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Pseudoconvex domains in the Hopf surface

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Abstract. With the aid of the technique of variation of domains developed in Memoirs of Amer. Math. Soc., Vol. 209, No. 984 (2011), we characterize the pseudoconvex domains with smooth boundary in Hopf surfaces which are not Stein.

1. Introduction.

Let $a \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ with $|a| > 1$ and let $\mathbb{H}_a$ be the *Hopf manifold with respect to* a, i.e., $\mathbb{H}_a = \mathbb{C}^n \setminus \{(0, \dots, 0)\}/\sim$ where $z' \sim z$ if and only if there exists $m \in \mathbb{Z}$ such that $z' = a^m z$ in $\mathbb{C}^n \setminus \{0\}$. In a previous paper [**1**] we showed that any pseudoconvex domain $D \subset \mathbb{H}_a$ with C^ω-smooth boundary which is not Stein is biholomorphic to $T_a \times D_0$ where D_0 is a Stein domain in $\mathbb{P}^{n-1}$ with C^ω-smooth boundary and T_a is a one-dimensional torus. This was achieved using the technique of variation of domains in a complex Lie group developed in [**1**] applied to $\mathbb{H}_a$ as a complex homogeneous space with transformation group $GL(n, \mathbb{C})$ (Theorem 6.5 in [**1**]).

For $a, b \in \mathbb{C}^*$ with $|b| \geq |a| > 1$ we let $\mathcal{H}_{(a,b)}$ be the *Hopf surface with respect to* (a, b), i.e., $\mathcal{H}_{(a,b)} = \mathbb{C}^2 \setminus \{(0,0)\}/\sim$, where $(z, w) \sim (z', w')$ if and only if there exists $n \in \mathbb{Z}$ such that $z' = a^n z$, $w' = b^n w$. We set $\boldsymbol{T}_a = T_a \times \{0\}$, $\boldsymbol{T}_b = \{0\} \times T_b$, and $\mathcal{H}^*_{(a,b)} = \mathcal{H}_{(a,b)} \setminus (\boldsymbol{T}_a \cup \boldsymbol{T}_b)$. For $(z, w) \in \mathbb{C}^2 \setminus \{(0,0)\}$ we denote by $[z, w]$ the corresponding point in $\mathcal{H}_{(a,b)}$.

We remark that $\mathcal{H}_{(a,b)}$ is not a complex Lie group. However, $\mathcal{H}^*_{(a,b)}$ is both a complex Lie group and a complex homogeneous space. With the aid of the aforementioned technique of variation of domains in [**1**], we can characterize the domains with C^ω-smooth boundary in $\mathcal{H}_{(a,b)}$ which are not Stein.

We set

$$\rho := \frac{\log |b|}{\log |a|} \geq 1 \tag{1.1}$$

and we define the holomorphic vector field

$$X_u := (\log |a|) z \frac{\partial}{\partial z} + (\log |b|) w \frac{\partial}{\partial w} \tag{1.2}$$

on $\mathbb{C}^2$. This induces a holomorphic vector field on $\mathcal{H}_{(a,b)}$ which we still write as X_u. These vector fields X_u are crucial and will be discussed in Section 3. We let $\widetilde{\sigma}_u$ be the integral curve of X_u with initial value at $[1, 1]$.

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To state our result, we divide the parameter space of pairs (a,b) into two disjoint sets following the discussion on p. 52 in [**2**]. We let

$$\boldsymbol{S} := \{(a,b)\in\mathbb{C}^*\times\mathbb{C}^* : 1<|a|\le|b|\} = S_1\cup S_2$$

where

$$S_1 := \{(a,b)\in\boldsymbol{S} : \text{ there do } \textit{not} \text{ exist positive integers } P,Q \text{ with } a^Q=b^P\}.$$

If $(a,b)\in S_1$, then $\mathcal{H}_{(a,b)}$ admits no nonconstant meromorphic functions. If $(a,b)\in S_2$, there exist positive integers P,Q such that $a^Q=b^P$; letting P be the minimal such integer, $\mathcal{H}_{(a,b)}$ admits the non-constant meromorphic function w^P/z^Q. Indeed, in this case any meromorphic function on $\mathcal{H}_{(a,b)}$ is a rational function of w^P/z^Q. For $(a,b)\in S_2$, since $\rho=\log|b|/\log|a|$ and $\tau:=(1/2\pi)(Q\arg a-P\arg b)$ are rational, we set

$$\rho := q/p,\ \ q\ge p\ge 1 \text{ and } (p,q)=1; \tag{1.3}$$

$$\tau := m/l,\ \ l\ge 1 \text{ and } (l,m)=\pm 1 \text{ or } \tau=0 \text{ (and we set } l=1). \tag{1.4}$$

We have the following decompositions of $\mathcal{H}:=\mathcal{H}_{(a,b)}$.

Proposition 1.1. *Let $\mathcal{H}:=\mathcal{H}_{(a,b)}$ be a Hopf surface.*

(α) *In case $(a,b)\in S_1$ we have*

$$\mathcal{H}=\Big(\bigcup_{c\in(0,\infty)}\Sigma_c\Big)\cup(\boldsymbol{T}_a\cup\boldsymbol{T}_b) \tag{1.5}$$

and this is a disjoint union. Here Σ_c is the closure of $[z_0,w_0]\widetilde{\sigma}_u$ with $c=|w_0|^{\log p}/|z_0|^{\log q}$ (and Σ_c is independent of the choice of $[z_0,w_0]$ provided $c=|w_0|^{\log p}/|z_0|^{\log q}$), and hence Σ_c is a real three-dimensional Levi-flat hypersurface in $\mathcal{H}^*:=\mathcal{H}^*_{(a,b)}$. We set $\Sigma_0=\boldsymbol{T}_a$ and $\Sigma_\infty=\boldsymbol{T}_b$ so that $\mathcal{H}=\bigcup_{c\in[0,\infty]}\Sigma_c$.

(β) *In case $(a,b)\in S_2$, with ρ and τ as in (1.3) and (1.4), we have*

$$\mathcal{H}=\Big(\bigcup_{c\in\mathbb{C}^*}\sigma_c\Big)\cup(\boldsymbol{T}_a\cup\boldsymbol{T}_b) \tag{1.6}$$

which is a disjoint union. Here $\sigma_c:=[z_0,w_0]\widetilde{\sigma}_u$ with $c=w_0^{pl}/z_0^{ql}$ (where σ_c is independent of the choice of $[z_0,w_0]$ provided $c=w_0^{pl}/z_0^{ql}$), and hence σ_c is compact curve in $\mathcal{H}^*$. We note that $\boldsymbol{T}_a=[z_0,0]\exp tX_u$ where $z_0\ne 0$ and $\boldsymbol{T}_b=[0,w_0]\exp tX_u$ where $w_0\ne 0$. We set $\sigma_0=\boldsymbol{T}_a$ and $\sigma_\infty=\boldsymbol{T}_b$ so that $\mathcal{H}=\bigcup_{c\in\mathbb{P}^1}\sigma_c$.

We can now state our main result.

Theorem 1.1. *Let D be a pseudoconvex domain in $\mathcal{H}_{(a,b)}$ with C^ω-smooth boundary. Suppose D is not Stein.*

Case a: $(a,b) \in S_1$.

D reduces to one of the following:

(a-1) *There exist* $0 < k_1 < k_2 < +\infty$ *such that* $D = \bigcup_{c\in(k_1,k_2)} \Sigma_c$.
(a-2′) *There exists a positive number k such that* $D = \bigcup_{c\in[0,k)} \Sigma_c$.
(a-2″) *There exists a positive number k such that* $D = \bigcup_{c\in(k,+\infty]} \Sigma_c$.

Case b: $(a,b) \in S_2$.

$D = \bigcup_{c\in\delta} \sigma_c$ *for some domain δ in $\mathbb{P}^1$ with smooth boundary.*

REMARK 1.1. In Case a, the Levi-flat hypersurfaces Σ_c for $c \neq 0, \infty$ are level sets of the logarithmically pluriharmonic function $s[z,w] := |w|^{\log|a|}/|z|^{\log|b|}$ on $\mathcal{H}^*$ (see (2.5)) and hence all these surfaces are biholomorphically equivalent in $\mathcal{H}^*$. In Case b, the compact curves σ_c are level sets of the meromorphic function $f[z,w] := w^{pl}/z^{ql}$ and for $c \in \mathbb{C}^*$ each σ_c is conformally equivalent to a torus $\mathbb{T}_{(a,b)}$. A detailed construction of $\mathbb{T}_{(a,b)}$ is discussed in Appendix A (Section 5).

The main idea behind the proof is this: starting with a pseudoconvex domain $D \subset \mathcal{H}$ with smooth boundary, we consider $D^* = D \cap \mathcal{H}^*$. We construct a natural plurisubharmonic exhaustion function using our c-Robin function techniques in [**1**]. It is natural to try to extend this function to D first as a plurisubharmonic function and then as an exhaustion function. The construction of a plurisubharmonic exhaustion function on D is the most delicate part in the proof of the theorem (see Section 4). Hirschowitz ([**3**], [**4**]) proved the existence of such a function on a pseudoconvex domain in an *infinitesimally homogeneous space*. However, a Hopf surface $\mathcal{H}_{(a,b)}$ with $a \neq b$ is *not* an infinitesimally homogeneous space – this essentially follows from the fact that any holomorphic vector field on $\mathcal{H}_{(a,b)}$ is of the form $c_1 z(\partial/\partial z) + c_2 w(\partial/\partial w)dw$ where $c_1, c_2 \in \mathbb{C}$ (cf. Example 2.15 on pp. 69–71 of [**2**]) – thus we cannot apply his result. We study obstructions to our resulting plurisubharmonic exhaustion function (or a modification of it) being strictly plurisubharmonic arising from the possible existence of certain holomorphic vector fields. As a by-product of this procedure, we also encounter an interesting class of Stein subdomains in $\mathcal{H}$ which we call *Nemirovskii-type* domains.

The outline of our paper is the following. In the next section, we briefly discuss properties of the Hopf surface $\mathcal{H}_{(a,b)}$, and in Section 3 we state without proof some preliminary results, including a classification in Lemma 3.1 of the holomorphic vector fields on $\mathcal{H}_{(a,b)}$ and their integral curves. This yields the decompositions (1.5) and (1.6) of $\mathcal{H}_{(a,b)}$ in Proposition 1.1. The proof of Theorem 1.1 is given in Section 4. At the end of this section we give an example of the aforementioned Nemirovskii-type domain. The proofs of the results in Section 3 are given at the end of the paper in Appendix A and Appendix B.

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2. Properties of the Hopf surface $\mathcal{H}_{(a,b)}$.

We write $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and $(\mathbb{C}^2)^* := \mathbb{C}^2 \setminus \{(0,0)\}$. Fix $a, b \in \mathbb{C}^*$ with $1 < |a| \le |b|$. For $(z,w), (z',w') \in (\mathbb{C}^2)^*$, we define the equivalence relation

$$(z,w) \sim (z',w') \quad \text{if and only if} \quad \exists\, n \in \mathbb{Z} \text{ such that } z' = a^n z,\ w' = b^n w.$$

The space $(\mathbb{C}^2)^*/\sim$ consisting of all equivalence classes

$$[z,w] := \{(a^n z, b^n w) : n \in \mathbb{Z}\}, \quad (z,w) \in (\mathbb{C}^2)^*$$

is called the *Hopf surface* $\mathcal{H} = \mathcal{H}_{(a,b)}$; it is a complex two-dimensional compact manifold.

For $z,\ z' \in \mathbb{C}^*$ we define $z \sim_a z'$ if and only if there exists $n \in \mathbb{Z}$ such that $z' = a^n z$ in $\mathbb{C}^*$. Then

$$T_a := \mathbb{C}^*/\sim_a \quad \text{and} \quad T_b := \mathbb{C}^*/\sim_b$$

are complex one-dimensional tori, and $\mathcal{H}$ contains two disjoint compact analytic curves $\boldsymbol{T}_a = T_a \times \{0\}$ and $\boldsymbol{T}_b = \{0\} \times T_b$. We have $\boldsymbol{T}_a \cup \boldsymbol{T}_b = \{(z,w) \in (\mathbb{C}^2)^* : zw = 0\}/\sim$ in $\mathcal{H}$; for simplicity we write $\boldsymbol{T}_a \cup \boldsymbol{T}_b = \{zw = 0\}$. We consider the subdomain $\mathcal{H}^*$ of $\mathcal{H}$ defined by

$$\mathcal{H}^* := \mathcal{H} \setminus \{zw = 0\}. \tag{2.1}$$

Thus $\mathcal{H}$ is a compactification of $\mathcal{H}^*$ by two disjoint one-dimensional tori. The set $\mathcal{H}^*$ is a complex Lie group and will play a crucial role in this work.

We give a more precise description of the Hopf surface. A fundamental domain for $\mathcal{H}$ is

$$\begin{aligned} \mathcal{F} &:= (\{|z| \le |a|\} \times \{|w| \le |b|\}) \setminus (\{|z| \le 1\} \times \{|w| \le 1\}) \\ &= E_1 \cup E_2 \Subset (\mathbb{C}^2)^*, \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} E_1 &= E_1' \times E_1'' := \{|z| \le |a|\} \times \{1 < |w| \le |b|\}, \\ E_2 &= E_2' \times E_2'' := \{1 < |z| \le |a|\} \times \{|w| \le |b|\}. \end{aligned}$$

For $k = 0, \pm 1, \ldots$ we set $\mathcal{F}_k := \mathcal{F} \cdot (a^k, b^k)$. Then $\mathcal{F}_0 = \mathcal{F}$; each $\mathcal{F}_k$ is a fundamental domain; and we have the disjoint union $(\mathbb{C}^2)^* = \bigcup_{n=-\infty}^{\infty} \mathcal{F}_n$.

The Hopf surface $\mathcal{H}$ is obtained by gluing the boundaries of $\partial\mathcal{F}$ in the following way: setting

$$\begin{aligned} L_a' &:= \{|z| \le |a|\} \times \{|w| = |b|\}, \quad L_1' = \{|z| \le 1\} \times \{|w| = 1\}; \\ L_b'' &:= \{|z| = |a|\} \times \{|w| \le |b|\}, \quad L_1'' = \{|z| = 1\} \times \{|w| \le 1\}, \end{aligned}$$

we have the identifications:

$$\begin{aligned}&(1)\ \ (z,w)\in L'_a \quad \text{with} \quad (z/a,w/b)\in L'_1;\\ &(2)\ \ (z,w)\in L''_b \quad \text{with} \quad (z/a,w/b)\in L''_1.\end{aligned}$$

We set

$$\mathcal{I}=\{(a^n,b^n)\in\mathbb{C}^*\times\mathbb{C}^*:n\in\mathbb{Z}\}\subset\mathbb{C}^*\times\mathbb{C}^*, \tag{2.3}$$

which is a discrete set in $(\mathbb{C}^2)^*$. For $D\subset\mathbb{C}^*\times\mathbb{C}^*$ we set

$$\widetilde{D}=D\cdot\mathcal{I}=\{(a^nz,b^nw)\in\mathbb{C}^*\times\mathbb{C}^*:(z,w)\in D,\ n\in\mathbb{Z}\}\subset\mathbb{C}^*\times\mathbb{C}^* \tag{2.4}$$

and

$$D/\sim\ =\{[z,w]\in\mathcal{H}:(z,w)\in\widetilde{D}\}\subset\mathcal{H}.$$

Therefore $\widetilde{D}/\sim\ = D/\sim$. We note that the subset $(\widetilde{D}/\sim)\cap\mathcal{F}$ in $(\mathbb{C})^*$ is equal to $\widetilde{D}\cap\mathcal{F}$, but it is *not* necessarily the same as $D\cap\mathcal{F}$.

We give an example of the action of the equivalence relation which will illustrate the difference between the Lie group $\mathcal{H}^*$ and the Hopf surface $\mathcal{H}$. Let $D=\mathbb{C}_z\times\{w\}$ where $w\neq 0$. As a subset of $\mathcal{H}^*$, the complex curve $D\cap(\mathbb{C}^*\times\mathbb{C}^*)/\sim$ is closed and is equivalent to $\mathbb{C}^*$. However, as a complex curve in $\mathcal{H}$, $D/\sim$ is not closed and is equivalent to $\mathbb{C}$. Moreover, if $|b|^{k-1}<|w|<|b|^k$, then $(0,w)\in\mathcal{F}_k$ and

$$D/\sim\ =D_0\cup D_1\cup D_2\cup\cdots$$

where

$$D_0=\{|z|<|a|^k\}\times\{w\},\quad D_n=\{|a|^{k-1}\leq|z|\leq|a|^k\}\times\{w/b^n\},\quad n=1,2,\ldots$$

Thus D_0 is a disk and $D_n,\ n=1,2,\ldots$ are annuli such that $D_{n+1}=D_n\cdot(1,1/b),\ n=1,2,\ldots$. Hence the $D_n,\ n=1,2,3,\ldots$ are conformally equivalent and, as $n\to\infty$, they wind around and converge to $\boldsymbol{T}_a$ in $\mathcal{H}$.

Following T. Ueda, we consider the following real-valued function $U[z,w]$ on $\mathcal{H}^*$:

$$U[z,w]=\frac{\log|z|}{\log|a|}-\frac{\log|w|}{\log|b|}\quad \text{for } [z,w]\in\mathcal{H}^*. \tag{2.5}$$

This has the following properties:

(1) $U[z,w]$ is a pluriharmonic function on $\mathcal{H}^*$ satisfying

$$\lim_{[z,w]\to\boldsymbol{T}_a}U[z,w]=+\infty\quad\text{and}\quad \lim_{[z,w]\to\boldsymbol{T}_b}U[z,w]=-\infty,$$

thus for any interval $I \Subset (-\infty, \infty)$, the subdomain $U^{-1}(I)$ of $\mathcal{H}^*$ is relatively compact in $\mathcal{H}^*$.

(2) $|U[z, w]| := \mathrm{Max}\,\{U[z, w], -U[z, w]\}$ is a plurisubharmonic exhaustion function for $\mathcal{H}^*$ which is pluriharmonic everywhere except on the Levi-flat set

$$\frac{\log|z|}{\log|a|} = \frac{\log|w|}{\log|b|}, \quad \text{i.e.,} \quad |w| = |z|^\rho \text{ in } \mathcal{H}^*.$$

(3) For $c \in (-\infty, +\infty)$, the level set

$$\mathcal{S}_c : \ U[z, w] = c$$

is equal to $|w| = k|z|^\rho$ where $k = e^{-c \log|b|} > 0$. Thus $\{k_2|z|^\rho \le |w| \le k_1|z|^\rho\}$ is equal to $U^{-1}([c_1, c_2])$ where $k_i = e^{-c_i \log|b|}$; while $\{|w| \le k|z|^\rho\}$ is equal to $U^{-1}([c, +\infty)) \cup \boldsymbol{T}_a$; and $\{|w| \ge k|z|^\rho\}$ is equal to $U^{-1}((-\infty, c]) \cup \boldsymbol{T}_b$ where $k = e^{-c \log|b|}$.

From (2) and (3), it is immediately clear that each of the domains D in (a-1), (a-2′) and (a-2″) in the statement of Theorem 1.1 contains a compact, Levi-flat hypersurface $\mathcal{S}_c$ for appropriate c; hence each such D is not Stein.

3. Preliminary results.

In this section, we discuss two basic results which we will need. The first concerns holomorphic vector fields in $\mathcal{H} = \mathcal{H}_{(a,b)}$, while the second concerns general pseudoconvex domains with C^ω-smooth boundary in $\mathbb{C}^2$.

We consider the linear space $\mathfrak{X}$ of all holomorphic vector fields X of the form

$$X = \alpha z \frac{\partial}{\partial z} + \beta w \frac{\partial}{\partial w}, \quad \alpha, \beta \in \mathbb{C}$$

in $\mathbb{C}^2$. Any such X clearly induces a holomorphic vector field on $\mathcal{H}$. The integral curve C of X with initial value $(z_0, w_0) \in \mathbb{C}^* \times \mathbb{C}^*$ is

$$(z_0, w_0) \exp tX = \begin{cases} z = z_0 e^{\alpha t}, \\ w = w_0 e^{\beta t}, \end{cases} \qquad t \in \mathbb{C}.$$

Therefore, if, for example, $\alpha \ne 0$, we can write

$$C : w = c_0 z^{\beta/\alpha} \quad \text{where } c_0 = w_0 / z_0^{\beta/\alpha}.$$

Regarding X as a holomorphic vector field on $\mathcal{H}$, the integral curve $[z_0, w_0] \exp tX$ of X in $\mathcal{H}$ with initial value $[z_0, w_0]$ is equal to $\{w = c_0\, z^{\beta/\alpha}\}/\sim$ in $\mathcal{H}^*$. We will often simply write $\exp tX := [1, 1] \exp tX$ in $\mathcal{H}$.

In particular, we recall the vector fields

$$X_u := (\log|a|) z \frac{\partial}{\partial z} + (\log|b|) w \frac{\partial}{\partial w}$$

from the introduction. The integral curve of X_u with initial value $(1,1)$ is

$$\exp tX_u = \begin{cases} z = e^{(\log|a|)t}, \\ w = e^{(\log|b|)t}, \end{cases} \qquad t \in \mathbb{C}.$$

Thus $w = z^\rho$ with $1^\rho = 1$. We set $\widetilde{\sigma}_u := \{\exp tX_u : t \in \mathbb{C}\}/\sim\ \subset \mathcal{H}^*$ and denote by $\widetilde{\Sigma}_u$ the closure of $\widetilde{\sigma}_u$ in $\mathcal{H}$. For future use, we define the linear subspace $\mathfrak{X}_u = \{cX_u : c \in \mathbb{C}\}$ of $\mathfrak{X}$.

The next lemma gives more precise information about the integral curves and will be crucial in the proof of the key Lemma 4.2.

Lemma 3.1. 1. *For $X_u = (\log|a|)z(\partial/\partial z) + (\log|b|)w(\partial/\partial w)$ we have*:

(1) *In case $(a,b) \in S_1$, $\widetilde{\sigma}_u$ is a non-compact curve in $\mathcal{H}$ and $\widetilde{\Sigma}_u = \{|w|^{\log|a|} = |z|^{\log|b|}\}/\sim$ is a real three-dimensional Levi-flat closed hypersurface in $\mathcal{H}$ with $\widetilde{\Sigma}_u \Subset \mathcal{H}^*$.*

(2) *In case $(a,b) \in S_2$, $\widetilde{\sigma}_u$ is a compact curve in $\mathcal{H}^*$ such that*
 i) $\widetilde{\sigma}_u = [z_0, w_0]\sigma_u$ *if and only if* $w_0^{pl} = z_0^{ql}$;
 ii) $\widetilde{\sigma}_u$, *as a Riemann surface, is equivalent to the torus $\mathbb{T}_{(a,b)}$ from Remark* 1.1.

2. *For $X = \alpha z(\partial/\partial z) + \beta w(\partial/\partial w) \notin \{cX_u : c \in \mathbb{C}\}$, the integral curve $\sigma := \{\exp tX : t \in \mathbb{C}\}/\sim$ in $\mathcal{H}^*$ is not relatively compact in $\mathcal{H}^*$. If we let Σ denote the closure of σ in $\mathcal{H}$, then*:

(1) *If $\alpha, \beta \neq 0$, we have $\Sigma \supset \boldsymbol{T}_a \cup \boldsymbol{T}_b$.*

(2) *If only one of α or β is not 0, e.g., $\alpha \neq 0$ and $\beta = 0$, we have $\Sigma \supset \boldsymbol{T}_a$ and $\Sigma \cap \boldsymbol{T}_b = \emptyset$.*

Remark 3.1. The decompositions of the Hopf surface $\mathcal{H} := \mathcal{H}_{(a,b)}$ in the two cases $(a,b) \in S_1$ or $(a,b) \in S_2$ given as (1.5) and (1.6) in Proposition 1.1 will essentially follow from Lemma 3.1. The precise proofs of Lemma 3.1 and Proposition 1.1 are in Appendix A.

We now turn to an elementary property of a pseudoconvex domain D with C^ω-smooth boundary in $\mathbb{C}^2$. In $\mathbb{C}^2 = \mathbb{C}_z \times \mathbb{C}_w$ we consider disks

$$\Delta_1 = \{|z| < r_1\}, \quad \Delta_2 = \{|w| < r_2\}$$

and the bidisk $\Delta = \Delta_1 \times \Delta_2$. Let D be a pseudoconvex domain with C^ω boundary in Δ. We do not assume D is relatively compact. Thus there exists a C^ω-smooth, real-valued function $\psi(z,w)$ on $\overline{\Delta}$ such that

$$D = \{(z,w) \in \Delta : \psi(z,w) < 0\};$$
$$\partial D \cap \Delta = \{(z,w) \in \Delta : \psi(z,w) = 0\},$$

and on $\psi(z,w) = 0$ we have both $\nabla_{(z,w)}\psi(z,w) \neq 0$ and the Levi form $\mathcal{L}\psi(z,w) \geq 0$. We write out this last condition: for

$$\mathcal{L}\psi(z,w) = \frac{\partial^2\psi}{\partial z\partial\overline{z}}\left|\frac{\partial\psi}{\partial w}\right|^2 - 2\Re\left\{\frac{\partial^2\psi}{\partial z\partial\overline{w}}\frac{\partial\psi}{\partial\overline{z}}\frac{\partial\psi}{\partial w}\right\} + \frac{\partial^2\psi}{\partial w\partial\overline{w}}\left|\frac{\partial\psi}{\partial z}\right|^2,$$

$$\text{we have} \quad \mathcal{L}\psi(z,w) \geq 0 \text{ on } \psi(z,w) = 0. \tag{3.1}$$

We may assume

$$\psi(0,0) = 0 \quad \text{and} \quad \frac{\partial\psi}{\partial w}(0,0) \neq 0$$

so that $\{w : \psi(0,w) = 0\}$ is a C^ω-smooth simple arc in Δ_2 passing through $w = 0$.

We set $\mathcal{S} := \partial D \cap \Delta$,

$$D(z) := \{w \in \Delta_2 : (z,w) \in D\} \subset \Delta_2; \text{ and}$$
$$S(z) := \{w \in \Delta_2 : (z,w) \in \mathcal{S}\} \subset \Delta_2,$$

so that $D = \bigcup_{z\in\Delta_1}(z, D(z)) \subset \Delta$ and $\mathcal{S} = \bigcup_{z\in\Delta_1}(z, S(z)) \subset \Delta$. Taking $r_1, r_2 > 0$ sufficiently small we can insure that

(i) for each $z \in \Delta_1$, $D(z)$ is a non-empty domain in Δ_2 and $S(z)$ is a C^ω-smooth open arc in Δ_2 connecting two points $a(z)$ and $b(z)$ on $\partial\Delta_2$;

(ii) $0 \in S(0)$.

We also need to assume the following condition for Lemma 3.2:

(iii) $\psi(z,0) \not\equiv 0$ in Δ_1, hence, for any disk $\delta_1 = \{|z| < r\} \subset \Delta_1$, there exists $z_0 \in \delta_1$ with $0 \notin S(z_0)$.

Under these three conditions we have the following.

Lemma 3.2. *For any disk $\delta_1 = \{|z| < r\} \subset \Delta_1$, there exists a disk $\delta_2 = \{|w| < r'\} \subset \Delta_2$ with*

$$\bigcup_{z\in\delta_1} S(z) \supset D(0) \cap \delta_2.$$

The proof of Lemma 3.2 is in Appendix B. This result will be used in proving Lemma 4.1.

4. Construction of the plurisubharmonic exhaustion function $-\lambda[z,w]$ on D.

Let $(\alpha,\beta) \in \mathbb{C}^* \times \mathbb{C}^*$. If we define

$$(\alpha,\beta) : [z,w] \in \mathcal{H} \mapsto [\alpha z, \beta w] \in \mathcal{H},$$

then (α,β) is an automorphism of $\mathcal{H}$. Thus $\mathbb{C}^* \times \mathbb{C}^*$ acts as a commutative group of automorphisms of $\mathcal{H}$ with identity element $e = (1,1)$. Although $\mathbb{C}^* \times \mathbb{C}^*$ is not transitive on $\mathcal{H}$, it is transitive on $\mathcal{H}^*$. Hence $\mathcal{H}^*$ *is a complex homogeneous space with*

Lie transformation group $\mathbb{C}^* \times \mathbb{C}^*$ *which acts transitively.* This is the setting of Chapter 6 of [**1**]. For any $[z,w] \in \mathcal{H}^*$ the isotropy subgroup $I_{[z,w]}$ of $\mathbb{C}^* \times \mathbb{C}^*$ is

$$\begin{aligned} I_{[z,w]} &:= \{(\alpha,\beta) \in \mathbb{C}^* \times \mathbb{C}^* : (\alpha,\beta)[z,w] = [z,w]\} \\ &= \{(a^n, b^n) \in \mathbb{C}^* \times \mathbb{C}^* : n \in \mathbb{Z}\} \\ &= \mathcal{I} \ \text{ in (2.3)}, \end{aligned}$$

and thus is independent of $[z,w] \in \mathcal{H}^*$. We have

$$\mathcal{H}^* = (\mathbb{C}^* \times \mathbb{C}^*)/\mathcal{I}.$$

In what follows we will generally consider the restriction to $\mathbb{C}^* \times \mathbb{C}^*$ of the Euclidean metric $ds^2 = |dz|^2 + |dw|^2$ on $\mathbb{C}^2$, and we fix a positive real-valued function $c(z,w)$ of class C^ω on $\mathbb{C}^2$. This allows us to define *c-harmonic functions* and thus a *c-Green function and c-Robin constant* associated to a smoothly bounded domain $\Omega \Subset \mathbb{C}^* \times \mathbb{C}^*$ and a point $p_0 \in \Omega$ (if $\Omega \not\Subset \mathbb{C}^* \times \mathbb{C}^*$ we define these by exhaustion); cf., Chapter 1 of [**1**]. Varying the point p_0 yields the *c-Robin function* for Ω. However, we remark that any Kähler metric dS^2 and positive function $C(z,w)$ of class C^ω on $\mathbb{C}^* \times \mathbb{C}^*$ gives rise to a C-Green function and hence a C-Robin function on Ω; this flexibility will be used in the 4^{th} case of the proof of Lemma 4.3. For simplicity, we will always take $c(z,w)$ (or $C(z,w)$) to be a positive constant.

In this section we always assume that $D \subset \mathcal{H}$ is a pseudoconvex domain with C^ω-smooth boundary in $\mathcal{H}$. Our first goal is to construct a plurisubharmonic exhaustion function for D. We note that

$$\text{if } D \supset \boldsymbol{T}_a \text{ or } D \supset \boldsymbol{T}_b, \text{ then } D \text{ is not Stein.}$$

We define

$$D^* := D \cap \{zw \neq 0\} \subset \mathcal{H}^*$$

(see (2.1)). The distinction between $D \subset \mathcal{H}$ and $D^* \subset \mathcal{H}^*$ will be very important. Since $(\alpha,\beta) \in \mathbb{C}^* \times \mathbb{C}^*$ defines an automorphism of $\mathcal{H}$, for $[z,w] \in \mathcal{H}$ we can define

$$D[z,w] = \{(\alpha,\beta) \in \mathbb{C}^* \times \mathbb{C}^* : (\alpha,\beta)[z,w] \in D\} \subset \mathbb{C}^* \times \mathbb{C}^*.$$

Equivalently, using the notation $D \cap \boldsymbol{T}_a = D_a \times \{0\}$, $D \cap \boldsymbol{T}_b = \{0\} \times D_b$, $\widetilde{D_a} = \{a^n z : z \in D_a,\ n \in \mathbb{Z}\} \subset \mathbb{C}^*_z$ and $\widetilde{D_b} = \{b^n w : w \in D_b,\ n \in \mathbb{Z}\} \subset \mathbb{C}^*_w$, we have

$$\begin{aligned} D[z,w] &= \left(\left(\frac{1}{z}, \frac{1}{w}\right) \cdot D^*\right) \cdot \mathcal{I} = \left(\frac{1}{z}, \frac{1}{w}\right) \cdot \widetilde{D^*} \quad \text{if } [z,w] \in \mathcal{H}^*; \\ D[z,0] &= \left(\frac{1}{z} D_a, \mathbb{C}^*\right) \cdot \mathcal{I} = \left(\frac{1}{z} \widetilde{D_a}\right) \times \mathbb{C}^*_w \qquad \text{if } [z,0] \in \boldsymbol{T}_a; \end{aligned}$$

$$D[0,w] = \left(\mathbb{C}^*, \frac{1}{w}D_b\right) \cdot \mathcal{I} = \mathbb{C}^*_z \times \left(\frac{1}{w}\widetilde{D_b}\right) \qquad \text{if } [0,w] \in \boldsymbol{T}_b.$$

We note the following:

(1) If $e \in D$ then $D[e] = \widetilde{D} \setminus \{zw = 0\} = \widetilde{D^*}$; and given $[z,w] \in \mathcal{H}$, $[z,w] \in D$ if and only if $e \in D[z,w]$ (recall the definition of $\widetilde{D}$ (and hence $\widetilde{D^*}$) in (2.4)).

(2) For each $[z,w] \in D$, $D[z,w]$ is an open set with C^ω boundary $\partial D[z,w]$ but it is not relatively compact in $\mathbb{C}^* \times \mathbb{C}^*$. We have

(i) $D[z,w] = D[z,w] \cdot \mathcal{I}$;

(ii) For $[z,w] \in D^*$ we define

$$D^*[z,w] = \{(\alpha,\beta) \in \mathbb{C}^* \times \mathbb{C}^* : (\alpha,\beta)[z,w] \in D^*\}.$$

Then $D[z,w] = D^*[z,w]$.

(3) (i) For $[z,w] \in D^*$ we have

$$D[z,w] = \widetilde{D^*} \cdot \left(\frac{1}{z}, \frac{1}{w}\right), \tag{4.1}$$

and for $[z,w], [z',w'] \in D^*$

$$D[z',w'] = \left(\frac{z}{z'}, \frac{w}{w'}\right) D[z,w]. \tag{4.2}$$

In particular, the sets $D[z,w]$ for $[z,w] \in D^*$ are biholomorphic in $\mathbb{C}^* \times \mathbb{C}^*$.

(ii) For any two points $[z,0], [z',0] \in D \cap \boldsymbol{T}_a$

$$D[z',0] = \left(\frac{z}{z'}, 1\right) D[z,0].$$

In particular, the sets $D[z,0]$ for $[z,0] \in D \cap \boldsymbol{T}_a$ are biholomorphic in $\mathbb{C}^* \times \mathbb{C}^*$.

(4) Fix $[z_0,0] \in D \cap \boldsymbol{T}_a$ and let $[z_n,w_n] \in D^*$ $(n = 1,2,\ldots)$ with $[z_n,w_n] \to [z_0,0]$ as $n \to \infty$ in $\mathcal{H}$. For $0 < r < R$, consider the product of annuli

$$\mathcal{A}(r,R) : \{r < |z| < R\} \times \{r < |w| < R\} \subset \mathbb{C}^* \times \mathbb{C}^*.$$

Then

$$\lim_{n\to\infty} \partial D[z_n,w_n] \cap \mathcal{A}(r,R) = \partial D[z_0,0] \cap \mathcal{A}(r,R) \tag{4.3}$$

in the Hausdorff metric as compact sets in $\mathbb{C}^* \times \mathbb{C}^*$.

We set

$$\mathcal{D} := \bigcup_{[z,w]\in D} ([z,w], D[z,w]). \tag{4.4}$$

This is a pseudoconvex domain in $D\times(\mathbb{C}^*\times\mathbb{C}^*)$ which we consider as a function-theoretic "parallel" variation

$$\mathcal{D}:[z,w]\in D\to D[z,w]\subset\mathbb{C}^*\times\mathbb{C}^*.$$

Since $e\in D[z,w]$ for $[z,w]\in D$, we have the c-Green function $g([z,w],(\xi,\eta))$ with pole at e and the c-Robin constant $\lambda[z,w]$ for $(D[z,w],e)$ with respect to the metric ds^2 on $\mathbb{C}^*\times\mathbb{C}^*$ and the function $c(z,w)>0$. We call $[z,w]\to\lambda[z,w]$ the *c-Robin function for D*.

The function $-\lambda[z,w]$ is a candidate to be a plurisubharmonic exhaustion function for D. To be precise, we have the following fundamental result.

LEMMA 4.1. 1. *$-\lambda[z,w]$ is a plurisubharmonic function on D.*
2. *We have the following*:
   (a) *For any $[z_0,w_0]\in\partial D^*$, $\lim_{[z,w]\to[z_0,w_0]}\lambda[z,w]=-\infty$.*
   (b) *If $\emptyset\neq\partial D\cap\boldsymbol{T}_a\neq\boldsymbol{T}_a$ then for any $[z_0,0]\in\partial D\cap\boldsymbol{T}_a$ we have $\lim_{[z,w]\to[z_0,0]}\lambda[z,w]=-\infty$ (and similarly if $\boldsymbol{T}_a$ is replaced by $\boldsymbol{T}_b$).*
3. *If $\partial D\not\supset\boldsymbol{T}_a$ and $\partial D\not\supset\boldsymbol{T}_b$, then $-\lambda[z,w]$ is a plurisubharmonic exhaustion function for D.*

PROOF. Note that 3. follows from 1. and 2. We divide the proof of 1. into two steps.

1^{st} step: *$-\lambda[z,w]$ is plurisubharmonic on D^*.*

Fix $[\zeta_0]=[z_0,w_0]\in D^*$. Let $\boldsymbol{a}\in\mathbb{C}^2\setminus\{0\}$ with $\|\boldsymbol{a}\|=1$ and let $B=\{|t|<r\}\subset\mathbb{C}_t$ be a small disk and let $(z(t),w(t))=\zeta_0+\boldsymbol{a}t$ be such that the complex line $l:t\in B\to[\zeta(t)]=[z(t),w(t)]=[\zeta_0]+\boldsymbol{a}t$ passing through $[\zeta_0]$ is contained in D^*. It suffices to prove that $-\lambda(t):=-\lambda[z(t),w(t)]$ is subharmonic on B, i.e.,

$$\frac{\partial^2\lambda(t)}{\partial t\partial\bar{t}}\leq 0\quad\text{on }B.$$

For brevity we write

$$\begin{aligned}D(t)&:=D[\zeta(t)]\subset\mathbb{C}^*\times\mathbb{C}^*\quad&&\text{for }t\in B;\\ g(t,(z,w))&:=g([\zeta(t)],(z,w))&&\text{for }(z,w)\in D[\zeta(t)].\end{aligned}$$

By (4.2) we have

$$D(t)=D[\zeta_0]\cdot\left(\frac{z_0}{z(t)},\frac{w_0}{w(t)}\right)\quad\text{in }\mathbb{C}^*\times\mathbb{C}^*.\tag{4.5}$$

We thus have the parallel variation of domains $D(t)$ in $\mathbb{C}^*\times\mathbb{C}^*$ with parameter $t\in B$:

$$\mathcal{D}|_B:t\in B\to D(t)\subset\mathbb{C}^*\times\mathbb{C}^*.$$

We write

$$\mathcal{D}|_B := \bigcup_{t\in B} (t, D(t)); \quad \partial\mathcal{D}|_B = \bigcup_{t\in B} (t, \partial D(t)) \quad \text{in } B \times (\mathbb{C}^* \times \mathbb{C}^*),$$

where again we identify the variation with the total space $\mathcal{D}|_B$. By (4.4), $\mathcal{D}|_B$ is a pseudoconvex domain in $B \times (\mathbb{C}^* \times \mathbb{C}^*)$ (and hence a Stein domain) such that $\partial\mathcal{D}|_B$ is C^ω smooth. Using the notation $\zeta = (z, w) \in \mathbb{C}^* \times \mathbb{C}^*$ and $g(t,\zeta) = g(t,(z,w))$, we have the following variation formula from Theorem 3.1 of [**1**]:

$$(\star) \qquad \begin{aligned} \frac{\partial^2 \lambda(t)}{\partial t \partial \bar{t}} = &- c_2 \int_{\partial D(t)} K_2(t,\zeta) \|\nabla_\zeta\ g(t,\zeta)\|^2 dS_\zeta \\ &- 4\, c_2 \iint_{D(t)} \left(\left| \frac{\partial^2 g(t,\zeta)}{\partial \bar{t} \partial z} \right|^2 + \left| \frac{\partial^2 g(t,\zeta)}{\partial \bar{t} \partial w} \right|^2 \right) dV_\zeta \\ &- 2\, c_2 \iint_{D(t)} c(\zeta) \left| \frac{\partial g(t,\zeta)}{\partial t} \right|^2 dV_\zeta. \end{aligned}$$

Here $1/c_2$ is the surface area of the unit sphere in $\mathbb{C}^2$, dV_ζ is the Euclidean volume element in $\mathbb{C}^2$;

$$K_2(t,\zeta) = \mathcal{L}(t,\zeta)/\|\nabla_\zeta \psi(t,\zeta)\|^3$$

where $\mathcal{L}(t,\zeta)$ is the "diagonal" Levi form defined by

$$\mathcal{L}(t,\zeta) = \frac{\partial^2 \psi}{\partial t \partial \bar{t}} \|\nabla_\zeta \psi\|^2 - 2\Re\left\{ \frac{\partial \psi}{\partial t} \left(\frac{\partial \psi}{\partial \bar{z}} \frac{\partial^2 \psi}{\partial \bar{t} \partial z} + \frac{\partial \psi}{\partial \bar{w}} \frac{\partial^2 \psi}{\partial \bar{t} \partial w} \right) \right\} + \left\| \frac{\partial^2 \psi}{\partial t} \right\|^2 \Delta_\zeta \psi;$$

and $\psi(t,\zeta)$ is a defining function of $\mathcal{D}|_B$. The quantity $K_2(t,\zeta)$ is independent of the defining function $\psi(t,\zeta)$ (cf., Chapter 3 of [**1**]). Since $\mathcal{D}|_B$ is pseudoconvex in $B \times (\mathbb{C}^* \times \mathbb{C}^*)$, following Theorem 3.2 of [**1**] we have $K_2(t,\zeta) \geq 0$ on $\partial\mathcal{D}|_B$ and hence $\partial^2\lambda(t)/\partial t\partial\bar{t} \leq 0$ on B, proving the first step.

Since $c(z,w) > 0$ in $\mathbb{C}^* \times \mathbb{C}^*$, the variation formula immediately implies the following rigidity result which will be useful later (cf., Lemma 4.1 of [**1**]).

REMARK 4.1. If $(\partial^2\lambda/\partial t\partial\bar{t})(0) = 0$, then $(\partial g/\partial t)(0,(z,w)) \equiv 0$ on $D(0)$, i.e.,

$$\left. \frac{\partial g([\zeta_0] + \boldsymbol{a}t, (z,w))}{\partial t} \right|_{t=0} \equiv 0 \text{ on } D[\zeta_0].$$

2nd step: *Plurisubharmonic extension of $-\lambda[z,w]$ to D.*

We fix a point of $D \cap [(T_a \times \{0\}) \cup (\{0\} \times T_b)]$, e.g., $[z_0, 0]$ with $z_0 \neq 0$. Let $[z_n, w_n] \in D^*$ $(n = 1, 2, \dots)$ with $[z_n, w_n] \to [z_0, 0]$ as $n \to \infty$. By (4.3)

$$\lim_{n\to\infty}(g([z_n,w_n],(\alpha,\beta))-g([z_0,0],(\alpha,\beta)))=0$$
$$\text{uniformly for } (\alpha,\beta) \text{ in } K\Subset D[z_0,0]\subset\mathbb{C}^*\times\mathbb{C}^*.$$

It follows that $\lim_{n\to\infty}\lambda[z_n,w_n]=\lambda[z_0,0]$, i.e., $\lambda[z,w]$ is continuous and finite at $[z_0,0]$. Hence $\lambda[z,w]$ is continuous and finite-valued on D. Since $D\cap \boldsymbol{T}_a$ is a complex line, it follows from the first step that $-\lambda[z,w]$ extends to be plurisubharmonic from $D^*\cap\boldsymbol{T}_a$ to $D\cap\boldsymbol{T}_a$. Hence $-\lambda[z,w]$ extends to be plurisubharmonic on D. □

We divide the proof of 2. in two steps; the first step is 2 (a).

1st step: *Fix $[z',w']\in\partial D^*$. If $[z,w]\in D\to[z',w']$ in $\mathcal{H}$, then $\lambda[z,w]\to-\infty$.*

Since $[z',w']\in\partial D^*$, we have $z'\neq 0$ and $w'\neq 0$. If $[z,w]\in D^*$ tends to $[z',w']$ in $\mathcal{H}$, then $\partial D[z,w]\subset\mathbb{C}^*\times\mathbb{C}^*$ tends to the single point e in the sense that if we define $d[z,w]=\operatorname{dist}(\partial D[z,w],e)>0$, where

$$\operatorname{dist}(\partial D[z,w],e):=\operatorname{Min}\big\{\sqrt{|\xi-1|^2+|\eta-1|^2}:(\xi,\eta)\in\partial D[z,w]\big\},$$

then $d[z,w]\to 0$ as $[z,w]\to[z',w']$. Indeed, let $[z,w]\in D$ approach $[z',w']$ in $\mathcal{H}$. By slightly deforming the fundamental domain $\mathcal{F}\subset\mathbb{C}^*\times\mathbb{C}^*$ if necessary, we may assume $(z',w'),(z,w)\in\mathcal{F}$. Since

$$\partial D[z,w]=\left\{\left(\frac{\alpha}{z},\frac{\beta}{w}\right)\in\mathbb{C}^*\times\mathbb{C}^*:[\alpha,\beta]\in\partial D\right\}$$

and $[z',w']\in\partial D^*$,

$$d[z,w]=\operatorname{dist}(\partial D[z,w],e)\le\sqrt{|z'/z-1|^2+|w'/w-1|^2}$$

which clearly tends to 0 as $[z,w]\to[z',w']$. Since $\partial D[z,w]$ is a *smooth* real three-dimensional hypersurface, it follows by standard potential-theoretic arguments that $-\lambda[z,w]\to+\infty$. □

It remains to prove 2 (b). Thus we assume $\emptyset\neq\partial D\cap\boldsymbol{T}_a\neq\boldsymbol{T}_a$.

2nd step: *Fix $[z_0,0]\in\partial D\cap\boldsymbol{T}_a$. If $[z,w]\in D\to[z_0,0]$ in $\mathcal{H}$, then $\lambda[z,w]\to-\infty$.*

For the proof of this step we require Lemma 3.2. Fix $p_0=[z_0,0]\in\partial D\cap\boldsymbol{T}_a$. We want to show

$$\lim_{[z,w]\to[z_0,0],\ [z,w]\in D}\lambda[z,w]=-\infty.$$

We take a sequence $\{[z_n,w_n]\}_n\subset D$ which converges to p_0 in $\mathcal{H}$. We show

$$\lim_{n\to\infty}\lambda[z_n,w_n]=-\infty. \tag{4.6}$$

From continuity of $\lambda[z,w]$ in D, it suffices to prove (4.6) for $[z_n,w_n]\in D^*$. Moreover, since $\partial D[z_n, w_n]$ is smooth, as in the end of the first step, we need only show

$$\lim_{n\to\infty} \mathrm{dist}(\partial D[z_n, w_n], e) = 0. \tag{4.7}$$

This is the key technical step and it is here where we will use Lemma 3.2 and the *pseudoconvexity* of the domain D in $\mathcal{H}$.

We may assume that $p_0 = [z_0, 0] \in \partial D$ lies in the fundamental domain $\mathcal{F}$ and we take a sufficiently small bidisk $\Delta = \Delta_1 \times \Delta_2$ with center $(z_0, 0)$ so that $\Delta \subset \mathcal{F}$. Let $\psi(z, w)$ be a defining function of D in Δ, i.e., $\psi(z, w) \in C^\omega(\Delta)$ with $D \cap \Delta = \{\psi(z, w) < 0\}$ and $\partial D \cap \Delta = \{\psi(z, w) = 0\}$. Since ∂D is smooth in $\mathcal{H}$, we have two cases:

$$\text{Case (c1)}: \ \frac{\partial \psi}{\partial z} \neq 0 \text{ on } \Delta; \qquad \text{Case (c2)}: \ \frac{\partial \psi}{\partial w} \neq 0 \text{ on } \Delta.$$

Apriori, we also have two cases relating to the behavior of $\psi(z, 0)$ on Δ_1:

$$\text{Case (d1)}: \ \psi(z, 0) \not\equiv 0 \text{ on } \Delta_1; \qquad \text{Case (d2)}: \ \psi(z, 0) \equiv 0 \text{ on } \Delta_1.$$

However, the hypothesis $\partial D \not\supset \boldsymbol{T}_a$ in 2 (b) together with the real-analyticity of ∂D imply that Case (d2) does not occur. Thus it suffices to prove (4.7) assuming that $\psi(z, 0) \not\equiv 0$ on Δ_1.

Proof of (4.7) in Case (c1). In this case, by taking a suitably smaller bidisk Δ if necessary, $l(0) := \{\psi(z, 0) = 0\}$ is a C^ω-smooth arc in Δ_1 passing through $z = z_0$ and $l(0) \times \{0\} \subset \partial D \cap \Delta$. For $w \in \Delta_2$,

$$l(w) := \{z \in \Delta_1 : (z, w) \in \partial D \cap \Delta\}$$

is a simple C^ω-smooth arc in Δ_1.

Fix $\varepsilon > 0$. Since $z_0 \neq 0$, we can find a disk $\delta_1 \subset \Delta_1$ with center z_0 such that

$$\left|\frac{z'}{z''} - 1\right| < \varepsilon \quad \text{for all } z', z'' \in \delta_1.$$

Now we take $\delta_2 : |w| < r < \varepsilon$ in Δ_2 so that each arc $l(w)$ passes through a certain point $\zeta(w)$ in δ_1. For sufficiently large n_0, if $n \geq n_0$ we have $(z_n, w_n) \in \delta_1 \times \delta_2$. Since $w_n \in \delta_2$, we have $\zeta(w_n) \in l(w_n) \cap \delta_1$ so that $(\zeta(w_n), w_n) \in \partial D$ in $\mathcal{H}$. Hence, $(\zeta(w_n)/z_n, w_n/w_n) = (\zeta(w_n)/z_n, 1) \in \partial D[z_n, w_n]$ in $\mathbb{C}^* \times \mathbb{C}^*$. Thus

$$\mathrm{dist}(\partial D[z_n, w_n], e) \leq \mathrm{dist}\left(\left(\frac{\zeta(w_n)}{z_n}, 1\right), e\right) = \left|\frac{\zeta(w_n)}{z_n} - 1\right| < \varepsilon \quad \text{for } n \geq n_0.$$

Proof of (4.7) in Case (c2). In this case, by taking a suitably smaller bidisk Δ if necessary, $S(z_0) := \{\psi(z_0, w) = 0\}$ is a C^ω-smooth arc in Δ_2 passing through $w = 0$ and $\{z_0\} \times S(z_0) \subset \partial D \cap \Delta$. For $z \in \Delta_1$,

$$S(z) := \{w \in \Delta_2 : (z, w) \in \partial D \cap \Delta\},$$

is a simple C^ω-smooth arc in Δ_2.

Fix $\delta_1 := \{|z - z_0| < r_1\} \Subset \Delta_1$. Case (d1) corresponds to the condition (iii) in Lemma 3.2, thus this lemma implies that there exists a disk $\delta_2 := \{|w| < r_2\}$ such that

$$\bigcup_{z \in \delta_1} S(z) \supset D(z_0) \cap \delta_2. \tag{4.8}$$

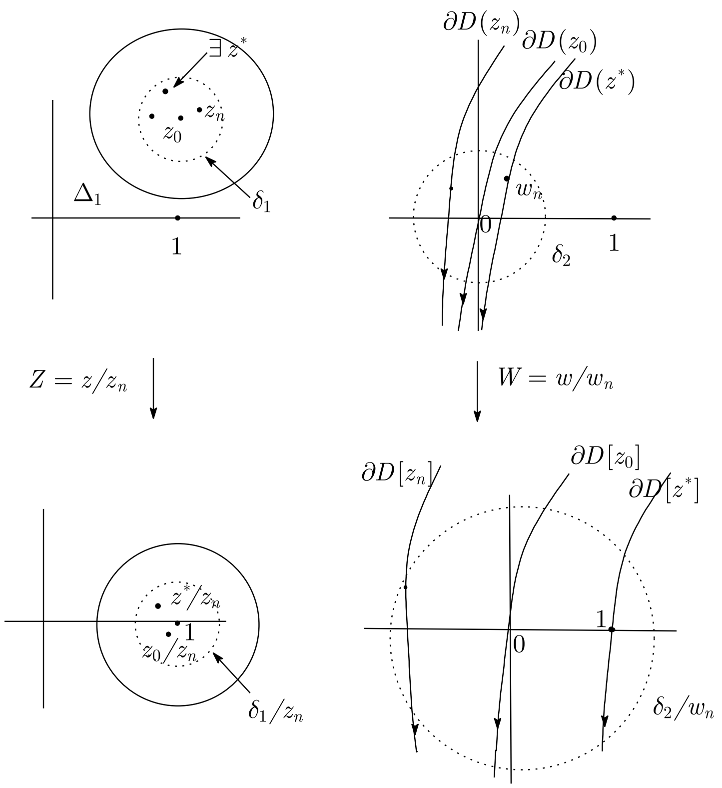
Fix $\varepsilon > 0$. Taking r_1 sufficiently small, we can insure that

$$|z'/z'' - 1| < \varepsilon \quad \text{for all} \ \ z', \ z'' \in \delta_1.$$

Take a disk $\delta_2 \subset \Delta_2$ satisfying (4.8). For sufficiently large n_0, if $n \geq n_0$ we have $(z_n, w_n) \in \delta_1 \times \delta_2$. We divide the points $w_n \in \delta_2$ into two types:

$$\text{Case (i): } w_n \in \delta_2 \cap D(z_0); \quad \text{Case (ii): } w_n \in \delta_2 \setminus D(z_0).$$

In Case (i), using (4.8) we can find $z^* \in \delta_1$ with $w_n \in S(z^*)$ so that $(z^*, w_n) \in \partial D$ in $\mathcal{H}$ (see w_n, $z^*, \partial D(z^*)$ in the figure below).



Thus, $(z^*/z_n, w_n/w_n) = (z^*/z_n, 1)$ in $\partial D[z_n, w_n]$ in $\mathbb{C}^* \times \mathbb{C}^*$ and hence

$$\mathrm{dist}(\partial D[z_n, w_n], e) \le \mathrm{dist}((z^*/z_n, 1), e) = |z^*/z_n - 1| < \varepsilon \quad \text{for all } n \ge n_0.$$

In Case (ii), let $\ell = [z_n, z_0]$ be a segment in δ_1. We can find $z^* \in \ell$ with $w_n \in \partial D(z^*)$. Indeed, as z goes from z_n to z_0 along ℓ, the arcs $\partial D(z) \cap \delta_2$ transform from $\partial D(z_n) \cap \delta_2$ to $\partial D(z_0) \cap \delta_2$ in a continuous fashion. Since $[z_n, w_n] \in D^*$, we can find $z^* \in \ell$ with $w_n \in \partial D(z^*)$.

Thus $(z^*, w_n) \in \partial D^*$, so that $(z^*/z_n, 1) \in \partial D^*[z_n, w_n]$, and hence

$$\mathrm{dist}(\partial D[z_n, w_n], e) \le \mathrm{dist}((z^*/z_n, 1), e) = |z^*/z_n - 1| < \varepsilon \quad \text{for all } n \ge n_0,$$

which is (4.7). This completes the proof of 2 (b) in Lemma 4.1. □

Remark 4.2. We offer a non-pseudoconvex example to explain the subtlety of the lemma, in particular, in proving (4.7). We encourage the reader to draw a picture to illustrate the following situation. Let D be a domain in $\mathcal{H}$ with smooth boundary but which is not pseudoconvex. We assume that $[z_0, 0] \in \partial D \cap \boldsymbol{T}_a$ where $1 < |z_0| < |a|$. We can find a bidisk $\delta := \delta_1 \times \delta_2 = \{|z - z_0| < r_1\} \times \{|w| < r_2\}$ with r_1, r_2 sufficiently small so that $D_1 := D \cap \delta$ is of the form $D_1 = \bigcup_{z \in \delta_1}(z, D_1(z))$ where $D_1(z) \subset \delta_2$ and $\partial D_1(z)$ is a non-empty smooth arc in δ_2. *We assume that, for each* $z \in \delta_1$

$$D_1(z) \supset D_1(z_0) \supset \delta_2 \cap \{\Re w > 0\} =: \delta_2^*$$

and it then follows from Hartogs theorem that D *is not pseudoconvex at* $[z_0, 0] \in \partial D$. We can find a sequence $\{(z_n, u_n)\}_n$ in D_1 with $u_n = \Re w_n > 0$ which converges to the point $(z_0, 0) \in \partial D$. Fix $r_1' : 0 < r_1' < r_1/|z_0|$. By definition

$$D[z_n, u_n] = (1/z_n, 1/u_n)\widetilde{D^*} \supset (1/z_n, 1/u_n)\delta_1 \times \delta_2^*$$

and for sufficiently large n, say $n \ge n_0$,

$$E := \{(z, w) \in \mathbb{C}^* \times \mathbb{C}^* : |z - 1| < r_1',\ |w - 1| < 1/2\} \subset (1/z_n, 1/u_n)\delta_1 \times \delta_2^*.$$

If we let A denote the c-Robin constant for the domain E in $\mathbb{C}^* \times \mathbb{C}^*$ and the point $e = (1, 1)$, it follows that $\lambda[z_n, u_n] > A$ for $n \ge n_0$, so that $-\lambda[z, w]$ is not an exhaustion function for D.

We next relate the possible absence of strict plurisubharmonicity of the function $-\lambda[z, w]$ on a pseudoconvex domain D in $\mathcal{H}$ *at a point in* D^* with existence of holomorphic vector fields on $\mathcal{H}$ with certain properties. This is in the spirit of, but does not follow from, Lemma 5.2 of [**1**]. Recall that if $(a, b) \in S_2$ (Case b of Theorem 1.1) we defined σ_c in (1.6) to be the integral curve $[z_0, w_0] \exp tX_u$ with $c = w_0^{pl}/z_0^{ql} \ne 0, \infty$ of $X_u := (\log|a|)z(\partial/\partial z) + (\log|b|)w(\partial/\partial w)$.

Lemma 4.2. *Let* D *be a pseudoconvex domain with* C^ω*-smooth boundary in* $\mathcal{H}$ *and* *let $\lambda[z,w]$ be the c-Robin function on D. Assume that there exists a point $p_0=[z_0,w_0]$ in D^* at which $-\lambda[z,w]$ is not strictly plurisubharmonic.*

(1) *There exists a holomorphic vector field $X=\alpha z(\partial/\partial z)dz+\beta w(\partial/\partial w)dw\neq 0$ on $\mathcal{H}$ such that if $[z,w]\in D^*$ (resp. ∂D^*), then the integral curve $I[z,w]:=[z,w]\exp tX$ in $\mathcal{H}$ is contained in D^* (resp. ∂D^*). We say X is a tangential vector field on ∂D^*.*

(2) *The form of the vector field X in* (1) *and the domain D are determined as follows*:
   - (i) *If $\partial D\not\supset \boldsymbol{T}_a$ and $\partial D\not\supset \boldsymbol{T}_b$, then $X=cX_u$ for some $c\neq 0$ with X_u in* (1.2). *If $(a,b)\in S_1$, D is of type* (a-1), (a-2′) *or* (a-2″) *in Theorem* 1.1. *If $(a,b)\in S_2$, $D=\bigcup_{c\in\delta}\sigma_c$ where δ is a relatively compact domain in $\mathbb{P}^1=\mathbb{C}\cup\{\infty\}$ with smooth boundary. In all cases, we have $\partial D\cap(\boldsymbol{T}_a\cup\boldsymbol{T}_b)=\emptyset$.*
   - (ii) *If $\partial D\supset \boldsymbol{T}_a$ and $\partial D\not\supset \boldsymbol{T}_b$, then we have two cases*:
     - (ii-a) *$X=cX_u$ for some $c\neq 0$ and D is of Case* b: *$D=\bigcup_{c\in\delta}\sigma_c$ where δ is a domain in $\mathbb{P}^1$ with smooth boundary $\partial\delta$ which contains 0 but not ∞.*
     - (ii-b) *$X=cz(\partial/\partial z)$ for some $c\neq 0$. Then D is a domain of "Nemirovskii type": $b>1$ and $D=\mathbb{C}_z\times\{Au+Bv<0\}/\sim$, where $A,B\in\mathbb{R}$ with $(A,B)\neq(0,0)$ (here $w=u+iv$).*
   - (ii′) *If $\partial D\supset \boldsymbol{T}_b$ and $\partial D\not\supset \boldsymbol{T}_a$, we have the result analogous to* (ii).
   - (iii) *If $\partial D\supset \boldsymbol{T}_a\cup\boldsymbol{T}_b$, then $X=cX_u$ for some c and D is of Case* b: *$D=\bigcup_{c\in\delta}\sigma_c$ where δ is a domain in $\mathbb{P}^1$ with smooth boundary $\partial\delta$ with $0,\infty\in\partial\delta$.*

Remark 4.3. With respect to the Nemirovskii-type domain in (ii-b), we recall Nemiroviskii's theorem in [**6**]. Let $a>1$ and let $\mathcal{H}=\mathcal{H}_{(a,a)}$. Then the domain $D=\mathbb{C}_z\times\{\Re w>0\}/\sim\ \subset\mathcal{H}$ is Stein and ∂D is Levi-flat. At the end of Section 4 we will discuss an explicit example of such a domain which will illustrate some of the ideas used in the proof of Theorem 1.1.

Proof. Since $-\lambda[z,w]$ is plurisubharmonic on D and is not strictly plurisubharmonic at $p_0=[z_0,w_0]\in D^*$, we can find a holomorphic vector field $X=\alpha z(\partial/\partial z)dz+\beta w(\partial/\partial w)dw\neq 0$ on $\mathcal{H}$ such that

$$\left.\frac{\partial^2\lambda[p_0\exp tX]}{\partial t\partial\bar t}\right|_{t=0}=0. \tag{4.9}$$

We shall show that this X is a tangential vector field on ∂D^*. Since $p_0\in D^*$, we can take a small disk $B=\{|t|<r\}$ with $p_0\exp tX\subset D^*$ for $t\in B$. We set $D(t)=D[p_0\exp tX]\subset\mathbb{C}^*\times\mathbb{C}^*$ so that $D(0)=D[p_0]$. We let $g(t,(z,w))$ (resp. $\lambda(t)$) denote the c-Green function $g([p_0\exp tX],(z,w))$ (resp. the c-Robin constant $\lambda[p_0\exp tX]$) for $(D(t),e)$ and $t\in B$. We set $\mathcal{D}|_B=\bigcup_{t\in B}(t,D(t))\subset B\times(\mathbb{C}^*\times\mathbb{C}^*)$ which we consider as the variation

$$\mathcal{D}|_B: t\in B\to D(t)=D[p_0\exp tX]\subset\mathbb{C}^*\times\mathbb{C}^*.$$

By (4.2) we have

$$\begin{aligned}D(t)&=D[p_0\exp tX]=D[[z_0,w_0]\exp tX]\\&=D[z_0,w_0]\exp(-tX)=D[z_0,w_0]\big(e^{-\alpha t},e^{-\beta t}\big)\quad\text{in }\mathbb{C}^*\times\mathbb{C}^*.\end{aligned}$$

Using the same reasoning as in the first step of the proof of Lemma 4.1 together with Remark 4.1 we see from (4.9) and the real analyticity of $\partial\mathcal{D}|_B = \bigcup_{t\in B}(t, \partial D(t))$ in $B \times (\mathbb{C}^* \times \mathbb{C}^*)$ that

$$\left.\frac{\partial g(t,(z,w))}{\partial t}\right|_{t=0} \equiv 0 \quad \text{on } D[z_0,w_0] \cup \partial D[z_0,w_0]. \tag{4.10}$$

For a fixed $t \in B$ we consider the automorphism

$$(Z,W) \to (z,w) = F(t,(Z,W))$$

of $\mathbb{C}^* \times \mathbb{C}^*$ where

$$F(t,(Z,W)) := (Z,W)\left(\frac{1}{z_0}, \frac{1}{w_0}\right)\exp(-tX) = \left(\frac{Ze^{-\alpha t}}{z_0}, \frac{We^{-\beta t}}{w_0}\right).$$

Then

$$(z,w) \to (Z,W) = F^{-1}(t,(z,w)) = \big(zz_0e^{\alpha t}, ww_0e^{\beta t}\big).$$

By (4.1) we have

$$D(t) = \widetilde{D^*}\left(\frac{1}{z_0}, \frac{1}{w_0}\right)\exp(-tX) = \widetilde{D^*}\left(\frac{e^{-\alpha t}}{z_0}, \frac{e^{-\beta t}}{w_0}\right) \quad \text{in } \mathbb{C}^* \times \mathbb{C}^*,$$

so that $D(t) = F(t, \widetilde{D^*})$. We note that $\widetilde{D^*} \subset \mathbb{C}^* \times \mathbb{C}^*$ is independent of $t \in B$. We set

$$G(t,(Z,W)) := g(t,(z,w)) \quad \text{where } (z,w) = F(t,(Z,W)), \quad (Z,W) \in \widetilde{D^*}.$$

Since

$$g(t,(z,w)) = G(t, F^{-1}(t,(z,w))) = G(t,(zz_0e^{\alpha t}, ww_0e^{\beta t})),$$

we have

$$\begin{aligned}\frac{\partial g}{\partial t}(t,(z,w)) &= \frac{\partial G}{\partial t}(t,(Z,W)) + \frac{\partial G}{\partial Z}(t,(Z,W))\alpha zz_0e^{\alpha t} + \frac{\partial G}{\partial W}(t,(Z,W))\beta ww_0e^{\beta t} \\ &= \frac{\partial G}{\partial t}(t,(Z,W)) + \alpha Z\frac{\partial G}{\partial Z}(t,(Z,W)) + \beta W\frac{\partial G}{\partial W}(t,(Z,W))\end{aligned}$$

where $(Z,W) = F^{-1}(t,(z,w))$. Since, for each $t \in B$,

$$G(t,(Z,W)) \equiv 0 \quad \text{on } \partial\widetilde{D^*}, \tag{4.11}$$

we have

$$\frac{\partial G}{\partial t}(t,(Z,W)) = 0 \quad \text{on } \partial\widetilde{D^*}.$$

It follows from (4.10) that

$$\alpha Z\frac{\partial G}{\partial Z}(0,(Z,W)) + \beta W\frac{\partial G}{\partial W}(0,(Z,W)) = 0 \quad \text{on } \partial\widetilde{D^*}.$$

Together with (4.11), this says that the holomorphic vector field

$$X = \alpha Z\frac{\partial}{\partial Z} + \beta W\frac{\partial}{\partial W},$$

considered as a vector field on $\mathbb{C}^* \times \mathbb{C}^*$, satisfies the property that for any $(z,w) \in \partial\widetilde{D^*}$, the integral curve $(z,w)\exp tX \subset \partial\widetilde{D^*}$ for all $t \in \mathbb{C}$. It follows that for any $(z,w) \in \widetilde{D^*}$, the integral curve $(z,w)\exp tX$ is contained in $\widetilde{D^*}$:

$$\widetilde{D^*}\exp tX = \widetilde{D^*}, \quad \text{for all } t \in \mathbb{C}.$$

Hence X is a tangential vector field on $\partial\widetilde{D^*}$.

This implies

$$D[[z,w]\exp tX] = D[z,w] \subset \mathbb{C}^* \times \mathbb{C}^*, \quad \text{for all } t \in \mathbb{C} \tag{4.12}$$

if $[z,w] \in D^*$ since

$$D[[z,w]\exp tX] = \widetilde{D^*}\left(\frac{1}{z},\frac{1}{w}\right)\exp(-tX) = \widetilde{D^*}\left(\frac{1}{z},\frac{1}{w}\right) = D[z,w].$$

But for $[z,w] \in D^*$ (resp. ∂D^*) it is clear that

$$[z,w]\exp tX \subset D^* \text{ (resp. } \partial D^*) \text{ in } \mathcal{H}$$

$$\text{if and only if}$$

$$(z,w)\exp tX \subset \widetilde{D^*} \text{(resp. } \partial\widetilde{D^*}) \text{ in } \mathbb{C}^* \times \mathbb{C}^*,$$

which proves that X, as a holomorphic vector field on $\mathcal{H}$, is a tangential vector field on ∂D^*, verifying (1) of Lemma 4.2.

To prove assertion (2) we first observe by (4.12)

$$\lambda[z,w] = \lambda[[z,w]\exp tX], \quad \text{for all } t \in \mathbb{C}$$

for any $[z,w] \in D^*$. In case (2)(i) in Lemma 4.2, from 3 in Lemma 4.1, the Robin function $-\lambda[z,w]$ is an exhaustion function on D, and it follows that

$$\{[z,w]\exp tX : t \in \mathbb{C}\} \Subset D \quad \text{for } [z,w] \in D^*. \tag{4.13}$$

We now prove (2) (i). First we show that $X = cX_u$ for some $c \neq 0$. If not, i.e., if $X \notin \{cX_u : c \in \mathbb{C}^*\}$, we take $[z,w] \in \partial D^*$ and let $\sigma = [z,w] \exp tX$ be the integral curve of X passing through $[z,w]$. From Lemma 3.1 part 2, the closure Σ of σ in $\mathcal{H}$ contains $\boldsymbol{T}_a$ or $\boldsymbol{T}_b$ (or both) which contradicts the hypothesis $\partial D \not\supset \boldsymbol{T}_a$ and $\partial D \not\supset \boldsymbol{T}_b$ of (2) (i) in Lemma 4.2. Thus $X = cX_u$ for some $c \neq 0$.

By (4.13), for $[z,w] \in D^*$ the closure of the integral curve $I[z,w] := [z,w] \exp tX_u$ is compactly contained in D and hence lies in D^*. It follows from (α) and (β) in Proposition 1.1 that we have

$$(\alpha^*) \quad D^* = \bigcup_{c \in I} \Sigma_c, \quad \text{where } I \text{ is an open interval in } (0,\infty)\text{; or}$$

$$(\beta^*) \quad D^* = \bigcup_{c \in \delta^*} \sigma_c, \quad \text{where } \delta^* \text{ is a domain in } \mathbb{C}^*.$$

We next show that if $D \cap \boldsymbol{T}_a \neq \emptyset$ then $D \supset \boldsymbol{T}_a$. Thus let $[z_0, 0] \in D \cap \boldsymbol{T}_a$. Let U, V be sufficiently small disks such that

$$(z_0, 0) \in U \times V =: U \times \{|w| < r\} \Subset D \cap E_2$$

where recall $E_2 = \{1<|z| \leq |a|\} \times \{|w| \leq |b|\} \subset \mathcal{F}$. We show that there exists r' with $0 < r' < r$ such that

$$G(r') := \{(z,w) \in E_2 : 1 < |z| < |a|,\ 0 < |w| < r'\} \subset D^*. \tag{4.14}$$

We prove (4.14) in the case (β^*); the proof in the case (α^*) is similar. We recall the non-constant meromorphic function $f[z,w] = w^{pl}/z^{ql}$ in $\mathcal{H}$ from Remark 1.1. Since this function vanishes on $\{w = 0\}$, if we set

$$\Delta := \{c = f[z,w] \in \mathbb{C}^* : (z,w) \in U \times \{0 < |w| < r\}\},$$

there exists $m > 0$ such that the punctured disk $\delta' = \{0 < |c| < m\}$ is contained in Δ. Clearly we can choose $r' > 0$ sufficiently small with $r' < r$ such that the corresponding set $G(r')$ satisfies $f(G(r')) \subset \delta'$. Combined with (β^*) this implies (4.14).

Suppose $D \not\supset \boldsymbol{T}_a$. Observe that $D(0) := D \cap \boldsymbol{T}_a$ is a domain in $\boldsymbol{T}_a$ whose boundary l consists of smooth real one-dimensional curves. For $|w| < r'$, we let $D(w) \subset \{1 < |z| < |a|\}$ denote the slice of D over w. Since ∂D is of class C^ω, each $\partial D(w)$ is a union of smooth real one-dimensional curves which approach ℓ as $w \to 0$. This contradicts (4.14); hence $D \supset \boldsymbol{T}_a$. A completely similar argument shows that if $D \cap \boldsymbol{T}_b \neq \emptyset$ then $D \supset \boldsymbol{T}_b$. Thus either $D = D^*$ as in (α^*) or (β^*) or D is the union of D^* with $\boldsymbol{T}_a$, $\boldsymbol{T}_b$ or $\boldsymbol{T}_a \cup \boldsymbol{T}_b$. If $D = D^*$ as in (α^*) then D is of type (a-1) in Case a; if $D = D^*$ as in (β^*) then D is as in Case b with $\delta \in \mathbb{C}^*$. We let D be the union of D^* with $\boldsymbol{T}_a, \boldsymbol{T}_b$ or $\boldsymbol{T}_a \cup \boldsymbol{T}_b$. The case $D = D^* \cup \boldsymbol{T}_a$ corresponds to (a-2$'$) in Case a and to δ in Case b with $0 \in \delta$ and $\infty \in \partial\delta$. The case $D = D^* \cup \boldsymbol{T}_b$ corresponds to (a-2$''$) in Case a and to δ in Case b with $\infty \in \delta$ and $0 \in \partial\delta$. For the last case $D = D^* \cup \boldsymbol{T}_a \cup \boldsymbol{T}_b$, in Case a we have $D = \mathcal{H}$ which does not occur, and in Case b, D corresponds to δ with $0, \infty \in \delta$.

To prove (2) (ii), we note that under the condition $\partial D \supset \boldsymbol{T}_a$ and $\partial D \not\supset \boldsymbol{T}_b$, from (2) of Lemma 3.1 we have either $X = cX_u$ with $c \neq 0$ or $X = \alpha z(\partial/\partial z)$ with $\alpha \neq 0$. Assume that $X = cX_u$ with $c \neq 0$. We conclude from (α^*) that D cannot be of the form in Case a, so that D^* is of the form (β^*). Since $\partial D \supset \boldsymbol{T}_a$ and $\partial D \not\supset \boldsymbol{T}_b$ we arrive at the conclusion in (2) (ii-a). On the other hand, if $X = \alpha z(\partial/\partial z)$ with $\alpha \neq 0$, we first observe from the facts that $\partial D \supset \boldsymbol{T}_a$ and ∂D is C^ω-smooth, for any $z_0 \in \mathbb{C}^*$ the slice of ∂D over $z = z_0$ contains a C^ω curve $C(z_0) \subset \mathbb{C}_w$ passing through the origin $w = 0$. We can find a sufficiently small disk $V := \{|w| < r_0\}$ so that $C(z_0)$ divides V into two parts V' and V'' with $\{z_0\} \times V' \subset D$ and $\{z_0\} \times V'' \subset \overline{D}^c$. We set $\widetilde{C}(z_0) := C(z_0) \cap V$. By (1) in Lemma 4.2 we conclude that $\mathbb{C}^* \times V' \subset D$ and $\mathbb{C}^* \times V'' \subset \overline{D}^c$. Thus $\mathbb{C}^* \times \widetilde{C}(z_0) \subset \partial D$, which implies $\partial D \cap (\mathbb{C}^* \times V) = \mathbb{C}^* \times \widetilde{C}(z_0)$ and $D \cap (\mathbb{C}^* \times V) = \mathbb{C}^* \times V'$.

We use this geometric set-up to show that b must be a positive real number (hence $b > 1$). To see this, fix a point $w_0 \in \widetilde{C}(z_0)$ (resp. V') with $w_0 \neq 0$. Since $(z_0, w_0) \in \partial D$ (resp. V'), we have $\mathbb{C}^* \times \{w_0\} \subset \partial D$ (resp. D). In particular, $(a^n z_0, w_0) \in \partial D$ (resp. D) for any $n \in \mathbb{Z}$. Hence $(z_0, w_0/b^n) \in \partial D$ (resp. D) for any $n \in \mathbb{Z}$. Since $|b| > 1$ we can take N sufficiently large so that $w_0/b^N \in V$. It follows that $w_0/b^n \in \widetilde{C}(z_0)$ (resp. V') for any $n \geq N$.

We first show that b is real. If not, let $b = |b|e^{i\phi}$ where $|b| > 1$ and $0 < |\phi| < \pi$. We set $w_0 = |w_0|e^{i\varphi_0}$. Let $\boldsymbol{n}_0 = e^{i\theta_0}$ be a unit normal vector to $\widetilde{C}(z_0)$ at $w = 0$ pointing in to V''. Since $\widetilde{C}(z_0)$ is smooth, we can find r_1 sufficiently small with $0 < r_1 < r_0$ so that the sector $\boldsymbol{e} := \{re^{i\theta} : 0 < r < r_1,\ |\theta - \theta_0| < 2\pi/3\}$ is contained in V''. For any $N' \in \mathbb{Z}$, it is clear that there exists $n' > N'$ satisfying

$$|(\varphi_0 - n'\phi) - \theta_0| < 2\pi/3 \text{ modulo } 2\pi. \tag{4.15}$$

We take $N' > N$ so that $|w_0|/|b|^{N'} < r_1$, and then we choose $n' > N'$ with property (4.15). Then $w_0/b^{n'} \in \boldsymbol{e} \subset V''$, which contradicts the fact that $w_0/b^{n'} \in \widetilde{C}(z_0)$. Thus b is real.

We next show b is positive. If not, we have $b < -1$. We take $w_1 \in V' \setminus \{0\}$ close to 0. Then $(z, w_1) \in D$ for all $z \in \mathbb{C}^*$. In particular, $(a^n z_0, w_1) \in D$ for any $n \in \mathbb{Z}$; hence $(z_0, w_1/b^n) \in (\{z_0\} \times V) \cap D$ for n sufficiently large. In other words, for $n > N$ we have $w_1/b^n \in V'$. Since $b < -1$ it follows that $\{w_1/b^n : n \geq N\}$ lies on a line L passing through $w = 0$. Moreover, if we take a sufficiently small disk $V_0 := \{|w| < r_0\} \subset V$, then $L \cap V_0$ intersects the smooth curve $\widetilde{C}(z_0)$ transversally. At the point $w = 0$, $L \cap V_0$ divides into two segments L' and L'' with $L' = (L \cap V_0) \cap V'$ and $L'' = (L \cap V_0) \cap V''$. Since $b < -1$, for n sufficiently large, if $w_1/b^n \in L'$ then $w_1/b^{n+1} \in L''$. This contradicts the fact that $w_1/b^m \in V'$ for all m sufficiently large. Thus $b > 1$.

Consequently,

$$w \in \widetilde{C}(z_0) \text{ (resp. } V') \longrightarrow w/b^n \in \widetilde{C}(z_0) \text{ (resp. } V') \quad \text{for } n = 1, 2, \ldots.$$

It follows from the smoothness of $\widetilde{C}(z_0)$ and the fact that $b > 1$ that $\widetilde{C}(z_0)$ is a line $Au + Bv = 0$ passing through $w = 0$, proving (2) (ii-b).

To verify (2) (iii), we show

$$X \in \{cX_u : c \in \mathbb{C}\} \cup \left\{\alpha z \frac{\partial}{\partial z} : \alpha \in \mathbb{C}\right\} \cup \left\{\beta w \frac{\partial}{\partial w} : \beta \in \mathbb{C}\right\}. \tag{4.16}$$

Once (4.16) is verified, we obtain 2 (iii) by repeating the arguments in 2 (i) and 2 (ii). Suppose $X = \alpha z(\partial/\partial z) + \beta w(\partial/\partial w) \notin \{cX_u : c \in \mathbb{C}\}$ where $\alpha \neq 0$, $\beta \neq 0$. We set $\beta/\alpha = A + iB$ where A, B are real numbers. To get a contradiction, we work in the case where A is irrational; the other cases are similar. Fix $z_0 \in \{1 < |z| < |a|\}$. Since $\partial D \supset \boldsymbol{T}_a \cup \boldsymbol{T}_b$ and ∂D is smooth, we can find a smooth curve ℓ in $\{|w| < |b|\}$ containing $w = 0$ such that $\{z_0\} \times \ell \subset \partial D$. We fix a disk $V := \{|w| < r\}$ with r sufficiently small so that ℓ divides V into two parts V' and V'' where $\{z_0\} \times V' \subset D$ and $\{z_0\} \times V'' \subset \overline{D}^c$. Let $w_0 \in V'$ and for $c = w_0/z_0^{A+Bi}$, we consider the integral curve $\sigma_c := \{w = cz^{A+iB}\}/\sim$ of X passing through (z_0, w_0) in $\mathcal{H}$. Using (1) in Lemma 4.2 we see that $\sigma_c \subset D$. On the other hand, by Remark 5.2 in Appendix A there is a point $(z_0, w(z_0)) \in \sigma_c$ with $w(z_0) \in V''$, which is a contradiction. This proves (4.16) and hence 2 (iii). □

Given a pseudoconvex domain D in $\mathcal{H}$ with C^ω-smooth boundary, under the various cases of (2) of Lemma 4.2, depending on the relationship between the tori $\boldsymbol{T}_a$, $\boldsymbol{T}_b$ and ∂D, we want to show that either D is Stein or D is the appropriate type of non-Stein domain in Theorem 1.1. This will be done in a series of lemmas. Before proceeding, we recall an important "rigidity" result from [**1**].

We let $\mathcal{D} : t \in B \to D(t) \subset M$ be a smooth variation of domains $D(t) \subset M$ over $B \subset \mathbb{C}$ where M is a complex Lie group of dimension $n \geq 1$. Here $D(t)$ need not be relatively compact in M but $\partial D(t)$ is assumed to be C^∞-smooth. Assume each domain $D(t)$ contains the identity element e. Let $g(t, z)$ and $\lambda(t)$ be the c-Green function and the c-Robin constant for $(D(t), e)$ associated to a Kähler metric and a positive, smooth function c on M. We have the following from [**1**]:

($\star$1) *Assume that the total space* $\mathcal{D} = \bigcup_{t\in B}(t, D(t))$ *is pseudoconvex in* $B \times M$. *If* $(\partial^2\lambda/\partial t\partial\bar{t})(0) = 0$, *then* $\partial g(t, z)/\partial t|_{t=0} \equiv 0$ *on* $D(0)$.

Next let $\psi(t, z)$ be a C^∞-defining function of $\mathcal{D}$ in a neighborhood of $\partial\mathcal{D} = \bigcup_{t\in B}(t, \partial D(t))$. Since $\partial D(t)$ is smooth, we have

$$\left(\frac{\partial\psi}{\partial z_1}(t, z), \ldots, \frac{\partial\psi}{\partial z_n}(t, z)\right) \neq (0, \ldots, 0)$$

for $(t, z) \in \partial\mathcal{D} = \{\psi(t, z) = 0\}$. We have a type of contrapositive of ($\star$1):

($\star$2) Assume that $\mathcal{D}$ is pseudoconvex in $B \times M$. If there exists a point $z_0 \in \partial D(0)$ with

$$\frac{\partial\psi}{\partial t}(0, z_0) \neq 0, \tag{4.17}$$

then $(\partial^2(-\lambda)/\partial t\partial\bar{t})(0) > 0$.

We prove this by contradiction; thus suppose $(\partial^2(-\lambda)/\partial t\partial\bar{t})(0) = 0$. By ($\star$1) we have $\partial g(t, z)/\partial t|_{t=0} \equiv 0$ on $\partial D(0)$. Since $-g(t, z)$ is a C^∞ defining function of $\mathcal{D}$, it follows that $\partial\psi(t,z)/\partial t|_{t=0} \equiv 0$ on $\partial D(0)$, which contradicts (4.17).

Returning to the case of a pseudoconvex domain D in $\mathcal{H}$ with C^ω-smooth boundary, we proved in Lemma 4.1 that under certain hypotheses on ∂D the function $-\lambda[z,w]$ is a plurisubharmonic exhaustion function for D. The next lemma shows that if ∂D hits, but does not contain, one of the tori $\boldsymbol{T}_a$ or $\boldsymbol{T}_b$, and D does not contain the other one, then D is Stein.

LEMMA 4.3. *Let D be a pseudoconvex domain in $\mathcal{H}$ with C^ω-smooth boundary. If $\emptyset \neq \partial D \cap \boldsymbol{T}_a \neq \boldsymbol{T}_a$ and $D \not\supset \boldsymbol{T}_b$, then D is Stein* (*and similarly if $\boldsymbol{T}_a$ and $\boldsymbol{T}_b$ are switched*).

The condition $D \not\supset \boldsymbol{T}_b$ separates into the following three cases:

$$\text{(c1)} \quad \partial D \cap \boldsymbol{T}_b = \emptyset, \quad \text{(c2)} \quad \emptyset \neq \partial D \cap \boldsymbol{T}_b \neq \boldsymbol{T}_b \ \text{ or } \ \text{(c3)} \quad \partial D \cap \boldsymbol{T}_b = \boldsymbol{T}_b.$$

PROOF. We first want to show that if $-\lambda[z,w]$ is not strictly plurisubharmonic in D, then there is point $p_0 = [z_0, w_0]$ in D^* at which $-\lambda[z,w]$ is not strictly plurisubharmonic; then we show this cannot occur so that D is Stein. Let $\psi[z,w]$ be a defining function for D defined in a neighborhood of ∂D. We divide the proof of the lemma in five cases related to $\psi[z,w]$ and the subcases (c1), (c2), (c3) of the condition $D \not\supset \boldsymbol{T}_b$.

1st case: Assume there exists $[z_0, 0] \in \partial D \cap \boldsymbol{T}_a$ with $z_0 \neq 0$ such that neither $\partial\psi/\partial z$ nor $\partial\psi/\partial w$ vanishes at $(z_0, 0)$ and assume case (c1).

Using ($\star$2), we first prove the following fact in this 1st case. Assume $(1,0) \in D \cap \boldsymbol{T}_a$. Then $-\lambda[z,w]$ is strictly subharmonic at $[1,0]$ in the direction $\boldsymbol{a} = (0,1)$, i.e.,

$$\frac{\partial^2(-\lambda)}{\partial\tau\partial\overline{\tau}}[1,\tau]\bigg|_{\tau=0} > 0.$$

To see this, we take a small disk $\delta := \{|\tau| < r\} \subset \mathbb{C}_\tau$ and consider the variation of domains

$$\mathfrak{D} : \tau \in \delta \to D(\tau) := D[1,\tau] \subset \mathbb{C}_Z^* \times \mathbb{C}_W^*.$$

Note that

$$D(\tau) = \left\{ \begin{array}{ll} \widetilde{D}^* \cdot (1, 1/\tau) & \text{if } \tau \in \delta \setminus \{0\}; \\ \widetilde{D}_a \times \mathbb{C}_W^* & \text{if } \tau = 0 \end{array} \right\}$$

(recall $D \cap \boldsymbol{T}_a = [D_a, 0]$). We let $\lambda(\tau) = \lambda[1,\tau]$ denote the c-Robin constant for $(D(\tau), (1,1))$. We set $\mathfrak{D} := \bigcup_{\tau\in\delta}(\tau, D(\tau))$ and $\partial\mathfrak{D} = \bigcup_{\tau\in\delta}(\tau, \partial D(\tau))$. For $\tau \in \delta \setminus \{0\}$, we consider the automorphism

$$F_\tau : \ (z,w) \in \mathbb{C}_z^* \times \mathbb{C}_w^* \to (Z,W) = \left(z, \frac{w}{\tau}\right) \in \mathbb{C}_Z^* \times \mathbb{C}_W^*.$$

From the definition of $D(\tau)$, we have $D(\tau) = F_\tau(\widetilde{D}^*)$. We let $\psi(z,w)$ be a defining function for ∂D in $\mathcal{H}$; to avoid notational issues we also regard $\psi(z,w)$ as a defining function of $\partial\widetilde{D}$. For $\tau\in\delta\setminus\{0\}$ we set

$$\Phi(\tau,(Z,W)) := \psi(Z,\tau W)$$

which is a defining function for $\partial\mathfrak{D}|_{\delta\setminus\{0\}}$. Setting $\Phi[0,(Z,W)] := \psi(Z,0)$, we see that $\Phi[\tau,(Z,W)]$ becomes a smooth defining function for the entire set $\partial\mathfrak{D}$. We focus on the special point $(z_0,1)$ in $\partial D(0)$. Then

$$\begin{aligned}\nabla_{(Z,W)}\Phi\big|_{(0,(z_0,1))} &= \left(\frac{\partial\Phi}{\partial Z},\frac{\partial\Phi}{\partial W}\right)\bigg|_{(0,(z_0,1))} = \left(\frac{\partial\psi}{\partial z},\frac{\partial\psi}{\partial w}\tau\right)\bigg|_{(0,(z_0,1))}\\ &= \left(\frac{\partial\psi}{\partial z}(z_0,0),0\right)\neq(0,0)\quad\text{by the condition of the 1}^{\text{st}}\text{ step.}\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial\Phi}{\partial\tau}\bigg|_{(0,(z_0,1))} &= \frac{\partial\psi}{\partial w}W\bigg|_{(0,(z_0,1))}\\ &= \frac{\partial\psi}{\partial w}(z_0,0)\neq 0\quad\text{by the condition of the 1}^{\text{st}}\text{ step.}\end{aligned}$$

It follows from $(\star 2)$ that $(\partial^2(-\lambda)/\partial\tau\partial\overline{\tau})[1,\tau]|_{\tau=0}>0$, as desired.

We next prove that $-\lambda[z,w]$ in D is strictly subharmonic at $[1,0]$ in *any* direction $\boldsymbol{a}=(a_1,a_2)\in\mathbb{C}^2\setminus\{0\}$ with $\|\boldsymbol{a}\|=1$ and $a_1\neq 0$, i.e.,

$$\frac{\partial^2(-\lambda)}{\partial\tau\partial\overline{\tau}}[1+a_1\tau,a_2\tau]\bigg|_{\tau=0}>0. \tag{4.18}$$

We use the same notation τ and $\psi(z,w)$ as in the case $\boldsymbol{a}=(1,0)$. We consider the variation of domains

$$\mathfrak{G}:\ \tau\in\delta\to G(\tau):=D[1+a_1\tau,a_2\tau]\subset\mathbb{C}^*_Z\times\mathbb{C}^*_W.$$

Note that

$$\begin{aligned}G(\tau) &= \left\{\begin{array}{ll}\widetilde{D}^*\cdot(1/(1+a_1\tau),1/(a_2\tau)) & \text{if }\tau\in\delta\setminus\{0\};\\ \widetilde{D}_a\times\mathbb{C}^*_W & \text{if }\tau=0\end{array}\right\}\quad\text{in case }a_2\neq 0,\\ G(\tau) &= [\widetilde{D}_a\cdot(1/(1+a_1\tau))]\times\mathbb{C}^*_W\quad\text{if }\tau\in\delta\qquad\qquad\text{in case }a_2=0.\end{aligned}$$

We let $\mu(\tau):=\lambda[1+a_1\tau,a_2\tau]$ denote the c-Robin constant for $(G(\tau),(1,1))$. Our claim (4.18) is that $(\partial^2(-\mu)/\partial\tau\partial\overline{\tau})(0)>0$.

We set $\mathfrak{G}:=\bigcup_{\tau\in\delta}(\tau,G(\tau))$ and $\partial\mathfrak{G}=\bigcup_{\tau\in\delta}(\tau,\partial G(\tau))$. Since $(\partial\psi/\partial z)(z_0,0)\neq 0$ and $a_1\neq 0$, we can find a point $W_0\in\mathbb{C}^*_W$ such that

$$a_1 z_0 \frac{\partial \psi}{\partial z}(z_0, 0) + a_2 W_0 \frac{\partial \psi}{\partial w}(z_0, 0) \neq 0.$$

We note that $(z_0, W_0) \in \partial G(0) = (\partial \widetilde{D}_a) \times \mathbb{C}^*_W$. We consider

$$\Psi(\tau, (Z, W)) := \psi((1 + a_1 \tau) Z, a_2 \tau W),$$

which is defined in a sufficiently small polydisk $\mathcal{V} := \delta_1 \times (U_1 \times V_1)$ of center $(0, (z_0, W_0))$ in $\delta \times \mathbb{C}^*_Z \times \mathbb{C}^*_W$. This is a defining function for $\partial \mathfrak{G}$ in $\mathcal{V}$. We have

$$\begin{aligned}
\nabla_{(Z,W)} \Psi \big|_{(0,(z_0,W_0))} &= \left(\frac{\partial \psi}{\partial z} \cdot (1 + a_1 \tau), \frac{\partial \psi}{\partial w} \cdot a_2 \tau \right) \bigg|_{(0,(z_0,W_0))} \\
&= \left(\frac{\partial \psi}{\partial z}(z_0, 0), 0 \right) \neq (0, 0); \\
\frac{\partial \Psi}{\partial \tau} \bigg|_{(0,(z_0,W_0))} &= \frac{\partial \psi}{\partial z} \cdot (a_1 Z) + \frac{\partial \psi}{\partial w} \cdot (a_2 W) \bigg]_{(0,(z_0,W_0))} \\
&= a_1 z_0 \frac{\partial \psi}{\partial z}(z_0, 0) + a_2 W_0 \frac{\partial \psi}{\partial w}(z_0, 0) \neq 0.
\end{aligned}$$

Using $(\star 2)$ we conclude that $(\partial^2(-\mu)/\partial \tau \partial \overline{\tau})(0) > 0$ which proves our claim.

A similar argument shows that $-\lambda[z, w]$ in D is strictly plurisubharmonic at any point $[z, 0] \in D \cap \boldsymbol{T}_a$. Hence, in case (c1), we conclude that if $-\lambda[z, w]$ is not strictly plurisubharmonic in D, there exists a point $p' = [z', w']$ in D^* at which $-\lambda[z, w]$ is not strictly plurisubharmonic. Now since $\partial D \not\supset \boldsymbol{T}_a$ and $\partial D \not\supset \boldsymbol{T}_b$, we are in case (2) (i) of Lemma 4.2. Hence we have $\partial D \cap (\boldsymbol{T}_a \cup \boldsymbol{T}_b) = \emptyset$. This contradicts $\partial D \cap \boldsymbol{T}_a \neq \emptyset$; thus D is Stein. □

2nd case: Assume there exists $[z_0, 0] \in \partial D \cap \boldsymbol{T}_a$ with $z_0 \neq 0$ such that neither $\partial \psi / \partial z$ nor $\partial \psi / \partial w$ vanishes at $(z_0, 0)$ and there exists $[0, w_0] \in \partial D \cap \boldsymbol{T}_b$ with $w_0 \neq 0$ such that neither $\partial \psi / \partial z$ nor $\partial \psi / \partial w$ vanishes at $(0, w_0)$, and assume case (c2).

Using the same argument as in the 1st case we see that $-\lambda[z, w]$ is strictly plurisubharmonic at any point $[0, w] \in D \cap \boldsymbol{T}_b$ and at any point $[z, 0] \in D \cap \boldsymbol{T}_a$. Thus there again exists a point $p' = [z', w']$ in D^* at which $-\lambda[z, w]$ is not strictly plurisubharmonic; and we similarly conclude that D is Stein. □

3rd case: Assume there exists $[z_0, 0] \in \partial D \cap \boldsymbol{T}_a$ with $z_0 \neq 0$ such that neither $\partial \psi / \partial z$ nor $\partial \psi / \partial w$ vanishes at $(z_0, 0)$ and assume case (c3).

Recall $\partial D \supset \boldsymbol{T}_b$ holds in case (c3). Here we need the function $U[z, w]$ on $\mathcal{H}^*$ defined in Section 2. Using 2 (a) of Lemma 4.1, i.e., for $[z_0, w_0] \in \partial D \setminus \boldsymbol{T}_b$,

$$-\lambda[z, w] \to \infty \quad \text{as } [z, w] \in D \to [z_0, w_0],$$

and property (1) of $U[z, w]$ we see that

$$s[z,w] := \max\{-\lambda[z,w], U[z,w]\} \tag{4.19}$$

is a well-defined plurisubharmonic exhaustion function for D. In order to prove that D is Stein, we use a result from Section 14 in [**7**]: it suffices to show that for any $K \Subset D$ there exists a Stein domain D_K with $K \Subset D_K \subset D$. To construct D_K, we take $m > \max_{[z,w]\in K}\{|-\lambda[z,w]|\}$ and consider

$$v[z,w] := \max\{-\lambda[z,w] + 2m,\ \varepsilon\, U[z,w]\} \tag{4.20}$$

where $\varepsilon > 0$ is chosen sufficiently small so that $v[z,w] = -\lambda[z,w]+2m$ on K. Again from property (1) of $U[z,w]$, $v[z,w]$ is a well-defined plurisubharmonic exhaustion function for D. We take $M > 1$ sufficiently large so that

$$K \Subset D(M) := \{[z,w] \in D : v[z,w] < M\} \quad \text{and} \quad \emptyset \neq \partial D(M) \cap \boldsymbol{T}_a \neq \boldsymbol{T}_a.$$

Note that $D(M) \Subset D$; thus $\partial D \supset \boldsymbol{T}_b$ implies that $\boldsymbol{T}_b \cap \overline{D(M)} = \emptyset$; also $\partial D(M)$ is piecewise smooth. We now have

$$\partial D(M) \cap \boldsymbol{T}_b = \emptyset \quad \text{and} \quad \emptyset \neq \partial D(M) \cap \boldsymbol{T}_a \neq \boldsymbol{T}_a. \tag{4.21}$$

We consider the c-Robin function $\lambda_M[z,w]$ for $D(M)$. Although $\partial D(M)$ is not smooth, by the construction of $\lambda_M[z,w]$ and the fact that $\partial D(M) \not\supset \boldsymbol{T}_a, \boldsymbol{T}_b$, it follows that $-\lambda_M[z,w]$ is a smooth plurisubharmonic exhaustion function for $D(M)$.

Let $D(M,M') := \{[z,w] \in D(M) : -\lambda_M[z,w] < M'\}$ and take $M' > 1$ sufficiently large so that

$$D(M,M') \Supset K \quad \text{and} \quad \emptyset \neq \partial D(M,M') \cap \boldsymbol{T}_a \not\supset \boldsymbol{T}_a.$$

Now since $-\lambda_M[z,w]$ is smooth we have that $D(M,M')$ is a pseudoconvex domain in $\mathcal{H}$ with smooth boundary; moreover we have

$$\partial D(M,M') \cap \boldsymbol{T}_b = \emptyset \quad \text{and} \quad \emptyset \neq \partial D(M,M') \cap \boldsymbol{T}_a \not\supset \boldsymbol{T}_a. \tag{4.22}$$

We can now apply the 1$^{\text{st}}$ case, where we assumed condition (c1), to $D(M,M')$ to conclude that $D(M,M')$ is Stein; hence D is Stein.

4$^{\text{th}}$ case: Assume one of $\partial\psi/\partial z, \partial\psi/\partial w$ vanishes identically on $\partial D \cap \boldsymbol{T}_a$ and assume case (c1).

To deal with this case we construct the C-Robin function $\Lambda[z,w]$ on D with respect to a positive constant function C on $\mathbb{P}^2 \supset \mathbb{C}^2$ and the restriction of the Fubini-Study metric dS^2 on $\mathbb{P}^2$ to $\mathbb{C}^* \times \mathbb{C}^*$. Note this metric is different than the Euclidean metric ds^2 on $\mathbb{C}^2$ restricted to $\mathbb{C}^* \times \mathbb{C}^*$; accordingly, $-\Lambda[z,w]$ is a smooth plurisubharmonic exhaustion function on D which is different from the function $-\lambda[z,w]$. Moreover, for any positive constant k the function $u_k[z,w] := -(\lambda[z,w] + k\Lambda[z,w])$ is a smooth plurisubharmonic exhaustion function for D. We claim that we can find a k and an increasing sequence

$\{M_n\}_{n=1,2,\dots}$ tending to $+\infty$ such that the increasing sequence of pseudoconvex domains $D_n = \{[z,w] \in D : u_k[z,w] < M_n\}$ satisfy the hypotheses of the 1st case. Clearly $\partial D_n \cap \boldsymbol{T}_b = \emptyset$ so that (c1) holds. It remains to select k and then the sequence M_n so that there exists $[z_n, 0] \in \partial D \cap \boldsymbol{T}_a$ with $z_n \neq 0$ such that neither $\partial\psi_n/\partial z$ nor $\partial\psi_n/\partial w$ vanishes at $(z_n, 0)$ where $\psi_n[z,w] := u_k[z,w] - M_n$. From the 1st case we conclude that each D_n is Stein and it follows from Section 14 of [**7**] that D is Stein.

5th case: Assume one of $\partial\psi/\partial z, \partial\psi/\partial w$ vanishes identically on $\partial D \cap \boldsymbol{T}_a$ and assume case (c2) or (c3).

The type of argument used to show a domain D in the 2nd or 3rd case, where we assume (c2) or (c3) of the condition $D \not\supset \boldsymbol{T}_b$, reduces to the 1st case, where we assume (c1) of this condition, allows us to deduce the 5th case from the 4th case. We leave the details to the reader. $\square$

We next turn to the situation where ∂D contains one of $\boldsymbol{T}_a$ or $\boldsymbol{T}_b$ but not both.

Lemma 4.4. *Let D be a pseudoconvex domain in $\mathcal{H}$ with C^ω-smooth boundary. If* (i) $\partial D \supset \boldsymbol{T}_a$ *and* (ii) $\partial D \cap \boldsymbol{T}_b \neq \boldsymbol{T}_b$, *then*

(1) *D is Stein or*
(2) *D is of Case* b *in Theorem* 1.1. *In fact, $D = \bigcup_{c\in\delta} \sigma_c$ with $0 \in \partial\delta$ and $\infty \notin \delta \cup \partial\delta$*

(*and similarly if $\boldsymbol{T}_a$ and $\boldsymbol{T}_b$ are switched as well as* 0 *and* ∞).

The condition (ii) separates into the following three cases:

$$(\tilde{c}\,1)\quad \emptyset \neq \partial D \cap \boldsymbol{T}_b \neq \boldsymbol{T}_b, \qquad (\tilde{c}\,2)\quad D \supset \boldsymbol{T}_b \quad \text{or} \quad (\tilde{c}\,3)\quad (\partial D \cup D) \cap \boldsymbol{T}_b = \emptyset.$$

Proof. We first treat the cases $(\tilde{c}\,1)$ and $(\tilde{c}\,3)$. We assume that D is not of Case b as in (2) and we show D is Stein. We proceed as in the proof of the 3rd case of Lemma 4.3 where we use the function $U[z,w]$ on $\mathcal{H}^*$ defined in Section 2. However, instead of (4.19) and (4.20) we use

$$s[z,w] := \max\{-\lambda[z,w], -U[z,w]\}$$

and

$$v[z,w] := \max\{-\lambda[z,w] + 2m,\ -\varepsilon\, U[z,w]\}.$$

We leave the details to the reader.

We next treat the case $(\tilde{c}\,2)$ in which $\partial D \supset \boldsymbol{T}_a$ and $D \supset \boldsymbol{T}_b$. In this setting we shall show that conclusion (2) in Lemma 4.4 holds.

Since $\boldsymbol{T}_b$ is compact in D, we can find a neighborhood V of $\boldsymbol{T}_b$ in D such that $\boldsymbol{T}_b \Subset V \Subset D$. Since $\Sigma_c := \{|w| = c|z|^\rho\}/\sim$ (or $\sigma_c := \{w = cz^\rho\}/\sim$) approaches $\boldsymbol{T}_b$ in $\mathcal{H}$ as $c \to \infty$, it follows that for c sufficiently large, the Levi-flat hypersurface Σ_c satisfies $\Sigma_c \Subset V \Subset D$ (or the compact torus σ_c satisfies $\sigma_c \Subset V \Subset D$). But $-\lambda[z,w]$ is a plurisubharmonic function on D (although not necessarily an exhaustion function);

hence $-\lambda[z,w]$ is not strictly plurisubharmonic at any point in Σ_c (or σ_c). From Lemma 4.2, we conclude that D is given as in case (2) (ii) of that lemma.

For simplicity, we complete the argument if $\Sigma_c \Subset V \Subset D$. We claim that (a,b) is of Case b in Theorem 1.1 and hence D is of the form in case (2) (ii-a) of Lemma 4.2, completing our proof. For if (a,b) is of Case a then from the proof of Lemma 4.2, we have (recall (α^*))

$$D^* = \bigcup_{c\in I} \Sigma_c, \quad \text{where } I=(r,R) \text{ is an open interval in } (0,\infty),$$

because D^* is connected. Since $D \supset \boldsymbol{T}_b$, $D = \bigcup_{c\in(r,\infty]} \Sigma_c$. However, since $\partial D \supset \boldsymbol{T}_a$, we must have $r=0$. Thus $D = \mathcal{H} \setminus \boldsymbol{T}_a$ which contradicts the smoothness of ∂D. □

Note in particular we have proved that the Nemirovskii-type domains in (2) (ii-b) of Lemma 4.2 are Stein. An entirely similar proof, which we omit, deals with the case where ∂D contains both $\boldsymbol{T}_a$ and $\boldsymbol{T}_b$.

LEMMA 4.5. *Let D be a pseudoconvex domain in $\mathcal{H}$ with C^ω-smooth boundary. If $\partial D \supset \boldsymbol{T}_a \cup \boldsymbol{T}_b$, then*

(1) *D is Stein or*
(2) *D is of type* b *in Theorem* 1.1. *More precisely, $D = \bigcup_{c\in\delta} \sigma_c$ with $0,\infty \in \partial\delta$.*

We suspect that under the hypotheses of Lemma 4.5 conclusion (2) must always hold, but we are unable to verify this.

We can now easily conclude with the proof of our main result.

PROOF OF THEOREM 1.1. Let D be a pseudoconvex domain in $\mathcal{H}$ with C^ω-smooth boundary which is not Stein. We consider three "symmetric" cases depending on the nature of $\partial D \cap \boldsymbol{T}_a$ or $\partial D \cap \boldsymbol{T}_b$.

1st case: $\partial D \supset \boldsymbol{T}_a$ (or $\partial D \supset \boldsymbol{T}_b$).

If $\partial D \supset \boldsymbol{T}_a$, we can have either $\partial D \cap \boldsymbol{T}_b \neq \boldsymbol{T}_b$ or $\partial D \supset \boldsymbol{T}_b$. If $\partial D \cap \boldsymbol{T}_b \neq \boldsymbol{T}_b$, from Lemma 4.4, $D = \bigcup_{c\in\delta} \sigma_c$ with $0 \in \partial\delta$ and $\infty \notin \delta \cup \partial\delta$. If $\partial D \supset \boldsymbol{T}_b$, this means $\partial D \supset \boldsymbol{T}_a \cup \boldsymbol{T}_b$; hence Lemma 4.5 implies $D = \bigcup_{c\in\delta} \sigma_c$ with $0,\infty \in \partial\delta$.

2nd case: $\partial D \cap \boldsymbol{T}_a = \emptyset$ (or $\partial D \cap \boldsymbol{T}_b = \emptyset$).

If $\partial D \cap \boldsymbol{T}_a = \emptyset$, we can have either $\partial D \cap \boldsymbol{T}_b \neq \boldsymbol{T}_b$ or $\partial D \supset \boldsymbol{T}_b$. If $\partial D \supset \boldsymbol{T}_b$, we are done by the 1st case. If $\partial D \cap \boldsymbol{T}_b \neq \boldsymbol{T}_b$, either

$$\text{(I)} \quad \partial D \cap \boldsymbol{T}_b = \emptyset \quad \text{or} \quad \text{(II)} \quad \emptyset \neq \partial D \cap \boldsymbol{T}_b \neq \boldsymbol{T}_b.$$

Note that if $\partial D \cap \boldsymbol{T}_b = \emptyset$, then in this 2nd case $\partial D \cap (\boldsymbol{T}_a \cup \boldsymbol{T}_b) = \emptyset$.

Let $\lambda[z,w]$ be the c-Robin function of D. From Lemma 4.1 we know that $-\lambda[z,w]$ is a plurisubharmonic exhaustion function on D. We shall prove that under our assumption that D is not Stein we can find a point $[z_0,w_0]$ in D^* at which $-\lambda[z,w]$ is not strictly plurisubharmonic.

In the setting of the 2^{nd} case with (I) $\partial D \cap \boldsymbol{T}_b = \emptyset$ we have three possible situations for D relative to $\boldsymbol{T}_a, \boldsymbol{T}_b$: (i) $D \cap (\boldsymbol{T}_a \cup \boldsymbol{T}_b) = \emptyset$; (ii) $D \cap \boldsymbol{T}_a = \emptyset$ and $D \supset \boldsymbol{T}_b$ (or the symmetric case with $\boldsymbol{T}_a, \boldsymbol{T}_b$ switched); and (iii) $D \supset \boldsymbol{T}_a \cup \boldsymbol{T}_b$.

In case (i) we are done since $D = D^*$ so that, by the assumption D is not Stein, there is a point $[z_0, w_0]$ in $D = D^*$ at which $-\lambda[z, w]$ is not strictly plurisubharmonic. By (2) (i) of Lemma 4.2, D is a domain of the type in Case (a-1) or Case b of Theorem 1.1 (in the latter situation, we have $D = \bigcup_{c\in\delta} \sigma_c$ where $\delta \subset \mathbb{C}^*$). For cases (ii) and (iii) we only give the proofs under the hypothesis of Case a of Theorem 1.1 ($(a, b) \in S_1$) as the proofs in Case b are similar. In case (ii), since $\boldsymbol{T}_b$ is compact in D, we can find a neighborhood V of $\boldsymbol{T}_b$ in D such that $\boldsymbol{T}_b \Subset V \Subset D$. The Levi-flat hypersurface Σ_c approaches $\boldsymbol{T}_b$ as $c \to \infty$; hence $\Sigma_c \Subset V \Subset D$ for c sufficiently large. Since $-\lambda[z, w]$ is a plurisubharmonic function on D, $-\lambda[z, w]$ is not strictly plurisubharmonic at points of Σ_c; thus we can find such a point in D^*. Recalling (α^*):

$$D^* = \bigcup_{c\in I} \Sigma_c, \quad \text{where } I \text{ is an open interval in } (0, \infty),$$

we see that D is of type (a-2$''$) in Theorem 1.1. In case (iii), similar reasoning as in case (ii) shows that $\Sigma_{c_0} \subset D$ for some $c_0 \neq 0, \infty$. It follows that $D = \bigcup_{c\in I} \Sigma_c$ where I is an interval in $[0, \infty]$. Since $D \supset \boldsymbol{T}_a \cup \boldsymbol{T}_b$, we have $I = [0, \infty]$, i.e., $D = \mathcal{H}$, which is absurd (note in Case b of case (iii) the conclusion is that $D = \bigcup_{c\in\delta} \sigma_c$ where $0, \infty \in \delta \subset \mathbb{P}^1$). This finishes the proof of the 2^{nd} case under situation (I).

To finish the proof of the 2^{nd} case, where $\partial D \cap \boldsymbol{T}_a = \emptyset$, it remains to deal with situation (II), i.e., $\partial D \cap \boldsymbol{T}_a = \emptyset$ and $\emptyset \neq \partial D \cap \boldsymbol{T}_b \neq \boldsymbol{T}_b$. Again, we give the proofs under the hypothesis $(a, b) \in S_1$ of Case a of Theorem 1.1 since the proofs in Case b are similar. Apriori, we separate this into two subcases:

$$\text{(c1)} \quad D \supset \boldsymbol{T}_a \quad \text{and} \quad \text{(c2)} \quad D \not\supset \boldsymbol{T}_a.$$

In case (c1), using the argument in case (ii) above we can find a neighborhood V of $\boldsymbol{T}_a$ in D such that $\boldsymbol{T}_b \Subset V \Subset D$ and hence $\Sigma_c \Subset V \Subset D$ for $c > 0$ sufficiently close to 0. Thus we obtain points in D^* at which $-\lambda[z, w]$ is not strictly plurisubharmonic. We now appeal to case (2) (i) of Lemma 4.2.

Now we observe that case (c2) cannot occur, for the assumptions $\emptyset \neq \partial D \cap \boldsymbol{T}_b \neq \boldsymbol{T}_b$ and $D \not\supset \boldsymbol{T}_a$ imply from Lemma 4.3 that D is Stein.

3^{rd} case: $\emptyset \neq \partial D \cap \boldsymbol{T}_a \neq \boldsymbol{T}_a$ (or $\emptyset \neq \partial D \cap \boldsymbol{T}_b \neq \boldsymbol{T}_b$).

If $\emptyset \neq \partial D \cap \boldsymbol{T}_a \neq \boldsymbol{T}_a$, from Lemma 4.3 we must have $D \supset \boldsymbol{T}_b$. Thus $\partial D \cap \boldsymbol{T}_b = \emptyset$ and we are done by the 2^{nd} case.

This completes the proof of Theorem 1.1. □

We end with an explicit example of the construction of both $D[z, w]$ and the c-Robin function $\lambda[z, w]$ for a specific Nemirovskii-type domain $D \subset \mathcal{H}$. We recall the fundamental domain $\mathcal{F} = E_1 \cup E_2 = (E_1' \cup E_1'') \cup (E_2' \cup E_2'')$ for $\mathcal{H}$ defined in (2.2). Let D be a subdomain of $\mathcal{F}$ defined by

$$D := (E_1' \times K_1'') \cup (E_2' \times K_2'') \subset E_1 \cup E_2$$

where (recall $b > 1$)

$$K_1'' := \{1 < |w| \leq b\} \cap \{\Re w > 0\} \quad \text{and} \quad K_2'' := \{|w| \leq b\} \cap \{\Re w > 0\}.$$

We note that ∂D, which can be written as

$$\{|z| \leq |a|\} \times \{\Re w = 0, 1 \leq |w| \leq b\} \cup \{1 \leq |z| \leq |a|\} \times \{\Re w = 0, |w| \leq |b|\},$$

is smooth in $\mathcal{H}$. To see that D is of Nemirovskii-type as in Lemma 4.2 (ii-b), setting

$$N = \mathbb{C}_z \times \{\Re w > 0\} \subset (\mathbb{C}^2)^*$$

we will show that

$$N/\sim\ = D \ \text{ in } \mathcal{H}, \quad \text{or equivalently,} \quad N = \widetilde{D} = D \cdot \mathcal{I} \ \text{ in } (\mathbb{C}^2)^* \tag{4.23}$$

(recall (2.3)). Hence $N \setminus (\mathbb{C}_z \times \{0\}) = \widetilde{D^*}$.

To prove (4.23), we show $N = \widetilde{D}$. Let $(z, w) \in N$. Then we have $z = a^n z_0$ and $w = b^m w_0$ for some $n, m \in \mathbb{Z}$ and $(z_0, w_0) \in \mathcal{F}$. Since $b > 1$, we have $\Re w_0 > 0$.

Case 1: $n \geq m$.

In this case we have $(z, w) \sim (z/a^n, w/b^n) = (z_0, b^{m-n} w_0) \in E_2' \times K_2'' \subset D$.

Case 2: $m \geq n$.

In this case we have $(z, w) \sim (z/a^m, w/b^m) = (a^{n-m} z_0, w_0) \in E_1' \times K_1'' \subset D$.

Hence $N \subset \widetilde{D} = D \cdot \mathcal{I}$. The converse is clear from the relations $D \subset N$ and $N \cdot \mathcal{I} = N$.

We turn to the study of the sets $D[z, w]$ and the c-Robin functions $\lambda[z, w]$ for $(D[z, w], e)$ with respect to the metric ds^2 on $\mathbb{C}^* \times \mathbb{C}^*$ and the function $c(z, w) > 0$. Recall $e = (1, 1)$. We put $\widetilde{K_1''} = \{\Re w > 0\}$. Let $w' \in K_2''$. We write $w' = |w'| e^{i\theta}$ where $-\pi/2 < \theta < \pi/2$ and define

$$\delta(w') := \{w = u + iv \in \mathbb{C}_w : (\cos\theta) u - (\sin\theta) v > 0\}. \tag{4.24}$$

We then have

$$\{\Re w > 0\} \cdot \frac{1}{w'} = \delta(w') \quad \text{in } \mathbb{C}_w,$$

so that $\operatorname{dist}(1, \partial\delta(w')) = \cos\theta$ for $|w'| \leq |b|$. Recalling the formulas

$$D[z, w] = \left(\left(\frac{1}{z}, \frac{1}{w} \right) \cdot D^* \right) \cdot \mathcal{I} \qquad \text{if } [z, w] \in D^*;$$

$$D[z, 0] = \left(\frac{1}{z} D_a, \mathbb{C}^* \right) \cdot \mathcal{I} = \left(\frac{1}{z} \widetilde{D_a} \right) \times \mathbb{C}_w^* \quad \text{if } [z, 0] \in D \cap \boldsymbol{T}_a;$$

$$D[0,w] = \left(\mathbb{C}^*, \frac{1}{w} D_b\right) \cdot \mathcal{I} = \mathbb{C}^*_z \times \left(\frac{1}{w}\,\widetilde{D_b}\right) \quad \text{if } [0,w] \in D \cap \boldsymbol{T}_b$$

where $D \cap \boldsymbol{T}_a = D_a \times \{0\}$, $D \cap \boldsymbol{T}_b = \{0\} \times D_b$, $\widetilde{D_a} = \{a^n z : z \in D_a,\ n \in \mathbb{Z}\} \subset \mathbb{C}^*_z$ and $\widetilde{D_b} = \{b^n w : w \in D_b,\ n \in \mathbb{Z}\} \subset \mathbb{C}^*_w$, in using the equality $\widetilde{D} = N$ we obtain the following:

If $(z', w') \in D^*$, then

$$D[z', w'] = \left(\frac{1}{z'}, \frac{1}{w'}\right) \widetilde{D^*} = \mathbb{C}^*_z \times \delta(w'),$$

while if $(0, w') \in D$, then

$$D[0, w'] = \mathbb{C}^*_z \times \frac{1}{w'}\,\widetilde{K''_1} = \mathbb{C}^*_z \times \delta(w').$$

Hence for any $[z,w] \in D$, we have

$$D[z,w] = \mathbb{C}^*_z \times \delta(w)$$

which is independent of z. It follows that $\lambda[z,w]$, $[z,w] \in D$ is independent of z.

We analyze the boundary behavior of $\lambda[z,w]$. We consider different cases:

(1) Let $[z_0, w_0] \in \partial D \setminus \boldsymbol{T}_a$; i.e., $z_0 \neq 0$, $w_0 = 0 + iv_0 \neq 0$. We let $[z,w] \in D$ approach $[z_0, iv_0]$. If $z \to z_0$ and $w \to iv_0$, then regarding (4.24) with $\theta = \pi/2$ we see that

$$D[z,w] = \mathbb{C}^*_z \times \delta(w) \text{ approaches } D[z_0, iv_0] = \mathbb{C}^*_z \times \{\Im w < 0\}.$$

In particular $e \in \partial(\mathbb{C}^*_z \times \{\Im w < 0\})$; thus as $[z,w]$ approaches $[z_0, iv_0]$, we have $\operatorname{dist}(\partial D[z,w], e)$ tends to 0 and $\lambda[z,w]$ tends to $-\infty$.

(2) Let $[z_0, 0] \in \partial D \cap \boldsymbol{T}_a = \boldsymbol{T}_a$ where $z_0 \neq 0$. We let $[z,w] \in D$ approach $[z_0, 0]$ in such a way that $z \to z_0$ arbitrarily but $w \to 0$ in an angular sector; i.e., writing $w = |w|e^{i\theta}$, there exists θ_0 with $0 < \theta_0 < \pi/2$ so that $|\theta| < \theta_0$ as $|w| \to 0$. As before we have $D[z,w] = \mathbb{C}^*_z \times \delta(w)$. It follows from (4.24) that $\operatorname{dist}(\partial D[z,w], e) \geq \cos\theta_0$ for $|w| \leq 1$. Let A be the c-Robin constant for the region

$$G(\theta_0) := \{(z,w) \in \mathbb{C}^*_z \times \mathbb{C}^*_w : |z-1|^2 + |w-1|^2 < \cos^2\theta_0\}$$

with pole e. Then A is finite and since $G(\theta_0) \subset D[z,w]$ for $|w| \leq 1$, clearly $\lambda[z,w] > A$. Thus $-\lambda[z,w]$ is not an exhaustion function due to its boundary behavior at $\boldsymbol{T}_a$.

Finally, we let $X := z(\partial/\partial z)$ and $p_0 = [z_0, w_0] \in D^*$. Then the integral curve for X with initial value p_0 is given by

$$\sigma := p_0 \exp tX = (\mathbb{C}^*_z \times \{w_0\})/\sim\ \subset \widetilde{D^*}/\sim\ = D^*.$$

Thus this example does indeed satisfy (1) and (2) (ii-b) of Lemma 4.2.

5. Appendix A: Proofs of Lemma 3.1 and Proposition 1.1.

We give the proof of Lemma 3.1 and simultaneously that of Proposition 1.1. We first prove 1. of the lemma; hence we recall that

$$X_u = (\log|a|)z\,\frac{\partial}{\partial z} + (\log|b|)w\,\frac{\partial}{\partial w};$$

the integral curve of X_u with initial value $(1,1)$ is

$$\exp tX_u = \begin{cases} z = e^{(\log|a|)t}, \\ w = e^{(\log|b|)t}, \end{cases} \qquad t \in \mathbb{C};$$

we set $\widetilde{\sigma}_u := \{\exp tX_u : t \in \mathbb{C}\}/\sim\ \subset \mathcal{H}^*$ and we denote by $\widetilde{\Sigma}_u$ the closure of $\widetilde{\sigma}_u$ in $\mathcal{H}$. Consider case (1) where we let $(a,b) \in S_1$. There are two subcases: $\rho = \log|b|/\log|a| > 1$ is irrational, or $\rho = q/p$ is rational, $(p,q) = 1$, and $\tau = (q\arg a - p\arg b)/2\pi$ is irrational.

In the first subcase, taking the closure in $\mathbb{C}^*_z \times \mathbb{C}^*_w$ we have

$$Cl[z^{\log|b|} = w^{\log|a|}] = \{|z|^{\log|b|} = |w|^{\log|a|}\}$$

so that

$$\widetilde{\Sigma}_u = \{|z|^{\log|b|} = |w|^{\log|a|}\}/\sim .$$

One can check that

$$\{|z|^{\log|b|} = |w|^{\log|a|}\}^{\sim} := \{|z|^{\log|b|} = |w|^{\log|a|}\}\cdot\mathcal{I} = \{|z|^{\log|b|} = |w|^{\log|a|}\};$$

it follows that $\widetilde{\Sigma}_u$ is an irreducible, compact, Levi-flat hypersurface in $\mathcal{H}^*$.

For any $(z_0,w_0) \in \mathbb{C}^*_z \times \mathbb{C}^*_w$, we have

$$Cl[[z_0w_0]\exp tX] = \left\{\frac{|z|^{\log|b|}}{|w|^{\log|a|}} = \frac{|z_0|^{\log|b|}}{|w_0|^{\log|a|}}\right\}\Big/\sim \quad \text{and} \tag{5.1}$$

$$\left\{\frac{|z|^{\log|b|}}{|w|^{\log|a|}} = \frac{|z_0|^{\log|b|}}{|w_0|^{\log|a|}}\right\}^{\sim} = \left\{\frac{|z|^{\log|b|}}{|w|^{\log|a|}} = \frac{|z_0|^{\log|b|}}{|w_0|^{\log|a|}}\right\}. \tag{5.2}$$

Indeed, since $\mathbb{C}^* \times \mathbb{C}^*$ are the group of automorphism of $\mathcal{H}$, letting $(\xi,\eta) \in \mathbb{C}^* \times \mathbb{C}^*$ we have

$$[\xi,\eta] \in Cl[[z_0w_0]\exp tX] = (z_0,w_0)\cdot Cl[[1,1]\exp tX] = (z_0,w_0)\cdot\widetilde{\Sigma}_u.$$

Equivalently, $[z_0^{-1}\xi, w_0^{-1}\eta] \in \widetilde{\Sigma}_u$. By the argument in the previous paragraph, this is equivalent to $|z_0^{-1}\xi|^{\log|b|} = |w_0^{-1}\eta|^{\log|a|}$, proving (5.1). The assertion (5.2) is easily checked and it yields the validity of the definition of Σ_c for $c \in (0,+\infty)$ in (α) of Proposition 1.1. This proves (α) as well as 1.(1) of Lemma 3.1 in case $(a,b) \in S_1$ and ρ

is irrational.

If $(a,b)\in S_1$ and ρ is rational while τ is irrational, setting $pr\{z^{q/p}\}$ for the principal q/p-th root, we have

$$\begin{aligned}\widetilde{\sigma}_u &= \{w = pr\{z^{q/p}\}\}/\sim\\ &= \bigcup_{n\in\mathbb{Z}}\{(a^n z, (a^n z)^{q/p})\}/\sim \qquad \text{(by analytic continuation)}\\ &= \bigcup_{n\in\mathbb{Z}}\{(z, b^{-n}((a^n z)^{q/p}))\}/\sim\\ &= \bigcup_{k=0}^{p-1}\bigcup_{n\in\mathbb{Z}}\{(z, pr\{z^{q/p}\}e^{2\pi i(n\tau/p+qk/p)})\}/\sim . \end{aligned}\tag{5.3}$$

Since τ is irrational, we similarly have

$$\widetilde{\Sigma}_u = \{|w| = |z|^{q/p}\}/\sim\ = \{|w|^{\log|a|} = |z|^{\log|b|}\}/\sim .$$

A similar argument as before verifies (5.1) and (5.2), finishing the proof of 1.(1) of Lemma 3.1 and (α) of Proposition 1.1.

We next prove 1.(2). Let $(a,b)\in S_2$ so that $\rho = \log|b|/\log|a| \geq 1$ is rational; we write $\rho := q/p$, $(p,q)=1$; and $\tau := (q\arg a - p\arg b)/2\pi$ is also rational; we write $\tau := m/l$, $l\geq 1$, $(l,m)=\pm 1$ ($l=1$ for $\tau = 0$), where $0\leq \arg a, \arg b < 2\pi$.

We consider the circle $A := \{e^{i\theta} : 0\leq\theta\leq 2\pi\}$ and an arc $B : t\in[0,1]\to\zeta(t) = r(t)e^{i\theta(t)}$ connecting 1 and a in $\mathbb{C}_z$ where $r(t),\theta(t)$ are increasing in t. We set

$$\gamma_n := \{e^{in\theta} : 0\leq\theta\leq 2\pi\} \text{ for } n=\pm1,\pm2,\ldots;$$
$$\zeta_n = a^{n-1}B \text{ for } n\geq 1 \quad\text{and}\quad \zeta_n = a^{n+1}(-B) \text{ for } n\leq -1$$

where $-B$ is the arc with the opposite orientation of B. We define

$$\zeta^{(n)} := \zeta_1\cdot\zeta_2\cdots\zeta_n \text{ for } n\geq 1;\quad \zeta^{(n)} := \zeta_{-1}\cdot\zeta_{-2}\cdots\zeta_{-n} \text{ for } n\leq -1$$

so that $\zeta^{(n)}$ is an arc connecting 1 and a^n in $\mathbb{C}^*_z$.

Given $k,s\in\mathbb{Z}$, we perform an analytic continuation of the principal value $pr\{z^{q/p}\}$ of $z^{q/p}$ from $z=1$ to a^s along the curve $\gamma_k\cdot\zeta^{(s)}$: we have

$$\begin{aligned}\widetilde{\sigma}_u &= \{(z, z^{q/p})\}/\sim\\ &= \{(z, |z|^{q/p}e^{i(q/p)(\mathrm{Arg} z + k2\pi)})\}/\sim \qquad \text{(by anal. cont. along } \gamma_k)\\ &= \{(z, pr\{z^{q/p}\}\cdot e^{2\pi ikq/p})\}/\sim\\ &= \{(a^s z, |a|^{sq/p}|z|^{q/p}e^{i(q/p)(s\arg a + \mathrm{Arg}\, z)}\cdot e^{2\pi ikq/p})\}/\sim \qquad \text{(by anal. cont. along } \zeta^{(s)})\end{aligned}$$

$$= \{(z, pr\{z^{q/p}\}b^{-s}\ |a|^{sq/p}e^{isq\arg a/p} \cdot e^{2\pi ikq/p})\}/\sim$$
$$= \{(z, pr\{z^{q/p}\}e^{2\pi ism/pl} \cdot e^{2\pi ikq/p})\} \quad \text{since } q\arg a - p\arg b = 2\pi m/l.$$

We set

$$w_{ks}(z) = pr\{z^{q/p}\}e^{2\pi i(sm/pl+kq/p)} \in \{w \in \mathbb{C}^* : |w| = |z|^{q/p}\}. \tag{5.4}$$

In particular we have

$$w_{ks}(1) = e^{2\pi i(sm/pl+kq/p)} \in \{w \in \mathbb{C}^* : |w| = 1\}. \tag{5.5}$$

Setting $\mathcal{W}(1) := \{w_{ks}(1) : 0 \le k \le p-1,\ 0 \le s \le l-1\} \subset \mathbb{C}^*_w$, we show

1) $(1, \mathcal{W}(1))$ consists of pl different points in the fundamental domain $\mathcal{F}$, and hence $\mathcal{W}(1) = \{e^{2\pi i(n/pl)} : 0 \le n \le pl-1\}$;
2) if $w \in \mathbb{C}^*_w$, then $[1, w] \in \sigma_u$ if and only if $w \in \mathcal{W}(1)$;
3) for $(z_0, w_0) \in \mathbb{C}^*_z \times \mathbb{C}^*_w$ we consider the integral curve $[z_0, w_0]\exp tX_u$ of X_u with initial value at $[z_0, w_0]$. Let $(z, w) \in \mathbb{C}^*_z \times \mathbb{C}^*_w$. Then

$$[z, w] \in [z_0, w_0]\exp tX_u \Longleftrightarrow w^{pl}/z^{ql} = w_0^{pl}/z_0^{ql} \quad \text{in } \mathbb{C}^*.$$

To prove 1), assume $w_{k_1\,s_1}(1) = w_{k_2\,s_2}(1)$ with $0 \le k_1,\ k_2 \le p-1$, $0 \le s_1,\ s_2 \le l-1$. Then we can find $N \in \mathbb{Z}$ with

$$\frac{(s_1 - s_2)m}{pl} + \frac{(k_1 - k_2)q}{p} = N.$$

Using $(l, m) = 1$ and $(p, q) = 1$ it follows that $k_1 = k_2$ and $s_1 = s_2$, which proves 1).

To prove 2), let $(1, w_0) \in \widetilde{\sigma}_u \cap \mathcal{F}$. Then w_0 is determined as follows: we can find $S \in \mathbb{Z}$ and a (not necessarily simple) curve C connecting 1 and a^S in $\mathbb{C}^*_z$ such that if we perform an analytic continuation $w = w_C(z)$ of $pr\{z^{q/p}\}$ along C, then the value w^* of $w_C(z)$ at the terminal point of C (which lies over a^S) satisfies $w_0 = b^{-S}w^*$. Since C is homotopic to the curve $\gamma_k \cdot \zeta^{(s)}$ for some $k, s \in \mathbb{Z}$, it follows from (5.5) and 1) that $w_0 = e^{2\pi i(sm+kql)/pl} \in \mathcal{W}(1)$.

To prove 3), we first consider the case where $(z_0, w_0) = (1, 1)$. Using arguments similar to 2), and using (5.4), for $(z, w) \in \mathbb{C}^*_z \times \mathbb{C}^*_w$ we have

$$[z, w] \in \widetilde{\sigma}_u \Longleftrightarrow w = pr\{z^{q/p}\}e^{2\pi i(n/pl)} \text{ for some } n \text{ with } 0 \le n \le pl-1.$$

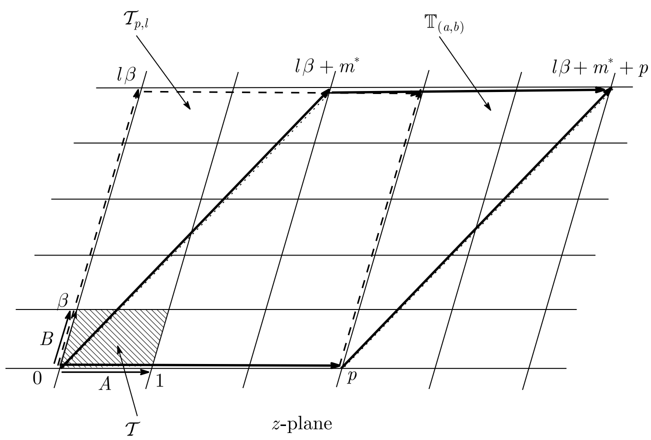
The point (z, w) in the right-hand-side satisfies $w^{pl} = z^{ql}$; conversely, if $(z, w) \in \mathbb{C}^*_z \times \mathbb{C}^*_w$ satisfies $w^{pl} = z^{ql}$ then it satisfies the right-hand-side of the displayed equivalence. Since $\sigma_u = [1, 1]\exp tX_u$, this shows that 3) is true for $(z_0, w_0) = (1, 1)$. For general $(z_0, w_0) \in \mathbb{C}^*_z \times \mathbb{C}^*_w$, fix (z, w) with $[z, w] \in [z_0, w_0]\exp tX_u$. Since any $(\alpha, \beta) \in \mathbb{C}^* \times \mathbb{C}^*$ induces an automorphism in $\mathcal{H}$, we have

$$[z,w]\in[z_0,w_0]\exp tX_u=(z_0,w_0)\cdot[1,1]\exp tX_u,$$

i.e., $[z_0^{-1}z, w_0^{-1}w]\in[1,1]\exp tX_u$. From the $(1,1)$ case we conclude that $(w_0^{-1}w)^{pl}=(z_0^{-1}z)^{ql}$, so that $w^{pl}/z^{ql}=w_0^{pl}/z_0^{ql}$.

We note that 3) guarantees the validity of the definition of σ_c in assertion (β) in Proposition 1.1 and proves (β). Furthermore 3) proves the equality $\{w^p=z^q\}/\sim\ =\{w^{pl}=z^{ql}\}/\sim$ and $\{w^{pl}=z^{ql}\}^{\sim}=\{w^{pl}=z^{ql}\}$ in $\mathbb{C}^*\times\mathbb{C}^*$.

We next prove 1.(2) ii) in Lemma 3.1. Here, $(a,b)\in S_2$. We show the curve $\widetilde{\sigma}_u$, as a Riemann surface, is equivalent to a torus $\mathbb{T}_{(a,b)}$. To construct $\mathbb{T}_{(a,b)}$ we begin with the annulus $\{1\le|z|\le|a|\}$. Identifying the inner boundary $A=\{e^{i\theta}:0\le\theta\le 2\pi\}$ with the outer boundary $\{a\,e^{i\theta}:0\le\theta\le 2\pi\}$, we get a torus $\mathcal{T}$. Recall that $B:t\in[0,1]\to\zeta(t)=r(t)e^{i\theta(t)}$ is an arc connecting 1 and a in $\mathbb{C}_z$. Let $\mathcal{T}_{p,l}$ be the covering space of $\mathcal{T}$ which covers the circle A p times and covers the arc B l times. We offer a realization of the tori $\mathcal{T}$ and $\mathcal{T}_{p,l}$ in the following figure:



Since $w_{k0}(1)$ for $0\le k\le p-1$ from (5.5) are p distinct points and $w_{p0}(1)=1=w_{00}(1)$, we can form the covering space $\mathcal{T}_{p,0}$ of $\mathcal{T}$ which covers A p times. Now $w_{0s}(1)$ for $0\le s\le l-1$ are l distinct points and $w_{0l}(1)=e^{2\pi im/p}$. If m/p is an integer, then $w_{0l}(1)=w_{00}(1)$, in which case the covering space $\mathcal{T}_{p,0}$ of $\mathcal{T}$ covers B l times. Since $w_{ks}(1)$ for $0\le k\le p-1$, $0\le s\le l-1$ are l distinct points by 1), it follows in this case that $\widetilde{\sigma}_u$ is equivalent to the torus $\mathcal{T}_{p,l}$. If, on the other hand, m/p is not an integer, there exists k with $1\le k\le p-1$ such that $w_{k0}(1)=w_{0l}(1)$. Setting $m^*:=p-k$, we have $1\le m^*\le p-1$. We perform an analytic continuation of $pr\{z^{q/p}\}$ along the closed curve $B^lA^{m^*}$ which traverses B l times and then A m^* times. In doing so, we return to $pr\{z^{q/p}\}$. Using 1), we see that σ_u is equivalent to the torus $\mathbb{T}_{(a,b)}$ pictured in the figure. This proves 1.(2) of Lemma 3.1.

Remark 5.1. As noted in the introduction, if $(a,b)\in S_2$ we have the non-constant meromorphic function $f[z,w]=w^P/z^Q$ on $\mathcal{H}$ with $a^Q=b^P$. We see that $P=pl$ and $Q=ql$; and for $c\in\mathbb{C}^*$ and $(z_0,w_0)\in\mathbb{C}_z^*\times\mathbb{C}_w^*$ with $f(z_0,w_0)=c$, the level curve $f(z,w)=c$ coincides with $([z_0,w_0]\exp tX_u)^{\widetilde{}}$.

We turn to 2. of Lemma 3.1 and we first prove 2.(1). Thus let

$$X = \alpha z\frac{\partial}{\partial z} + \beta w\frac{\partial}{\partial w} \notin \{cX_u : c \in \mathbb{C}\}$$

with $\alpha, \beta \neq 0$. Considering X as a vector field in $\mathbb{C}_z^* \times \mathbb{C}_w^*$, the integral curve $\{\exp tX : t \in \mathbb{C}\}$ of X with initial value $e = (1,1)$ in $\mathbb{C}_z^* \times \mathbb{C}_w^*$ is $w = z^{\beta/\alpha}$. Let $\beta/\alpha = A + Bi$ where A, B are real. Then

$$w = z^{A+Bi} = e^{(A+Bi)\log z}.$$

Fix $z \in \mathbb{C}^*$ and let $\operatorname{Log} z = \log|z| + i\theta$ $(0 \leq \theta < 2\pi)$ be the principal value. By analytic continuation, over z we have

$$\begin{aligned} w_n(z) &= e^{(A+Bi)(\log|z|+i(\theta+2n\pi))} \\ &= e^{A(\log|z|+i\theta)}e^{[-B(\theta+2n\pi)]}e^{i(A2n\pi+B\log|z|)}, \quad n \in \mathbb{Z}. \end{aligned} \tag{5.6}$$

We first assume $B \neq 0$, e.g., $B > 0$. Then $|w_n(z)| = (|z|^A e^{-B\theta})e^{-2nB\pi}$, $n \in \mathbb{Z}$. Hence $\lim_{n\to+\infty}|w_n(z)| = 0$ in $\mathbb{C}_w$; thus

$$\lim_{n\to+\infty}(z, w_n(z))/\sim\; = [z, 0] \in \boldsymbol{T}_a \ \text{ in } \mathcal{H}.$$

Since $z \in \mathbb{C}^*$ is arbitrary, we have $\boldsymbol{T}_a \subset \Sigma$, the closure of $\sigma = \{w = z^{A+Bi}\}/\sim$ in $\mathcal{H}$.

Since $w = z^{A+Bi}$ can be written as

$$z = w^{A'+iB'} \quad \text{where } A' = A/(A^2+B^2),\ B' = -B/(A^2+B^2) < 0,$$

we similarly have $\boldsymbol{T}_b \subset \Sigma$. This proves 2.(1) in case $B \neq 0$.

We next assume $B = 0$ and $A \neq \rho$. Since the proof is similar, we shall prove 2.(1) assuming $-\infty < A < \rho$. For $z \in \mathbb{C}^*$ we have $\operatorname{Log} z = \log|z| + i\theta$ $(0 \leq \theta < 2\pi)$. By analytic continuation of $w(z) = z^A = e^{A(\log|z|+i\arg z)}$ along an arbitrary path l from z to $a^k z$ where $k \in \mathbb{Z}$ is arbitrary, we have

$$w(a^k z) = (a^k z)^A = |a^k z|^A e^{iA\arg a^k z} = |a^k z|^A e^{iA(k\arg a+\theta+2n\pi)}, \quad n \in \mathbb{Z}.$$

Thus $p_k := (a^k z, w(a^k z)) \in \sigma$. In $\mathcal{H}^*$ the point p_k coincides with

$$(z, w(a^k z)/b^k)/\sim\; = (z, \widetilde{w}_k(z))/\sim\; \in \sigma \tag{5.7}$$

where $\widetilde{w}_k(z) := |a^A/b|^k|z|^A e^{ik(A\arg a-\arg b)}e^{iA(\theta+2n\pi)} \in \mathbb{C}_z^*$.

Using $\rho = \log|b|/\log|a|$,

$$|\widetilde{w}_k(z)| = |z|^A(|a|^{kA}/|b|^k) = |z|^A(|a|^{A-\rho})^k. \tag{5.8}$$

Since $A < \rho$ and $|a| > 1$, it follows that $\lim_{k\to+\infty}|\widetilde{w}_k(z)| = 0$, so that $[z, 0] \in \Sigma$. Since

$z \in \mathbb{C}^*$ is arbitrary, we have $\Sigma \supset \boldsymbol{T}_a$.

Since $w = z^A$ can be written as $z = w^{1/A}$, we have by analytic continuation $q_k := ((b^k w)^{1/A}, b^k w) \in \sigma$ for any $k \in \mathbb{Z}$. In $\mathcal{H}^*$, the point q_k coincides with $((b^k w)^{1/A}/a^k, w)/\sim\ =: (\widetilde{z}_k(w), w)/\sim$. Since $|\widetilde{z}_k(w)| = |w|^{1/A}(|a|^{\rho-A})^{k/A}$, we have $\lim_{k\to-\infty} |\widetilde{z}_k(w)| = 0$ if $A > 0$ and $\lim_{k\to+\infty} |\widetilde{z}_k(w)| = 0$ if $A < 0$. Since $w \in \mathbb{C}^*$ is arbitrary, we have $\Sigma \supset \boldsymbol{T}_b$, which proves 2.(1).

Finally, to prove 2.(2), let $X = \alpha z(\partial/\partial z) \neq 0$. Then the integral curve σ of X passing through $[1,1]$ in $\mathcal{H}$ is given by $\{(e^{\alpha t}, 1) : t \in \mathbb{C}\}/\sim\ = \mathbb{C}^*_z \times \{1\}/\sim$. In the fundamental domain $\mathcal{F}$,

$$\sigma = (\{0 < |z| \le |a|\}, 1) \cup (\{1 < |z| \le |a|\}, 1/b) \cup (\{1 < |z| \le |a|\}, 1/b^2) + \cdots,$$

so that $\Sigma = (\{|z| \le 1\}, 1) \bigcup_{n=1}^{\infty} (\{1 \le |z| \le |a|\}, 1/b^n) \cup \boldsymbol{T}_a$, proving 2.(2). □

We end this appendix with a remark. Let $X = \alpha z(\partial/\partial z) + \beta w(\partial/\partial w) \notin \{cX_u : c \in \mathbb{C}\}$ with $\alpha \neq 0$, $\beta \neq 0$ and set $\beta/\alpha = A + Bi$ as in the proof of 2.(1). Fix $(z_0, w_0) \in \mathbb{C}^* \times \mathbb{C}^*$ and for $c = w_0/z_0^{A+Bi}$ consider the integral curve $\sigma_c = \{w = cz^{A+Bi}\}/\sim$ of X passing through $[z_0, w_0]$ in $\mathcal{H}$. For each $z' \in \{1 < |z| < |a|\}$ we consider the set of all points $w_k(z')$, $k = 1, 2, \dots$ in $\{|w| < |b|\}$ with $[z', w_k(z')] \in \sigma_c$. The following fact was used to prove (2) (iii) in Lemma 4.2.

Remark 5.2. If A is irrational, then there exists a subsequence $\{w_{k_j}(z')\}_{j=1,2,\dots}$ with the properties that $\lim_{j\to\infty} |w_{k_j}(z')| = 0$ and the closure of the set $\{\arg w_{k_j}(z')\}_{j=1,2,\dots}$ modulo 2π is equal to $[0, 2\pi]$.

Proof. Since $\sigma_c = \{w = cz^{A+Bi}\}/\sim$ and $\sigma = \{w = z^{A+Bi}\}/\sim$ where σ is defined in the proof of 2.(1), it suffices to prove the result using $\sigma_c = \sigma$. If $B \neq 0$, we can assume $B > 0$. Since A is irrational, formula (5.6) gives the result. If $B = 0$ we have $A \neq \rho$, and we can assume $-\infty < A < \rho$. In this case, since A is irrational, formulas (5.7) and (5.8) imply the result. □

6. Appendix B: Proof of Lemma 3.2.

We give the proof of Lemma 3.2. The lemma is local, hence we may assume from (i) and (ii) that the unit outer normal vector of the curve $\partial D(0)$ in Δ_2 is $(0,1)$; i.e., $\partial D(0)$ is tangent to the u-axis at $w = 0$ where $w = u + iv$. Thus, we may assume that $\psi(z, w)$ has the following Taylor expansion about the origin $(z, w) = (z, (u, v)) = (0, (0, 0))$:

$$\psi(z, w) = v + p_0(z) + p_1(z)u + p_2(z)u^2 + \cdots \tag{6.1}$$

where each $p_i(z)$, $i = 0, 1, 2, \dots$ is a C^ω-smooth real-valued function and

$$p_0(0) = 0 \quad \text{and} \quad p_1(0) = 0.$$

We may further assume that formula (6.1) holds on $(z, u) \in \Delta_1 \times (-r_2, r_2)$ where $\Delta_2 = \{|w| < r_2\}$. Thus we write

$$D = \{v + p_0(z) + p_1(z)u + p_2(z)u^2 + \cdots < 0 : (z,w) \in \Delta_1 \times \Delta_2\};$$
$$\mathcal{S} = \partial D = \{v + p_0(z) + p_1(z)u + p_2(z)u^2 + \cdots = 0 : (z,w) \in \Delta_1 \times \Delta_2\},$$

or equivalently,

$$D : v < -(p_0(z) + p_1(z)u + p_2(z)u^2 + \cdots) \quad \text{in } \Delta_1 \times \Delta_2, \tag{6.2}$$

and, for each $z \in \Delta_1$,

$$S(z) : v = -(p_0(z) + p_1(z)u + p_2(z)u^2 + \cdots) \quad \text{in } \Delta_2.$$

In particular, $-ip_0(z) \in S(z)$. By condition (iii) we have

$$p_0(z) \not\equiv 0 \quad \text{on } \Delta_1. \tag{6.3}$$

Since $\psi(z,w)$ satisfies the Levi condition (3.1) on $\psi(z,w) = 0$, using the notation

$$\psi(z,w) = \frac{w - \overline{w}}{2i} + p_0(z) + p_1(z)\frac{w + \overline{w}}{2} + p_2(z)(\frac{w + \overline{w}}{2})^2 + \cdots,$$

on points $(z,w) = (z, u + iv)$ with $\psi(z, u + iv) = 0$ we obtain

$$\begin{aligned}
\mathcal{L}\psi(z,w) = {} & \left(\frac{\partial^2 p_0(z)}{\partial z \partial \overline{z}} + \frac{\partial^2 p_1(z)}{\partial z \partial \overline{z}}u + \frac{\partial p_2(z)}{\partial z \partial \overline{z}}u^2 + \cdots\right)\left|\frac{1}{2i} + \frac{1}{2}p_1(z) + p_2(z)u + \cdots\right|^2 \\
& - 2\Re\Big\{\left(\frac{1}{2}\frac{\partial p_1(z)}{\partial z} + \frac{\partial p_2(z)}{\partial \overline{z}}u + \cdots\right)\left(\frac{\partial p_0(z)}{\partial \overline{z}} + \frac{\partial p_1(z)}{\partial \overline{z}}u + \frac{\partial p_2(z)}{\partial \overline{z}}u^2 + \cdots\right) \\
& \qquad \times \left(\frac{1}{2i} + \frac{1}{2}p_1(z) + p_2(z)u + \cdots\right)\Big\} \\
& + \left(\frac{1}{2}p_2(z) + 3p_3(z)u + \cdots\right)\left|\frac{\partial p_0(z)}{\partial z} + \frac{\partial p_1(z)}{\partial z}u + \frac{\partial p_2(z)}{\partial z}u^2 + \cdots\right|^2 \geq 0.
\end{aligned}$$

In particular,

$$\begin{aligned}
\mathcal{L}\psi(z, 0 + iv) = {} & \frac{1}{4}\ (1 + p_1(z)^2)\frac{\partial^2 p_0(z)}{\partial z \partial \overline{z}} \\
& - \frac{1}{2}\Re\Big\{\frac{\partial p_1(z)}{\partial z}\frac{\partial p_0(z)}{\partial \overline{z}}(-i + p_1(z))\Big\} + \frac{1}{2}p_2(z)\left|\frac{\partial p_0(z)}{\partial z}\right|^2 \geq 0 \\
& \text{on } v + p_0(z) = 0 \text{ for } z \in \Delta_1.
\end{aligned}$$

Since this expression for $\mathcal{L}\psi(z, 0 + iv)$ is independent of v, we have

$$(1+p_1(z)^2)\frac{\partial^2 p_0(z)}{\partial z\partial\overline{z}} - 2\Re\left\{\frac{\partial p_1(z)}{\partial z}\frac{\partial p_0(z)}{\partial\overline{z}}(-i+p_1(z))\right\} + 2p_2(z)\left|\frac{\partial p_0(z)}{\partial z}\right|^2 \geq 0$$
$$\text{for } z\in\Delta_1. \quad (6.4)$$

This formula will be used later on in the proof.

Claim. *To prove the lemma, it suffices to show that for $r_1>0$ sufficiently small and $\delta_1=\{|z|<r_1\}$,*

$(\diamondsuit)$ *there exists $z^*\in\delta_1$ such that $p_0(z^*)>0$.*

Indeed, if $(\diamondsuit)$ is true, consider the segment $[0,z^*]$ in δ_1 and the set

$$\boldsymbol{s} := \bigcup_{z\in[0,z^*]} S(z)\subset\Delta_2.$$

The arc $S(z)$ in Δ_2 varies continuously with $z\in\Delta_1$. Hence it follows from $0\in S(0)$, $-ip_0(z^*)\in S(z^*)$, $-p_0(z^*)<0$ and (6.2) that there exists a sufficiently small disk $\delta_2\subset\Delta_2$ centered at $w=0$ with $D(0)\cap\delta_2\subset\boldsymbol{s}$.

Thus we turn to the proof of $(\diamondsuit)$. We have two cases, depending on whether $(\partial p_0/\partial z)(0)$ vanishes:

Case (i): $(\partial p_0/\partial z)(0)\neq 0$.

Since $p_0(0)=0$, we have

$$p_0(x+iy)=ax+by+O(|z|^2)\quad\text{near } z=0$$

with $(a,b)\neq(0,0)$. It is clear that there exist $z^*\in\delta_1$ which satisfies $(\diamondsuit)$.

Case (ii): $(\partial p_0/\partial z)(0)=0$.

In this case, we have the following Taylor expansion of $p_0(z)$ about $z=0$:

$$(1)\quad p_0(z)=\Re\{a_{20}z^2\}+a_{11}z\overline{z}+\cdots+J_{2n-1}+J_{2n}+O(|z|^{2n+1})\quad\text{near } z=0,$$

where

$$J_{2n-1}=\Re\left\{\sum_{k=0}^{n-1}a_{2n-1-k,k}\,z^{2n-1-k}\,\overline{z}^k\right\},\quad J_{2n}=\Re\left\{\sum_{k=0}^{n-1}a_{2n-k,k}\,z^{2n-k}\,\overline{z}^k\right\}+a_{nn}|z|^{2n}.$$

Here a_{ij} is, in general, a complex number for $i\neq j$; while a_{ii} is real.

1^{st} step: Since $(\partial p_0/\partial z)(0)=0$ and $p_0(0)=p_1(0)=0$, inequality (6.4) reduces to

$$\frac{\partial^2 p_0}{\partial z\partial\overline{z}}(0)\geq 0,\quad\text{i.e.,}\quad a_{11}\geq 0.$$

If $a_{11} > 0$, (1) implies that

$$\frac{\partial^2 p_0}{\partial z \partial \overline{z}}(z) = a_{11} + O(|z|) \geq \frac{a_{11}}{2} > 0 \quad \text{near } z = 0.$$

Thus $p_0(z)$ is strictly subharmonic on a sufficiently small disk $\delta_1' := \{|z| < r'\} \subset \delta_1$; hence there exists z^* with $|z^*| = r'/2$ and $p_0(z^*) > p_0(0) = 0$, proving $(\diamondsuit)$.

If $a_{11} = 0$, then (1) becomes, for $z = re^{i\theta}$,

$$p_0(z) = \Re\{a_{20}z^2\} + O(|z|^3) = |z|^2 \Re\{a_{20}e^{2i\theta} + O(|z|)\} \quad \text{near } z = 0.$$

If $a_{20} = |a_{20}|e^{i\theta_0} \neq 0$, then for $z^* \in \delta_1$ of the form $z^* = r^* e^{-i\theta_0/2} \neq 0$ with r^* sufficiently small, we have

$$p_0(z^*) = (r^*)^2(|a_{20}| + O(|z^*|)) \geq (r^*)^2 \frac{|a_{20}|}{2} > 0,$$

which proves $(\diamondsuit)$.

Thus it suffices to prove $(\diamondsuit)$ in the following two cases when $n \geq 2$:

$$\text{Case (I)}: \ p_0(z) = J_{2n-1}(z) + O(|z|^{2n}) \ \text{ near } z = 0$$

where

$$J_{2n-1}(z) := \Re\{a_{2n-1}z^{2n-1} + a_{2n-2}z^{2n-2}\overline{z} + \cdots + a_n z^n \overline{z}^{n-1}\} \quad \text{in } \mathbb{C}_z;$$

a_i is, in general, a complex number; and

$$(a_{2n-1}, a_{2n-2}, \ldots, a_n) \neq (0, 0, \ldots, 0). \tag{6.5}$$

$$\text{Case (II)}: \ p_0(z) = J_{2n}(z) + O(|z|^{2n+1}) \ \text{ near } z = 0$$

where

$$J_{2n}(z) := \Re\{a_{2n}z^{2n} + a_{2n-1}z^{2n-1}\overline{z} + \cdots + a_{n+1}z^{n+1}\overline{z}^{n-1}\} + a_n|z|^{2n} \quad \text{in } \mathbb{C}_z;$$

a_i for $n+1 \leq i \leq 2n$ is, in general, a complex number; a_n is a real number; and

$$(a_{2n}, a_{2n-1}, \ldots, a_{n+1}, a_n) \neq (0, 0, \ldots, 0, 0). \tag{6.6}$$

We first assume Case (I). Setting $z = |z|e^{i\theta}$, we have

$$J_{2n-1}(z) = |z|^{2n-1}\Re\{a_{2n-1}e^{i(2n-1)\theta} + a_{2n-2}e^{i(2n-3)\theta} + \cdots + a_n e^{i\theta}\} \quad \text{in } \mathbb{C}_z.$$

We consider the polynomial in Z defined by

$$g(Z) := a_{2n-1}Z^{2n-1} + a_{2n-2}Z^{2n-3} + \cdots + a_n Z.$$

Note that $g(Z) \not\equiv 0$ by (6.5). Thus $g(Z) \neq 0$ for all Z with $|Z| = r$ for some $0 < r < 1$. Since $g(0) = 0$, by the argument principle $\int_{|Z|=r} d\arg g(Z) \geq 1$, hence there exists $0 \leq \theta' < 2\pi$ such that $\Re\, g(re^{i\theta'}) > 0$. By the maximum principle for the harmonic function $\Re\, g(Z)$ on $\{|Z| \leq 1\}$, there exists $0 \leq \theta^* \leq 2\pi$ such that

$$A := \Re\, g(e^{i\theta^*}) \geq \Re\, g(re^{i\theta'}) > 0.$$

Since $J_{2n-1}(z) = |z|^{2n-1} g(e^{i\theta})$, we have

$$\begin{aligned} p_0(|z|e^{i\theta^*}) &= |z|^{2n-1}A + O(|z|^{2n}) \quad \text{for } 0 < |z| \ll 1 \\ &\geq |z|^{2n-1}A/2 > 0 \qquad\quad \text{for } 0 < |z| \ll 1, \end{aligned}$$

showing that $(\diamondsuit)$ is true in Case (I).

We next assume Case (II). For $z = |z|e^{i\theta}$

$$\frac{\partial^2 p_0(z)}{\partial z \partial \overline{z}} = |z|^{2n-2}\big(\Re\{(*)\} + n^2 a_n + O(|z|)\big) \tag{6.7}$$

where

$$(*) = (2n-1)a_{2n-1}e^{i(2n-2)\theta} + (2n-2)2 \cdot a_{2n-2}e^{i(2n-4)\theta} + \cdots + (n+1)(n-1)a_{n+1}e^{i2\theta}.$$

We substitute this in (6.4) to obtain

$$\begin{aligned} &(1 + O(1)^2)|z|^{2n-2}\big(\Re\{(*)\} + n^2 a_n + O(|z|)\big) \\ &\quad - 2\Re\big\{O(1)O(|z|^{2n-1})(-i + O(1))\big\} + 2\,O(|z|)\,O(|z|^{2n-1})^2 \geq 0 \end{aligned}$$

for $|z|$ sufficiently small. Dividing both sides by $(1 + O(1)^2)|z|^{2n-2} > 0$ with $|z| > 0$ and then letting $|z| \to 0$, we have

$$\Re\{(*)\} + n^2\, a_n \geq 0 \quad \text{for all } 0 \leq \theta < 2\pi. \tag{6.8}$$

We substitute this in the definition of $p_0(z)$ in Case (II) to obtain

$$\begin{aligned} p_0(z) \geq |z|^{2n}\Re\bigg\{ &a_{2n}e^{i2n\theta} + a_{2n-1}\left(1 - \frac{2n-1}{n^2}\right)e^{i(2n-2)\theta} \\ &+ a_{2n-2}\left(1 - \frac{(2n-2)2}{n^2}\right)e^{i(2n-4)\theta} \\ &+ \cdots + a_{n+1}\left(1 - \frac{(n+1)(n-1)}{n^2}\right)e^{i2\theta}\bigg\} + O(|z|^{2n+1}) \end{aligned}$$

for $|z|$ sufficiently small.

We divide the proof of Case (II) in two subcases:

$$\text{Case (II-1):} \quad (a_{2n}, a_{2n-1}, \dots, a_{n+1}) \neq (0, 0, \dots, 0);$$
$$\text{Case (II-2):} \quad (a_{2n}, a_{2n-1}, \dots, a_{n+1}) = (0, 0, \dots, 0).$$

From (6.6), $a_n \neq 0$ in Case (II-2). In Case (II-1) we consider the polynomial

$$g(Z) = a_{2n}Z^{2n} + a_{2n-1}\left(1 - \frac{2n-1}{n^2}\right)Z^{2n-2} + a_{2n-2}\left(1 - \frac{(2n-2)2}{n^2}\right)Z^{2n-4}$$
$$+ \cdots + a_{n+1}\left(1 - \frac{(n+1)(n-1)}{n^2}\right)Z^2.$$

Since $n \geq 2$, we have $(1 - (2n-k)k/n^2) \neq 0$ for $k = 1, 2, \dots, n-1$ so that $g(Z) \not\equiv 0$ on $\mathbb{C}_Z$ and $g(0) = 0$. By the same reasoning as in Case (I) we have the existence of $0 \leq \theta^* < 2\pi$ and $A > 0$ with

$$p_0(|z|e^{i\theta^*}) \geq |z|^{2n}A/2 > 0 \quad \text{for } 0 < |z| \ll 1,$$

which proves $(\diamondsuit)$ in Case (II-1).

In Case (II-2) we have $(*) = 0$ in (6.7) and hence $a_n \geq 0$ from (6.8); thus $a_n > 0$. Using (6.7) we have

$$\frac{\partial^2 p_0(z)}{\partial z \partial \bar{z}} \geq |z|^{2n-2}a_n + O(|z|^{2n-2}) \geq |z|^{2n-2}a_n/2 \geq 0$$

for z in a sufficiently small disk δ centered at $z = 0$. In other words, $p_0(z)$ is subharmonic on δ and is strictly subharmonic in $\delta \setminus \{0\}$. Thus, for a given $0 < r < r_0$, we can find $0 \leq \theta^* < 2\pi$ with $p_0(re^{i\theta^*}) > 0$, which proves $(\diamondsuit)$ in Case (II-2). This completes the proof of $(\diamondsuit)$. $\square$

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