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Linking pairing and Hopf fibrations on S^3

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Abstract. This article studies the asymptotic linking pairing lk on the space of exact 2-forms $B^2(S^3)$ on the 3-sphere S^3 through the geometry of Hopf fibrations. Mitsumatsu [**7**] tried to apply this pairing to 3-dimensional contact topology. He considered a positive definite subspace $P(\xi)$ in $(B^2(M), lk)$ associated with a contact structure ξ on a closed 3-manifold M. Further he introduced an invariant of ξ, called the analytic torsion. We investigate the case of the standard contact structure on S^3 and construct a positive definite subspace of arbitrary large dimension in the lk-orthogonal complement of $P(\xi)$. This shows that the analytic torsion is infinite. Also we show that it is infinite even for any closed contact 3-manifold.

1. Introduction.

The asymptotic linking pairing lk is a symmetric bi-linear form on the space of homology-free vector fields on 3-manifolds. This is a generalization of the linking number of knots or links. It is realized as a pairing on exact 2-forms, see [**1**], [**2**], and also Section 2 for the definitions. The purpose of this paper is to study the linking pairing on the three sphere S^3 through the geometry of Hopf fibrations.

In [**7**], Mitsumatsu tried to apply this pairing to the theory of foliations and contact structures on a closed 3-manifold M. In the space of exact 2-forms $(B^2(M), lk)$, a foliation $\mathcal{F}$ defines the null subspace $N(\mathcal{F})$ and a positive contact structure ξ defines the positive definite subspace $P(\xi)$. Roughly speaking, he studied the maximality of $N(\mathcal{F})$ and $P(\xi)$ as null or positive definite subspaces. To define the "signature" of lk, he considered if the dimension of $N(\mathcal{F})^{\perp_{lk}}/N(\mathcal{F})$ is bounded and succeeded in determining it for the case of algebraic Anosov foliations. Here $N(\mathcal{F})^{\perp_{lk}}$ means the lk-orthogonal complement of $N(\mathcal{F})$. In particular, the space $(N(\mathcal{F})^{\perp_{lk}}/N(\mathcal{F}), lk)$ has important meanings, which is related to the theory of secondary characteristic classes of foliations. To apply the similar way in contact topology, he studied the supremum of the dimensions of positive definite subspaces in $(P(\xi)^{\perp_{lk}}, lk)$. It is called *the analytic torsion of* ξ and is denoted by $\mathrm{Tor}^{\mathrm{an}}(M, \xi)$. This is interesting from viewpoints of the correspondence between foliations and contact structures, and also the theory of secondary characteristics. Mitsumatsu showed the criterion of tightness of ξ by using $\mathrm{Tor}^{\mathrm{an}}(M, \xi)$ (Theorem 2.4). A sketch of the proof is given in Section 2.2. It is important to study the space $P(\xi)^{\perp_{lk}}$ and the quantity $\mathrm{Tor}^{\mathrm{an}}(M, \xi)$; however there are few known results.

We focus the case of the 3-sphere S^3 and the standard positive contact structure ξ_{st}. Our main result is the following:

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Theorem 1.1. *There exists a positive definite subspace in* $P(\xi_{\mathrm{st}})^{\perp_{lk}}$ *with respect to* lk *of arbitrary large dimension, that is, the analytic torsion of* (S^3, ξ_{st}) *is infinite.*

This problem can be replaced into the space of homology-free vector fields on S^3 by Arnold's theorem (Theorem 2.1). We calculate the linking number of a special class of homology-free vector fields on S^3. As a Lie group, S^3 contains many subgroups which are isomorphic to S^1. Each of them defines a different Hopf fibration. In Section 3, we show that for each fiber in a Hopf fibration, there are two special Hopf fibers in any other Hopf fibrations (Theorem 3.1). From such observations, we construct vector fields which give positive definite subspaces of arbitrary large dimensions (Proposition 3.4). Theorem 1.1 follows by combining Proposition 3.4 and Theorem 2.1. Further, such a positive definite subspace can be locally constructed for any closed contact 3-manifold. We will give proofs of Theorem 1.1 and Corollary 1.2 in Section 4.

Corollary 1.2. *For any closed contact* 3*-manifold, the analytic torsion is infinite.*

From the standpoint of Theorem 2.4, our result implies that the criterion is not applicable without major modification. On the other hand, our result indicates the difference of the subspaces associated with foliations and contact structures in $(B^2(M), lk)$; For algebraic Anosov foliations, the dimensions of $N(\mathcal{F})^{\perp_{lk}}/N(\mathcal{F})$ are bounded ([**7**]). That is, $N(\mathcal{F})$ has finite codimension to a maximal null subspace. However, once such a foliation deform to contact structures, the corresponding quantities $\mathrm{Tor}^{\mathrm{an}}(M,\xi)$ become infinite. That is, $P(\xi)$ has infinite codimension to a maximal positive definite subspace. This phenomenon that occurs infinite dimension seems to express a part of "deformation quantization".

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2. Linking pairing.

2.1. Two definitions of the linking paring.

We review the definition of the linking pairing and related topics. Let M be a closed oriented 3-manifold. Fix a volume form $d\,\mathrm{vol}$ on M. There is a one to one correspondence between the space of differential 2-forms $\Omega^2(M)$ and the space of C^∞-vector fields $\mathfrak{X}(M)$. Each 2-form is obtained by $\iota_X d\,\mathrm{vol}$ for a vector field X. Under this correspondence, the space of divergence-free vector fields $\mathfrak{X}_d(M)$ is isomorphic to the space of closed 2-forms $Z^2(M)$. Let $\mathfrak{X}_h(M)$ denote the space of vector fields corresponding to the space of exact 2-forms $B^2(M)$. The elements of $\mathfrak{X}_h(M)$ are called *homology-free* vector fields. Arnold [**1**] introduced the pairing on $B^2(M)$;

$$lk(d\alpha_1, d\alpha_2) = \int_M \alpha_1 \wedge d\alpha_2.$$

This is called the *asymptotic linking pairing* (or the *asymptotic Hopf invariant*), briefly we say the linking pairing. This pairing is an indefinite symmetric bi-linear form on $B^2(M)$.

There is an another interpretation of the linking pairing lk as the *asymptotic linking number* of homology-free vector fields X_1 and X_2. Let $\phi_1^{t_1}$ and $\phi_2^{t_2}$ denote the flows generated by X_1 and X_2. Choose a Riemannian metric and two points p_1 and p_2. Once $\phi_i^{T_i}(p_i)$ is close to the initial point p_i, where T_i is such a return time, we connect them with a short geodesic for each $i = 1, 2$. These closed curves are denoted by $k_1(T_1; p_1)$ and $k_2(T_2; p_2)$. For almost all (p_1, p_2) and (T_1, T_2), the closed curves $k_1(T_1; p_1)$ and $k_2(T_2; p_2)$ are disjoint and the linking number is defined. In particular, we can choose a short geodesic connecting $\phi_1^{T_1}(p_1)$ and p_1 so that the intersection number with Seifert surfaces of $k_2(T_2; p_2)$ is uniformly bounded. Arnold showed that the limit

$$lk(X_1, X_2; p_1, p_2) = \lim_{T_1, T_2 \to \infty} \frac{1}{T_1 T_2} \operatorname{link}(k_1(T_1; p_1), k_2(T_2; p_2))$$

exists for almost all (p_1, p_2) and it is integrable on $M \times M$ with respect to $d\,\mathrm{vol} \times d\,\mathrm{vol}$. The *asymptotic linking number* of X_1 and X_2 is defined by

$$lk(X_1, X_2) = \iint_{(p_1, p_2) \in M \times M} lk(X_1, X_2; p_1, p_2) \quad d\,\mathrm{vol}(p_1) d\,\mathrm{vol}(p_2).$$

For more details, see [**1**], [**2**] and also [**5**] for related topics. This can be interpreted in the context of Schwartzman's asymptotic cycles ([**8**], [**9**] and [**5**]). In this paper, we will consider only if all trajectories are closed. Thus the computation is not much complicated.

Arnold connected the two definitions of lk. We will use this theorem in Section 4.

THEOREM 2.1 (Arnold [**1**], [**2**]). *The linking pairing and the asymptotic linking number coincide on S^3;*

$$lk(d\alpha_1, d\alpha_2) = lk(X_1, X_2),$$

where $d\alpha_i = \iota_{X_i} d\,\mathrm{vol}$ $(i = 1, 2)$.

2.2. A relationship to contact topology.

To close this section, we state a relationship to contact topology. First, recall the definition of a contact structure.

Let α be a *contact form* on an oriented 3-manifold M, that is, α is a nonsingular 1-form and $\alpha \wedge d\alpha$ vanishes nowhere. A contact form is said to be *positive* if the volume form $\alpha \wedge d\alpha$ gives the orientation of M. A contact form α determines a non-integrable plane field ξ by Ker α, which is called the *contact plane field* or the *contact structure.*

There is a dichotomy in 3-dimensional contact topology; A contact structure contains an overtwisted disk or not. The overtwisted disk is an embedded disk $D \subset (M, \xi)$ such that $\xi_p = T_p D$ for each $p \in \partial D$, that is, ξ does not twist along ∂D. A contact structure which admits no overtwisted disks is said to be *tight.* Darboux's theorem tells that any contact structure is locally tight. See [**3**] and [**4**] for more details.

Let $P(\xi)$ denote the space of exact 2-form $d(\phi\alpha)$ for any smooth function ϕ, where α determines a positive contact structure ξ on M.

Proposition 2.2 ([**7**]). *The space $P(\xi)$ is a positive definite subspace in $(B^2(M), lk)$.*

Proof. This follows from the easy calculation:

$$lk(d(\phi\alpha), d(\phi\alpha)) = \int_M \phi\alpha \wedge d(\phi\alpha) = \int_M \phi^2 \alpha \wedge d\alpha > 0$$

for any smooth function ϕ which is not identically zero. □

In order to measure the maximality of $P(\xi)$ as a positive definite subspace in $(B^2(M), lk)$, Mitsumatsu [**7**] introduced the following definitions.

Definition 2.3. The analytic torsion $\mathrm{Tor}^{\mathrm{an}}(M, \xi)$ of a positive contact structure ξ is defined by the supremum of the dimensions of positive definite subspaces in the lk-orthogonal complement $P(\xi)^{\perp_{lk}}$ of $P(\xi)$ in $(B^2(M), lk)$.

He gave a criterion for the tightness of a contact structure.

Theorem 2.4 (Mitsumatsu [**7**]). *If* $\mathrm{Tor}^{\mathrm{an}}(M, \xi)$ *is bounded, then ξ is tight.*

Outline of the proof. Following [**7**], we mention to the outline of his proof. First we recall *the Giroux torsion* of ξ. This torsion is defined by the supremum length $n \in (1/2)\mathbb{Z}$ such that there is a contact embedding

$$\varphi_n : (T^2 \times [0, 1], \zeta_n) \hookrightarrow (M, \xi),$$

where $\zeta_n = \mathrm{Ker}\{\cos(2\pi nt)dx - \sin(2\pi nt)dy\}$ and (x, y, t) is the coordinate of $T^2 \times [0, 1]$. Mitsumatsu considered the total length of a disjoint union (possibly connected) of contact embeddings

$$\coprod_{i=1}^{m} \varphi_{n_i} : \coprod_{i=1}^{m} (T^2 \times [0, 1],\ \zeta_{n_i}) \hookrightarrow (M, \xi),$$

where each image is mutually disjoint in (M, ξ). The supremum of the total length $\sum_{i=1}^{m} n_i$ is called the *twisting invariant* $\mathrm{Tw}(M, \xi)$ of (M, ξ). By definition, the twisting invariant is greater than or equal to the Giroux torsion. It is known that the Giroux torsion of an overtwisted contact structure is infinite. Thus, if $\mathrm{Tw}(M, \xi)$ is bounded, then ξ is tight.

Suppose that there is a contact embedding φ_n of length n. We can extend φ_n to

$$\widetilde{\varphi}_n : (T^2 \times (-\epsilon, 1 + \epsilon), \zeta_n) \hookrightarrow (M, \xi)$$

for small $\epsilon > 0$. Next we separate $\widetilde{\varphi}_n$ into $2n$ disjoint contact embeddings of length 1/2.

Choose open sets $\{U_i\}$ of M which cover each of connected components of the image and which are mutually disjoint. Further we take a 1-form λ_i on $T^2 \times [0,1]$ as

$$\lambda_i = \sin(2\pi n_i t)dx + \cos(2\pi n_i t)dy.$$

This can be extended to a global 1-form $\tilde{\lambda_i}$ on M in such a way that the following properties are satisfied

(i) $\mathrm{supp}(\tilde{\lambda_i}) \subset U_i$,
(ii) $d\tilde{\lambda_i} \in P(\xi)^{\perp_{lk}}$, $\quad lk(d\tilde{\lambda_i}, d\tilde{\lambda_i}) > 0 \quad$ and $\quad lk(d\tilde{\lambda_i}, d\tilde{\lambda_j}) = 0$ for $i \neq j$.

To achieve this, we need to carefully deform λ_i. This reduces to a problem of ordinary differential equations.

Once this is done, the vector space P spanned by such exact 2-forms $\{d\tilde{\lambda_i}\}$ is a positive definite subspace in $P(\xi)^{\perp_{lk}}$, where $\dim P = 2n$. We do this procedure for all possible contact embeddings, then we have

$$\mathrm{Tor}^{\mathrm{an}}(M,\xi) \quad \geq \quad 2 \cdot \mathrm{Tw}(M,\xi) \quad \geq \quad 2 \cdot \text{Giroux torsion}.$$

Therefore if $\mathrm{Tor}^{\mathrm{an}}(M,\xi)$ is bounded, then ξ is tight.

REMARK 2.5. Similarly, for a foliation $\mathcal{F}$ defined by a 1-form ω, we set $N(\mathcal{F}) = \{d(\phi\omega) \mid \phi \in C^\infty(M)\}$. This is a null subspace in $(B^2(M), lk)$. Mitsumatsu [**7**] showed that, for an algebraic Anosov foliation, the dimension of $N(\mathcal{F})^{\perp_{lk}}/N(\mathcal{F})$ is determined by using the leafwise cohomology of $\mathcal{F}$. In particular, this is bounded. Compare Theorem 1.1. See also Sections 3, 4 and 5 in [**6**] as the brief summary of [**7**].

3. Linking of fibers of Hopf fibrations.

This section deals with the geometry of Hopf fibrations on S^3. For the basic geometry of S^3, we refer to e.g. [**10**]. Our first aim is to prove Theorem 3.1. To do this, we consider our problem in the oriented Grassman manifold $\widetilde{\mathrm{Gr}}(4,2)$ rather than S^3. In $\widetilde{\mathrm{Gr}}(4,2)$, we will understand the linking number of *any* two Hopf fibers. Second, we calculate the linking pairing of Hopf fibrations for concrete examples in Section 3.2.

3.1. Hopf fibrations on S^3.

We regard the unit sphere $S^3 \subset \mathbb{C}^2 \cong \mathbb{R}^4$ as the Lie group $SU(2)$. Throughout this paper, we fix the correspondence between the unit sphere S^3 and the group $SU(2)$ where $\boldsymbol{e}_1 = {}^t(1,0,0,0)$ in $S^3 \subset \mathbb{R}^4$ is identified with the unit e in $SU(2)$. The Lie algebra $\mathfrak{su}(2)$ is isomorphic to the real vector space spanned by $i,j,k \in \mathbb{H}$. Let $S^2(i,j,k)$ denote the set of pure unit quaternions. Each $u \in S^2(i,j,k)$ generates the one-parameter subgroup $\mathcal{O}_u \subset SU(2)$, which is isomorphic to S^1. The right (resp. left) action of $\mathcal{O}_u$ on S^3 is defined by the multiplication $x \mapsto x \cdot g$ (resp. $x \mapsto g \cdot x$) in $SU(2)$. The right (resp. left) action gives the principal S^1-bundle $h_u : S^3 \to \mathbb{P}_u$ (resp. ${}_uh : S^3 \to {}_u\mathbb{P}$), called the *right-handed* (resp. *left-handed*) Hopf fibration *for* u. In order to distinguish the underlying space $\mathbb{P}^1$, we use the notation $\mathbb{P}_u$ (resp. ${}_u\mathbb{P}$). The space $\mathbb{P}_u$ (resp. ${}_u\mathbb{P}$) is regarded as the set of complex lines for the positive (resp. negative) complex structure determined by u.

We fix the standard metric and the orientation on S^3 induced from $\mathbb{R}^4$. We assign the orientation of S^3 by the outward normal vector in $\mathbb{R}^4$. The orientations of Hopf fibers are given by the actions, so that the linking number of two fibers is determined.

Hopf fibrations have the following property: Each fiber of h_u (resp. ${}_uh$) links positively (resp. negatively) once to the others. In contrast, the following result describes the linking of fibers in different Hopf fibrations.

Theorem 3.1. *Fix a right-handed Hopf fibration $h_u : S^3 \to \mathbb{P}_u$ and a fiber γ of h_u. For each right-handed Hopf fibration $h_v : S^3 \to \mathbb{P}_v$ where v is neither u nor $-u$, the following statements hold*:

(i) *There exist precisely two fibers $\delta^+(\gamma)$ and $\delta^-(\gamma)$ of h_v, depending on the choice of γ, which satisfy the following condition*: *The fiber γ links positively* (*resp. negatively*) *once with $\delta^+(\gamma)$ (resp. $\delta^-(\gamma)$), where $\gamma, \delta^+(\gamma)$ and $\delta^-(\gamma)$ are mutually parallel with respect to the standard metric on S^3.*

(ii) *There is the circle $C(\gamma)$ separating $\mathbb{P}_v$ into two open disks $D^+(\gamma)$ and $D^-(\gamma)$ centered at $\delta^+(\gamma)$ and $\delta^-(\gamma)$ respectively. Each oriented fiber over $D^+(\gamma)$* (*resp. $D^-(\gamma)$*) *links positively* (*resp. negatively*) *once with γ.*

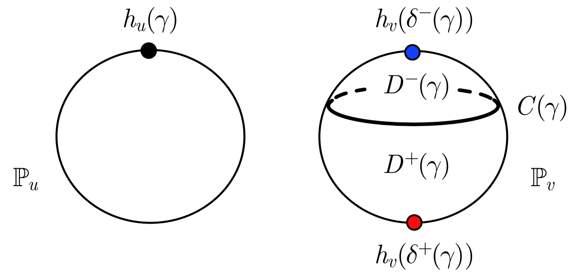

Figure 1.

The closed disk $\overline{D^-(\gamma)}$ (resp. $\overline{D^+(\gamma)}$) degenerates to the point $\delta^-(\gamma)$ (resp. $\delta^+(\gamma)$) when $v = u$ (resp. $-u$). It will be understood that the similar statement holds if we replace a right-handed Hopf fibration by a left-handed one. To prove Theorem 3.1, we consider 2-dimensional oriented subspaces in $\mathbb{R}^4$ instead of oriented Hopf fibers in S^3. In what follows, we regard ${}_u\mathbb{P}$ and $\mathbb{P}_v$ as submanifolds in the oriented Grassman manifold $\widetilde{\mathrm{Gr}}(4,2)$.

Proposition 3.2. *For each u and v in $S^2(i,j,k)$, two submanifolds ${}_u\mathbb{P}$ and $\mathbb{P}_v$ intersect at one point* L *in $\widetilde{\mathrm{Gr}}(4,2)$. Conversely, for each* L *in $\widetilde{\mathrm{Gr}}(4,2)$, there exist u and v such that ${}_u\mathbb{P} \cap \mathbb{P}_v = \{\mathrm{L}\}$.*

Proof. Let us find a common invariant subspace in $\mathbb{R}^4$ by the left action of $\mathcal{O}_u$ and the right action of $\mathcal{O}_v$. Since the right and left actions commute, these actions are simultaneous diagonalizable as elements of $SO(4)$. Hence we can find two invariant 2-dimensional subspaces in $\mathbb{R}^4$ under both actions. On one of the invariant subspaces, which is denoted by L, the induced orientations by both actions coincide. On the other one, which is denoted by L', those are opposite. Of course, $\mathrm{L}' = \mathrm{L}^\perp$ (= the orthogonal complement of L in $\mathbb{R}^4$). This invariant subspace L is the desired element.

Conversely, take L in $\widetilde{\mathrm{Gr}}(4,2)$ and the orthogonal complement $\mathrm{L}^\perp$ whose orientation is given by L and $\mathbb{R}^4$. We choose the orthonormal basis $\langle \boldsymbol{e}_1, \boldsymbol{e}_2 \rangle$ of L and $\langle \boldsymbol{e}_3, \boldsymbol{e}_4 \rangle$ of $\mathrm{L}^\perp$. The positive complex structure J^+ associated with $\mathrm{L} \oplus \mathrm{L}^\perp$ determines the element v in $S^2(i,j,k)$ and the right action of $\mathcal{O}_v$. Here J^+ maps as $\boldsymbol{e}_1 \mapsto \boldsymbol{e}_2$ and $\boldsymbol{e}_3 \mapsto \boldsymbol{e}_4$. Similarly, the negative complex structure J^-, which maps as $\boldsymbol{e}_1 \mapsto \boldsymbol{e}_2$ and $\boldsymbol{e}_3 \mapsto -\boldsymbol{e}_4$, determines the element u in $S^2(i,j,k)$ and the left action of $\mathcal{O}_u$. Thus the latter statement follows. □

Let ${}_u\mathrm{L}_v$ denote the intersection point of ${}_u\mathbb{P}$ and $\mathbb{P}_v$ in $\widetilde{\mathrm{Gr}}(4,2)$. Also $-\mathrm{L}$ denotes the subspace L with the opposite orientation. Notice that $-({}_u\mathrm{L}_v) = {}_{-u}\mathrm{L}_{-v}$. From Proposition 3.2, it follows that $\widetilde{\mathrm{Gr}}(4,2)$ is diffeomorphic to ${}_u\mathbb{P} \times \mathbb{P}_v$.

Next, we consider the actions on $\widetilde{\mathrm{Gr}}(4,2)$. The action of $SU(2)$ on S^3 itself is linearly extended on $\mathbb{R}^4$. This induces the right and left actions on $\widetilde{\mathrm{Gr}}(4,2)$. The homomorphism $\tau : SU(2) \times SU(2) \twoheadrightarrow SO(4)$ is defined by $\tau(g,h) = L_{g^{-1}} R_h$, where L and R denote the left and right translations on $\widetilde{\mathrm{Gr}}(4,2)$. The kernel is isomorphic to the cyclic group of order 2, so that $SO(4) \cong SU(2) \times_{\mathbb{Z}/2\mathbb{Z}} SU(2)$. See e.g. [**10**, p. 107]. The action of $SU(2) \times SU(2)$ on $\widetilde{\mathrm{Gr}}(4,2) \cong {}_u\mathbb{P} \times \mathbb{P}_u$ is described as follows:

Proposition 3.3. *The action of $SU(2) \times SU(2)$ preserves the horizontal and vertical foliations of $\widetilde{\mathrm{Gr}}(4,2) \cong {}_u\mathbb{P} \times \mathbb{P}_u$. The right action of $SU(2)$ preserves each horizontal leaf although it moves vertical leaves through the foliation. More precisely, a vertical leaf $\mathbb{P}_u$ moves to another one $\mathbb{P}_{\mathrm{Ad}_g(u)}$ for some g in $SU(2)$, where* Ad : $SU(2) \twoheadrightarrow SO(3)$ *is the adjoint representation. This statement also holds if we replace right by left and horizontal by vertical respectively.*

Proof. Any element L in $\mathbb{P}_u$ is obtained as follows. Take $x \in S^3 \cap \mathrm{L} \subset \mathbb{R}^4$. Let $\boldsymbol{x}$ denote the position vector for x in $\mathbb{R}^4$. The differential of the right action of $\mathcal{O}_u$ yields the left invariant vector field $\{\boldsymbol{v}_x = d/dt|_{t=0}\ x \cdot \exp(tu)\}_{x \in S^3}$. Then L is spanned by $\boldsymbol{x}$ and $\boldsymbol{v}_x$, denoted by $\mathbb{R}\langle \boldsymbol{x}, \boldsymbol{v}_x \rangle$. Thus the left action preserves the right quotient space $\mathbb{P}_u$. (Similarly, the right action preserves the left quotient spaces.) For each g in $SU(2)$, we have

$$\tau(g,g)\ \mathrm{L} = \mathbb{R}\langle \boldsymbol{g^{-1} x g}, \mathrm{Ad}_g(\boldsymbol{v}_x) \rangle,$$

so that $\tau(g,g)$ L belongs to $\mathbb{P}_{\mathrm{Ad}_g(u)}$. We have

$$\tau(e,g)\mathbb{P}_u = \tau(g,g)\mathbb{P}_u = \mathbb{P}_{\mathrm{Ad}_g(u)}.$$

Hence the right action preserves both foliations as desired. □

Here we write the action of $SU(2) \times SU(2)$ explicitly. Since $\{{}_u\mathrm{L}_v\} = {}_u\mathbb{P} \cap \mathbb{P}_v$, we have

$$\tau(g,h)\ {}_u\mathrm{L}_v = {}_{\mathrm{Ad}_g(u)}\mathrm{L}_{\mathrm{Ad}_h(v)}.$$

The fixed point sets of the left and right actions are obtained as follows:

$$\mathrm{Fix}(\tau(g,e)) = {}_u\mathbb{P} \sqcup {}_{-u}\mathbb{P} \quad \text{and} \quad \mathrm{Fix}(\tau(e,g)) = \mathbb{P}_u \sqcup \mathbb{P}_{-u} \ \text{ if } \ g \in \mathcal{O}_u.$$

Take L_1 and L_2 in $\widetilde{\mathrm{Gr}}(4,2)$. Let $\mathrm{Int}(\mathrm{L}_1, \mathrm{L}_2)$ denote the algebraic intersection number of L_1 and L_2 in $\mathbb{R}^4$. This is equal to the linking number of their restrictions on S^3 if it is defined.

PROOF OF THEOREM 3.1. We consider $\mathrm{L} \in \widetilde{\mathrm{Gr}}(4,2)$ corresponding to the oriented Hopf fiber γ in Theorem 3.1. After transformations of $SU(2) \times SU(2)$, we may assume that $\mathrm{L} = {}_u\mathrm{L}_u$. The intersection number is invariant under such transformations. Let E denote the set of all L in $\widetilde{\mathrm{Gr}}(4,2)$ for which $\dim(\mathrm{L} \cap {}_u\mathrm{L}_u)$ is at least 1. We determine the set E. First, we show that the diagonal set $\Delta = \{{}_v\mathrm{L}_v \mid v \in S^2(i,j,k)\}$ is contained in E. Recall the correspondence between S^3 and $SU(2)$. Choose i in $S^2(i,j,k)$ and ${}_i\mathrm{L}_i$ $(= \mathbb{R}\langle \boldsymbol{e}_1, \boldsymbol{e}_2\rangle)$. For each g in $SU(2)$, we have

$${}_i\mathrm{L}_i \cap \tau(g,g)_i\mathrm{L}_i = \mathbb{R}\langle \boldsymbol{e}_1\rangle \subset \mathbb{R}^4.$$

By the surjectivity of Ad, it follows that any element of Δ includes $\mathbb{R}\langle \boldsymbol{e}_1\rangle$. Thus Δ is included in E. Take g in $\mathcal{O}_u \subset SU(2)$. We have

$$\tau(g,e)({}_u\mathrm{L}_u \cap {}_v\mathrm{L}_v) = {}_{\mathrm{Ad}_g(u)}\mathrm{L}_u \cap {}_{\mathrm{Ad}_g(v)}\mathrm{L}_v = {}_u\mathrm{L}_u \cap {}_{\mathrm{Ad}_g(v)}\mathrm{L}_v.$$

Hence ${}_{\mathrm{Ad}_g(v)}\mathrm{L}_v$ is contained in E for such g. Conversely, any element of E is obtained in such a way. The left action of $\mathcal{O}_u$ on E is free except ${}_u\mathrm{L}_u$ and ${}_{-u}\mathrm{L}_{-u}$ and the quotient space $\mathcal{O}_u\backslash E$ is equal to the diagonal set Δ. Observe that E is obtained by collapsing the product bundle $\Delta \times S^1$ along $\{{}_u\mathrm{L}_u\} \times S^1 \cup \{{}_{-u}\mathrm{L}_{-u}\} \times S^1$. The general fiber $\{{}_{\mathrm{Ad}_g(v)}\mathrm{L}_v \mid g \in \mathcal{O}_u\}$ of E corresponds to the circle $C(\gamma) \subset \mathbb{P}_v$ in Theorem 3.1, where v is neither u nor $-u$.

The left action of $\mathcal{O}_u$ on $\mathbb{P}_v$ preserves $C(\gamma)$ and fixes ${}_u\mathrm{L}_v$ and ${}_{-u}\mathrm{L}_v$, which are desired elements $\delta^-(\gamma)$ and $\delta^+(\gamma)$ respectively. In fact, the intersection numbers are obtained by

$$\mathrm{Int}({}_u\mathrm{L}_u, {}_u\mathrm{L}_v) = -1, \quad \mathrm{Int}({}_u\mathrm{L}_u, {}_{-u}\mathrm{L}_v) = -\mathrm{Int}({}_u\mathrm{L}_u, {}_u\mathrm{L}_{-v}) = +1$$

since ${}_u\mathrm{L}_u, {}_u\mathrm{L}_v$ and ${}_u\mathrm{L}_{-v}$ belong to ${}_u\mathbb{P}$. These restrictions on S^3 are mutually parallel. Thus the statement (i) follows. The set E decomposes $\widetilde{\mathrm{Gr}}(4,2)$ into two components. The circle $C(\gamma)$ decomposes $\mathbb{P}_v$ into two disks $D^-(\gamma)$ and $D^+(\gamma)$ which contain $\delta^-(\gamma)$ and $\delta^+(\gamma)$ respectively. By the continuity of $\mathrm{Int}({}_u\mathrm{L}_u, \cdot) : \widetilde{\mathrm{Gr}}(4,2)\backslash E \to \{+1, -1\}$, the statement (ii) follows.

As seeing Figure 2, we can understand the linking numbers of any Hopf fibers. The thick line means the set E. The "+" (resp. "−") chamber surrounded by the thick line is the set of L in $\widetilde{\mathrm{Gr}}(4,2)$ for which $\mathrm{Int}({}_u\mathrm{L}_u, \mathrm{L}) = +1$ (resp. $= -1$). The intersection $E \cap \mathbb{P}_v$ is equal to $C(\gamma)$. Compare with Figure 1.

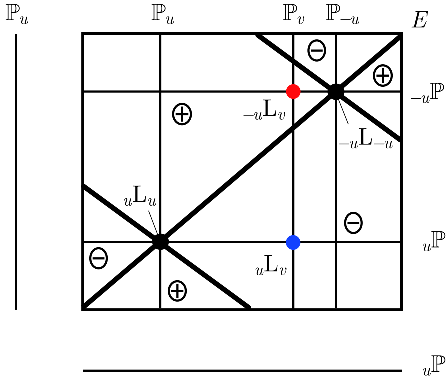

Figure 2. The signs of intersection numbers with $_u\mathrm{L}_u$ in $\widetilde{\mathrm{Gr}}(4,2) \simeq {_u\mathbb{P}} \times \mathbb{P}_u$.

3.2. Linking quadratic forms and Hopf fibrations.

Let us compute the linking numbers of flows related to Hopf fibrations. Further we try to compute the rank of lk as a quadratic form.

Example 1. We consider a right-handed Hopf fibration h_u. Fix the standard metric and the volume form $d\,\mathrm{vol}$ on S^3. (The volume of the unit sphere is $2\pi^2$.) Let X_u denote the left invariant vector field on S^3 such that $(X_u)_e = u$, which is called the Hopf vector field for u. The vector field gives the Hopf fibration h_u. The self-linking number of X_u is given by

$$\begin{aligned} lk(X_u, X_u) &= \iint_{(x,y)\in S^3\times S^3} lk(X_u, X_u; x, y)\, d\,\mathrm{vol}(x)\, d\,\mathrm{vol}(y) \\ &= \frac{+1}{2\pi\cdot 2\pi}\cdot \mathrm{vol}(S^3)\cdot \mathrm{vol}(S^3) = \pi^2. \end{aligned}$$

This is also obtained by the calculation on $B^2(S^3)$. Let X_i, X_j, X_k denote Hopf vector fields for i, j, k, which give the basis of $\mathfrak{su}(2)$. The dual basis of $\mathfrak{su}(2)^*$ is denoted by $\{X_i^*, X_j^*, X_k^*\}$ and it is easy to see that this satisfies the following relations:

$$dX_i^* = X_j^* \wedge X_k^*, \quad dX_j^* = X_k^* \wedge X_i^*, \quad dX_k^* = X_i^* \wedge X_j^*,$$

and $X_i^* \wedge X_j^* \wedge X_k^*$ is a half of the standard volume form $d\,\mathrm{vol}$ on S^3. Since lk (over $\mathfrak{X}_h(M)$) is invariant under volume preserving diffeomorphisms, we have

$$lk(X_u, X_u) = lk(X_i, X_i) = lk(dX_i^*, dX_i^*) = \int_{S^3} X_i^* \wedge dX_i^* = \int_{S^3} X_i^* \wedge X_j^* \wedge X_k^* = \pi^2.$$

The second equality follows by the Arnold's theorem (Theorem 2.1). These calculations are known in [**1**] and [**2**].

Next we consider a smooth function f such that $X_u(f) = 0$, where $X_u(\cdot)$ means the differential by X_u. Then fX_u is a homology-free vector field. Set $V_{X_u} = \{fX_u \mid$

$X_u(f)=0\}$ in $\mathfrak{X}_h(S^3)$. The self-linking number of fX_u is given by

$$
\begin{aligned}
lk(fX_u, fX_u) &= \iint_{S^3\times S^3} f(x)f(y)lk(X_u, X_u; x, y)d\,\mathrm{vol}(x)d\,\mathrm{vol}(y) \\
&= \left\{\frac{1}{2\pi}\int_{S^3} f(x)d\,\mathrm{vol}(x)\right\}^2.
\end{aligned}
$$

The computation shows that the quadratic form lk has rank one over V_{X_u} and it is positive definite. Notice that it is easy to compute the rank of $lk|_{V_{X_u}}$ on $\mathfrak{X}_h(S^3)$, although it seems to be difficult to do that on $B^2(S^3)$.

EXAMPLE 2. Choose u and v in $S^2(i,j,k)$. We consider the linking number of two Hopf fibrations h_u and h_v. Let X_u and X_v denote the Hopf vector fields for u and v. Also let θ denote the angle between u and v. We have

$$
\begin{aligned}
lk(X_u, X_v) &= \int_{x\in S^3}\left\{\int_{y\in S^3} lk(X_u, X_v; x, y)d\,\mathrm{vol}(y)\right\}d\,\mathrm{vol}(x) \\
&= \int_{x\in S^3}\left\{\int_{y\in h_v^{-1}(D^+(h_u(x)))} lk(X_u, X_v; x, y)d\,\mathrm{vol}(y)\right. \\
&\qquad \left. + \int_{y\in h_v^{-1}(D^-(h_u(x)))} lk(X_u, X_v; x, y)d\,\mathrm{vol}(y)\right\}d\,\mathrm{vol}(x) \\
&= \mathrm{vol}(S^3)\cdot\left\{\frac{+1}{2\pi\cdot 2\pi}\cdot\frac{(\pi-\theta)^2}{(\pi-\theta)^2+\theta^2}\cdot\mathrm{vol}(S^3)\right. \\
&\qquad \left. + \frac{-1}{2\pi\cdot 2\pi}\cdot\frac{\theta^2}{(\pi-\theta)^2+\theta^2}\cdot\mathrm{vol}(S^3)\right\} \\
&= \frac{(\pi-\theta)^2-\theta^2}{(\pi-\theta)^2+\theta^2}\cdot\pi^2.
\end{aligned}
$$

Here, the area of $D^+(h_u(x))\subset\mathbb{P}_v$ is equal to $(\pi-\theta)^2/((\pi-\theta)^2+\theta^2)$ and the area of $D^-(h_u(x))\subset\mathbb{P}_v$ is equal to $\theta^2/((\pi-\theta)^2+\theta^2)$. If u and v are perpendicular, then we have $lk(X_u, X_v)=0$.

EXAMPLE 3. Assume that u and v are perpendicular. We consider laminar flows contained in h_u and h_v. Here a Hopf fiber γ in S^3 is identified with the base point $h_u(\gamma)$ on $\mathbb{P}_u$. By abuse of notation, we write γ instead of $h_u(\gamma)$. For each γ in $\mathbb{P}_u$, the disks $D^+(\gamma)$ and $D^-(\gamma)$ are hemispheres in $\mathbb{P}_v$ (see Theorem 3.1). We choose $2(2m+1)$-points $\gamma_1,\gamma_2,\ldots,\gamma_{2(2m+1)}$ on a great circle in $\mathbb{P}_u$ so that these points divide the circle $2(2m+1)$-th equally as shown in Figure 3. Then the points $\delta_1,\delta_2,\ldots,\delta_{2(2m+1)}$ in $\mathbb{P}_v$ are determined by $\delta_i=\delta^+(\gamma_i)$.

By the configuration of γ_i's and δ_j's, the corresponding orbits link with each other in S^3. In fact, it suffices to choose points so that $\delta_1,\delta_2,\ldots,\delta_{2(2m+1)}$ do not lie on the great circles $\mathcal{C}(\gamma_1),\mathcal{C}(\gamma_2),\ldots,\mathcal{C}(\gamma_{2(2m+1)})$.

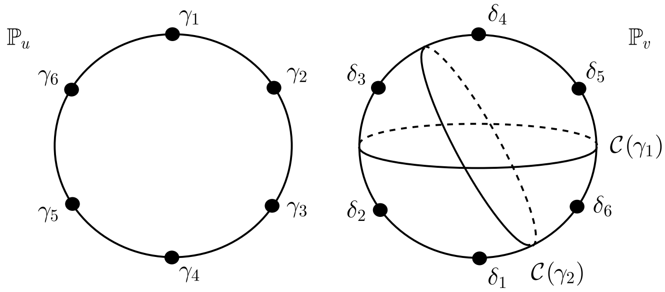

Figure 3. The case of $m = 1$: $\{\gamma_1, \gamma_2, \ldots, \gamma_6, \delta_1, \delta_2, \ldots, \delta_6\}$.

Let us regard $\gamma_1, \ldots, \gamma_{2(2m+1)}, \delta_1, \ldots, \delta_{2(2m+1)}$ as atomic measures on $\mathbb{P}_u$ and $\mathbb{P}_v$. We put

$$\omega_i := \gamma_i + (-\gamma_{2m+1+i}) \quad \text{and} \quad \tau_i := \delta_i + (-\delta_{2m+1+i}) \quad \text{for} \quad 1 \leq i \leq 2m+1.$$

Let Q_m denote the $\mathbb{R}$-vector space spanned by

$$\{\omega_1, \ldots, \omega_{2m+1}, \tau_1, \ldots, \tau_{2m+1}\}.$$

Now we compute a matrix associated with the quadratic form lk, which is called the *linking matrix*. First, observe that

$$lk(\omega_i, \omega_j) = 0 \quad \text{and} \quad lk(\tau_i, \tau_j) = 0$$

for $1 \leq i, j \leq 2m+1$. The self-linking number of γ_i is defined by X_u and it is equal to 1. Second, notice that $lk(\gamma_i, \delta_{2m+1+j}) = -lk(\gamma_i, \delta_j)$ by the configuration. Then we have

$$lk(\omega_i, \tau_j) = 4 \cdot lk(\gamma_i, \delta_j).$$

Hence the linking matrix Lk_m over the basis $\{\omega_1, \ldots, \omega_{2m+1}, \tau_1, \ldots, \tau_{2m+1}\}$ is obtained by

$$\mathrm{Lk}_m = \left(\begin{array}{c|c} \mathbf{0} & A_m \\ \hline A_m & \mathbf{0} \end{array} \right),$$

where the diagonal part consists of only 0. The anti-diagonal part A_m is shown in Figure 4. The part denoted by "1" (resp. "-1") consists of only 1 (resp. -1). After the conjugation by elementary transformations, it follows that the maximal rank of positive (and also negative) definite subspaces in Q_m is equal to $2m+1$. The elementary transformations are roughly showed in Figure 5. (The Figures 4 and 5 are listed in the last page.)

Such discrete orbits $\omega_1, \ldots, \omega_{2m+1}$, $\tau_1, \ldots, \tau_{2m+1}$ are approximated by a smooth vector field supported on a small tubular neighborhood invariant under the actions of $\mathcal{O}_u$ and $\mathcal{O}_v$. This is interpreted as diffusing atomic measures on $\mathbb{P}_u \sqcup \mathbb{P}_v$. Thus it follows that there is a positive definite subspace of rank $2m+1$ in $V_{X_u} + V_{X_v} \subset \mathfrak{X}_h(S^3)$. As increasing m arbitrary, we obtain the following result.

PROPOSITION 3.4. *There exist positive and negative definite subspaces in* $V_{X_u}+V_{X_v}$ (*with respect to* lk) *of arbitrary large dimensions.*

4. Proofs of Theorem 1.1 and Corollary 1.2.

Recall that the standard positive contact structure on S^3 can be expressed by elements of $\mathfrak{su}(2)^*$. Let X_i^*, X_j^*, X_k^* denote as in Section 3.2, Example 1. The plane field defined by the kernel of X_k^* gives a positive contact structure on S^3, which is denoted by ξ_{st}. We consider the lk-orthogonal complement $P(\xi_{\mathrm{st}})^{\perp_{lk}}$ of $P(\xi_{\mathrm{st}}) = d(\phi X_k^*) \mid \phi \in C^\infty(S^3)\}$. Then we have the next lemma.

LEMMA 4.1. *The space* $P(\xi_{\mathrm{st}})^{\perp_{lk}}$ *has the following expression*:

$$P(\xi_{\mathrm{st}})^{\perp_{lk}} = f dX_i^* + g dX_j^* \mid X_i(f) + X_j(g) = 0,\ \ f, g \in C^\infty(S^3)\}.$$

PROOF. Take any element $d\lambda \in P(\xi_{\mathrm{st}})^{\perp_{lk}}$. This is written by $d\lambda = f dX_i^* + g dX_j^* + h dX_k^*$ for f, g and h in $C^\infty(S^3)$. By $lk(dX_k^*, d\lambda) = 0$, we have $h \equiv 0$. Further, since $f dX_i^* + g dX_j^*$ belongs to $B^2(S^3) = Z^2(S^3)$, we obtain the equality $X_i(f) + X_j(g) = 0$. □

We set $P_{X_i} = \{f dX_i^* \mid X_i(f) = 0\} \subset P(\xi_{\mathrm{st}})^{\perp_{lk}}$. By the above expression, the linear subspace $P_{X_i} + P_{X_j}$ is contained in $P(\xi_{\mathrm{st}})^{\perp_{lk}}$. Notice that P_{X_i} is isomorphic to V_{X_i} under the correspondence $B^2(S^3) \cong \mathfrak{X}_h(S^3)$. By Proposition 3.4 and Theorem 2.1, it follows that there exists a positive definite subspace in $P_{X_i} + P_{X_j} \subset P(\xi_{\mathrm{st}})^{\perp_{lk}}$ of arbitrary large dimension. Thus Theorem 1.1 follows.

At the end, we show Corollary 1.2. Our construction was done in a (sufficiently large) open ball $\mathbb{B} \subset (S^3 \backslash \{p_0\}, \xi_{\mathrm{st}}) \cong (\mathbb{R}^3, \xi_{\mathbb{R}^3})$. Here p_0 is a point in S^3 and $\xi_{\mathbb{R}^3}$ is the standard contact structure on $\mathbb{R}^3$. More precisely, the open ball $\mathbb{B}$ includes the support of any exact 2-form contained in such a positive definite subspace. By Darboux's theorem, there is a contact embedding $\varphi : (\mathbb{B}, \xi_{\mathrm{st}}) \to (M, \xi)$. Notice that lk (over B^2) is invariant under orientation preserving diffeomorphisms. The induced map

$$\varphi_* : \big(B^2_{\mathrm{cpt}}(\mathbb{B}), lk\big) \to (B^2(M), lk)$$

preserves the pairing lk, where B^2_{cpt} means the space of exact 2-forms with compact supports. In particular, the map

$$\varphi_* : (P(\xi_{\mathrm{st}})^{\perp_{lk}} \cap B^2_{\mathrm{cpt}}(\mathbb{B}), lk) \to (P(\xi)^{\perp_{lk}}, lk)$$

is defined. By Theorem 1.1, we obtain a positive definite subspace in $P(\xi)^{\perp_{lk}}$ of arbitrary large dimension. Thus Corollary 1.2 follows.

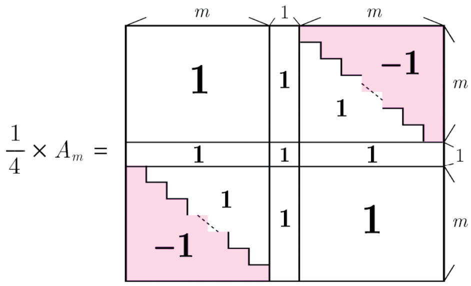

Figure 4. The anti-diagonal part of Lk_m.

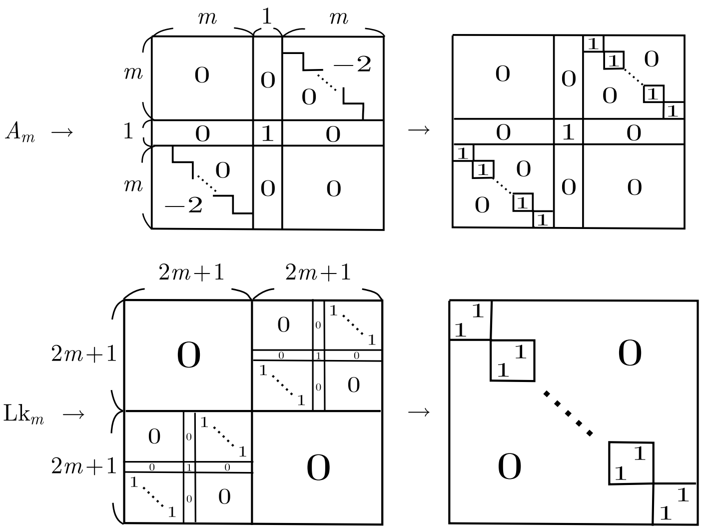

Figure 5. The conjugation of elementary transformations of Lk_m.

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