lift of c_i contained in $V(\tilde{c}_{i-1})$. Then we have $V(\tilde{c}_i) \subset V(\tilde{c}_{i-1})$ ($\forall i > 1$), and the lift $\{\tilde{c}_i\}$ is determined uniquely by the choice of $\tilde{c}_1$.

Let $x \in \Pi(\xi)$ be an arbitrary point, and let $\{c_i\}$ be a topological chain representing ξ such that $c_i \to x$. Let $\{\tilde{c}_i\}$ be a lift of $\{c_i\}$ and $\tilde{x}$ a lift of x. Then since $c_i \to x$, there is a sequence of integers n_i such that $T^{n_i}(\tilde{c}_i) \to \tilde{x}$. Let us show that n_i is identical for any large i.

Since $\Pi(\xi)$ is compact and does not separate the two ends of $\mathbb{A}$, there is a simple closed curve Γ such that $\Pi(\xi)$ is contained in the open disc E bounded by Γ. Assume that there are infinitely many i such that $n_{i+1} \neq n_i$. For any large i, the cross cuts c_i and c_{i+1} are contained in E. Consider a simple path γ joining c_i to c_{i+1} in $V(c_i) \setminus V(c_{i+1})$. Then γ, starting and ending in E, must wind the annulus $\mathbb{A}$ since $n_{i+1} \neq n_i$. Thus there is a cross cut c_i' contained in Γ which separates c_{i+1} and c_i. Passing to a further subsequence, we may assume $\mathrm{Cl}(c_i')$ are disjoint, since c_i' are disjoint open intervals of a single curve Γ. We also have $\mathrm{diam}(c_i') \to 0$. Thus $\{c_i'\}$ is a topological chain contained in Γ, which is clearly equivalent to $\{c_i\}$. Thus any accumulation point of $\{c_i'\}$ must be contained in the principal point set $\Pi(\xi)$. This contradicts the choice of Γ: $\Gamma \cap \Pi(\xi) = \emptyset$.

Now one can assume, changing the lift $\tilde{x}$ of x if necessary, that $\tilde{c}_i \to \tilde{x}$ for the lift $\{\tilde{c}_i\}$. By the definition of the topology of the prime end compactification (Section 2), the family $\{V(c_i)\}$ forms a fundamental neighbourhood system of the prime end $\xi \in U_\infty^*$. Then it follows immediately that $\{V(\tilde{c}_i)\}$ forms a fundamental neighbourhood system of a lift $\tilde{\xi}$ of ξ. On the other hand, we have $(\tilde{h}_\infty^*)^q \circ T^{-p}(\tilde{\xi}) = \tilde{\xi}$. Thus $\{\tilde{h}^q \circ T^{-p}(V(\tilde{c}_i))\}$ is also a fundamental neighbourhood system of $\tilde{\xi}$. That is, $\{\tilde{h}^q \circ T^{-p}(\tilde{c}_i)\}$ and $\{\tilde{c}_i\}$ are equivalent in the sense that for any i, there is j such that $\tilde{c}_j \subset V(\tilde{h}^q \circ T^{-p}(\tilde{c}_i))$ and $\tilde{h}^q \circ T^{-p}(\tilde{c}_j) \subset V(\tilde{c}_i)$.

Let $\tilde{\Pi}(\xi)$ be the lift of $\Pi(\xi)$ which contains the point $\tilde{x}$. The set $\tilde{\Pi}(\xi)$ is characterized as the set of the limit points of lifts of topological chains which are equivalent to $\{\tilde{c}_i\}$.

Since $\{\tilde{h}^q \circ T^{-p}(\tilde{c}_i)\}$ is equivalent to $\{\tilde{c}_i\}$, and $\tilde{h}^q \circ T^{-p}(\tilde{c}_i) \to \tilde{h}^q \circ T^{-p}(\tilde{x})$, we have $\tilde{h}^q \circ T^{-p}(\tilde{x}) \in \tilde{\Pi}(\xi)$. But then since $h^q(\Pi(\xi)) = \Pi(\xi)$, we have $\tilde{h}^q \circ T^{-p}(\tilde{\Pi}(\xi)) = \tilde{\Pi}(\xi)$.

Finally by the Cartwright-Littlewood theorem, there is a fixed point $\tilde{y}$ of $\tilde{h}^q \circ T^{-p}$ in the corresponding lift of $\hat{\Pi}(\xi)$, completing the proof of Theorem 1.5.

5. Accessible case.

This section is devoted to the proof of Theorem 1.6. Let $h \in \mathcal{H}$ be a homeomorphism satisfying $\mathrm{rot}(\tilde{h}, \infty) = \alpha$ for some lift $\tilde{h}$ and $\alpha \in \mathbb{R}$ such that $-\infty$ is accessible from U_∞. By changing the coordinates of $\mathbb{A}$, one may assume that h satifies

$$h(\theta, t) = (\theta, t-1), \quad \forall (\theta, t) \in B,$$

where $B = \{(\theta, t) \in \mathbb{A} \mid t \leq 0\}$. Clearly $B \subset U_{-\infty}$. Let

$$Z = \mathbb{A} \setminus (U_\infty \cup U_{-\infty}).$$

We shall show that $\lim_{i\to\infty} i^{-1}\Pi_1(\tilde{h}^i(z)) = \alpha$ for any $z \in \pi^{-1}(Z)$. Clearly this implies (1) of Theorem 1.6.

Let V be the unbounded component of $U_\infty \cap (\mathbb{A} \setminus B)$. Notice that $V \subset h(V)$. It is an essential open subannulus of $\mathbb{A}$. Let $\{c_\nu\}$ be the family of cross cuts of U_∞ contained in $\partial B \cap \mathrm{Cl}(V)$ and let V_ν be the connected component of $U_\infty \setminus c_\nu$ which is disjoint from V. The components V_ν are mutually disjoint open discs, which may intersect $\mathbb{A} \setminus B$. The cross cut c_ν is called the *gate* of V_ν. Since $U_\infty = h(U_\infty)$ and $V \subset h(V)$, we have $h(\cup_\nu V_\nu) \subset \cup_\nu V_\nu$.

A component V_ν is said to be *accessible* if $-\infty$ is accessible from V_ν. This means that there is a path $\gamma : (-\infty, 0] \to V_\nu$ such that $\Pi_2 \circ \gamma(t) \to -\infty$ as $t \to -\infty$, where $\Pi_2 : \mathbb{A} \to \mathbb{R}$ is the projection onto the second factor (the hight function). There is an accessible component by the assumption. For any V_ν, there exists $V_{\nu'}$ such that $h(V_\nu) \subset V_{\nu'}$, and if V_ν is accessible, so is $V_{\nu'}$.

Choose a sequence V_i ($i \in \mathbb{N}$) from the family $\{V_\nu\}$ as follows. Let V_1 be any accessible component. For $i > 1$, let V_i be the component such that $h(V_{i-1}) \subset V_i$. Then any V_i is accessible. The sequence $\{V_i\}$ may be all distinct or eventually periodic, that is, there is $p \in \mathbb{N}$ such that $V_{i+p} = V_i$ for any large i.

To the gate c_i of V_i is associated a closed interval $\hat{c}_i$ in the set of prime ends ∂U_∞^*, defined by

$$\hat{c}_i = \mathrm{Cl}_{U_\infty^*}(V_i) \cap \partial U_\infty^*.$$

Since $h(V_{i-1}) \subset V_i$, we have $h_\infty^*(\hat{c}_{i-1}) \subset \hat{c}_i$. The cyclic orders of the family $\{c_i\}$ in ∂B and $\{\hat{c}_i\}$ in ∂U_∞^* are the same, and there is a homeomorphism $\varphi : \partial B \to \partial U_\infty^*$ such that $\varphi(\mathrm{Cl}(c_i)) = \hat{c}_i$ ($\forall i$).

Fix once and for all a lift $\tilde{V}_i$ of V_i to $\tilde{\mathbb{A}}$ in the following way. Let $\tilde{V}_1$ be any lift of V_1, and for $i > 1$, $\tilde{V}_i$ the unique lift of V_i which satisfies $\tilde{h}(\tilde{V}_{i-1}) \subset \tilde{V}_i$ for the prescribed lift $\tilde{h}$. The gate of $\tilde{V}_i$ is denoted by $\tilde{c}_i$, that is, $\tilde{c}_i$ is the frontier of $\tilde{V}_i$ in $\pi^{-1}(U_\infty)$. It is a lift of c_i. A closed interval $\tilde{\hat{c}}_i$ of $\partial\tilde{U}^*_\infty = \pi^{-1}(\partial U^*_\infty)$ is defined by

$$\tilde{\hat{c}}_i = \mathrm{Cl}(\tilde{V}_i) \cap \partial\tilde{U}^*_\infty.$$

It is a lift of $\hat{c}_i$, and the map $\tilde{h}^*_\infty$ defined on $\partial\tilde{U}^*_\infty$ as an extension of $\tilde{h}$, satisfy $\tilde{h}^*_\infty(\tilde{\hat{c}}_{i-1}) \subset \tilde{\hat{c}}_i$.

Denote by T the generator of the covering transformations of both $\tilde{\mathbb{A}}$ and $\partial\tilde{U}^*_\infty$. There is a lift

$$\tilde{\varphi} : \pi^{-1}(\partial B) \to \partial\tilde{U}^*_\infty$$

of φ such that $\tilde{\varphi}(T^j(\mathrm{Cl}(\tilde{c}_i)) = T^j(\tilde{\hat{c}}_i)$ $(\forall i \in \mathbb{N},\ \forall j \in \mathbb{Z})$. We identify $\partial\tilde{U}^*_\infty$ with $\pi^{-1}(\partial B)$ by $\tilde{\varphi}^{-1}$, and then with $\mathbb{R}$ by Π_1. Thus T is the right translation by 1.

Let us denote the interval $\tilde{\hat{c}}_i = [a_i, b_i]$, where a_i and b_i are real numbers by the above identification. Recall that $\alpha = \mathrm{rot}(\tilde{h}, \infty)$ is, by definition, the rotation number of $\tilde{h}^*_\infty : \partial\tilde{U}^*_\infty \to \partial\tilde{U}^*_\infty$. Since $\tilde{h}^*_\infty(\tilde{\hat{c}}_{i-1}) \subset \tilde{\hat{c}}_i$ and the length of each $\tilde{\hat{c}}_i$ is always less than 1, we have

$$\alpha = \lim_{i\to\infty} i^{-1} a_i. \tag{5.1}$$

Below we consider (a_i, b_i) to be the interval $\tilde{c}_i \subset \pi^{-1}(\partial B)$ by the above identification. It is important that (5.1) still holds.

Our aim is to show that $\lim_{i\to\infty} \Pi_1 \circ \tilde{h}^i(z) = \alpha$ for any $z \in \pi^{-1}(Z)$. But we shall show only $\lim_{i\to\infty} \Pi_1 \circ \tilde{h}^i(z) \le \alpha$, the other inequality being shown similarly.

Let us denote by Γ_i the set of all the simple curves $l : \mathbb{R} \to \tilde{U}_\infty$ such that

(1) $\Pi_2 \circ l(t) \to \pm\infty$ as $t \to \pm\infty$, and
(2) $l(t) \in \tilde{V}_i$ for all negative t.

Since $\tilde{V}_i$ is the lift of an accessible component, Γ_i is nonempty for any $i \in \mathbb{N}$.

Definition 5.1. Let $z \in \pi^{-1}(Z)$. We say $z \le \tilde{V}_i$ if there is $l \in \Gamma_i$ such that z lies on the left side of l.

See Figure 5.

Lemma 5.2. *If $z \le \tilde{V}_{i-1}$ for $z \in \pi^{-1}(Z)$ and $i > 1$, then $\tilde{h}(z) \le \tilde{V}_i$.*

Proof. If $l \in \Gamma_{i-1}$, then $\tilde{h}(l) \in \Gamma_i$. The lemma follows from this. □

Lemma 5.3. *There is $M > 0$ such that if $z \le \tilde{V}_i$ $(z \in \pi^{-1}(Z))$, then $\Pi_1(z) \le a_i + M$.*

Proof. We shall show the following.

(1) There is $M > 0$ such that if $z \le \tilde{V}_1$ $(z \in \pi^{-1}(Z))$, then $\Pi_1(z) \le a_1 + M - 1$.

Let us explain why this is sufficient. Considering the action of covering transformations, (1) implies the following.

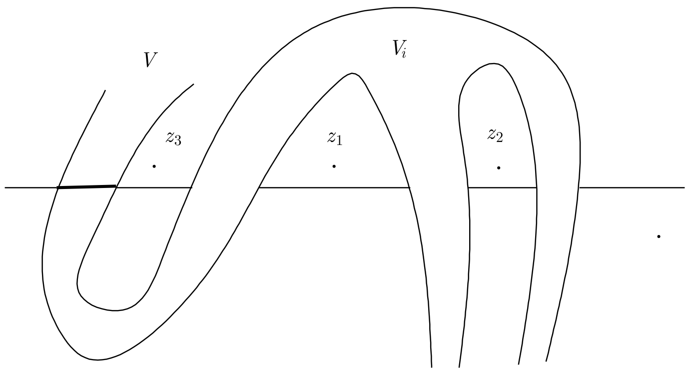

Figure 5. $z_1 \leq V_i$, $z_2 \leq V_i$, $z_3 \not\leq V_i$.

(2) If $z \leq T^n(\tilde{V}_1)$ ($z \in \pi^{-1}(Z)$, $n \in \mathbb{Z}$) under the similar definition, then $\Pi_1(z) \leq a_1 + n + M - 1$.

To deduce the lemma from (2), let n be the integer such that $a_1 + n - 1 \leq a_i < a_1 + n$. The last inequality means that the interval $T^n(\tilde{c}_1)$ lies on the right of $\tilde{c}_i$ in $\pi^{-1}(\partial B)$, and therefore $z \leq \tilde{V}_i$ implies that $z \leq T^n(\tilde{V}_1)$. Then by (2), we have

$$\Pi_1(z) \leq a_1 + n + M - 1 \leq a_i + M.$$

Let us start the proof of (1). Let δ be a simple curve in V joining $\pi(a_1)$ to $\pi(b_1)$ which is not homotopic to $\pi([a_1, b_1])$, and let $\gamma = \pi([b_1, a_1 + 1]) \subset \partial B$. Choose δ so that the concatenation $\delta \cdot \gamma$ is a simple closed curve which bounds a closed disc D containing Z in its interior. This is possible because Z is a compactum not separating both ends of $\mathbb{A}$. There is a lift $\tilde{D}$ of D which is bounded by the concatenation $\tilde{\delta} \cdot \tilde{\gamma}$, where $\tilde{\delta}$ is a lift of δ and $\tilde{\gamma} = [b_1, a_1 + 1]$. Let $Z_0 = \pi^{-1}(Z) \cap \tilde{D}$. Then we have $\pi^{-1}(Z) = \coprod_{i \in \mathbb{Z}} T^i(Z_0)$.

We shall show that the point $z \in \pi^{-1}(Z)$ satisfying $z \leq \tilde{V}_1$ is contained in $T^i(Z_0)$ for some $i \leq 0$. Clearly this is sufficient for our purpose since Z_0 is compact. Assume the contrary, say, $z \in T(Z_0)$. Since $z \leq \tilde{V}_1$, there is a curve l in Γ_1 which contains $z \in T(Z_0)$ on its left side. Let t_0 be the smallest value such that $l(t_0) \in (a_1, b_1)$. The curve l is homotopic in the family Γ_1 to a curve, still denoted by l, such that $l(t_0, \infty)$ is contained in $\pi^{-1}(V)$. It can further be homotoped so that $l(t_0, \infty)$ does not intersect the disc $T(\tilde{D})$, since $\pi^{-1}(V)$ is simply connected.

The other half of the curve, $l((-\infty, t_0))$, is contained in $\tilde{V}_1$. It must intersect $[b_1 + 1, a_1 + 2]$, the lower boundary of $T(\tilde{D})$, since a point $z \in T(D)$ is still on the left side of the new curve l.

Consider the curve $T \circ l$. The two curves $l(-\infty, t_0)$ and $T \circ l(-\infty, t_0)$ must intersect. See Figure 6. But the former is contained in $\tilde{V}_1$ while the latter in $T(\tilde{V}_1)$. Since $\tilde{V}_1 \cap T(\tilde{V}_1) \neq \emptyset$, this is impossible. $\square$

To finish, let $z \in \pi^{-1}(Z)$. One may assume $z \leq \tilde{V}_1$ by replacing z by $T^{-n}(z)$ if necessary. Then by successive use of Lemma 5.2, we have $\tilde{h}^i(z) \leq \tilde{V}_i$ for any $i \in \mathbb{N}$. Then by Lemma 5.3, we have

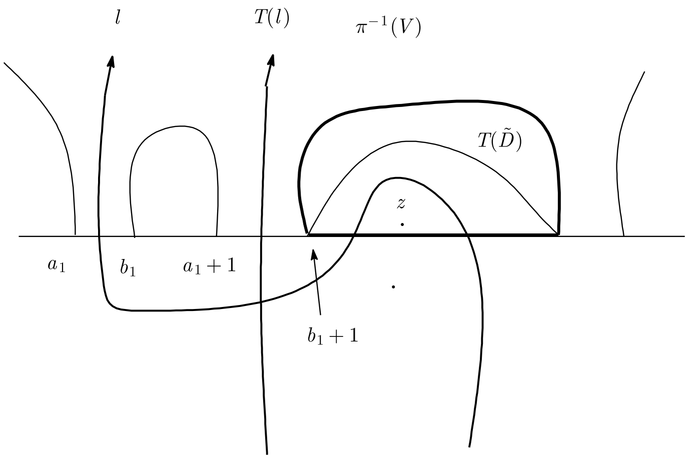

Figure 6.

$$\Pi_1(\tilde{h}^i(z)) \leq a_i + M,$$

showing that

$$\lim_{i\to\infty} i^{-1}\Pi_1(\tilde{h}^i(z)) \leq \lim i^{-1}a_i = \alpha,$$

completing the proof of Theorem 1.6 (1).

To show (2), just consider a lift of a point in Z accessible from $U_{-\infty}$. Details are left to the reader.

6. Appendix: C^∞ complete Lyapunov functions.

We fix $h \in \mathcal{H}$. Here is a criterion of the chain recurrent set C and a chain transitive class in terms of attractors and repellors ([**4**]). A subset A_i in S^2 is called an *attractor* if there is an open neighbourhood V_i of A_i such that $h(\mathrm{Cl}(V_i)) \subset V_i$ and $\bigcap_{j\geq 0} f^j(\mathrm{Cl}(V_i)) = A_i$. The set V_i is called an *isolating block* of A_i, and the set $A_i^* = \bigcap_{j\geq 0} f^{-j}(S^2 \setminus V_i)$ the *dual repellor* of A. The totality of attractors is at most countable, and we denote it by $\{A_i\}_{i\in I}$. Then we have ([**4**])

$$C = \bigcap_{i\in I}(A_i \cup A_i^*).$$

For x, $y \in C$, we also have

$$x \sim y \iff \forall i \in I, \quad \text{either} \quad x,\, y \in A_i \text{ or } x,\, y \in A_i^*.$$

We begin with the following well known fact due to H. Whitney.

Lemma 6.1. *For any closed subset P in S^2, there is a C^∞ function $\varphi_P : S^2 \to [0,1]$ such that $\varphi_P^{-1}(0) = P$.*

Lemma 6.2. *For any disjoint closed subsets P and Q of S^1, there is a C^∞ function $\psi : S^2 \to [0,1]$ such that $\psi^{-1}(0) = P$ and $\psi^{-1}(1) = Q$.*

Proof. The function φ_P in Lemma 6.1 can easily be modified so as to satisfy $Q \subset \varphi_P^{-1}(1)$. Define a function φ_Q replacing the roles of P and Q, and set

$$\psi = 2^{-1}(\varphi_P + 1 - \varphi_Q). \qquad \square$$

Recall that $\{A_i\}_{i\in I}$ is the family of the attractors of h.

Lemma 6.3. *For each $i \in I$, there is a C^∞ function $H_i : S^2 \to [0,1]$ such that*

(1) *$H_i^{-1}(0) = A_i$ and $H_i^{-1}(1) = A_i^*$.*
(2) *For any $x \in S^2 \setminus (A_i \cup A_i^*)$, we have $H_i(h(x)) < H_i(x)$.*

Proof. Let V_i be an isolating block of A_i. Then for any $j \in \mathbb{Z}$, there is a C^∞ function $\psi_j : S^2 \to [0,1]$ such that $\psi_j^{-1}(0) = f^j(\mathrm{Cl}(V_i))$ and $\psi_j^{-1}(1) = S^2 \setminus f^{j-1}(V_i)$. Choose a sequence $c_j > 0$ such that

$$\sum_{j\in\mathbb{N}} c_j \|\psi_j\|_{|j|} < \infty,$$

where $\|\cdot\|_{|j|}$ denotes the $C^{|j|}$ norm. Then the function $\sum_{j\in\mathbb{Z}} c_j\psi_j$ is a C^∞ function, and after normalized it satisfies the conditions of Lemma 6.3. $\square$

Proof of Proposition 3.2. By an appropriate choice of positive numbers a_i, the function $H = \sum_{i\in I} a_i H_i$ is a C^∞ function satisfying (1) and (2) of Definition 3.1. If the indexing set I is infinite, set $I = \mathbb{N}$ and choose a_i such that $a_{i+1} < 3^{-1}a_i$ $(\forall i)$. Then we obtain that $H(C)$ is closed and that the Lebesgue measure of $H(C)$ is zero. $\square$

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