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Robustness of noninvertible dichotomies

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Abstract. We establish the robustness of exponential dichotomies for evolution families of linear operators in a Banach space, in the sense that the existence of an exponential dichotomy persists under sufficiently small linear perturbations. We note that the evolution families may come from nonautonomous differential equations involving unbounded operators. We also consider the general case of a noninvertible dynamics, thus including several classes of functional equations and partial differential equations. Moreover, we consider the general cases of nonuniform exponential dichotomies and of dichotomies that may exhibit stable and unstable behaviors with respect to arbitrary asymptotic rates $e^{c\rho(t)}$ for some function $\rho(t)$.

1. Introduction.

Our main aim is to establish the robustness of exponential dichotomies for evolution families $T(t,s)$ of linear operators in a Banach space. This means that all sufficiently small linear perturbations of an exponential dichotomy are also exponential dichotomies. We consider simultaneously the general cases of:

1. *nonautonomous dynamics* (of course including the case of autonomous dynamics), which in particular may be defined by an nonautonomous differential equation involving unbounded operators;
2. *noninvertible dynamics*, thus including several classes of functional equations such as delay equations and partial differential equations such as parabolic equations;
3. *nonuniformly hyperbolic dynamics*, in which the notion of (uniform) exponential dichotomy is replaced by the more general notion of nonuniform exponential dichotomy;
4. *arbitrary growth rates*, with dichotomies that may exhibit stable and unstable behaviors with respect to arbitrary asymptotic rates $e^{c\rho(t)}$ for some function $\rho(t)$.

In particular, we give a unified proof of the robustness property that includes all these general situations simultaneously.

The notion of exponential dichotomy, essentially introduced by Perron in [**17**], plays a central role in a substantial part of the theory of differential equations and dynamical systems. In particular, the existence of an exponential dichotomy for a linear equation $x' = A(t)x$, or more generally for an evolution family, causes the existence of stable and unstable invariant manifolds for the nonlinear differential equation $x' = A(t)x + f(t,x)$, or respectively for an appropriate perturbation of the evolution family, up to

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mild additional assumptions on the nonlinear part $f(t,x)$ of the vector field. Moreover, the local instability of trajectories caused by the existence of an exponential dichotomy influences the global behavior of the system. In particular, this instability is one of the main mechanisms responsible for the occurrence of stochastic behavior, especially in the presence of a nontrivial recurrence caused by the existence of a finite invariant measure. Certainly, the theory of exponential dichotomies and its applications are widely developed. We refer to the books [**10**], [**12**], [**13**], [**15**], [**21**] for details and references.

In view of the central role played by the notion of exponential dichotomy in a substantial part of the theory of differential equations and dynamical systems, not surprisingly the study of robustness has a long history. In particular, in the case of (uniform) exponential dichotomies the robustness was discussed by Massera and Schäffer [**14**] (building on earlier work of Perron [**17**]; see also [**15**]), Coppel [**9**], and in the case of Banach spaces by Dalec′kiĭ and Kreĭn [**11**], with different approaches and successive generalizations. For more recent works we refer to [**8**], [**16**], [**18**], [**19**] and the references therein. In particular, Chow and Leiva [**8**] and Pliss and Sell [**18**] considered the context of linear skew-product semiflows and gave examples of applications in the infinite-dimensional setting, including to parabolic partial differential equations and functional differential equations. In the general case of a noninvertible dynamics we refer in particular to Räbiger, Schnaubelt, Rhandi and Voigt [**20**] for a class of perturbations of evolutions families with bounded growth rate. We emphasize that all these works consider only the case of uniform exponential dichotomies.

The study of robustness in the general setting of a nonuniform exponential behavior was initiated in our work [**3**], although only for an invertible dynamics. In [**5**] we considered the more general case of arbitrary growth rates using an elaboration of the approach in [**3**]. More recently, in [**6**] we obtained a much shorter proof of the robustness property using Lyapunov functions, at the expense of considering only an invertible dynamics and without obtaining explicit formulas for the projections in the stable and unstable spaces. Building on classical work of Perron on the admissibility property, in [**7**] we suggested an alternative approach to the study of robustness of a nonuniformly hyperbolic dynamics using a characterization of hyperbolicity in terms of the existence of bounded solutions of a linear dynamics under bounded perturbations.

As we already mentioned, we consider simultaneously the general cases of nonautonomous dynamics, noninvertible dynamics, nonuniformly hyperbolic dynamics, and arbitrary growth rates, with emphasis on the noninvertibility and the nonuniform hyperbolicity of the dynamics when compared to former work. Incidentally, it should be noted that a nonuniformly hyperbolic dynamics (which is not uniformly hyperbolic) cannot be autonomous (since then it would automatically be uniform), and thus, considering a nonuniform exponential behavior requires considering dynamics that are not necessarily autonomous (we note that by definition a uniformly hyperbolic dynamics is a particular case of nonuniformly hyperbolic dynamics).

Concerning the nonuniform hyperbolicity of the dynamics, we note that the existence of a uniform exponential dichotomy is a strong requirement and it is of interest to look for more general types of hyperbolic behavior. In comparison with the notion of uniform exponential dichotomy, the notion of nonuniform exponential dichotomy is a much weaker requirement. In particular, in the case of $\mathbb{R}^n$ essentially any linear equation $x' = A(t)x$

with nonzero Lyapunov exponents has a nonuniform exponential dichotomy (we refer to [**4**] for details). On the other hand, as a consequence of Oseledets' multiplicative ergodic theorem, from the point of view of ergodic theory the nonuniformity in the dichotomies of "most" of these equations is arbitrarily small. We refer to [**1**] for a detailed exposition of the nonuniform hyperbolicity theory.

We also consider exponential dichotomies that may exhibit stable and unstable behaviors with asymptotic rates of the form $e^{c\rho(t)}$ for an arbitrary function $\rho(t)$. The main motivation are those linear equations for which all Lyapunov exponents are infinite (either $+\infty$ or $-\infty$), and thus to which one is not able, at least without further modifications, to apply the existing stability theory. This gives rise to the notion of ρ-nonuniform exponential dichotomy, which turns out to be rather common. In particular, we showed in [**2**] that for ρ in a large class of functions, any linear ordinary differential equation in a finite-dimensional space, with two blocks having asymptotic rates $e^{c\rho(t)}$ respectively with c negative and positive, has a ρ-nonuniform exponential dichotomy.

2. Robustness of nonuniform exponential dichotomies.

We establish in this section the robustness of nonuniform exponential dichotomies. We first recall the notion of dichotomy for a dynamics that need not be invertible.

We denote by $\mathcal{B}(X)$ the space of bounded linear operators in a Banach space X. Let $T(t,s)$ be an evolution family of linear operators in $\mathcal{B}(X)$ for $t,s \in \mathbb{R}$ with $t \geq s$. This means hat:

1. $T(t,t) = \mathrm{Id}$, and
$$T(t,\tau)T(\tau,s) = T(t,s), \quad t \geq \tau \geq s;$$
2. $(t,s,x) \mapsto T(t,s)x$ is continuous on $\{(t,s,x) \in \mathbb{R}^2 \times X : t \geq s\}$.

Consider an increasing differentiable function $\rho\colon \mathbb{R} \to \mathbb{R}$ with
$$\rho(0) = 0 \quad \text{and} \quad \lim_{t \to \pm\infty} \rho(t) = \pm\infty.$$

We say that the evolution family $T(t,s)$ admits a *ρ-nonuniform exponential dichotomy* if:

1. there exist projections $P(t)\colon X \to X$ for each $t \in \mathbb{R}$ satisfying
$$T(t,s)P(s) = P(t)T(t,s), \quad t \geq s,$$
such that the map
$$\overline{T}(t,s) := T(t,s)|\ker P(s)\colon\ \ker P(s) \to \ker P(t)$$
is invertible for each $t \geq s$;
2. there exist constants $\lambda, D > 0$ and $a \geq 0$ such that

$$\|T(t,s)P(s)\| \le De^{-\lambda(\rho(t)-\rho(s))+a|\rho(s)|}, \quad t \ge s, \tag{1}$$

and

$$\|T(t,s)Q(s)\| \le De^{-\lambda(\rho(s)-\rho(t))+a|\rho(s)|}, \quad s \ge t, \tag{2}$$

where $Q(t) = \mathrm{Id} - P(t)$ for each $t \in \mathbb{R}$, and where

$$T(t,s) = \overline{T}(s,t)^{-1} | \ker P(s), \quad t \le s. \tag{3}$$

We then define stable and unstable subspaces for each $t \in \mathbb{R}$ respectively by

$$E(t) = P(t)X \quad \text{and} \quad F(t) = Q(t)X.$$

We also consider the perturbed equation

$$u(t) = T(t,s)u(s) + \int_s^t T(t,\tau)B(\tau)u(\tau)\,d\tau, \quad t \ge s, \tag{4}$$

where $B\colon \mathbb{R} \to \mathcal{B}(X)$ is strongly continuous (this means that $t \mapsto B(t)x$ is continuous for each $x \in X$). We always assume that equation (4) defines an evolution family $\hat{T}(t,s)$ of bounded linear operators. For example, if $T(t,s)$ has bounded growth, that is,

$$\|T(t,s)\| \le Ke^{c(t-s)}, \quad t \ge s,$$

for some constants $K, c > 0$, and the function $t \mapsto B(t)$ is bounded, then equation (4) defines an evolution family $\hat{T}(t,s)$ (see [**22**] for details and references).

The following is our robustness result for nonuniform exponential dichotomies.

Theorem 1. *Let $T(t,s)$ be an evolution family admitting a ρ-nonuniform exponential dichotomy with $\lambda > 2a > 0$, and let $B\colon \mathbb{R} \to \mathcal{B}(X)$ be a continuous function satisfying*

$$\|B(t)\| \le \delta e^{-3a|\rho(t)|}\rho'(t), \quad t \in \mathbb{R},$$

such that equation (4) *also defines an evolution family $\hat{T}(t,s)$. If δ is sufficiently small, then $\hat{T}(t,s)$ admits a ρ-nonuniform exponential dichotomy, with the constants λ and a replaced respectively by λ and $2a$.*

Proof. We divide the proof of the theorem into several steps.

Step 1: *Construction of bounded solutions I.* The first step is the construction of bounded solutions of equation (4) into the future and into the past, already with an appropriate exponential rate.

We first construct solutions into the future. Set

$$I = \{(t,s) \in \mathbb{R} \times \mathbb{R} : t \ge s\}. \tag{5}$$

We consider the Banach space

$$\mathcal{C} = \big\{U\colon I \to \mathcal{B}(X) : U \text{ is continuous and } \|U\| < +\infty\big\}$$

with the norm

$$\|U\| = \sup\big\{\|U(t,s)\|e^{\lambda(\rho(t)-\rho(s))-a|\rho(s)|} : (t,s) \in I\big\}.$$

Lemma 1. *If δ is sufficiently small, then there is a unique $U \in \mathcal{C}$ such that*

$$\begin{aligned} U(t,s) = T(t,s)P(s) &+ \int_s^t T(t,\tau)P(\tau)B(\tau)U(\tau,s)\,d\tau \\ &- \int_t^\infty T(t,\tau)Q(\tau)B(\tau)U(\tau,s)\,d\tau \end{aligned}$$

for every $(t,s) \in I$. Moreover, for each $\xi \in X$ the function $u(t) = U(t,s)\xi$ is a solution of equation (4)*.*

Proof of the lemma. For the first property, we show that the operator L defined for each $U \in \mathcal{C}$ by

$$\begin{aligned} (LU)(t,s) = T(t,s)P(s) &+ \int_s^t T(t,\tau)P(\tau)B(\tau)U(\tau,s)\,d\tau \\ &- \int_t^\infty T(t,\tau)Q(\tau)B(\tau)U(\tau,s)\,d\tau \end{aligned} \tag{6}$$

has a unique fixed point in $\mathcal{C}$. We first note that

$$\begin{aligned} &\int_t^\infty \|T(t,\tau)Q(\tau)B(\tau)U(\tau,s)\|\,d\tau \\ &\quad\le D\delta e^{-\lambda(\rho(t)-\rho(s))+a|\rho(s)|}\|U\| \int_t^\infty e^{-2\lambda(\rho(\tau)-\rho(t))}\rho'(\tau)\,d\tau \\ &\quad\le \frac{D}{2\lambda}\delta e^{-\lambda(\rho(t)-\rho(s))+a|\rho(s)|}\|U\| < +\infty. \end{aligned}$$

Therefore, $(LU)(t,s)$ is well defined, and

$$\begin{aligned} \|(LU)(t,s)\| \le \|T(t,s)P(s)\| &+ \int_s^t \|T(t,\tau)P(\tau)\|\cdot\|B(\tau)\|\cdot\|U(\tau,s)\|\,d\tau \\ &+ \int_t^\infty \|T(t,\tau)Q(\tau)\|\cdot\|B(\tau)\|\cdot\|U(\tau,s)\|\,d\tau \end{aligned}$$

$$
\begin{aligned}
&\leq De^{-\lambda(\rho(t)-\rho(s))+a|\rho(s)|} + D\delta e^{-\lambda(\rho(t)-\rho(s))+a|\rho(s)|}\|U\| \int_s^t e^{-a|\rho(\tau)|}\rho'(\tau)\,d\tau \\
&\quad + D\delta e^{-\lambda(\rho(t)-\rho(s))+a|\rho(s)|}\|U\| \int_t^\infty e^{-2\lambda(\rho(\tau)-\rho(t))}\rho'(\tau)\,d\tau \\
&\leq De^{-\lambda(\rho(t)-\rho(s))+a|\rho(s)|} + \frac{D}{a}\delta e^{-\lambda(\rho(t)-\rho(s))+a|\rho(s)|}\|U\| \\
&\quad + \frac{D}{2\lambda}\delta e^{-\lambda(\rho(t)-\rho(s))+a|\rho(s)|}\|U\|. \qquad (7)
\end{aligned}
$$

This implies that

$$
\|LU\| \leq D + \delta D\left(\frac{1}{a} + \frac{1}{2\lambda}\right)\|U\| < +\infty, \qquad (8)
$$

and thus we have a well-defined operator $L\colon \mathcal{C} \to \mathcal{C}$. Using (6) and proceeding in a similar manner to that in (7) we also obtain

$$
\|LU_1 - LU_2\| \leq \delta D\left(\frac{1}{a} + \frac{1}{2\lambda}\right)\|U_1 - U_2\|,
$$

for every $U_1, U_2 \in \mathcal{C}$. Therefore, for any sufficiently small δ the operator L is a contraction, and there exists a unique $U \in \mathcal{C}$ such that $LU = U$.

Finally, we note that

$$
\begin{aligned}
U(t,s) - T(t,s)U(s,s) &= T(t,s)P(s) - T(t,s)P(s) + \int_s^t T(t,\tau)P(\tau)B(\tau)U(\tau,s)\,d\tau \\
&\quad + \int_s^t T(t,\tau)Q(\tau)B(\tau)U(\tau,s)\,d\tau \\
&= \int_s^t T(t,\tau)B(\tau)U(\tau,s)\,d\tau \qquad (9)
\end{aligned}
$$

for each $t \geq s$. This completes the proof of the lemma. □

We also show that the bounded solutions constructed in Lemma 1 have a certain invariance property. Later on this will correspond to the invariance of the stable subspaces of the perturbed dynamics.

Lemma 2. *If δ is sufficiently small, then*

$$
U(t,\tau)U(\tau,s) = U(t,s), \quad t \geq \tau \geq s.
$$

Proof of the lemma. We first note that

$$
\begin{aligned}
U(t,\tau)U(\tau,s) = T(t,s)P(s) &+ \int_s^\tau T(t,\sigma)P(\sigma)B(\sigma)U(\sigma,\tau)U(\tau,s)\,d\sigma \\
&+ \int_\tau^t T(t,\sigma)P(\sigma)B(\sigma)U(\sigma,\tau)U(\tau,s)\,d\sigma \\
&- \int_t^\infty T(t,\sigma)Q(\sigma)B(\sigma)U(\sigma,\tau)U(\tau,s)\,d\sigma.
\end{aligned}
$$

Given $s \in \mathbb{R}$, we set

$$I_s = \big\{(t,\tau) \in \mathbb{R} \times \mathbb{R} : t \ge \tau \ge s\big\},$$

and we consider the Banach space

$$\mathcal{C}_s = \big\{H \colon I_s \to \mathcal{B}(X) : H \text{ is continuous and } \|H\|_s < +\infty\big\}$$

with the norm

$$\|H\|_s = \sup\big\{\|H(t,\tau)\|e^{-2a|\rho(t)|} : (t,\tau) \in I_s\big\}.$$

Writing

$$h(t,\tau) = U(t,\tau)U(\tau,s) - U(t,s)$$

for $t \ge \tau \ge s$ (with s fixed), we obtain $L_1 h = h$, where

$$(L_1 H)(t,\tau) = \int_\tau^t T(t,\sigma)P(\sigma)B(\sigma)H(\sigma,s)\,d\sigma - \int_t^\infty T(t,\sigma)Q(\sigma)B(\sigma)H(\sigma,s)\,d\sigma.$$

for each $H \in \mathcal{C}_s$ and $(t,\tau) \in I_s$. Since $\rho(s) \le \rho(\tau)$ (note that $s \le \tau$ and that ρ is an increasing function), we have

$$
\begin{aligned}
&\int_\tau^t \|T(t,\sigma)P(\sigma)\| \cdot \|B(\sigma)\| \cdot \|H(\sigma,s)\|\,d\sigma + \int_t^\infty \|T(t,\sigma)Q(\sigma)\| \cdot \|B(\sigma)\| \cdot \|H(\sigma,s)\|\,d\sigma \\
&\quad\le \frac{D}{\lambda}\delta\|H\|_s + \frac{D}{\lambda}\delta\|H\|_s = \frac{2D}{\lambda}\delta\|H\|_s.
\end{aligned}
$$

This shows that $(L_1 H)(t,\tau)$ is well defined, and that

$$\|L_1 H\|_s \le \frac{2D}{\lambda}\|H\|_s < +\infty.$$

We thus obtain an operator $L_1 \colon \mathcal{C}_s \to \mathcal{C}_s$. Moreover, for each $H_1, H_2 \in \mathcal{C}_s$ and $t \ge \tau$, we have

$$
\begin{aligned}
&\|(L_1H_1)(t,\tau)-(L_1H_2)(t,\tau)\| \\
&\quad\le \int_\tau^t \|T(t,\sigma)P(\sigma)\|\cdot\|B(\sigma)\|\cdot\|H_1(\sigma,s)-H_2(\sigma,s)\|\,d\sigma \\
&\qquad + \int_t^\infty \|T(t,\sigma)Q(\sigma)\|\cdot\|B(\sigma)\|\cdot\|H_1(\sigma,s)-H_2(\sigma,s)\|\,d\sigma \\
&\quad\le \frac{D}{\lambda}\delta\|H_1-H_2\|_s+\frac{D}{\lambda}\delta\|H_1-H_2\|_s=\frac{2D}{\lambda}\|H_1-H_2\|_s.
\end{aligned}
$$

Therefore,

$$\|L_1H_1-L_1H_2\|\le\frac{2D}{\lambda}\|H_1-H_2\|_s.$$

This shows that for δ sufficiently small the operator L_1 is a contraction, and thus there exists a unique $H\in\mathcal{C}_s$ such that $L_1H=H$. Since $0\in\mathcal{C}_s$ also satisfies this identity, we have $H=0$. Now we show that $h\in\mathcal{C}_s$. Indeed, it follows from Lemma 1, together with the inequalities $\rho(t)\ge\rho(\tau)\ge\rho(s)$ (since $t\ge\tau\ge s$) and $2a<\lambda$ that

$$
\begin{aligned}
\|U(t,\tau)U(\tau,s)\| &\le \|U(t,\tau)\|\cdot\|U(\tau,s)\| \\
&\le \|U\|^2e^{-\lambda(\rho(t)-\rho(s))+a(|\rho(\tau)|+|\rho(s)|)} \\
&\le \|U\|^2e^{-\lambda(\rho(t)-\rho(s))}e^{a(\rho(t)-\rho(\tau))}e^{a(\rho(t)-\rho(s))}e^{2a|\rho(t)|} \\
&\le \|U\|^2e^{(2a-\lambda)(\rho(t)-\rho(s))}e^{2a|\rho(t)|},
\end{aligned}
$$

and

$$
\begin{aligned}
\|U(t,s)\| &\le \|U\|e^{-\lambda(\rho(t)-\rho(s))+a|\rho(s)|} \\
&\le \|U\|e^{(a-\lambda)(\rho(t)-\rho(s))+a|\rho(t)|}\le\|U\|e^{2a|\rho(t)|}
\end{aligned}
$$

for $t\ge\tau\ge s$. This shows that $h\in\mathcal{C}_s$, and by the uniqueness of the fixed point of L_1 we conclude that $h=0$. $\square$

We emphasize that Lemma 2 does not follow immediately from Lemma 1. Indeed, since the operator

$$U(t,t)=P(t)-\int_t^\infty T(t,\tau)Q(\tau)B(\tau)U(\tau,t)\,d\tau$$

need not be the identity, the solutions of equation (4) in $[\tau,+\infty)$ given by $t\mapsto U(t,\tau)U(\tau,s)\xi$ and $t\mapsto U(t,s)\xi$ a priori need not have the same initial condition at time τ, and thus we cannot deduce in this manner the identity in Lemma 2.

Step 2: *Construction of bounded solutions II.* Now we construct bounded solutions into the past. We consider the set

$$J = \{(t,s) \in \mathbb{R} \times \mathbb{R} : t \leq s\}, \tag{10}$$

and the Banach space

$$\mathcal{D} = \big\{V \colon J \to \mathcal{B}(X) : V \text{ is continuous and } \|V\| < +\infty\big\},$$

with the norm

$$\|V\| = \sup\big\{\|V(t,s)\|e^{-\lambda(\rho(t)-\rho(s))-a|\rho(s)|} : (t,s) \in J\big\}.$$

Lemma 3. *If δ is sufficiently small, then there is a unique $V \in \mathcal{D}$ such that*

$$V(t,s) = T(t,s)Q(s) + \int_{-\infty}^{t} T(t,\tau)P(\tau)B(\tau)V(\tau,s)\,d\tau$$
$$- \int_{t}^{s} T(t,\tau)Q(\tau)B(\tau)V(\tau,s)\,d\tau.$$

for every $(t,s) \in J$. Moreover, for each $\xi \in X$ the function $t \mapsto V(t,s)\xi$ is a solution of equation (4).

Proof of the lemma. For the first property, we show that the operator M defined for each $V \in \mathcal{D}$ by

$$(MV)(t,s) = T(t,s)Q(s) + \int_{-\infty}^{t} T(t,\tau)P(\tau)B(\tau)V(\tau,s)\,d\tau$$
$$- \int_{t}^{s} T(t,\tau)Q(\tau)B(\tau)V(\tau,s)\,d\tau \tag{11}$$

has a unique fixed point in $\mathcal{D}$. We first note that

$$\begin{aligned}
&\int_{-\infty}^{t} \|T(t,\tau)P(\tau)\| \cdot \|B(\tau)\| \cdot \|V(\tau,s)\|\,d\tau \\
&\quad\leq D\delta e^{\lambda(\rho(t)-\rho(s))+a|\rho(s)|}\|V\| \int_{-\infty}^{t} e^{2\lambda(\rho(\tau)-\rho(t))}\rho'(\tau)\,d\tau \\
&\quad\leq \frac{D}{2\lambda}\delta e^{\lambda(\rho(t)-\rho(s))+a|\rho(s)|}\|V\|.
\end{aligned}$$

Therefore, $(MV)(t,s)$ is well defined, and

$$\|(MV)(t,s)\| \leq \|T(t,s)Q(s)\| + \int_{-\infty}^{t} \|T(t,\tau)P(\tau)\| \cdot \|B(\tau)\| \cdot \|V(\tau,s)\|\,d\tau$$
$$+ \int_{t}^{s} \|T(t,\tau)Q(\tau)\| \cdot \|B(\tau)\| \cdot \|V(\tau,s)\|\,d\tau$$

$$\begin{aligned}
&\le De^{\lambda(\rho(t)-\rho(s))+a|\rho(s)|} \\
&\quad + D\delta e^{\lambda(\rho(t)-\rho(s))+a|\rho(s)|}\|V\| \int_{-\infty}^{t} e^{2\lambda(\rho(\tau)-\rho(t))}\rho'(\tau)\,d\tau \\
&\quad + D\delta e^{\lambda(\rho(t)-\rho(s))+a|\rho(s)|}\|V\| \int_{t}^{s} e^{-a\rho(\tau)}\rho'(\tau)\,d\tau \\
&\le De^{\lambda(\rho(t)-\rho(s))+a|\rho(s)|} + \frac{D}{2\lambda}\delta e^{\lambda(\rho(t)-\rho(s))+a\rho(s)}\|V\| \\
&\quad + \frac{D}{a}\delta e^{\lambda(\rho(t)-\rho(s))+a|\rho(s)|}\|V\|. \qquad (12)
\end{aligned}$$

This implies that

$$\|MV\| \le D + \delta D\left(\frac{1}{2\lambda} + \frac{1}{a}\right)\|V\| < +\infty, \qquad (13)$$

and we thus have a well-defined operator $M\colon \mathcal{D} \to \mathcal{D}$. Using (11) and proceeding in a similar manner to that in (12) we obtain

$$\|MV_1 - MV_2\| \le \delta D\left(\frac{1}{2\lambda} + \frac{1}{a}\right)\|V_1 - V_2\|$$

for every $V_1, V_2 \in \mathcal{D}$. Therefore, for any sufficiently small δ the operator M is a contraction, and there is a unique $V \in \mathcal{D}$ such that $MV = V$.

Moreover, we have

$$\begin{aligned}
&V(s,s) - T(s,t)V(t,s) \\
&= Q(s) + \int_{-\infty}^{s} T(s,\tau)P(\tau)B(\tau)V(\tau,s)\,d\tau - T(s,t)T(t,s)Q(s) \\
&\quad - T(s,t)\left(\int_{-\infty}^{t} T(t,\tau)P(\tau)B(\tau)V(\tau,s)\,d\tau + \int_{t}^{s} T(t,\tau)Q(\tau)B(\tau)V(\tau,s)\,d\tau\right) \\
&= Q(s) + \int_{-\infty}^{t} T(s,\tau)P(\tau)B(\tau)V(\tau,s)\,d\tau + \int_{t}^{s} T(s,\tau)P(\tau)B(\tau)V(\tau,s)\,d\tau \\
&\quad - Q(s) - \int_{-\infty}^{t} T(s,\tau)P(\tau)B(\tau)V(\tau,s)\,d\tau + \int_{t}^{s} T(s,\tau)Q(\tau)B(\tau)V(\tau,s)\,d\tau \\
&= \int_{t}^{s} T(s,\tau)P(\tau)B(\tau)V(\tau,s)\,d\tau + \int_{t}^{s} T(s,\tau)Q(\tau)B(\tau)V(\tau,s)\,d\tau \\
&= \int_{t}^{s} T(s,\tau)B(\tau)V(\tau,s)\,d\tau
\end{aligned}$$

for each $t \le s$. This completes the proof of the lemma. $\square$

Similarly, we establish a corresponding invariance property for the solutions constructed in Lemma 3.

LEMMA 4. *If δ is sufficiently small, then*

$$V(t,\tau)V(\tau,s)=V(t,s), \quad t\le\tau\le s.$$

PROOF OF THE LEMMA. The argument is analogous to that in the proof of Lemma 2. We have

$$\begin{aligned}V(t,\tau)V(\tau,s)=T(t,s)Q(s)&-\int_\tau^s T(t,\sigma)Q(\sigma)B(\sigma)V(\sigma,s)\,d\sigma\\&+\int_{-\infty}^t T(t,\sigma)P(\sigma)B(\sigma)V(\sigma,\tau)V(\tau,s)\,d\sigma\\&-\int_t^\tau T(t,\sigma)Q(\sigma)B(\sigma)V(\sigma,\tau)V(\tau,s)\,d\sigma.\end{aligned}$$

Given $s\in\mathbb{R}$, we set

$$J_s=\bigl\{(t,\tau)\in\mathbb{R}\times\mathbb{R}: t\le\tau\le s\bigr\},$$

and we consider the Banach space

$$\mathcal{D}_s=\bigl\{\bar H\colon J_s\to\mathcal{B}(X):\bar H \text{ is continuous and } \|\bar H\|_s<+\infty\bigr\}$$

with the norm

$$\|\bar H\|_s=\sup\bigl\{\|\bar H(t,\tau)\|e^{-2a|\rho(t)|}:(t,\tau)\in J_s\bigr\}.$$

Writing

$$\bar h(t,s)=V(t,\tau)V(\tau,s)-V(t,s)$$

for $t\le\tau\le s$ (with s fixed), we obtain $M_1\bar h=\bar h$, where

$$(M_1\bar H)(t,\tau)=\int_{-\infty}^t T(t,\sigma)P(\sigma)B(\sigma)\bar H(\sigma,\tau)\,d\sigma-\int_t^\tau T(t,\sigma)Q(\sigma)B(\sigma)\bar H(\sigma,\tau)\,d\sigma$$

for each $\bar H\in\mathcal{D}_s$ and $(t,\tau)\in J_s$. Proceeding in a similar manner to that in the proof of Lemma 2, one can show that 0 is the unique fixed point of M_1 in $\mathcal{D}_s$, and since $\bar h\in\mathcal{D}_s$ we conclude that $\bar h=0$. □

Step 3: *Characterization of the bounded solutions.* In the following two lemmas we show that all bounded solutions of equation (4) are those constructed in Lemmas 1 and 3. We emphasize that this holds regardless of any a priori rate assumed for the bounded solutions. In other words, any bounded solution in a semi-interval has the exponential rate already incorporated in the spaces $\mathcal{C}$ and $\mathcal{D}$.

We start with the solutions into the future.

Lemma 5. *Given $s \in \mathbb{R}$, if $y\colon [s,+\infty) \to X$ is a bounded solution of equation* (4) *with $y(s) = \xi$, then*

$$y(t) = T(t,s)P(s)\xi + \int_s^t T(t,\tau)P(\tau)B(\tau)y(\tau)\,d\tau - \int_t^\infty T(t,\tau)Q(\tau)B(\tau)y(\tau)\,d\tau,$$

that is, $y(t) = U(t,s)\xi$ for $t \geq s$.

Proof of the lemma. For each $t \geq s$ we have

$$P(t)y(t) = T(t,s)P(s)\xi + \int_s^t T(t,\tau)P(\tau)B(\tau)y(\tau)\,d\tau \tag{14}$$

and

$$Q(t)y(t) = T(t,s)Q(s)\xi + \int_s^t T(t,\tau)Q(\tau)B(\tau)y(\tau)\,d\tau.$$

The last formula can be written in the form

$$Q(s)\xi = T(s,t)Q(t)y(t) - \int_s^t T(s,\tau)Q(\tau)B(\tau)y(\tau)\,d\tau, \tag{15}$$

using the notation in (3). Since y is bounded, we have

$$\|T(s,t)Q(t)y(t)\| \leq CDe^{-\lambda(\rho(t)-\rho(s))+a|\rho(t)|},$$

where

$$C = \sup\bigl\{\|y(t)\| : t \geq s\bigr\} < +\infty.$$

Therefore, taking limits in (15) when $t \to +\infty$ we obtain

$$Q(s)\xi = -\int_s^{+\infty} T(s,\tau)Q(\tau)B(\tau)y(\tau)\,d\tau.$$

In particular, replacing (s,ξ) by $(t,y(t))$ the identity yields

$$Q(t)y(t) = -\int_t^{+\infty} T(t,\tau)Q(\tau)B(\tau)y(\tau)\,d\tau.$$

Adding this equation to (14) yields the desired statement. $\square$

Now we consider the solutions into the past.

LEMMA 6. *Given $s \in \mathbb{R}$, if $y\colon (-\infty, s] \to X$ is a bounded solution of equation (4) with $y(s) = \xi$, then*

$$y(t) = T(t,s)Q(s)\xi + \int_{-\infty}^{t} T(t,\tau)P(\tau)B(\tau)y(\tau)\,d\tau - \int_{t}^{s} T(t,\tau)Q(\tau)B(\tau)y(\tau)\,d\tau,$$

that is, $y(t) = V(t,s)\xi$ for $t \le s$.

PROOF OF THE LEMMA. For each $t \le s$ we have

$$P(s)\xi = T(s,t)P(t)y(t) + \int_{t}^{s} T(s,\tau)P(\tau)B(\tau)y(\tau)\,d\tau \tag{16}$$

and

$$Q(s)\xi = T(s,t)Q(t)y(t) + \int_{t}^{s} T(s,\tau)Q(\tau)B(\tau)y(\tau)\,d\tau. \tag{17}$$

Since y is bounded, we have

$$\|T(s,t)P(t)y(t)\| \le CDe^{-\lambda(\rho(s)-\rho(t))+a|\rho(t)|},$$

where

$$C = \sup\bigl\{\|y(t)\| : t \le s\bigr\} < +\infty.$$

Since $\lambda > a$, taking limits in (16) when $t \to -\infty$ we thus obtain

$$P(s)\xi = \int_{-\infty}^{s} T(s,\tau)P(\tau)B(\tau)y(\tau)\,d\tau.$$

Replacing (s,ξ) by $(t,y(t))$ in this identity we obtain

$$P(t)y(t) = \int_{-\infty}^{t} T(t,\tau)P(\tau)B(\tau)y(\tau)\,d\tau. \tag{18}$$

On the other hand, by (17) we have

$$Q(t)y(t) = T(t,s)Q(s)\xi - \int_{t}^{s} T(t,\tau)Q(\tau)B(\tau)y(\tau)\,d\tau. \tag{19}$$

Adding (18) and (19) yields the desired identity. $\square$

Step 4: *Construction of invariant subspaces.* Now we construct stable and unstable invariant subspaces for the perturbed equation. For this we observe that the operators $U(t,t)$ and $V(t,t)$ are projections (by Lemmas 2 and 4), which motivates the introduction of the stable and unstable subspaces respectively as their images. Indeed, we know that the stable and unstable subspaces should correspond to the bounded solutions respectively into the future and into the past. However, since the perturbed dynamics need not be invertible this requires a special care when establishing the invariance of the subspaces.

As motivated above, for each $t \in \mathbb{R}$ we consider the linear subspaces

$$\hat{E}(t) = \operatorname{Im} U(t,t) \quad \text{and} \quad \hat{F}(t) = \operatorname{Im} V(t,t). \tag{20}$$

Lemma 7. *For each $t \in \mathbb{R}$ we have*

$$\hat{E}(t) = \hat{T}(t,s)\hat{E}(s) \quad \textit{and} \quad \hat{F}(t) = \hat{T}(t,s)\hat{F}(s),$$

provided that δ is sufficiently small.

Proof of the lemma. By Lemma 1, for each $\xi \in X$ the function $t \mapsto U(t,s)\xi$, $t \geq s$ is a solution of equation (4) with initial condition at time s equal to $U(s,s)\xi$. Therefore, $U(t,s) = \hat{T}(t,s)U(s,s)$, where $\hat{T}(t,s)$ is the evolution operator associated to equation (4). Hence, by Lemma 2,

$$\begin{aligned}
\hat{T}(t,s)\hat{E}(s) &= \operatorname{Im} U(t,s) \\
&= \operatorname{Im} \big(U(t,t)U(t,s)\big) \\
&= U(t,t) \operatorname{Im} U(t,s) \subset \hat{E}(t)
\end{aligned}$$

for each $t \geq s$. Similarly, by Lemma 3, the function $t \mapsto V(t,s)\xi$, $t \leq s$ is a solution of equation (4), and hence,

$$V(s,s) = \hat{T}(s,t)V(t,s). \tag{21}$$

This implies that

$$\begin{aligned}
\hat{F}(s) &= \hat{T}(s,t) \operatorname{Im} V(t,s) \\
&= \hat{T}(s,t) \operatorname{Im} \big(V(t,t)V(t,s)\big) \\
&\subset \hat{T}(s,t)\hat{F}(t)
\end{aligned}$$

for each $t \leq s$.

Now we establish the reverse inclusions. For this we use the characterization of the bounded solutions given in Lemmas 5 and 6. Take $x \in \hat{E}(t)$. We must show that $x \in \hat{T}(t,s)\hat{E}(s)$ for each $t \geq s$. Take $y \in \hat{T}(t,s)^{-1}x$. Then $\hat{T}(\tau,s)y = \hat{T}(\tau,t)x$ for each $\tau \geq t$. Since $x \in \hat{E}(t) = \operatorname{Im} U(t,t)$ we have $x \in U(t,t)z$ for some $z \in X$, and hence,

$$\hat{T}(\tau,s)y=\hat{T}(\tau,t)U(t,t)z=U(\tau,t)z.$$

In particular, this shows that $[s,+\infty)\ni\tau\mapsto\hat{T}(\tau,s)y$ is bounded. Thus, it follows from Lemma 5 that $\hat{T}(\tau,s)y=U(\tau,s)w$ for some $w\in X$. In particular,

$$y=\hat{T}(s,s)y=U(s,s)w\in\hat{E}(s).$$

Therefore, $x=\hat{T}(t,s)y\in\hat{T}(t,s)\hat{E}(s)$. This establishes the first identity in the lemma. For the second identity, take $x\in\hat{T}(s,t)\hat{F}(t)$. We must show that $x\in\hat{F}(s)$ for each $t\le s$. Take $y\in\hat{T}(s,t)^{-1}x$. Then $\tau\mapsto V(\tau,t)y$, $\tau\le t$ is a bounded solution of equation (4) and hence,

$$(-\infty,s]\ni\tau\mapsto\begin{cases}V(\tau,t)y, & \tau\le t,\\ \hat{T}(s,t)V(t,t)y, & t\le\tau\le s\end{cases}$$

is also a bounded solution of the equation. Hence, it follows from Lemma 6 that $V(\tau,t)y=V(\tau,s)z$, $\tau\le t$ for some $z\in X$. In particular, by (21),

$$x=\hat{T}(s,t)V(t,t)y=\hat{T}(s,t)V(t,s)z=V(s,s)z.$$

This shows that $x\in\hat{F}(s)$, which completes the proof of the lemma. $\square$

Step 5: *Exponential bounds along $\hat{E}(t)$ and $\hat{F}(t)$.* Here we obtain the required exponential bounds from the exponential rates incorporated in the spaces $\mathcal{C}$ and $\mathcal{D}$.

A technical point is that we first need to show that the dynamics is invertible along the spaces $\hat{F}(t)$. Since $V(s,s)^2=V(s,s)$, restricting identity (21) to $\hat{F}(s)$ yields

$$\mathrm{Id}_{\hat{F}(s)}=V(s,s)|\hat{F}(s)=\hat{T}(s,t)V(t,s)|\hat{F}(s).$$

This implies that the operator $\hat{T}(s,t)|\hat{F}(t)$ is invertible, with

$$\big(\hat{T}(s,t)|\hat{F}(t)\big)^{-1}=V(t,s)|\hat{F}(s).$$

Moreover, it follows from Lemma 7 that

$$\hat{T}(t,s)|\hat{E}(s)=U(t,s)|\hat{E}(s)\colon\hat{E}(s)\to\hat{E}(t),\quad t\ge s,$$

and

$$\big(\hat{T}(s,t)|\hat{F}(t)\big)^{-1}=V(t,s)|\hat{F}(s)\colon\hat{F}(s)\to\hat{F}(t),\quad t\le s.$$

Therefore, since $U\in\mathcal{C}$ we obtain

$$\big\|\hat{T}(t,s)|\hat{E}(s)\big\|\le Ke^{-\lambda(\rho(t)-\rho(s))+a|\rho(s)|},\quad t\ge s,\tag{22}$$

and since $V \in \mathcal{D}$ we obtain

$$\left\| \left(\hat{T}(s,t)|\hat{F}(t)\right)^{-1} \right\| \le K e^{-\lambda(\rho(s)-\rho(t))+a|\rho(s)|}, \quad t \le s, \tag{23}$$

for some constant $K > 0$. Namely, we can take

$$K = D \Big/ \left(1 - \delta D \left(\frac{1}{a} + \frac{1}{2\lambda}\right)\right)$$

Step 6: *Construction of projections.* Now we use the results in the former lemmas to show that $\hat{E}(t)$ and $\hat{F}(t)$ form a direct sum, which automatically allows one to define the required projections.

We start with an auxiliary statement about the operators

$$S_s = U(s,s) + V(s,s). \tag{24}$$

Lemma 8. *If δ is sufficiently small, then S_s is invertible for every $s \in \mathbb{R}$.*

Proof of the lemma. We have

$$\begin{aligned} S_s &= U(s,s) + V(s,s) \\ &= P(s) - \int_s^\infty T(s,\tau)Q(\tau)B(\tau)U(\tau,s) + Q(s) + \int_{-\infty}^s T(s,\tau)P(\tau)B(\tau)V(\tau,s), \end{aligned}$$

and hence,

$$S_s - \mathrm{Id} = -\int_s^\infty T(s,\tau)Q(\tau)B(\tau)U(\tau,s)\,d\tau + \int_{-\infty}^s T(s,\tau)P(\tau)B(\tau)V(\tau,s)\,d\tau.$$

Therefore, using Lemmas 1 and 3 we obtain

$$\begin{aligned} \|S_s - \mathrm{Id}\,\| &\le \int_s^\infty \|T(s,\tau)Q(\tau)\| \cdot \|B(\tau)\| \cdot \|U(\tau,s)\|\,d\tau \\ &\quad + \int_{-\infty}^s \|T(s,\tau)P(\tau)\| \cdot \|B(\tau)\| \cdot \|V(\tau,s)\|\,d\tau \\ &\le D\delta\|U\| \int_s^\infty e^{-2\lambda(\rho(\tau)-\rho(s))}\rho'(\tau)\,d\tau + D\delta\|V\| \int_{-\infty}^s e^{2\lambda(\rho(\tau)-\rho(s))}\rho'(\tau)\,d\tau \\ &\le \frac{\delta D}{2\lambda}\big(\|U\| + \|V\|\big). \end{aligned}$$

Moreover, it follows from (8) and (13) that

$$\|U\| \le D \Big/ \left(1 - \delta D \left(\frac{1}{a} + \frac{1}{2\lambda}\right)\right)$$

and

$$\|V\| \leq D \bigg/ \left(1 - \delta D\left(\frac{1}{a} + \frac{1}{2\lambda}\right)\right).$$

This implies that for δ sufficiently small (independently of s), the operator S_s is invertible. □

LEMMA 9. *For each $t \in \mathbb{R}$ we have $\hat{E}(t) \oplus \hat{F}(t) = X$, provided that δ is sufficiently small.*

PROOF OF THE LEMMA. Let $\xi \in \hat{E}(t) \cap \hat{F}(t)$. It follows from (22) and (23) that

$$\frac{1}{K} e^{\lambda(\rho(t)-\rho(s))-a|\rho(t)|}\|\xi\| \leq \|\hat{T}(t,s)\xi\| \leq K e^{-\lambda(\rho(t)-\rho(s))+a|\rho(s)|}\|\xi\| \tag{25}$$

for each $t \geq s$. Since $a < \lambda$ this is only possible if $\xi = 0$. Therefore, $\hat{E}(t) \cap \hat{F}(t) = \{0\}$. Moreover, since the operator S_t is invertible, we have

$$X = S_t X = \operatorname{Im} U(t,t) + \operatorname{Im} V(t,t) = \hat{E}(t) + \hat{F}(t).$$

This concludes the proof of the lemma. □

It follows from Lemma 9 that for each $t \in \mathbb{R}$ any $x \in X$ can be written in a unique form $x = y_t + z_t$ with $y_t \in \hat{E}(t)$ and $z_t \in \hat{F}(t)$. We can thus define a projection $\hat{P}(t)\colon X \to X$ by $\hat{P}(t)x = y_t$. We also write $\hat{Q}(t) = \mathrm{Id} - \hat{P}(t)$ and thus $\hat{Q}(t)x = z_t$. Clearly,

$$\hat{E}(t) = \hat{P}(t)X \quad \text{and} \quad \hat{F}(t) = \hat{Q}(t)X.$$

LEMMA 10. *For each $t \geq s$ we have $\hat{P}(t)\hat{T}(t,s) = \hat{T}(t,s)\hat{P}(s)$, provided that δ is sufficiently small.*

PROOF OF THE LEMMA. The statement follows readily from the definition of $\hat{P}(t)$ and Lemmas 7 and 9. □

Step 7: *Bounds for the norms of the projections.* In order to complete the proof of the theorem we need to given a bound for the norms of the projections $\hat{P}(t)$ and $\hat{Q}(t)$. We first establish an auxiliary result. Set

$$\alpha_t^B = \inf\big\{\|x - y\| : x \in \hat{E}(t), y \in \hat{F}(t),\ \|x\| = \|y\| = 1\big\}. \tag{26}$$

LEMMA 11. *It δ is sufficiently small, then there exists a constant $c > 0$ such that*

$$\alpha_t^B \geq c e^{-a|\rho(t)|} \tag{27}$$

for each $t \in \mathbb{R}$.

Proof of the lemma. Given $x \in \hat{E}(t)$ and $y \in \hat{F}(t)$, there exist $\bar{x} \in E(t)$ and $\bar{y} \in F(t)$ such that

$$x = U(t,t)\bar{x} = (\mathrm{Id} + G_E(t))\bar{x} \quad \text{and} \quad y = V(t,t)\bar{y} = (\mathrm{Id} + G_F(t))\bar{y},$$

where

$$G_E(t) = -\int_t^\infty T(t,\tau)Q(\tau)B(\tau)U(\tau,t)\,d\tau$$

and

$$G_F(t) = \int_{-\infty}^t T(t,\tau)P(\tau)B(\tau)V(\tau,t)\,d\tau.$$

We have

$$\begin{aligned} \|G_E(t)\|e^{a|\rho(t)|} &\le \int_t^\infty \|T(t,\tau)Q(\tau)\| \cdot \|B(\tau)\| \cdot \|U(\tau,t)\|\,d\tau \\ &\le D\delta\|U\| \int_t^\infty e^{2(a-\lambda)(\rho(\tau)-\rho(t))}\rho'(\tau)\,d\tau \le \mu\|U\|, \end{aligned}$$

where

$$\mu = \frac{\delta D}{2(\lambda - a)}. \tag{28}$$

This implies that

$$\|G_E(t)\| \le \mu\|U\|e^{-a|\rho(t)|}. \tag{29}$$

Furthermore,

$$\begin{aligned} \|G_F(t)\|e^{a|\rho(t)|} &\le \int_{-\infty}^t \|T(t,\tau)P(\tau)\| \cdot \|B(\tau)\| \cdot \|V(\tau,t)\|\,d\tau \\ &\le D\delta\|V\| \int_{-\infty}^t e^{2(a-\lambda)(\rho(t)-\rho(\tau))}\rho'(\tau)\,d\tau \le \mu\|V\|. \end{aligned}$$

This implies that

$$\|G_F(t)\| \le \mu\|V\|e^{-a|\rho(t)|}. \tag{30}$$

Hence,

$$\bigl(1 - \mu\|U\|e^{-a|\rho(t)|}\bigr)\|\bar{x}\| \le \|x\| \le \bigl(1 + \mu\|U\|e^{-a|\rho(t)|}\bigr)\|\bar{x}\| \tag{31}$$

and

$$\big(1-\mu\|V\|e^{-a|\rho(t)|}\big)\|\bar{y}\| \le \|y\| \le \big(1+\mu\|V\|e^{-a|\rho(t)|}\big)\|\bar{y}\|. \tag{32}$$

On the other hand, setting $s=t$ in (1) and (2) we obtain

$$\|P(t)\| \le De^{a|\rho(t)|} \quad \text{and} \quad \|Q(t)\| \le De^{a|\rho(t)|}.$$

We recall that (see for example [**4**])

$$\frac{1}{\|P(t)\|} \le \alpha_t^0 \le \frac{2}{\|P(t)\|} \quad \text{and} \quad \frac{1}{\|Q(t)\|} \le \alpha_t^0 \le \frac{2}{\|Q(t)\|} \tag{33}$$

for each $t \in \mathbb{R}$. Therefore,

$$\alpha_t^0 \ge \frac{1}{D}e^{-a|\rho(t)|}, \quad t \in \mathbb{R}.$$

Now we observe that

$$\left\|\frac{\bar{x}}{\|\bar{x}\|} - \frac{\bar{y}}{\|\bar{y}\|}\right\| \le \frac{\|(\bar{x}-\bar{y})\|\bar{y}\| + \bar{y}(\|\bar{y}\|-\|\bar{x}\|)\|}{\|\bar{x}\|\cdot\|\bar{y}\|} \le \frac{2}{\|\bar{x}\|}\|\bar{x}-\bar{y}\|.$$

Therefore, by (31) and (32),

$$\begin{aligned}
\|x-y\| &= \|\bar{x}-\bar{y}+G_E(t)\bar{x}-G_F(t)\bar{y}\| \\
&\ge \|\bar{x}-\bar{y}\| - \|G_E(t)\|\cdot\|\bar{x}\| - \|G_F(t)\|\cdot\|\bar{y}\| \\
&\ge \frac{\|\bar{x}\|}{2}\left\|\frac{\bar{x}}{\|\bar{x}\|} - \frac{\bar{y}}{\|\bar{y}\|}\right\| - \frac{\|G_E(t)\|}{1-\mu\|U\|e^{-a\rho(t)}}\|x\| - \frac{\|G_F(t)\|}{1-\mu\|V\|e^{-a|\rho(t)|}}\|y\| \\
&\ge \frac{\|x\|}{2(1+\mu\|U\|e^{-a|\rho(t)|})}\left\|\frac{\bar{x}}{\|\bar{x}\|} - \frac{\bar{y}}{\|\bar{y}\|}\right\| \\
&\quad - \frac{\mu\|U\|e^{-a|\rho(t)|}}{1-\mu\|U\|e^{-a|\rho(t)|}}\|x\| - \frac{\mu\|V\|e^{-a|\rho(t)|}}{1-\mu\|V\|e^{-a|\rho(t)|}}\|y\|.
\end{aligned}$$

Taking the infimum over all vectors x, y with $\|x\| = \|y\| = 1$ we obtain

$$\begin{aligned}
\alpha_t^B &\ge \frac{1}{2(1+\mu\|U\|e^{-a|\rho(t)|})}\alpha_t^0 - \frac{\mu\|U\|e^{-a|\rho(t)|}}{1-\mu\|U\|e^{-a|\rho(t)|}}\|x\| - \frac{\mu\|V\|e^{-a|\rho(t)|}}{1-\mu\|V\|e^{-a|\rho(t)|}}\|y\| \\
&\ge \frac{e^{-a|\rho(t)|}}{4D(1+\mu\|U\|)} - \frac{\mu\|U\|e^{-a|\rho(t)|}}{1-\mu\|U\|} - \frac{\mu\|V\|e^{-a|\rho(t)|}}{1-\mu\|V\|}.
\end{aligned}$$

Taking δ sufficiently small (see (28)) yields inequality (27). This concludes the proof of the lemma. □

In a similar manner to that in (33), one can also show that

$$\frac{1}{\|\hat{P}(t)\|} \le \alpha_t^B \le \frac{2}{\|\hat{P}(t)\|} \quad \text{and} \quad \frac{1}{\|\hat{Q}(t)\|} \le \alpha_t^B \le \frac{2}{\|\hat{Q}(t)\|},$$

and hence, it follows from Lemma 11 that

$$\|\hat{P}(t)\| \le \frac{2}{\alpha_t^B} \le \frac{2}{c} e^{a|\rho(t)|} \tag{34}$$

and

$$\|\hat{Q}(t)\| \le \frac{2}{\alpha_t^B} \le \frac{2}{c} e^{a|\rho(t)|} \tag{35}$$

for each $t \in \mathbb{R}$. Since

$$\|\hat{T}(t,s)\hat{P}(s)\| \le \|\hat{T}(t,s)|\hat{E}(s)\| \cdot \|\hat{P}(s)\|, \quad t \ge s,$$

and

$$\|\hat{T}(t,s)\hat{Q}(s)\| \le \|\big(\hat{T}(s,t)|\hat{F}(t)\big)^{-1}\| \cdot \|\hat{Q}(s)\|, \quad t \le s,$$

the theorem follows now readily from (34) and (35) together with inequalities (22) and (23). $\square$

3. The case of uniform exponential dichotomies.

We consider in this section the particular case of uniform exponential dichotomies. We emphasize that in Theorem 1 we assume that $a > 0$. Indeed, some arguments in the proof of Theorem 1 break down when $a = 0$ (that is, in the uniform setting). Nevertheless, with some relatively small modifications the same method applies.

The following is our robustness result for uniform exponential dichotomies.

Theorem 2. *Let $T(t,s)$ be an evolution family admitting a ρ-uniform exponential dichotomy, and let $B\colon \mathbb{R} \to \mathcal{B}(X)$ be a continuous function satisfying*

$$\|B(t)\| \le \delta\rho'(t), \quad t \in \mathbb{R},$$

such that equation (4) also defines an evolution family $\hat{T}(t,s)$. For each $\varepsilon > 0$ sufficiently small, if δ is sufficiently small, then $\hat{T}(t,s)$ admits a ρ-uniform exponential dichotomy with the constant λ replaced by $\lambda - \varepsilon$.

Proof. The strategy of the proof is the same as in Theorem 1, and thus we only explain what changes are necessary.

We first construct bounded solutions. Given $\varepsilon \in (0, \lambda/2)$, we consider the Banach space

$$\mathcal{C} = \big\{U \colon I \to \mathcal{B}(X) : U \text{ is continuous and } \|U\| < +\infty\big\}$$

with the norm

$$\|U\| = \sup \big\{\|U(t,s)\| e^{(\lambda-\varepsilon)(\rho(t)-\rho(s))} : (t,s) \in I\big\},$$

and with I as in (5).

Lemma 12. *If δ is sufficiently small, then there is a unique $U \in \mathcal{C}$ such that*

$$\begin{aligned} U(t,s) = T(t,s)P(s) &+ \int_s^t T(t,\tau)P(\tau)B(\tau)U(\tau,s)\,d\tau \\ &- \int_t^\infty T(t,\tau)Q(\tau)B(\tau)U(\tau,s)\,d\tau \end{aligned}$$

for every $(t,s) \in I$. Moreover, for each $\xi \in X$ the function $u(t) = U(t,s)\xi$ is a solution of equation (4).

Proof of the lemma. Again we use (6) to define an operator L for each $U \in \mathcal{C}$. We first note that

$$\begin{aligned} &\int_t^\infty \|T(t,\tau)Q(\tau)B(\tau)U(\tau,s)\|\,d\tau \\ &\le D\delta\|U\| \int_t^\infty e^{-\lambda(\rho(\tau)-\rho(t))+(-\lambda+\varepsilon)(\rho(\tau)-\rho(s))}\rho'(\tau)\,d\tau \\ &= D\delta\|U\| e^{(-\lambda+\varepsilon)(\rho(t)-\rho(s))} \int_t^\infty e^{(-2\lambda+\varepsilon)(\rho(\tau)-\rho(t))}\rho'(\tau)\,d\tau \\ &\le \frac{D\delta}{2\lambda-\varepsilon} e^{(-\lambda+\varepsilon)(\rho(t)-\rho(s))}\|U\| < +\infty. \end{aligned} \tag{36}$$

Therefore, $(LU)(t,s)$ is well defined, and

$$\begin{aligned} \|(LU)(t,s)\| &\le \|T(t,s)P(s)\| + \int_s^t \|T(t,\tau)P(\tau)\| \cdot \|B(\tau)\| \cdot \|U(\tau,s)\|\,d\tau \\ &\quad + \int_t^\infty \|T(t,\tau)Q(\tau)\| \cdot \|B(\tau)\| \cdot \|U(\tau,s)\|\,d\tau \\ &\le De^{-\lambda(\rho(t)-\rho(s))} + D\delta\|U\| \int_s^t e^{-\lambda(\rho(t)-\rho(\tau))+(-\lambda+\varepsilon)(\rho(\tau)-\rho(s))}\rho'(\tau)\,d\tau \\ &\quad + \frac{D\delta}{2\lambda-\varepsilon} e^{(-\lambda+\varepsilon)(\rho(t)-\rho(s))}\|U\| \end{aligned}$$

$$= De^{(-\lambda+\varepsilon)(\rho(t)-\rho(s))} + D\delta\|U\|e^{(-\lambda+\varepsilon)(\rho(t)-\rho(s))}\int_s^t e^{-\varepsilon(\rho(t)-\rho(\tau))}\rho'(\tau)\,d\tau$$
$$+ \frac{D\delta}{2\lambda-\varepsilon}e^{(-\lambda+\varepsilon)(\rho(t)-\rho(s))}\|U\|.$$

This implies that

$$\|LU\| \le D + \delta D\left(\frac{1}{\varepsilon} + \frac{1}{2\lambda-\varepsilon}\right)\|U\| < +\infty,$$

and thus we have a well-defined operator $L\colon \mathcal{C} \to \mathcal{C}$. We show in a similar manner that

$$\|LU_1 - LU_2\| \le D + \delta D\left(\frac{1}{\varepsilon} + \frac{1}{2\lambda-\varepsilon}\right)\|U_1 - U_2\|$$

for every $U_1, U_2 \in \mathcal{C}$. Therefore, for any sufficiently small δ the operator L is a contraction, and there exists a unique $U \in \mathcal{C}$ such that $LU = U$. The last property in the lemma can be obtained as in (9). □

Similarly, we consider the Banach space

$$\mathcal{D} = \big\{U\colon J \to \mathcal{B}(X) : V \text{ is continuous and } \|V\| < +\infty\big\}$$

with the norm

$$\|V\| = \sup\big\{\|V(t,s)\|e^{(-\lambda+\varepsilon)(\rho(t)-\rho(s))} : (t,s) \in J\big\},$$

and with J as in (10). We can then make appropriate modifications in the proof of Lemma 3 to obtain the following.

Lemma 13. *If δ is sufficiently small, then there is a unique $V \in \mathcal{D}$ such that*

$$V(t,s) = T(t,s)Q(s) + \int_{-\infty}^t T(t,\tau)P(\tau)B(\tau)V(\tau,s)\,d\tau$$
$$- \int_t^s T(t,\tau)Q(\tau)B(\tau)V(\tau,s)\,d\tau.$$

for every $(t,s) \in J$. Moreover, for each $\xi \in X$ the function $t \mapsto V(t,s)\xi$ is a solution of equation (4)*.*

Moreover, repeating arguments in the proofs of Lemmas 2 and 4 (with $a = 0$) we also obtain the following.

Lemma 14. *If δ is sufficiently small, then*

$$U(t,\tau)U(\tau,s) = U(t,s), \quad t \ge \tau \ge s,$$

and

$$V(t,\tau)V(\tau,s) = V(t,s), \quad t \le \tau \le s.$$

The remaining arguments in Steps 3–5 require no change, and thus we also obtain the characterization of the bounded solutions given by Lemmas 5 and 6, as well as the candidates for the stable and unstable subspaces $\hat{E}(t)$ and $\hat{F}(t)$ in (20). Moreover, we also have the estimates in (22) and (23) with $a = 0$.

To construct projections we first establish the invertibility of the operators S_s in (24).

LEMMA 15. *If δ is sufficiently small, then S_s is invertible for every $s \in \mathbb{R}$.*

PROOF OF THE LEMMA. It follows from (36) and the corresponding inequality for $V(t,s)$ that

$$\begin{aligned}\|S_s - \mathrm{Id}\,\| \le \int_s^\infty &\|T(s,\tau)Q(\tau)\| \cdot \|B(\tau)\| \cdot \|U(\tau,s)\|\,d\tau \\ &+ \int_{-\infty}^s \|T(s,\tau)P(\tau)\| \cdot \|B(\tau)\| \cdot \|V(\tau,s)\|\,d\tau \\ &\le \frac{\delta D}{2\lambda - \varepsilon}\bigl(\|U\| + \|V\|\bigr).\end{aligned}$$

Since

$$\|U\| \le D \Big/ \left(1 - \delta D\left(\frac{1}{\varepsilon} + \frac{1}{2\lambda - \varepsilon}\right)\right)$$

and

$$\|V\| \le D \Big/ \left(1 - \delta D\left(\frac{1}{\varepsilon} + \frac{1}{2\lambda - \varepsilon}\right)\right),$$

we conclude that for δ sufficiently small (independently of s), the operator S_s is invertible. $\square$

LEMMA 16. *For each $t \in \mathbb{R}$ we have $\hat{E}(t) \oplus \hat{F}(t) = X$, provided that δ is sufficiently small.*

PROOF OF THE LEMMA. Inequality (25) holds with $a = 0$, and thus we have again $\hat{E}(t) \cap \hat{F}(t) = \{0\}$. The remaining argument is also identical. $\square$

One can now repeat arguments in the proof of Lemma 11 to obtain the following lower bound for α_t^B in (26).

LEMMA 17. *If δ is sufficiently small, then there exists a constant $c > 0$ such that*

$$\alpha_t^B \geq c \quad for \quad t \in \mathbb{R}. \tag{37}$$

Proof of the lemma. It is sufficient to observe that inequalities (29) and (30) are replaced by

$$\|G_E(t)\| \leq \frac{D\delta}{2\lambda - \varepsilon}\|U\|$$

(set $t = s$ in (36)), and

$$\|G_F(t)\| \leq \frac{D\delta}{2\lambda - \varepsilon}\|V\|.$$

Proceeding as in the proof of Lemma 11 we then obtain

$$\alpha_t^B \geq \frac{1}{4D(1 + D\delta\|U\|/(2\lambda - \varepsilon))} - \frac{D\delta\|U\|/(2\lambda - \varepsilon)}{1 - D\delta\|U\|/(2\lambda - \varepsilon)} - \frac{D\delta\|V\|/(2\lambda - \varepsilon)}{1 - D\delta\|V\|/(2\lambda - \varepsilon)}.$$

Taking δ sufficiently small yields inequality (37). □

The statement in Theorem 2 follows now from combining the above lemmas. □

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