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Spin representations of twisted central products of double covering finite groups and the case of permutation groups

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Abstract. Let S be a finite group with a character, sgn, of order 2, and S' its central extension by a group $Z = \langle z\rangle$ of order 2. A representation π of S' is called *spin* if $\pi(z\sigma') = -\pi(\sigma')$ $(\sigma' \in S')$, and the set of all equivalence classes of spin irreducible representations (= IRs) of S' is called the *spin dual* of S'. Take a finite number of such triplets $(S'_j, z_j, \mathrm{sgn}_j)$ $(1 \le j \le m)$. We define twisted central product $S' = S'_1 \hat{*} S'_2 \hat{*} \cdots \hat{*} S'_m$ as a double covering of $S = S_1 \times \cdots \times S_m$, $S_j = S'_j/\langle z_j\rangle$, and for spin IRs π_j of S'_j, define twisted central product $\pi = \pi_1 \hat{*} \pi_2 \hat{*} \cdots \hat{*} \pi_m$ as a spin IR of S'. We study their characters and prove that the set of spin IRs π of this type gives a complete set of representatives of the spin dual of S'. These results are applied to the case of representation groups S' for $S = \mathfrak{S}_n$ and $\mathfrak{A}_n$, and their (Frobenius-)Young type subgroups.

Introduction.

Let S' be a central extension of a finite group S by a central subgroup $Z = \langle z\rangle$ of order 2. A representation π of S' is called *spin* if $\pi(z\sigma') = -\pi(\sigma')$ $(\sigma' \in S')$, and the set of equivalence classes of spin irreducible representations (= IRs) is denoted by $\widehat{S'}^{\mathrm{spin}}$, and is called the *spin dual* of S'. The category of such (S', z) is denoted by $\mathscr{G}$. Suppose moreover that S has a character, denoted by sgn, of order 2, and extend it to S' through $S' \to S$. The subcategory consisting of all such triplets (S', z, sgn) is denoted by $\mathscr{G}'$. For $(S', z) \in \mathscr{G} \setminus \mathscr{G}'$, we add the superfluous datum $\mathrm{sgn} \equiv 1$ for convenience.

A typical example in $\mathscr{G}'$ is given by $S = \mathfrak{S}_n$, n-th symmetric group, and its representation group $\widetilde{\mathfrak{S}}_n = \mathfrak{T}'_n$ in the notation in [**15**, Section 3], and that of $\mathscr{G} \setminus \mathscr{G}'$ is given by $S = \mathfrak{A}_n$, n-th alternating group, and $\mathfrak{B}_n := \Phi_{\mathfrak{S}}^{-1}(\mathfrak{A}_n)$ with the canonical homomorphism $\Phi_{\mathfrak{S}} : \widetilde{\mathfrak{S}}_n \to \mathfrak{S}_n$. $\mathfrak{B}_n$ is a representation group of $\mathfrak{A}_n$ for $n \ge 4, \ne 6, 7$ [**15**, Section 4].

We study here the twisted central product $S' = S'_1 \hat{*} S'_2 \hat{*} \cdots \hat{*} S'_m \in \mathscr{G}$ of $(S'_j, z_j, \mathrm{sgn}_j) \in \mathscr{G}$ $(j \in \boldsymbol{I}_m = \{1, 2, \ldots, m\})$ and the twisted central product $\pi = \pi_1 \hat{*} \pi_2 \hat{*} \cdots \hat{*} \pi_m$ of spin IRs π_j of S'_j as a spin IR of S'. They turn out to be non-twisted central products if all $(S'_j, z_j, \mathrm{sgn}_j)$ but with at most one exception belong to $\mathscr{G} \setminus \mathscr{G}'$. The main idea comes from Schur's fundamental work [**15**] on spin (projective) representations of $\mathfrak{S}_n$

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and $\mathfrak{A}_n$, and the present paper clarifies the generality and the validity of his method, which at the first glance seems to be very mysterious. Important examples are the case of (Frobenius-)Young type subgroups $\mathfrak{S}_{\boldsymbol{\nu}} = \mathfrak{S}_{\nu_1} \times \mathfrak{S}_{\nu_2} \times \cdots \times \mathfrak{S}_{\nu_m} \hookrightarrow \mathfrak{S}_n$ and the corresponding Schur-Young type subgroup $\widetilde{\mathfrak{S}}_{\boldsymbol{\nu}} := \Phi_{\mathfrak{S}}^{-1}(\mathfrak{S}_{\boldsymbol{\nu}})$, where $\boldsymbol{\nu} = (\nu_j)_{j \in \boldsymbol{I}_m}$ is a partition of $n : \nu_1 + \nu_2 + \cdots + \nu_m = n$. The latter is canonically isomorphic to the twisted central product of $\widetilde{\mathfrak{S}}_{\nu_j}$ $(j \in \boldsymbol{I}_m)$. Also important is the case of $\mathfrak{A}_n \cap \mathfrak{S}_{\boldsymbol{\nu}}$ and $\mathfrak{B}_n \cap \widetilde{\mathfrak{S}}_{\boldsymbol{\nu}}$.

Our main results here are

(1) by the method of taking the twisted central product $\pi_1 \hat{*} \pi_2 \hat{*} \cdots \hat{*} \pi_m$ of spin IRs π_j of S'_j, we obtain a complete set of representatives of spin dual of the twisted central product $S'_1 \hat{*} S'_2 \hat{*} \cdots \hat{*} S'_m$ of $(S'_j, z_j, \mathrm{sgn}_j)$, $j \in \boldsymbol{I}_m$, and
(2) we give a formula for calculating the characters of twisted central product $\pi_1 \hat{*} \pi_2 \hat{*} \cdots \hat{*} \pi_m$, and
(3) applying these methods to the case of Schur-Young type subgroups of $\widetilde{\mathfrak{S}}_n$ and $\mathfrak{B}_n$, we obtain a complete parametrization of their spin duals, and certain results on their characters.

We observe that, contrary to the cases of ordinary tensor products and of non-twisted central products, the associativity law does not hold in general for the method of constructing twisted central product of π_j's, that is, even though $(\pi_1 \hat{*} \pi_2) \hat{*} \pi_3$ and $\pi_1 \hat{*} (\pi_2 \hat{*} \pi_3)$ are mutually equivalent, their intertwining operators are non-trivial and to be determined by explicit calculations (see Section 3).

The category $\mathscr{G}$ was first introduced by Hoffman-Humphreys [**6**], and the twisted central product of representations is studied in [**7**, Sections 3–5] and also in [**8**] (in modular cases) in case $m = 2$. Here we study it from a more general standpoint following ideas of Schur and clarifies the situation for general $m \geq 2$ over $\boldsymbol{C}$.

Our aim in the future is to apply these results to our study of spin (projective) representations of complex reflection groups $G(m, p, n)$, $p|m$, $4 \leq n \leq \infty$, succeeding to [**2**], [**3**] and [**4**] (cf. Section 10.5 below).

The present paper is organized as follows.

In Section 1, the twisted central product $S' = S'_1 \hat{*} S'_2 \hat{*} \cdots \hat{*}\, S'_m$ of groups $(S'_j, z_j, \mathrm{sgn}_j) \in \mathscr{G}$ $(j \in \boldsymbol{I}_m)$ are studied. In Section 2, the twisted central product $\pi_1 \hat{*} \pi_2 \hat{*} \cdots \hat{*} \pi_m$ of spin representations π_j of S'_j, as a spin representation of S', are defined and studied. Here $\pi_1 \hat{*} \pi_2 \hat{*} \cdots \hat{*}\, \pi_m$ is irreducible if so are all π_j's. In Section 3, properties of these twisted central products are examined.

In Section 4, a formula for calculating the character of $\pi_1 \hat{*} \pi_2 \hat{*} \cdots \hat{*} \pi_m$ is given, and the support of the character is evaluated. In Section 5, the completeness of the set of spin IRs of type $\pi_1 \hat{*} \pi_2 \hat{*} \cdots \hat{*} \pi_m$ of S' is proved, that is, any spin IR of S' is equivalent to someone of the above form. In Section 6, the similar results are given for the normal subgroup $B' := \{\sigma' \in S'; \mathrm{sgn}(\sigma') = 1\}$ of index 2 of S'.

In Section 7, we explain the intimate relations of the present study to the study of projective representations of finite groups in general, and to our study on projective IRs in the case of complex reflection groups in particular.

In Section 8, spin representations of $\widetilde{\mathfrak{S}}_n$ and those of $\mathfrak{B}_n$ are reviewed for preparing definitions and notations for Sections 9–10. In Section 9 and Section 10, the general theory in Sections 1–6 is applied to the case of Schur-Young type subgroups $\widetilde{\mathfrak{S}}_{\boldsymbol{\nu}}$ of $\widetilde{\mathfrak{S}}_n$

and to the case of their normal subgroups $\mathfrak{B}_n \cap \widetilde{\mathfrak{S}}_{\boldsymbol{\nu}}$ of index 2 defined by $\mathrm{sgn}(\cdot) = 1$.

1. Twisted central products.

1.1. Categories of double covering finite groups.

Suppose a finite group S' has a central element z of order 2 and a one-dimensional character of order ≤ 2, denoted by sgn, such that $\mathrm{sgn}(z) = 1$.

Denote by $\mathscr{G}$ the category of all such triplets (S', z, sgn) (after [**6**]), and by $\mathscr{G}'$ the subcategory of all such (S', z, sgn) that the order of sgn is exactly 2. We denote (S', z, sgn) simply by S' if there is no danger of misunderstanding.

Put $S := S'/Z$, $Z := \{e, z\}$, then $\{e\} \to Z \to S' \xrightarrow{\Phi} S \to \{e\}$ is exact, where e denotes the identity element and Φ denotes the natural homomorphism $S' \to S$. The group S' is a double covering of S, and sgn induces on S a character, denoted by sgn_S or again by sgn if there is no danger of confusion. In the case where sgn is of order 1 (or trivial), the symbol sgn is superfluous and (S', z, sgn) means simply a double covering (S', z) of S.

A representation π of S' is called *spin representation* of S' (also of S) if $\pi(z) = -I$, where I denotes the identity operator. A representation π with $\pi(z) = I$ is reduced to a linear representation of $S = S'/Z$.

For a representation π of S' (or of S), the product representation $\mathrm{sgn} \cdot \pi$ is called its *associate representation.* In case $\pi \cong \mathrm{sgn} \cdot \pi$ (equivalent), π is called self-associate. For a character χ of S' (or of S), $\mathrm{sgn} \cdot \chi$ is called its *associate character*, and in case $\chi = \mathrm{sgn} \cdot \chi$, it is called *self-associate* (in [**15**, Section 14], Schur called this kind of character as *zweiseitige Charakter*).

An element $\sigma' \in S'$ is called *even* or *odd* according as $\mathrm{sgn}(\sigma') = 1$ or $= -1$, and we put $\mathrm{ord}(\sigma') = 0$ or $= 1$ accordingly (the symbol $\mathrm{ord}(\cdot)$ is an abbreviation of the word order, and this is commonly used in our papers [**2**]–[**4**]). For $\sigma = \Phi(\sigma') \in S$, it is called *even* or *odd* according to σ', and we put $\mathrm{ord}(\sigma) := \mathrm{ord}(\sigma')$, $\mathrm{sgn}(\sigma) := \mathrm{sgn}(\sigma')$.

Notation 1.1. For $(S', z, \mathrm{sgn}) \in \mathscr{G}$, put $B' := \mathrm{sgn}^{-1}(\{1\})$, $C' := \mathrm{sgn}^{-1}(\{-1\})$, and $B := \mathrm{sgn}_S^{-1}(\{1\})$, $C := \mathrm{sgn}_S^{-1}(\{-1\})$. Then, $S' = B' \sqcup C'$, $S = B \sqcup C$.

Example 1.1. For $n \geq 2$, let $\mathfrak{S}_n$ be the n-th symmetric group, and $s_i = (i\ \ i{+}1)$, $i \in \boldsymbol{I}_{n-1} := \{1, 2, \ldots, n-1\}$, be simple transpositions. Further let $\widetilde{\mathfrak{S}}_n$ be the double covering group of $\mathfrak{S}_n$ with a central subgroup $Z = \{e, z\}$ of order 2 and a canonical homomorphism $\Phi_{\mathfrak{S}} : \widetilde{\mathfrak{S}}_n \to \mathfrak{S}_n$ such that $\{e\} \to Z \to \widetilde{\mathfrak{S}}_n \xrightarrow{\Phi_{\mathfrak{S}}} \mathfrak{S}_n \to \{e\}$ is exact, and with generators $\{z, r_i; i \in \boldsymbol{I}_{n-1}\}$ satisfying a set of fundamental relations

$$z^2 = e, \quad zr_i = r_i z \ (i \in \boldsymbol{I}_{n-1}), \quad r_i^2 = e, \quad (r_i r_{i+1})^3 = e, \quad r_i r_j = z r_j r_i \ (|i - j| \geq 2),$$

and $\Phi_{\mathfrak{S}}(r_i) = s_i$ $(i \in \boldsymbol{I}_{n-1})$ (in more detail, cf. Theorem 8.1 below). For $n \geq 4$, $\widetilde{\mathfrak{S}}_n$ is a representation group of $\mathfrak{S}_n$ which is introduced and denoted by $\mathfrak{T}'_n$ in [**15**, Section 3]. Put $(S', z, \mathrm{sgn}) = (\widetilde{\mathfrak{S}}_n, z, \mathrm{sgn})$ for $n \geq 2$. Then they are typical elements of the category $\mathscr{G}'$, and $B' := \Phi_{\mathfrak{S}}^{-1}(\mathfrak{A}_n)$, $C' = \mathfrak{C}_n := \widetilde{\mathfrak{S}}_n \setminus \mathfrak{B}_n$, $S = \mathfrak{S}_n$, $B = \mathfrak{A}_n$, $C = \mathfrak{S}_n \setminus \mathfrak{A}_n$. Note that $(B', z, \mathrm{sgn}|_{B'})$ belongs to $\mathscr{G}$. For $n = 1$, put $\widetilde{\mathfrak{S}}_1 := \{e, z\}$, then $\widetilde{\mathfrak{S}}_1 \in \mathscr{G} \setminus \mathscr{G}'$.

Definition 1.1. For $(S'_j, z_j, \mathrm{sgn}_j) \in \mathscr{G}$, $j \in \boldsymbol{I}_m = \{1, 2, \dots, m\}$, we define their *twisted central product* in two steps.

First step. Prepare a central element z of order 2, and consider a set $\mathfrak{H}$ of elements expressed as $z^a \sigma'_1 \sigma'_2 \cdots \sigma'_m$ ($a = 0, 1, \sigma'_j \in S'_j$ ($j \in \boldsymbol{I}_m$)). Preserving the product rule in each S'_j, we introduce in $\mathfrak{H}$ a product by

$$\sigma'_j \sigma'_k := z^{\mathrm{ord}(\sigma'_j)\,\mathrm{ord}(\sigma'_k)} \sigma'_k \sigma'_j \;\; (j \neq k,\;\, j, k \in \boldsymbol{I}_m), \tag{1.1}$$

and accordingly, for $b = 0, 1$ and $\sigma''_j \in S'_j$ ($j \in \boldsymbol{I}_m$),

$$\begin{aligned} &\left(z^a \sigma'_1 \sigma'_2 \cdots \sigma'_m\right)\left(z^b \sigma''_1 \sigma''_2 \cdots \sigma''_m\right) \\ &\quad := z^{a+b+\sum_{j>k} \mathrm{ord}(\sigma'_j)\,\mathrm{ord}(\sigma''_k)} \cdot \left(\sigma'_1 \sigma''_1\right)\left(\sigma'_2 \sigma''_2\right) \cdots \left(\sigma'_m \sigma''_m\right). \end{aligned} \tag{1.2}$$

Then $\mathfrak{H}$ becomes a group. Define a one-dimensional character sgn of $\mathfrak{H}$ by

$$\mathrm{sgn}\left(z^a \sigma'_1 \sigma'_2 \cdots \sigma'_m\right) := \prod_{j \in \boldsymbol{I}_m} \mathrm{sgn}_j(\sigma'_j). \tag{1.3}$$

Second step. Let Z' be the central subgroup of $\mathfrak{H}$ generated by $z_j z^{-1} = z_j z$ ($j \in \boldsymbol{I}_m$), and take the quotient group $\overline{\mathfrak{H}} := \mathfrak{H}/Z'$. Then a character, denoted again by sgn, is induced from that on $\mathfrak{H}$, and the triplet $(\overline{\mathfrak{H}}, z, \mathrm{sgn})$ is an element of the category $\mathscr{G}$. $\overline{\mathfrak{H}}$ is called the *twisted central product* of S'_j ($j \in \boldsymbol{I}_m$) and is denoted by $S'_1 \hat{*} S'_2 \hat{*} \cdots \hat{*} S'_m$.

Note that, when all S'_j ($j \in \boldsymbol{I}_m$) are from $\mathscr{G} \setminus \mathscr{G}'$ with at most one exception, then $S'_1 \hat{*} S'_2 \hat{*} \cdots \hat{*} S'_m$ is actually a non-twisted central product.

Each group S'_j is contained in $S' := S'_1 \hat{*} S'_2 \hat{*} \cdots \hat{*} S'_m$ isomorphically and its image is identified with S'_j. Then $S'_j \cap S'_k = Z := \langle z \rangle$ ($j \neq k$), and each element $\sigma' \in S'$ is expressed as $\sigma' = \sigma'_1 \sigma'_2 \cdots \sigma'_m$ ($\sigma'_j \in S'_j$). Introduce notations $S = S'/Z$, $S_j := S'_j/\langle z_j \rangle$, then S is canonically isomorphic to the direct product $S_1 \times S_2 \times \cdots \times S_m$, and is identified with it. Furthermore we put

$$\begin{cases} B'_j := \{\sigma'_j \in S'_j;\, \mathrm{sgn}_j(\sigma'_j) = 1\}, & C'_j := \{\sigma'_j \in S'_j;\, \mathrm{sgn}_j(\sigma'_j) = -1\}; \\ B_j := \{\sigma_j \in S_j;\, \mathrm{sgn}_j(\sigma_j) = 1\}, & C_j := \{\sigma_j \in S_j;\, \mathrm{sgn}_j(\sigma_j) = -1\}. \end{cases} \tag{1.4}$$

Each $(B'_j, z_j, \mathrm{sgn}_j|_{B'_j})$ is an element of $\mathscr{G}$ with trivial $\mathrm{sgn}_j|_{B'_j}$, and the product $B'_1 B'_2 \cdots B'_m = \{b'_1 b'_2 \cdots b'_m;\, b'_j \in B'_j \;(j \in \boldsymbol{I}_m)\}$ in S' is a group isomorphic to the non-twisted central product $B'_1 \hat{*} B'_2 \hat{*} \cdots \hat{*} B'_m$.

1.2. Conjugacy in a twisted central product.

Proposition 1.1. *Express two elements of a twisted central product* $S' := S'_1 \hat{*} \cdots \hat{*} S'_m$ *as* $\sigma' = \sigma'_1 \cdots \sigma'_m$, $\sigma'' = \sigma''_1 \cdots \sigma''_m$ ($\sigma'_j, \sigma''_j \in S'_j$ ($j \in \boldsymbol{I}_m$)).

(i) σ' *and* σ'' *coincide with each other if and only if*

$$\sigma'_j = z^{a_j} \sigma''_j \;\; (j \in \boldsymbol{I}_m), \quad a_1 + \cdots + a_m \equiv 0 \pmod 2.$$

(ii) *If σ' and σ'' are conjugate to each other under S', then for each $j \in \boldsymbol{I}_m$, σ'_j is conjugate to $z^{a_j}\sigma''_j$, with some $a_j = 0,1$, under S'_j. Conversely if σ'_j is conjugate to $z^{a_j}\sigma''_j$ under S'_j, then σ' and $z^a\sigma''$ are conjugate to each other under S' with some $a = 0,1$.*

PROOF. (ii) Suppose that σ' and σ'' are conjugate to each other under $\xi = \xi_1\cdots\xi_m$ ($\xi_j \in S'_j$), that is, $\xi\sigma'\xi^{-1} = \sigma''$. The left hand side equals $\xi\sigma'\xi^{-1} = z^a(\xi_1\sigma'_1\xi_1^{-1})\cdots(\xi_m\sigma'_m\xi_m^{-1})$, with

$$a \equiv \sum_{j\neq k \text{ in } \boldsymbol{I}_m} \operatorname{ord}(\sigma'_k)\operatorname{ord}(\xi_j) + \sum_{j<k \text{ in } \boldsymbol{I}_m} \operatorname{ord}(\xi_j)\operatorname{ord}(\xi_k) \pmod 2.$$

Accordingly $\xi_j\sigma'_j\xi_j^{-1} = z^{a_j}\sigma''_j$ ($\exists a_j,\ j \in \boldsymbol{I}_m$), with $a_1 + \cdots + a_m \equiv a$.

Converse discussion is also valid. □

EXAMPLE 1.2. For a $\sigma' \in \widetilde{\mathfrak{S}}_n$, put $\sigma = \Phi(\sigma') \in \mathfrak{S}_n$. After Schur [**15**], we call σ' (and σ) of *the first kind* or *the second kind* if σ' is conjugate under $\widetilde{\mathfrak{S}}_n$ to $z\sigma'$ or not. Decompose σ into a product of mutually disjoint cycles σ_j as $\sigma = \sigma_1\sigma_2\cdots\sigma_t$ and let ℓ_j be the length of cycle σ_j. Adjoining cycles of length 1 for conveniences, we assume $\ell_1+\ell_2+\cdots+\ell_t = n$. We call $\sigma' \in \mathfrak{B}_n$ (or $\sigma \in \mathfrak{A}_n$) of *the third kind* if ℓ_j's are all different. We know from [**15**, Section 7, Satz IV] and [**15**, Section 9, Satz V] the following:

(1.2.1). $\sigma' \in \widetilde{\mathfrak{S}}_n$ is of the second kind if and only if one of the following conditions holds:

(2-ev) $\operatorname{sgn}(\sigma) = 1$ and ℓ_j's are all odd (or $\operatorname{sgn}(\sigma_j) = 1$);
(2-od) $\operatorname{sgn}(\sigma) = -1$ and ℓ_j's are all different.

(1.2.2). For $\sigma' \in \mathfrak{B}_n$, σ' and $z\sigma'$ are conjugate under $\widetilde{\mathfrak{S}}_n$ but not under $\mathfrak{B}_n$ if and only if it is of the third kind.

A complete set of conjugacy classes of $\widetilde{\mathfrak{S}}_n$ (or of $\mathfrak{B}_n$) is given as follows. For a subinterval $K = [p,q] = \{p, p+1, \ldots, q\}$ of $\boldsymbol{I}_n$, put $\sigma'_K := r_pr_{p+1}\cdots r_{q-1}$ ($\sigma_K = e$ if $p = q$), then $\Phi_{\mathfrak{S}}(\sigma'_K)$ is the cyclic permutation $(p\ \ p+1\ \ldots\ q)$. For a partition

$$\boldsymbol{\ell} := (\ell_1, \ell_2, \ldots, \ell_t),\quad \ell_1 \geq \ell_2 \geq \cdots \geq \ell_t > 0,\quad \ell_1 + \ell_2 + \cdots + \ell_t = n, \tag{1.5}$$

of n, put $M_0 = 0$, $M_i = \ell_1 + \cdots + \ell_i$ ($i \in \boldsymbol{I}_t$), and let $K_i := [M_{i-1}, M_i]$ ($i \in \boldsymbol{I}_t$) be subinterval of $\boldsymbol{I}_n$ and call $\sigma'_{\boldsymbol{\ell}} := \sigma'_{K_1}\sigma'_{K_2}\cdots\sigma'_{K_t}$ the *standard element* of $\widetilde{\mathfrak{S}}_n$ corresponding to $\boldsymbol{\ell}$. Let $L(\sigma') = L(\sigma)$ be the length of σ with respect to simple transpositions $s_i = (i\ \ i+1)$ ($i \in \boldsymbol{I}_{n-1}$), then $L(\sigma) = \sum_{j\in\boldsymbol{I}_t}(\ell_j - 1)$. Put $\boldsymbol{s}(\boldsymbol{\ell}) := \sharp\{j \in \boldsymbol{I}_t; \ell_j \text{ even}\}$, then $\operatorname{sgn}(\sigma) = (-1)^{\boldsymbol{s}(\boldsymbol{\ell})}$. From (1.2.1) and (1.2.2) we obtain the following:

(1.2.3). A complete set of representatives of conjugacy classes of $\widetilde{\mathfrak{S}}_n$ is given by

$$\{\sigma'_{\boldsymbol{\ell}}, z\sigma'_{\boldsymbol{\ell}}; \boldsymbol{s}(\boldsymbol{\ell}) = 0\} \bigsqcup \{\sigma'_{\boldsymbol{\ell}}, z\sigma'_{\boldsymbol{\ell}}; \boldsymbol{s}(\boldsymbol{\ell}) \text{ odd, } \ell_j \text{ all different}\} \bigsqcup \{\sigma'_{\boldsymbol{\ell}}; \text{other } \boldsymbol{\ell}\}.$$

Such a set for $\mathfrak{B}_n$ is given by

$$\big\{\sigma'_{\boldsymbol{s}(\boldsymbol{\ell})}, z\sigma'_{\boldsymbol{s}(\boldsymbol{\ell})}; \boldsymbol{s}(\boldsymbol{\ell})=0\big\} \bigsqcup \big\{\sigma'_{\boldsymbol{s}(\boldsymbol{\ell})}, z\sigma'_{\boldsymbol{s}(\boldsymbol{\ell})}; \boldsymbol{s}(\boldsymbol{\ell})>0 \text{ even, } \ell_j \text{ all different}\big\}$$
$$\bigsqcup \big\{\sigma'_{\boldsymbol{s}(\boldsymbol{\ell})}; \boldsymbol{s}(\boldsymbol{\ell})>0 \text{ even, } \ell_j\text{'s have multiplicities}\big\}.$$

Example 1.3. Take an *ordered* decomposition $\boldsymbol{\nu}=(\nu_j)_{j\in \boldsymbol{I}_m}$, $|\boldsymbol{\nu}|:=\nu_1+\nu_2+\cdots+\nu_m=n$ (here we do not assume the order of *large* or *small* among ν_j's), and take a subgroup of (Frobenius-)Young type of $\mathfrak{S}_n$ as $\mathfrak{S}_{\boldsymbol{\nu}}:=\mathfrak{S}_{\nu_1}\times\mathfrak{S}_{\nu_2}\times\cdots\times\mathfrak{S}_{\nu_m}$, then the full inverse image $\widetilde{\mathfrak{S}}_{\boldsymbol{\nu}}:=\Phi_{\mathfrak{S}}^{-1}\big(\mathfrak{S}_{\nu_1}\times\cdots\times\mathfrak{S}_{\nu_m}\big)$ is called a *subgroup of Schur-Young type* of $\widetilde{\mathfrak{S}}_n$. Here $\mathfrak{S}_{\nu_j}$ is identified with the symmetric group $\mathfrak{S}_{J_j}$ acting on the sub-interval $J_j:=[\nu_1+\cdots+\nu_{j-1}+1, \nu_1+\cdots+\nu_j]$ of $\boldsymbol{I}_n=[1,n]=\{1,2,\dots,n\}$ (in particular, $J_1=[1,\nu_1]$). In this case, $\widetilde{\mathfrak{S}}_{\boldsymbol{\nu}}$ is naturally isomorphic to the twisted central product of $\widetilde{\mathfrak{S}}_{\nu_j}=\Phi_{\mathfrak{S}}^{-1}(\mathfrak{S}_{\nu_j})$ $(j\in\boldsymbol{I}_m)$, where $\widetilde{\mathfrak{S}}_{\nu_j}=\{e,z\}$ if $\nu_j=1$: $\widetilde{\mathfrak{S}}_{\boldsymbol{\nu}}\cong\widetilde{\mathfrak{S}}_{\nu_1}\hat{*}\cdots\hat{*}\widetilde{\mathfrak{S}}_{\nu_m}$.

A complete set of representatives of conjugacy classes of $\widetilde{\mathfrak{S}}_{\boldsymbol{\nu}}$ is given by using (1.2.3).

2. Spin representations of twisted central product groups.

2.1. Spin representations of double covering groups.

Let $(S',z,\mathrm{sgn})\in\mathscr{G}$. If sgn is of order 2, $B'=\{\sigma'\in S';\mathrm{ord}(\sigma')=0\}$ is a normal subgroup of S' of index 2, and $C'=\{\sigma'\in S';\mathrm{ord}(\sigma')=-1\}=\kappa'B'$ for any $\kappa'\in C'$. When sgn is trivial, $B'=S'$, $C'=\emptyset$.

Let $\widehat{S'}^{\mathrm{spin}}$ be the set of all equivalence classes of irreducible spin representations of S', and call it the *spin dual* of S'. This is nearly a half of the total dual $\widehat{S'}$ as is shown in the following proposition. Denote by $[\pi]$ the equivalence class of π.

Proposition 2.1. *For the spin dual $\widehat{S'}^{\mathrm{spin}}$ of a group $S'\in\mathscr{G}$, there holds*

$$\sum_{[\pi]\in\widehat{S'}^{\mathrm{spin}}}(\dim\pi)^2=\frac{1}{2}\,|S'|=|S|. \tag{2.1}$$

Proof. Note that $\widehat{S'}=\widehat{S'}^{\mathrm{spin}}\sqcup\widehat{S}$. Then the above formula comes from the following two equalities for S' and S:

$$\sum_{\pi\in\widehat{S'}}(\dim\pi)^2=|S'|,\qquad \sum_{\pi_0\in\widehat{S}}(\dim\pi_0)^2=|S|=\frac{1}{2}|S'|. \qquad\square$$

For $(S',z,\mathrm{sgn})\in\mathscr{G}'$, an intimate relation between IRs of S' and those of its normal subgroup B' is given as follows. We prepare a notation: for a representation π of S', denote by $\mathrm{Res}^{S'}_{B'}\pi$ the restriction $\pi|_{B'}$.

Proposition 2.2. *Let π be a spin IR of $S'\in\mathscr{G}'$.*

(i) *Assume that π is non-self-associate, that is, $\pi\not\cong\mathrm{sgn}\cdot\pi$. Then the restriction*

$\mathrm{Res}^{S'}_{B'}\pi$ *of* π *onto* B' *remains to be irreducible. Denote it by* ρ*, then* ${}^{\kappa'}\rho \cong \rho$ *for any* $\kappa' \in C'$*, where* $({}^{\kappa'}\rho)(\tau') := \rho({\kappa'}^{-1}\tau'\kappa')$ $(\tau' \in B')$*, and for the induced representation of* ρ *from* B' *to* S'*,*

$$\mathrm{Ind}^{S'}_{B'}\rho \cong \pi \oplus (\mathrm{sgn}\cdot\pi).$$

(ii) *Assume that* π *is self-associate, that is,* $\pi \cong \mathrm{sgn}\cdot\pi$*. Then the restriction* $\pi|_{B'}$ *splits into a direct sum of two non-equivalent IRs* ρ *and* ρ'*, and* ρ' *is equivalent to* ${}^{\kappa'}\rho$ *for any* $\kappa' \in C'$*:*

$$\mathrm{Res}^{S'}_{B'}\pi = \rho \oplus \rho', \quad \rho' \cong {}^{\kappa'}\rho.$$

In this case, $\mathrm{Ind}^{S'}_{B'}\rho \cong \mathrm{Ind}^{S'}_{B'}\rho' \cong \pi$*.*

Proof. (1) Let ρ be an irreducible component of the restriction $\pi|_{B'}$. Let $V(\pi)$ and $V(\rho)$ be their representation spaces. Then, since $S' = B' \sqcup C'$, $C' = \kappa' B'$, we have only two cases as follows: (Case 1) $V(\pi) = V(\rho)$, $\pi(\kappa')V(\rho) = V(\rho)$, and (Case 2) $V(\pi) = V(\rho) + \pi(\kappa')V(\rho)$ (direct sum).

On the subspace $\pi(\kappa')V(\rho)$, B' acts according to ${}^{\kappa'}\rho$. In fact, $\pi(\tau')(\pi(\kappa')w) = \pi(\kappa')(({}^{\kappa'}\rho)(\tau')w)$ $(\tau' \in B',\, w \in V(\rho))$. Hence, in Case 1, we have necessarily ${}^{\kappa'}\rho \cong \rho$.

In Case 2, we have ${}^{\kappa'}\rho \not\cong \rho$. In fact, if ${}^{\kappa'}\rho \cong \rho$, as is proved by calculations, there exist non-trivial intertwining operators of π. This contradicts the irreducibility of π. So ${}^{\kappa'}\rho \not\cong \rho$.

(2) On the other hand, consider $\Pi := \mathrm{Ind}^{G'}_{B'}\rho$. We realize this as follows. The space $V(\Pi)$ consists of $V(\rho)$-valued functions φ on G' satisfying the homogeneity condition

$$\varphi(\tau'\sigma') = \rho(\tau')\varphi(\sigma') \quad (\tau' \in B', \sigma' \in S'), \tag{2.2}$$

and the operator $\Pi(\sigma'_0)$ is given by $\Pi(\sigma'_0)\varphi(\sigma') := \varphi(\sigma'\sigma'_0)$ $(\sigma'_0, \sigma' \in S')$. Taking a complete set of representatives of the coset space $B'\backslash S'$ as $\{e, {\kappa'}^{-1}\}$, we define a map

$$\Psi : V(\Pi) \ni \varphi \mapsto \big(\varphi(e), \varphi({\kappa'}^{-1})\big) \in V(\rho) \otimes V(\rho).$$

Then, for $\varphi' := \Pi(\tau')\varphi$ and $\varphi'' := \Pi(\kappa')\varphi$,

$$\begin{aligned}
\big(\varphi'(e), \varphi'({\kappa'}^{-1})\big) &= \big(\varphi(\tau'), \varphi({\kappa'}^{-1}\tau')\big) = \big(\rho(\tau')\varphi(e), ({}^{\kappa'}\rho)(\tau')\varphi({\kappa'}^{-1})\big),\\
\big(\varphi''(e), \varphi''({\kappa'}^{-1})\big) &= \big(\varphi(\kappa'), \varphi(e)\big) = \big(\rho({\kappa'}^{2})\varphi({\kappa'}^{-1}), \varphi(e)\big).
\end{aligned}$$

Therefore $\Pi' := \Psi \cdot \Pi \cdot \Psi^{-1}$ is expressed in a matrix form as

$$\Pi'(\tau') = \begin{pmatrix} \rho(\tau') & O \\ O & ({}^{\kappa'}\rho)(\tau') \end{pmatrix}, \quad \Pi'(\kappa') = \begin{pmatrix} O & \rho(\tau'_0) \\ I & O \end{pmatrix}, \tag{2.3}$$

where $\tau'_0 := {\kappa'}^2 \in B'$, and I denote the identity operator on $V(\rho)$.

Now, in Case 1, there exists an intertwining operator A on $V(\rho)$ such that $({}^{\kappa'}\rho)(\tau') = A^{-1}\rho(\tau')A$ $(\tau' \in B')$. Then, for $\tau' \in B'$,

$$\begin{cases} \rho\big({\tau'_0}^{-1}\tau'\tau'_0\big) = A^{-1}\rho\big({\kappa'}^{-1}\tau'\kappa'\big)A = A^{-2}\rho(\tau')A^2, \\ \rho\big({\tau'_0}^{-1}\tau'\tau'_0\big) = \rho\big(\tau'_0\big)^{-1}\rho(\tau')\rho\big(\tau'_0\big), \end{cases}$$

and since ρ is irreducible, it follows that $A^2\rho(\tau'_0)^{-1}$ is a scalar operator. Replacing A by its scalar multiple appropriately, we obtain $A^2 = \rho(\tau'_0)$. Transform Π' to $\Pi'' = \Phi \cdot \Pi' \cdot \Phi^{-1}$ with $\Phi = \mathrm{diag}(I, A)$, then

$$\Pi''(\tau') = \begin{pmatrix} \rho(\tau') & O \\ O & \rho(\tau') \end{pmatrix}, \quad \Pi''(\kappa') = \begin{pmatrix} O & A \\ A & O \end{pmatrix}. \tag{2.4}$$

A basis of the space of intertwining operators of Π'' is given by two projections

$$P_\pm := \frac{1}{\sqrt{2}}(\boldsymbol{I} \pm Q) \quad \text{with} \quad \boldsymbol{I} := \begin{pmatrix} I & O \\ O & I \end{pmatrix}, \quad Q := \begin{pmatrix} O & I \\ I & O \end{pmatrix}.$$

For two irreducible components $\Pi''_\pm := P_\pm \Pi'' P_\pm$ of Π'', we have by simple calculations $\Pi''_-(\tau') = \Pi''_+(\tau')$, $\Pi''_-(\tau'\kappa') = -\Pi''_-(\tau'\kappa') = \mathrm{sgn}(\tau'\kappa') \cdot \Pi''_-(\tau'\kappa')$ for $\tau' \in B'$.

Lastly in Case 2, since $\rho \not\cong {}^{\kappa'}\rho$, we see from (2.3) that Π' is irreducible. For the character $\chi_{\Pi'}$ of Π', we see immediately from (2.3) that $\chi_{\Pi'} = 0$ on $C' = \kappa' B'$, and so $\chi_{\Pi'} = \mathrm{sgn} \cdot \chi_{\Pi'}$. This means that $\Pi' \cong \mathrm{sgn} \cdot \Pi'$.

Summarizing altogether, we see that Case 1 and Case 2 correspond exactly to the assertions (i) and (ii) of the proposition respectively. $\square$

Proposition 2.3. *In the converse way, take a spin IR ρ of B' and put $\Pi := \mathrm{Ind}_{B'}^{S'}\rho$. Then $\Pi|_{B'} \cong \rho \oplus {}^{\kappa'}\rho$ for any $\kappa' \in C'$.*

- (i) *Assume that ${}^{\kappa'}\rho \cong \rho$. Then Π is equivalent to a direct sum of non-equivalent spin IRs π and $\mathrm{sgn} \cdot \pi$: $\Pi = \mathrm{Ind}_{B'}^{S'}\rho \cong \pi \oplus (\mathrm{sgn} \cdot \pi)$.*
- (ii) *Assume that ${}^{\kappa'}\rho \not\cong \rho$. Then Π is irreducible. Denoted it by π, then $\pi \cong \mathrm{sgn} \cdot \pi$.*

Proof. Starting from a spin IR ρ of B', we can read the part (2) of the above proof for Proposition 2.2 as a proof of this proposition. $\square$

For an $S' \in \mathscr{G}$, we introduce in the set of its representations an equivalence relation $\overset{\mathrm{ass}}{\sim}$ obtained by adding to the usual equivalence $\cong$ a new relation (new if $S' \in \mathscr{G}'$) that π and its associate $\mathrm{sgn} \cdot \pi$ are mutually equivalent, that is,

$$\pi \overset{\mathrm{ass}}{\sim} \pi' \overset{\mathrm{def}}{\Longleftrightarrow} \pi' \cong \pi \text{ or } \pi' \cong \mathrm{sgn} \cdot \pi.$$

The equivalence class of spin IR π under $\overset{\mathrm{ass}}{\sim}$ is denoted by $[\pi]_{\mathrm{ass}}$ and is called the *associate equivalence class* of π. If π is self-associate or non-self-associate, then $[\pi]_{\mathrm{ass}} = [\pi]$ or

$[\pi]_{\rm ass} = [\pi] \sqcup [\operatorname{sgn}\cdot\pi]$ correspondingly. Denote by ${}^{\rm ass}\widehat{S'}^{\rm spin}$ the set of all equivalence classes of spin IRs under $\stackrel{\rm ass}{\sim}$, which is a quotient of $\widehat{S'}^{\rm spin}$.

For the set of spin IRs of B', introduce an equivalence relation $\stackrel{\rm out}{\sim}$ defined by $\rho' \stackrel{\rm out}{\sim} \rho \stackrel{\rm def}{\Longleftrightarrow} \rho' \cong \rho$ or $\rho' \cong {}^{\kappa'}\rho$ ($\exists\, \kappa' \in C'$). This means that ρ' is equivalent to ${}^{x}\rho$ for some $x \in \operatorname{Ad}(S')|_{B'} = \{\operatorname{Ad}(\sigma')|_{B'}; \sigma' \in S'\}$ which may contain outer automorphisms of B', where $\operatorname{Ad}(\sigma')(s') := \sigma' s' {\sigma'}^{-1}$ ($s' \in S'$). The equivalence class of ρ for $\stackrel{\rm out}{\sim}$ is denoted by $[\rho]_{\rm out}$. Then, $[\rho]_{\rm out} = [\rho]$ or $[\rho]_{\rm out} = [\rho] \sqcup [{}^{\kappa'}\rho]$ according as $\rho \cong {}^{\kappa'}\rho$ or not. The set of all such equivalence classes is denoted by ${}^{\rm out}\widehat{B'}^{\rm spin}$.

Noting that Frobenius reciprocity law says that, for any IRs π_0 of S' and ρ_0 of B', $[\operatorname{Ind}_{B'}^{S'}\rho_0 : \pi_0] = [\operatorname{Res}_{B'}^{S'}\pi_0 : \rho_0]$, we obtain from Propositions 2.2 and 2.3, the following theorem.

THEOREM 2.4. *Let* $(S', z, \operatorname{sgn}) \in \mathscr{G}'$.

(i) *There exists a natural bijective correspondence between the sets of equivalence classes* ${}^{\rm ass}\widehat{S'}^{\rm spin}$ *and* ${}^{\rm out}\widehat{B'}^{\rm spin}$, *and it is given by* $\operatorname{Res}_{B'}^{S'}$ *and* $\operatorname{Ind}_{B'}^{S'}$ *as follows. For* $[\pi]_{\rm ass}$, *take the equivalence class of* ${}^{\rm out}\widehat{B'}^{\rm spin}$ *which is represented by irreducible components of* $\operatorname{Res}_{B'}^{S'}$. *Conversely for* $[\rho]_{\rm out}$, *take the equivalence class of* ${}^{\rm ass}\widehat{S'}^{\rm spin}$ *which is represented by irreducible components of* $\operatorname{Ind}_{B'}^{S'}\rho$.

(ii) *When* π *runs over a complete set of representatives of* ${}^{\rm ass}\widehat{S'}^{\rm spin}$, *the set of irreducible components of* $\operatorname{Res}_{B'}^{S'}\pi$ *forms a complete set of representatives of the spin dual* $\widehat{B'}^{\rm spin}$ *of* B'.

For a self-associate spin IR π, the difference of characters of two IRs ρ, ρ' of B' is called the *complement* of the character χ_π (and of π too), after the terminology *Komplement* in [**15**, Section 16], and is denoted by δ_π:

$$\delta_\pi(\tau') := \chi_\rho(\tau') - \chi_{\rho'}(\tau') = \chi_\rho(\tau') - \chi_\rho\big({\kappa'}^{-1}\tau'\kappa'\big) \quad (\tau' \in B'). \tag{2.5}$$

Here we should specify ρ (from ρ'). We have irreducible characters of B' as

$$\chi_\rho = \frac{1}{2}(\chi_\pi + \delta_\pi), \quad \chi_{\rho'} = \frac{1}{2}(\chi_\pi - \delta_\pi). \tag{2.6}$$

We extend δ_π from B' to the whole $S' = B' \sqcup C'$ by putting $\delta_\pi(\kappa') = 0$ for $\kappa' \in C$.

LEMMA 2.5. *Let* π *be a self-associate spin representation of* $S' \in \mathscr{G}'$ *on a vector space* $V(\pi)$. *Then there exists a linear operator* H *on* $V(\pi)$ *satisfying*

$$H^2 = I, \quad \operatorname{tr}(H) = 0, \tag{2.7}$$

$$\begin{cases} H\pi(\tau') = \pi(\tau')H & (\tau' \in B'), \\ H\pi(\kappa') = -\pi(\kappa')H & (\kappa' \in C'). \end{cases} \tag{2.8}$$

In case π is irreducible, H is unique up to the sign $\pm$. Choose the sign appropriately, then

$$\mathrm{tr}\big(\pi(\tau')H\big) = \delta_\pi(\tau') \ (\tau' \in B'), \quad \mathrm{tr}\big(\pi(\kappa')H\big) = 0 \ (\kappa' \in C'). \tag{2.9}$$

Proof. Suppose π is irreducible. As is seen in part (2) of the proof of Proposition 2.2, π is realized in the form of (2.3), so that we have an expression of π by matrices as

$$\pi(\tau') = \begin{pmatrix} \rho(\tau') & 0_p \\ 0_p & \rho'(\tau') \end{pmatrix}, \quad \pi(\kappa') = \begin{pmatrix} 0_p & q(\kappa') \\ q'(\kappa') & 0_p \end{pmatrix} \quad (\tau' \in B',\ \kappa' \in C'),$$

where $p = \dim \rho = \dim \rho'$, 0_p and E_p are respectively zero matrix and identity matrix of degree p, and $q(\kappa')$, $q'(\kappa')$ are matrices determined by κ'. Then we can take as H the following: $H = \pm\,\mathrm{diag}(E_p, -E_p)$.

Suppose π is not irreducible. Then, we see from Theorem 2.4 that, by irreducible decomposition, π is a direct sum of self-associate IRs π_j $(j \in J)$ and pairs $\pi_k \oplus (\mathrm{sgn}\cdot\pi_k)$ $(k \in K)$ of non-self-associate IRs. For each π_j $(j \in J)$, H_j on $V(\pi_j)$ is taken as above.

For a pair $\pi'_k := \pi_k \oplus (\mathrm{sgn}\cdot\pi_k)$, put $V(\pi'_k) := V_k^{(0)} \oplus V_k^{(1)}$, $V_k^{(\alpha)} := V((\mathrm{sgn})^\alpha \cdot \pi_k) = V(\pi_k)$, $\alpha = 0, 1$. By Proposition 2.2 (i), $\rho_k := \pi_k|_{B'}$ is irreducible, and so we can take as H_k the interchange of $V_k^{(0)}$ and $V_k^{(1)}$ as $H_k(v \oplus v') = v' \oplus v$ $(v, v' \in V(\pi_k))$. In fact, for $\tau' \in B'$, $\kappa' \in C'$, $\pi'_k(\tau') = \pi_k(\tau') \oplus \pi_k(\tau')$, $\pi'_k(\kappa') = \pi_k(\kappa') \oplus (-\pi_k(\kappa'))$.

The sum H of H_j $(j \in J)$ and H_k $(k \in K)$ satisfies the condition (2.7)–(2.8). □

2.2. Construction of spin representations of twisted central product.

Let $(S'_j, z_j, \mathrm{sgn}_j)$ $(j \in \boldsymbol{I}_m)$ be elements of $\mathscr{G}$. Taking a spin representation π_j of S'_j for each $j \in \boldsymbol{I}_m$, we construct a spin representation of the twisted central product group $S' := S'_1 \hat{*} S'_2 \hat{*} \cdots \hat{*} S'_m$, following the method of Schur [**15**, Section 27]. Since we can add easily the part of $(S'_j, z_j, \mathrm{sgn}_j) \in \mathscr{G} \setminus \mathscr{G}'$, with sgn_j trivial and $S'_j = B'_j$, later by a (non-twisted) tensor product process, here we restrict ourselves to the case where all sgn_j are of order 2, i.e., $(S'_j, z_j, \mathrm{sgn}_j) \in \mathscr{G}'$ $(j \in \boldsymbol{I}_m)$, for simplicity.

Assume that, among π_j's given, π_i $(i \in \boldsymbol{I}_m^{\mathrm{sa}})$ are self-associate, and π_j $(j \in \boldsymbol{I}_m^{\mathrm{nsa}})$ are non-self-associate with $\boldsymbol{I}_m = \boldsymbol{I}_m^{\mathrm{sa}} \sqcup \boldsymbol{I}_m^{\mathrm{nsa}}$. Express these sets of indexes as

$$\boldsymbol{I}_m^{\mathrm{sa}} = \{i_1, \ldots, i_r\},\ i_1 < \cdots < i_r, \quad \boldsymbol{I}_m^{\mathrm{nsa}} = \{j_1, \ldots, j_s\},\ j_1 < \cdots < j_s, \tag{2.10}$$

where $r + s = m$.

First we prepare matrices $F_1, \ldots, F_s$ of size $2^{s'}$, $s' = [s/2]$, with $F_0 = E_{2^{s'}}$, satisfying

$$\begin{cases} F_j^2 = F_0 & (j \in \boldsymbol{I}_s), \\ F_j F_k = -F_k F_j & (j, k \in \boldsymbol{I}_s,\ j \neq k). \end{cases} \tag{2.11}$$

As an example, we can take as follows: put

$$\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad b := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad c := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.12}$$

and with the notation $x^{\otimes k} = \underbrace{x \otimes x \otimes \cdots \otimes x}_{k\text{-times tensor product}}$ (ignored if $k = 0$),

$$\begin{cases} F_{2k-1} := c^{\otimes(k-1)} \otimes a \otimes \varepsilon^{\otimes(s'-k)} & (k \in \boldsymbol{I}_{s'}), \\ F_{2k} := c^{\otimes(k-1)} \otimes b \otimes \varepsilon^{\otimes(s'-k)} & (k \in \boldsymbol{I}_{s'}), \\ F_{2s'+1} := c^{\otimes s'}. \end{cases} \tag{2.13}$$

Then, for self-associate π_j ($j \in \boldsymbol{I}_m^{\mathrm{sa}}$), we choose a linear operator H_j on $V(\pi_j)$ satisfying the following condition: for $\tau'_j \in B'_j$, $\kappa'_j \in C'_j = S'_j \setminus B'_j$,

$$\begin{cases} H_j^2 = I_j, \quad \mathrm{tr}(H_j) = 0, \\ H_j \pi_j(\tau'_j) = \pi_j(\tau'_j) H_j, \quad H_j \pi_j(\kappa'_j) = -\pi_j(\kappa'_j) H_j, \end{cases} \tag{2.14}$$

where I_j denotes the identity operator on $V(\pi_j)$. The operator H_j is uniquely determined up to a multiplicative sign if π_j is irreducible.

Now we give operators $\pi(\sigma')$ for $\sigma' \in S' = S'_1 \hat{*} S'_2 \hat{*} \cdots \hat{*} S'_m$ on the space $V(\pi) := V_0 \otimes V(\pi_1) \otimes \cdots \otimes V(\pi_m)$ with $V_0 := \boldsymbol{C}^{2^{s'}}$ on which F_j's act.

Formula 2.1. As notation, we use symbols $\tau'_j \in B'_j$, $\kappa'_j \in C'_j$.

(1) Case of $\tau'_j \in B'_j$ for some $j \in \boldsymbol{I}_m$,

$$\pi(\tau'_j) := F_0 \otimes I_1 \otimes \cdots \otimes I_{j-1} \otimes \pi_j(\tau'_j) \otimes I_{j+1} \otimes \cdots \otimes I_m. \tag{2.15}$$

(2) Case of $\kappa'_j \in C'_j$ for some $j = i_p \in \boldsymbol{I}_m^{\mathrm{sa}}$,

$$\pi\big(\kappa'_{i_p}\big) := F_0 \otimes \Big(\bigotimes_{k \in \boldsymbol{I}_m} X_k\Big), \tag{2.16}$$

where

$$X_i = \begin{cases} H_{i_q} & \big(i = i_q \in \boldsymbol{I}_m^{\mathrm{sa}},\ q < p\big), \\ \pi_{i_p}\big(\kappa'_{i_p}\big) & \big(i = i_p,\ \text{in } \boldsymbol{I}_m^{\mathrm{sa}}\big), \\ I_i & \big(i \in \boldsymbol{I}_m,\ \text{otherwise}\big). \end{cases}$$

(3) Case of $\kappa'_j \in C'_j$ for some $j = j_p \in \boldsymbol{I}_m^{\mathrm{nsa}}$,

$$\pi\big(\kappa'_{j_p}\big) := F_p \otimes \Big(\bigotimes_{k \in \boldsymbol{I}_m} X_k\Big), \tag{2.17}$$

where

$$X_i = \begin{cases} H_i & \big(i \in \boldsymbol{I}_m^{\mathrm{sa}}\big), \\ \pi_{j_p}(\kappa'_{j_p}) & \big(i = j_p,\ \text{in}\ \boldsymbol{I}_m^{\mathrm{nsa}}\big), \\ I_i & \big(i \in \boldsymbol{I}_m^{\mathrm{nsa}},\ \text{otherwise}\big). \end{cases}$$

We illustrate this formula in a special case where $\boldsymbol{I}_m^{\mathrm{sa}} = \boldsymbol{I}_r$, $\boldsymbol{I}_m^{\mathrm{nsa}} = \boldsymbol{I}_m \setminus \boldsymbol{I}_r$. In Case (2) above, for $j \in \boldsymbol{I}_r = \boldsymbol{I}_m^{\mathrm{sa}}$,

$$\pi\big(\kappa'_j\big) = F_0 \otimes H_1 \otimes \cdots \otimes H_{j-1} \otimes \pi_j(\kappa'_j) \otimes I_{j+1} \otimes \cdots \otimes I_m. \tag{2.18}$$

In Case (3) above, for $j = r + p \in \boldsymbol{I}_m \setminus \boldsymbol{I}_r = \boldsymbol{I}_m^{\mathrm{nsa}}$,

$$\pi\big(\kappa'_j\big) = F_p \otimes H_1 \otimes \cdots \otimes H_r \otimes I_{r+1} \otimes \cdots \otimes I_{j-1} \otimes \pi_j(\kappa'_j) \otimes I_{j+1} \otimes \cdots \otimes I_m. \tag{2.19}$$

In general, we can easily show that the above set of operators satisfies

$$\begin{cases} \pi\big(\tau'_k\big)\pi\big(\tau'_{k'}\big) = \pi\big(\tau'_{k'}\big)\pi\big(\tau'_k\big) & \big(k,\ k' \in \boldsymbol{I}_m^{\mathrm{sa}}\big), \\ \pi\big(\tau'_j\big)\pi\big(\kappa'_k\big) = \pi\big(\kappa'_k\big)\pi\big(\tau'_j\big) & \big(k \in \boldsymbol{I}_m^{\mathrm{sa}},\ j \in \boldsymbol{I}_m^{\mathrm{nsa}}\big), \\ \pi\big(\kappa'_j\big)\pi\big(\kappa'_{j'}\big) = -\pi\big(\kappa'_{j'}\big)\pi\big(\kappa'_j\big) & \big(j, j' \in \boldsymbol{I}_m^{\mathrm{nsa}},\ j \neq j'\big). \end{cases}$$

This means that this set of operators define a spin representation of the twisted central product $S' = S'_1 \hat{*} S'_2 \hat{*} \cdots \hat{*} S'_m$. We call it the *twisted central product* of $\pi_1, \ldots, \pi_m$ and denote it by $\pi_1 \hat{*} \pi_2 \hat{*} \cdots \hat{*} \pi_m$.

Note 2.1. The formulas (2.18)–(2.19) are in a sense generic. However we should keep in mind that the order of components π_i's here has a significant meaning for us. As will be seen in the next section (Section 3), the above construction of twisted central product $\pi_1 \hat{*} \pi_2 \hat{*} \cdots \hat{*} \pi_m$ of spin representations π_i's does not satisfy in general the commutativity nor the associativity, unlike the usual tensor product operation. After calculating their characters in Section 4, we know however that any permutation of components π_i's do not affect their equivalence, that is, $\pi_{i_1} \hat{*} \pi_{i_2} \hat{*} \cdots \hat{*} \pi_{i_m}$ of $S'_{i_1} \hat{*} S'_{i_2} \hat{*} \cdots S'_{i_m} \cong S'_1 \hat{*} S'_2 \hat{*} \cdots \hat{*} S'_m$ is equivalent to $\pi_1 \hat{*} \pi_2 \hat{*} \cdots \hat{*} \pi_m$, where $i_1, i_2, \ldots, i_m$ is any permutation of $1, 2, \ldots, m$.

In this connection, see also Section 5.2.

Summarizing, we have the following theorem.

Theorem 2.6. *Let* $(S'_j, z_j, \mathrm{sgn}_j)$ $(j \in \boldsymbol{I}_m)$ *be elements of* $\mathscr{G}'$. *For spin representations* π_j *of* S'_j *for* $j \in \boldsymbol{I}_m$, *Formula* 2.1 *above defines a spin representation* $\pi = \pi_1 \hat{*} \pi_2 \hat{*} \cdots \hat{*} \pi_m$ *of* $S' = S'_1 \hat{*} S'_2 \hat{*} \cdots \hat{*} S'_m$, *for each choice of* H_j *in* (2.14) *for self-associate* π_j*'s. Moreover* $\dim \pi = 2^{[s/2]} \prod_{j \in \boldsymbol{I}_m} \dim \pi_j$. *When all* π_j*'s are irreducible,* π *is irreducible.*

We omit here a direct proof of irreducibility (cf. Proposition 5.1 (iii) for another proof). The following lemma shows a peculiar property of the twisted central product $\pi_1 \hat{*} \pi_2 \hat{*} \cdots \hat{*} \pi_m$.

PROPOSITION 2.7. *According that the number s of non-self-associate π_j's is even or odd, the twisted central product $\pi = \pi_1 \hat{*} \pi_2 \hat{*} \cdots \hat{*} \pi_m$ of $S' = S'_1 \hat{*} S'_2 \hat{*} \cdots \hat{*} S'_m$ is self-associate or non-self-associate.*

PROOF. We prove this for $m = 2$ and $s = 2$. The general case will become clear in occasion of calculating characters later in Section 4.1. Note that $B' = B'_1 B'_2 \sqcup C'_1 C'_2$, $C' = B'_1 C'_2 \sqcup C'_1 B'_2$, where $C'_1 C'_2 := \{\kappa'_1 \kappa'_2 ; \kappa'_j \in C'_j \ (j \in \boldsymbol{I}_2)\}$ etc. For $s = 2$, we have $2^{s'} = 2^{[s/2]} = 2$, and for $\sigma' = \tau'_1 \kappa'_2 \in B'_1 C'_2$ or $\sigma' = \kappa'_1 \tau'_2 \in C'_1 B'_2$ respectively,

$$\pi(\sigma') = \pi\big(\tau'_1 \kappa'_2\big) = F_2 \otimes \pi_1\big(\tau'_1\big) \otimes \pi_2\big(\kappa'_2\big), \quad \text{or}$$
$$\pi(\sigma') = \pi\big(\kappa'_1 \tau'_2\big) = F_1 \otimes \pi_1\big(\kappa'_1\big) \otimes \pi_2\big(\tau'_2\big).$$

Hence, for the character χ_π of π, we have $\chi_\pi(\sigma') = 0$ for $\sigma' \in C'$. This means that $\chi_\pi = \chi_{\mathrm{sgn} \cdot \pi}$, and so $\pi \cong \mathrm{sgn} \cdot \pi$, that is, π is self-associate. □

3. Remarks on the twisted central products.

3.1. On the twisted product of groups $S'_1 \hat{*} S'_2 \hat{*} \cdots \hat{*} S'_m$.

The associativity law holds for the twisted central product $S'_1 \hat{*} S'_2 \hat{*} \cdots \hat{*} S'_m$ of groups as is seen from the very Definition 1.1, that is, $(S'_1 \hat{*} S'_2) \hat{*} S'_3 = S'_1 \hat{*} (S'_2 \hat{*} S'_3)$.

Moreover, since $S'_1 \hat{*} S'_2$ and $S'_2 \hat{*} S'_1$ are naturally isomorphic, the commutativity law is also valid for the twisted central product of groups.

3.2. On the associativity for the twisted central product $\pi_1 \hat{*} \cdots \hat{*} \pi_m$.

We ask if the associativity law holds for the twisted central product of spin representations $\pi_1 \hat{*} \pi_2 \hat{*} \cdots \hat{*} \pi_m$.

EXAMPLE 3.1. For spin irreducible representations π_j of S'_j for $j \in \boldsymbol{I}_3$, assume that π_1 is self-associate and π_2, π_3 are non-self-associate. Put $\pi := (\pi_1 \hat{*} \pi_2) \hat{*} \pi_3$, $\pi' := \pi_1 \hat{*} (\pi_2 \hat{*} \pi_3)$. Then, noting that $\pi_1 \hat{*} \pi_2$ is non-self-associate and $\pi_2 \hat{*} \pi_3$ is self-associate (by Proposition 2.7), we obtain the following, with F_j's of degree 2. Put $V_j := V(\pi_j)$ $(j \in \boldsymbol{I}_3)$, and $V_0 = \boldsymbol{C}^2$ on which F_j's act, then the representation spaces are

$$V(\pi) = V_0 \otimes \big(V_1 \otimes V_2\big) \otimes V_3, \quad V(\pi') = V_1 \otimes \big(V_0 \otimes V_2 \otimes V_3\big),$$

and let $R_{ij} : V_i \otimes V_j \to V_j \otimes V_i$ be the linear operator permuting components as $R_{ij}(v_i \otimes v_j) := v_j \otimes v_i$ $(v_p \in V_p)$. For $\kappa'_j \in C'_j$ $(j \in \boldsymbol{I}_3)$,

$$\begin{cases} \pi(\kappa'_1) = F_1 \otimes \big(\pi_1(\kappa'_1) \otimes I_2\big) \otimes I_3, \\ \pi(\kappa'_2) = F_1 \otimes \big(H_1 \otimes \pi_2(\kappa'_2)\big) \otimes I_3, \\ \pi(\kappa'_3) = F_2 \otimes \big(I_1 \otimes I_2\big) \otimes \pi_3(\kappa'_3), \end{cases} \tag{3.1}$$

and π is self-associate with the operator $H = F_3 \otimes I_1 \otimes I_2 \otimes I_3$. Moreover

$$\begin{cases} \pi'(\kappa'_1) = \pi_1(\kappa'_1) \otimes \big(F_0 \otimes I_2 \otimes I_3\big), \\ \pi'(\kappa'_2) = H_1 \otimes \big(F_1 \otimes \pi_2(\kappa'_2) \otimes I_3\big), \\ \pi'(\kappa'_3) = H_1 \otimes \big(F_2 \otimes I_2 \otimes \pi_3(\kappa'_3)\big), \end{cases} \tag{3.2}$$

and π' is self-associate with the operator $H' = H_1 \otimes (F_3 \otimes I_2 \otimes I_3)$.

Comparing these formulas for π and π', it is difficult to say that they are *the same*, but they are actually mutually equivalent as seen below.

Proposition 3.1. *Let π_1 be self-associate, and π_2, π_3 be non-self-associate. Then two spin IRs $\pi = (\pi_1 \hat{*} \pi_2) \hat{*} \pi_3$ and $\pi' = \pi_1 \hat{*} (\pi_2 \hat{*} \pi_3)$ of $S' = S'_1 \hat{*} S'_2 \hat{*} S'_3$ are mutually equivalent, and an intertwining operator $T : V(\pi) \to V(\pi')$ between them such as $\pi'(\sigma')T = T\pi(\sigma')$ $(\sigma' \in S')$ is given by*

$$T = (R_{01}X) \otimes I_2 \otimes I_3, \quad X = \frac{1}{2}(F_0 + F_1) \otimes I_1 + \frac{1}{2}(F_0 - F_1) \otimes H_1,$$

with $X^2 = F_0 \otimes I_1$, $T^{-1} = (XR_{10}) \otimes I_2 \otimes I_3$.

3.3. On the commutativity for the twisted central product $\pi_1 \hat{*} \cdots \hat{*} \pi_m$.

Here we check if the commutativity law holds for the construction of $\pi_1 \hat{*} \cdots \hat{*} \pi_m$. So we compare $\pi := \pi_1 \hat{*} \pi_2$ and $\pi' := \pi_2 \hat{*} \pi_1$.

Example 3.2. Assume that π_1, π_2 are both self-associate. Then $V(\pi) = V_1 \otimes V_2$ and $V(\pi') = V_2 \otimes V_1$, and for $\kappa'_j \in C'_j$,

$$\begin{cases} \pi(\kappa'_1) = \pi_1(\kappa'_1) \otimes I_2, & \pi\text{: self-associate,} \\ \pi(\kappa'_2) = H_1 \otimes \pi_2(\kappa'_2), & H = H_1 \otimes H_2. \end{cases}$$

$$\begin{cases} \pi'(\kappa'_1) = H_2 \otimes \pi_1(\kappa'_1), & \pi'\text{: self-associate,} \\ \pi'(\kappa'_2) = \pi_2(\kappa'_2) \otimes I_1, & H' = H_2 \otimes H_1. \end{cases}$$

An intertwining operator T, which proves the equivalence between π and π', is given as follows. Put

$$T' := \frac{1}{2}(I_1 + H_1) \otimes I_2 + \frac{1}{2}(I_1 - H_1) \otimes H_2,$$

a linear transformation on $V(\pi)$. Take $T = R_{12}T' : V(\pi) \to V(\pi')$, then we obtain an intertwining relation as $\pi'(\sigma') \cdot T = T \cdot \pi(\sigma')$ $(\sigma' \in S')$. Since T is not trivial, we can say that the commutativity law fails to hold for $\pi_1 \hat{*} \pi_2$ and $\pi_2 \hat{*} \pi_1$.

Example 3.3. Assume that π_1 and π_2 are both non-self-associate. Then, $V(\pi) = V_0 \otimes V_1 \otimes V_2$, $V(\pi') = V_0 \otimes V_2 \otimes V_1$ with $V_0 = \boldsymbol{C}^2$, and for $\kappa'_j \in C'_j$,

$$\begin{cases} \pi(\kappa_1') = F_1 \otimes \pi_1(\kappa_1') \otimes I_2, & \pi\text{: self-associate,} \\ \pi(\kappa_2') = F_2 \otimes I_1 \otimes \pi_2(\kappa_2'), & H = F_3 \otimes I_1 \otimes I_2. \end{cases}$$

$$\begin{cases} \pi'(\kappa_1') = F_2 \otimes I_2 \otimes \pi_1(\kappa_1'), & \pi'\text{: self-associate,} \\ \pi'(\kappa_2') = F_1 \otimes \pi_2(\kappa_2') \otimes I_1, & H' = F_3 \otimes I_2 \otimes I_1. \end{cases}$$

Put $T := (1/\sqrt{2})(F_1 + F_2) \otimes R_{12}$. Then T gives an equivalence between π and π' as $\pi'(\sigma')T = T\,\pi(\sigma')$ $(\sigma' \in S')$. This time again, T is not trivial, and we cannot say that the commutativity holds for $\pi_1 \hat{*} \pi_2$ and $\pi_2 \hat{*} \pi_1$.

4. Characters of the twisted central product.

4.1. Characters of the twisted central product.

To calculate the character of $\pi_1 \hat{*} \cdots \hat{*} \pi_m$, it can be seen from Formula 2.1 that, without loss of generality, we may assume that π_j $(j \in \boldsymbol{I}_r)$ are self-associate and π_j $(j \in \boldsymbol{I}_m \setminus \boldsymbol{I}_r)$ are non-self-associate. Put $s := m - r$. An element σ' of $S' = S_1' \hat{*} S_2' \hat{*} \cdots \hat{*} S_m'$ is expressed as $\sigma' = \sigma_1' \cdots \sigma_r' \cdot \sigma_{r+1}' \cdots \sigma_m'$, $\sigma_j' \in S_j'$ $(j \in \boldsymbol{I}_m)$. To calculate the character $\chi_\pi(\sigma') = \mathrm{tr}(\pi(\sigma'))$ of $\pi = \pi_1 \hat{*} \cdots \hat{*} \pi_m$, we see easily from Formula 2.1, or more exactly from the formula (2.18)–(2.19), the following two facts.

(4.1). Assume that σ_j' is from $C_j' = S_j' \setminus B_j'$ for some $j \in \boldsymbol{I}_r$. Put $\kappa_j' = \sigma_j'$ and insert it in Formula 2.1 (2), then, in the expression of

$$\pi(\sigma') = \pi(\sigma_1') \cdots \pi(\sigma_j') \cdots \pi(\sigma_{r+1}') \cdots \pi(\sigma_m')$$

as a tensor product along the form of the space $V(\pi) = V_0 \otimes V(\pi_1) \otimes \cdots \otimes V(\pi_m)$, its component on $V(\pi_j)$ is either $\pi_j(\kappa_j')$ or $\pi_j(\kappa_j')H_j$, and in any case its trace is zero by Lemma 2.5. Hence $\mathrm{tr}\,\pi(\sigma') = 0$.

Therefore, if $\mathrm{tr}\,\pi(\sigma') \neq 0$, then necessarily $\sigma_j' \in B_j'$ for any $j \in \boldsymbol{I}_r$.

(4.2). Assume that $\sigma_j' \in C_j'$ for some $j \in \boldsymbol{I}_m \setminus \boldsymbol{I}_r$. Then, in the expression of $\pi(\sigma')$ in the form of tensor product, its component on the space V_0 is the product of F_i over such $i \in \boldsymbol{I}_s$ that $\sigma_j' \in C_j'$ for $j = r + i \in \boldsymbol{I}_m \setminus \boldsymbol{I}_r$. Then we apply the following lemma on the trace of the products of F_i's, and see that, if $\mathrm{tr}\,\pi(\sigma') \neq 0$, then s is odd and $\sigma_j' \in C_j'$ for all $j \in \boldsymbol{I}_m \setminus \boldsymbol{I}_r$.

LEMMA 4.1. *For a product $F_1^{a_1} F_2^{a_2} \cdots F_s^{a_s}$, $a_j \geq 0$, of F_i's, its trace is non-zero only in the following two cases*:

(1) $a_j \equiv 0 \pmod{2, \forall j \in \boldsymbol{I}_s}$, *in this case,*

$$\mathrm{tr}\big(F_1^{a_1} F_2^{a_2} \cdots F_s^{a_s}\big) = \mathrm{tr}\big(E_{2^{[s/2]}}\big) = 2^{[s/2]}.$$

(2) $a_j \equiv 1 \pmod{2, \forall j \in \boldsymbol{I}_s}$ *and s is odd, and in this case,*

$$\mathrm{tr}\big(F_1^{a_1} F_2^{a_2} \cdots F_s^{a_s}\big) = \mathrm{tr}\big(F_1 F_2 \cdots F_s\big) = \mathrm{tr}(abc)^{[s/2]} = (2i)^{[s/2]}.$$

Thus we obtain the following result for the character χ_π of the twisted central product $\pi = \pi_1 \hat{*} \cdots \hat{*} \pi_m$.

Lemma 4.2. *For $\sigma' = \sigma'_1 \sigma'_2 \cdots \sigma'_m \in S'_1 \hat{*} S'_2 \hat{*} \cdots \hat{*} S'_m$ $(\sigma'_j \in S'_j)$, the character $\chi_\pi(\sigma') = \operatorname{tr} \pi(\sigma')$ is zero except the following two cases*:

Case (1): $\sigma'_j = \tau'_j \in B'_j$ $(\forall j \in \boldsymbol{I}_m) : \pi(\sigma') = E_{2^{[s/2]}} \otimes \pi_1(\tau'_1) \otimes \cdots \otimes \pi_m(\tau'_m)$,

$$\chi_\pi(\sigma') = 2^{[s/2]} \cdot \chi_{\pi_1}(\tau'_1) \cdots \chi_{\pi_m}(\tau'_m). \tag{4.1}$$

Case (2): *s is odd, and $\sigma'_j = \tau'_j \in B'_j$ $(\forall j \in \boldsymbol{I}_r)$ and $\sigma'_j = \kappa'_j \in C'_j$ $(\forall j \in \boldsymbol{I}_m \setminus \boldsymbol{I}_r)$* : $\pi(\sigma') = (F_1 F_2 \cdots F_s) \otimes \big(\bigotimes_{j \in \boldsymbol{I}_r} \pi_j(\tau'_j) H_j\big) \otimes \big(\bigotimes_{j \in \boldsymbol{I}_m \setminus \boldsymbol{I}_r} \pi_j(\kappa'_j)\big)$,

$$\chi_\pi(\sigma') = (2i)^{[s/2]} \cdot \prod_{j \in \boldsymbol{I}_r} \delta_{\pi_j}(\tau'_j) \cdot \prod_{j \in \boldsymbol{I}_m \setminus \boldsymbol{I}_r} \chi_{\pi_j}(\kappa'_j). \tag{4.2}$$

For $\alpha = 0, 1$, put $\pi_j^{(\alpha)} := (\mathrm{sgn}_j)^\alpha \cdot \pi_j$. Then, by assumption on the indexing, $\pi_j^{(1)} \cong \pi_j^{(0)}$ $(j \in \boldsymbol{I}_r)$, $\pi_j^{(1)} \not\cong \pi_j^{(0)}$ $(j \in \boldsymbol{I}_m \setminus \boldsymbol{I}_r)$. Noting this, we obtain the following results from the character formula above.

Proposition 4.3. *Assume $s = m - r$ be odd.*

(i) *$\pi = \pi_1 \hat{*} \cdots \hat{*} \pi_m$ is non-self-associate, and $\operatorname{tr}(\pi(\sigma')) \neq 0$ only in the following two cases*:
Case (1-odd): *$\sigma' = \tau'_1 \cdots \tau'_m$ with $\tau'_j \in B'_j$ $(j \in \boldsymbol{I}_m)$. In this case the character χ_π is given by the formula* (4.1).
Case (2-odd): *$\sigma' = \tau'_1 \cdots \tau'_r \kappa'_{r+1} \cdots \kappa'_m$ with $\tau'_j \in B'_j$ $(j \in \boldsymbol{I}_r)$, $\kappa'_j \in C'_j$ $(j \in \boldsymbol{I}_m \setminus \boldsymbol{I}_r)$. In this case, the character χ_π is given by the formula* (4.2).

(ii) *For $\alpha_j = 0, 1$ $(j \in \boldsymbol{I}_m)$,*

$$\pi_1^{(\alpha_1)} \hat{*} \cdots \hat{*} \pi_m^{(\alpha_m)} \cong (\mathrm{sgn})^{\alpha_{r+1} + \cdots + \alpha_m} \cdot \pi_1 \hat{*} \cdots \hat{*} \pi_m. \tag{4.3}$$

Proposition 4.4. *Assume $s = m - r$ be even. Then $\pi = \pi_1 \hat{*} \cdots \hat{*} \pi_m$ is self-associate, and for $\alpha_j = 0, 1$ $(j \in \boldsymbol{I}_m)$,*

$$\pi_1^{(\alpha_1)} \hat{*} \cdots \hat{*} \pi_m^{(\alpha_m)} \cong \pi_1 \hat{*} \cdots \hat{*} \pi_m. \tag{4.4}$$

As an operator H in (2.7)–(2.8) *for π, we have*

$$H = F_{s+1} \otimes H_1 \otimes \cdots \otimes H_r \otimes I_{r+1} \otimes \cdots \otimes I_m$$

(if $s = 0$, the term of F_{s+1} is absent). For the complement δ_π of π, if $\delta_\pi(\sigma') \neq 0$ for $\sigma' = \sigma'_1 \cdots \sigma'_m$, then $\sigma'_j = \tau'_j \in B'_j$ $(j \in \boldsymbol{I}_r)$, $\sigma'_j = \kappa'_j \in C'_j$ $(j \in \boldsymbol{I}_m \setminus \boldsymbol{I}_r)$, and

$$\pi(\sigma')H=(F_1\cdots F_sF_{s+1})\otimes\Big(\bigotimes_{j\in\boldsymbol{I}_r}\pi_j(\tau'_j)H_j\Big)\otimes\Big(\bigotimes_{j\in\boldsymbol{I}_m\setminus\boldsymbol{I}_r}\pi_j(\kappa'_j)\Big),$$

$$\delta_\pi(\sigma')=(2i)^{[s/2]}\cdot\prod_{j\in\boldsymbol{I}_r}\delta_{\pi_j}(\tau'_j)\cdot\prod_{\boldsymbol{I}_m\setminus\boldsymbol{I}_r}\chi_{\pi_j}(\kappa'_j). \tag{4.5}$$

4.2. Supports of characters and complements of $\boldsymbol{\pi_1\hat{*}\cdots\hat{*}\pi_m}$.

Let $(S'_j,z_j,\mathrm{sgn}_j)\in\mathscr{G}$. For the twisted central product of spin IRs $\pi=\pi_1\hat{*}\cdots\hat{*}\pi_m$, which is again irreducible and spin, we evaluate the supports of its character χ_π, and of its complement δ_π when π is self-associate. This gives us very important information on the property of π. We summarize the result in a form of a table. We prepare some notation for subsets of the twisted central product $S'=S'_1\hat{*}\cdots\hat{*}S'_m$ and of their components S'_j's:

$$\begin{cases}B'=\{\sigma'\in S';\mathrm{sgn}(\sigma')=1\},\\ C'=\{\sigma'\in S';\mathrm{sgn}(\sigma')=-1\},\end{cases}\qquad \begin{cases}\mathcal{B}(\pi_j):=\{\tau'_j\in B'_j;\chi_{\pi_j}(\tau'_j)\neq 0\},\\ \mathcal{C}(\pi_j):=\{\kappa'_j\in C'_j;\chi_{\pi_j}(\kappa'_j)\neq 0\},\\ \mathcal{D}(\pi_j):=\{\tau'_j\in B'_j;\delta_{\pi_j}(\tau'_j)\neq 0\},\end{cases}$$

$$\begin{cases}\mathcal{B}(\pi_1,\ldots,\pi_r):=\mathcal{B}(\pi_1)\cdots\mathcal{B}(\pi_r),\\ \mathcal{C}(\pi_{r+1},\ldots,\pi_m):=\mathcal{C}(\pi_{r+1})\cdots\mathcal{C}(\pi_m),\\ \mathcal{D}(\pi_1,\ldots,\pi_r):=\mathcal{D}(\pi_1)\cdots\mathcal{D}(\pi_r).\end{cases}$$

Table 4.1. Supports of $\boldsymbol{\chi_\pi}$ and $\boldsymbol{\delta_\pi}$ for $\boldsymbol{\pi=\pi_1\hat{*}\cdots\hat{*}\pi_m}$.
π_i ($i\in\boldsymbol{I}_r$) self-associate, π_j ($j\in\boldsymbol{I}_m\setminus\boldsymbol{I}_r$) non-self-associate, $m=r+s$.

- Case of s odd: χ_π is non-self-associate, and

$$\mathrm{supp}(\chi_\pi)\cap B'\subset\mathcal{B}(\pi_1,\ldots,\pi_m),$$
$$\mathrm{supp}(\chi_\pi)\cap C'\subset\mathcal{D}(\pi_1,\ldots,\pi_r)\mathcal{C}(\pi_{r+1},\ldots,\pi_m).$$

- Case of s even: χ_π is self-associate, with $H=F_{s+1}\otimes(H_1\otimes\cdots\otimes H_r)\otimes(I_{r+1}\otimes\cdots\otimes I_{r+s})$, and

$$\mathrm{supp}(\chi_\pi)\subset\mathcal{B}(\pi_1,\ldots,\pi_m)\subset B',$$
$$\mathrm{supp}(\delta_\pi)\subset\mathcal{D}(\pi_1,\ldots,\pi_r)\mathcal{C}(\pi_{r+1},\ldots,\pi_m)\subset B'.$$

4.3. Sets of representatives of twisted central products $\boldsymbol{\pi_1\hat{*}\cdots\hat{*}\pi_m}$.

Taking into account the equivalence relations (4.3) in case s odd and (4.4) in case s even, let us choose sets of representatives of $\pi=\pi_1\hat{*}\cdots\hat{*}\pi_m$ for $S'=S'_1\hat{*}\cdots\hat{*}S'_m$.

In Section 2.1, for an $S'\in\mathscr{G}$, the equivalence relation $\overset{\mathrm{ass}}{\sim}$ is introduced as $\pi\overset{\mathrm{ass}}{\sim}\pi'\overset{\mathrm{def}}{\Longleftrightarrow}\pi'\cong\pi$ or $\pi'\cong\mathrm{sgn}\cdot\pi$, and the set of associated equivalence classes $[\pi]_{\mathrm{ass}}$ of spin IRs is denoted by ${}^{\mathrm{ass}}\widehat{S'}^{\mathrm{spin}}$. Take a complete set of representatives of spin IRs $\Omega(S')=\Omega^{\mathrm{sa}}(S')\sqcup\Omega^{\mathrm{nsa}}(S')$ of ${}^{\mathrm{ass}}\widehat{S'}^{\mathrm{spin}}$, where $\{[\pi]_{\mathrm{ass}}=[\pi];\pi\in\Omega^{\mathrm{sa}}(S')\}$ covers the associate equivalence classes of self-associate spin IRs, and $\{[\pi]_{\mathrm{ass}}=[\mathrm{sgn}\cdot\pi]_{\mathrm{ass}};\pi\in\Omega^{\mathrm{nsa}}(S')\}$ covers the associate equivalence classes of non-self-associate spin IRs.

Denote by $\ell^2(B')$ (resp. $\ell^2(S')$) the ℓ^2-space on B' (resp. on S') with respect to the normalized invariant measure on B' (resp. on S'). A function f on B' (resp. on S') is called *spin* if $f(z\sigma') = -f(\sigma')$ $(\sigma' \in B')$ (resp. $\sigma' \in S'$).

Lemma 4.5. (i) *The set of spin functions on the subgroup B' given by*

$$\Big\{ \tfrac{1}{\sqrt{2}} \chi_\pi \Big|_{B'},\ \tfrac{1}{\sqrt{2}} \delta_\pi \,;\, \pi \in \Omega^{\mathrm{sa}}(S') \Big\} \bigsqcup \big\{ \chi_\pi|_{B'} \,;\, \pi \in \Omega^{\mathrm{nsa}}(S') \big\} \tag{4.6}$$

gives an orthonormal basis of the subspace of $\ell^2(B')$ consisting of spin central functions on B'.

(ii) *The set of functions on the group S' given by characters as*

$$\big\{ \chi_\pi \,;\, \pi \in \Omega^{\mathrm{sa}}(S') \big\} \bigsqcup \big\{ \chi_\pi,\ \mathrm{sgn} \cdot \chi_\pi \,;\, \pi \in \Omega^{\mathrm{nsa}}(S') \big\} \tag{4.7}$$

gives an orthonormal basis of the subspace of $\ell^2(S')$ consisting of spin central functions on S'.

(iii) *Denote by $\chi_\pi|'_{B'}$ (resp. $\chi_\pi|'_{C'}$) the trivial extension of $\chi_\pi|'_{B'}$ (resp. $\chi_\pi|'_{C'}$) by putting 0 outside of B' (resp. C'). Then, χ_π in the first subset of (4.7) can be replaced by $\chi_\pi|'_{B'}$, and the pair $\{\chi_\pi,\ \mathrm{sgn} \cdot \chi_\pi\}$ in the second subset can be replaced by the pair $\{\sqrt{2}\,\chi_\pi|'_{B'},\ \sqrt{2}\,\chi_\pi|'_{C'}\}$.*

Proof. (i) This comes from Theorem 2.4 (i).

(ii) The assertion comes from the definition of $\Omega^{\mathrm{sa}}(S')$ and $\Omega^{\mathrm{nsa}}(S')$.

(iii) The assertion is affirmed by the fact that $\chi_\pi|_{C'} = 0$ for $\pi \in \Omega^{\mathrm{sa}}(S')$, and $(\mathrm{sgn} \cdot \chi_\pi)|_{C'} = -\chi_\pi|_{C'}$ for $\pi \in \Omega^{\mathrm{nsa}}(S')$. □

Now let $S' = S'_1 \hat{*} \cdots \hat{*} S'_m$. For each S'_j $(j \in \boldsymbol{I}_m)$, take a complete set of representatives $\Omega(S'_j) = \Omega^{\mathrm{sa}}(S'_j) \sqcup \Omega^{\mathrm{nsa}}(S'_j)$ of ${}^{\mathrm{ass}}\widehat{S'_j}^{\mathrm{spin}}$ as above. For a set $(\pi_j)_{j \in \boldsymbol{I}_m}$ of $\pi_j \in \Omega(S'_j)$, let s be the number of non-self-associate π_j's, that is, $s := \sharp\{j \in \boldsymbol{I}_m ; \pi_j \in \Omega^{\mathrm{nsa}}(S'_j)\}$, where $\sharp A$ denotes the order of a set A. Denote by $\Omega^{\mathrm{odd}}(S')$ the set of the twisted central products $\pi = \pi_1 \hat{*} \cdots \hat{*} \pi_m$ with s odd (and so π is non-self-associate), and by $\Omega^{\mathrm{even}}(S')$ the set of $\pi = \pi_1 \hat{*} \cdots \hat{*} \pi_m$ with s even (and so π is self-associate), and put

$$\Omega(S') = \Omega^{\mathrm{odd}}(S') \sqcup \Omega^{\mathrm{even}}(S').$$

The following lemma follows from the equivalence relations (4.3)–(4.4).

Lemma 4.6. *Let $S' = S'_1 \hat{*} \cdots \hat{*} S'_m$ with $S'_j \in \mathscr{G}$. For a set $(\pi_j)_{j \in \boldsymbol{I}_m}$ of spin IR π_j of S'_j for $j \in \boldsymbol{I}_m$, let $\pi = \pi_1 \hat{*} \cdots \hat{*} \pi_m$ be their twisted central product. If π is non-self-associate, then it is equivalent to π' or to $\mathrm{sgn} \cdot \pi'$ for an element $\pi' \in \Omega^{\mathrm{odd}}(S')$. If π is self-associate, then it is equivalent to an element $\pi' \in \Omega^{\mathrm{even}}(S')$.*

To prove the completeness of the set $\mho_{\mathrm{tw}}(S')$ of spin IRs obtained as twisted central products, we need the following.

PROPOSITION 4.7. *The set of spin IRs of S' given by*

$$\{\pi', \ \mathrm{sgn}\cdot\pi' \,;\, \pi' \in \Omega^{\mathrm{odd}}(S')\} \bigsqcup \Omega^{\mathrm{even}}(S') \tag{4.8}$$

consists of mutually inequivalent IRs.

To prove this, since two sets $\{\pi', \ \mathrm{sgn}\cdot\pi' \,;\, \pi' \in \Omega^{\mathrm{odd}}(S')\}$ and $\Omega^{\mathrm{even}}(S')$ consist of mutually inequivalent IRs, it is sufficient for us to prove that two sets of characters

$$\{\chi_{\pi'}, \ \mathrm{sgn}\cdot\chi_{\pi'} \,;\, \pi' \in \Omega^{\mathrm{odd}}(S')\} \quad \text{and} \quad \{\chi_{\pi'} \,;\, \pi' \in \Omega^{\mathrm{even}}(S')\}$$

each consist of mutually orthogonal elements. For this, we should calculate explicitly using character formulas (4.1) and (4.2), and so it is convenient to postpone the calculations in the next section (Section 5.2).

5. Completeness of the set of spin IRs $\pi_1\hat{*}\cdots\hat{*}\pi_m$.

Let $(S'_j, z_j, \mathrm{sgn}_j) \in \mathscr{G}$ for $j \in \boldsymbol{I}_m$, and take their twisted central product $S' = S'_1\hat{*}\cdots\hat{*}S'_m$. In this section, we prove the completeness of the set $\mho_{\mathrm{tw}}(S')$ of spin IRs obtained as twisted central products $\pi = \pi_1\hat{*}\pi_2\hat{*}\cdots\hat{*}\pi_m$ of spin IRs π_j of S'_j $(j \in \boldsymbol{I}_m)$, that is to say, any spin IR of S' is equivalent to someone in $\mho_{\mathrm{tw}}(S')$. The essential part of the proof is the case where all S'_j's are from the category $\mathscr{G}'$, and we treat this case hereafter. Moreover, choosing certain 'standard' $\pi = \pi_1\hat{*}\pi_2\hat{*}\cdots\hat{*}\pi_m$, we obtain a subset of $\mho_{\mathrm{tw}}(S')$, which gives a complete set of representatives of the spin dual $\widehat{S'}^{\mathrm{spin}}$ of S', and so obtain a parametrization of $\widehat{S'}^{\mathrm{spin}}$.

5.1. Equalities for characters of $\pi_1\hat{*}\cdots\hat{*}\pi_m$.

First we give some equalities for the norm of χ_π, $\pi = \pi_1\hat{*}\pi_2\hat{*}\cdots\hat{*}\pi_m$ in $\ell^2(S')$, which in turn confirm the irreducibility of π. For convenience, we use the normalized invariant measure μ_G on a finite group G and the integral notation in place of the sum notation as follows. Denote by $|G|$ the order of G, then for a function f on G,

$$\int_G f(g)d\mu_G(g) := \frac{1}{|G|}\sum_{g\in G} f(g).$$

PROPOSITION 5.1. *Let π_j be a spin IR of S'_j for $j \in \boldsymbol{I}_m$, and s be the number of non-self-associate π_j's. Let $\pi = \pi_1\hat{*}\cdots\hat{*}\pi_m$ be the twisted central product representation of $S' = S'_1\hat{*}\cdots\hat{*}S'_m$.*

(i) *If s is odd, then*

$$\int_{B'} \left|\chi_\pi(\sigma')\right|^2 d\mu_{S'}(\sigma') = \int_{C'} \left|\chi_\pi(\sigma')\right|^2 d\mu_{S'}(\sigma') = \frac{1}{2}. \tag{5.1}$$

(ii) *If s is even, then*

$$\int_{S'} \left|\chi_\pi(\sigma')\right|^2 d\mu_{S'}(\sigma') = \int_{B'} \left|\chi_\pi(\sigma')\right|^2 d\mu_{S'}(\sigma') = 1. \tag{5.2}$$

(iii) *The spin representation* $\pi = \pi_1 \hat{*} \cdots \hat{*} \pi_m$ *of* $S' = S'_1 \hat{*} \cdots \hat{*} S'_m$ *is irreducible.*

For the proof, we apply the following lemma.

Lemma 5.2. *On the twisted central product* S', *there holds the following multiple integral formula: for a function* f *on* S',

$$\int_{S'} f(\sigma') d\mu_{S'}(\sigma') = \int_{S'_1} \cdots \int_{S'_m} f(\sigma'_1 \sigma'_2 \cdots \sigma'_m) d\mu_{S'_1}(\sigma'_1) \cdots d\mu_{S'_m}(\sigma'_m).$$

Proof of Proposition. By the results in Section 4, we may assume that, among π_j's, π_j $(j \in \boldsymbol{I}_r)$ are self-associate, and π_j $(j \in \boldsymbol{I}_m \setminus \boldsymbol{I}_r)$ are non-self-associate. Then, for $j \in \boldsymbol{I}_r$, since π_j is irreducible and $\pi_j \cong \mathrm{sgn} \cdot \pi_j \iff \chi_{\pi_j} = \mathrm{sgn} \cdot \chi_{\pi_j} \iff \chi_{\pi_j} = 0$ on C'_j,

$$\int_{S'_j} \left|\chi_{\pi_j}(\sigma'_j)\right|^2 d\mu_{S'_j}(\sigma'_j) = \int_{B'_j} \left|\chi_{\pi_j}(\tau'_j)\right|^2 d\mu_{S'_j}(\tau'_j) = 1,$$

and since $(1/2)(\chi_{\pi_j} \pm \delta_{\pi_j})$ are characters of non-equivalent IRs of B', and $\mu_{S'_j}|_{B'} = (1/2)\mu_{B'}$,

$$\int_{S'_j} \left|\delta_{\pi_j}(\sigma'_j)\right|^2 d\mu_{S'_j}(\sigma'_j) = \int_{B'_j} \left|\delta_{\pi_j}(\tau'_j)\right|^2 d\mu_{S'_j}(\tau'_j) = 1.$$

For $j \in \boldsymbol{I}_m \setminus \boldsymbol{I}_r$, since $\pi_j \not\cong \mathrm{sgn} \cdot \pi_j$, we have the orthogonality $\chi_{\pi_j} \perp \mathrm{sgn} \cdot \chi_{\pi_j}$, and so

$$\int_{B'_j} \left|\chi_{\pi_j}(\tau'_j)\right|^2 d\mu_{S'}(\tau'_j) = \int_{C'_j} \left|\chi_{\pi_j}(\kappa'_j)\right|^2 d\mu_{S'}(\kappa'_j) = \frac{1}{2}.$$

(i) If $s = m - r$ is odd, then π is non-self-associate, and

$$\chi_\pi\left(\tau'_1 \tau'_2 \cdots \tau'_m\right) = 2^{[s/2]} \cdot \prod_{j \in \boldsymbol{I}_m} \chi_{\pi_j}(\tau'_j) \quad \text{for } \tau'_j \in B'_j \ (j \in \boldsymbol{I}_m),$$

$$\chi_\pi\left(\tau'_1 \cdots \tau'_r \kappa'_{r+1} \cdots \kappa'_m\right) = (2i)^{[s/2]} \cdot \prod_{j \in \boldsymbol{I}_r} \delta_{\pi_j}(\tau'_j) \cdot \prod_{j \in \boldsymbol{I}_m \setminus \boldsymbol{I}_r} \chi_{\pi_j}(\kappa'_j),$$
$$\text{for } \tau'_j \in B'_j \ (j \in \boldsymbol{I}_r), \quad \kappa'_j \in C'_j \ (j \in \boldsymbol{I}_m \setminus \boldsymbol{I}_r),$$

and $\chi_\pi(\sigma') = 0$ elsewhere. Therefore we have

$$\int_{B'} \left|\chi_\pi(\sigma')\right|^2 d\mu_{S'}(\sigma') = 2^{s-1} \prod_{j \in \boldsymbol{I}_m \setminus \boldsymbol{I}_r} \int_{B'_j} \left|\chi_{\pi_j}(\tau'_j)\right|^2 d\mu_{S'_j}(\tau'_j) = \frac{1}{2},$$

$$\int_{C'} \left|\chi_\pi(\sigma')\right|^2 d\mu_{S'}(\sigma') = 2^{s-1} \prod_{j\in \boldsymbol{I}_m\setminus \boldsymbol{I}_r} \int_{C'_j} \left|\chi_{\pi_j}(\kappa'_j)\right|^2 d\mu_{S'_j}(\kappa'_j) = \frac{1}{2}.$$

(ii) If $s = m - r$ is even, then π is self-associate, and

$$\chi_\pi\big(\tau'_1\tau'_2\cdots\tau'_m\big) = 2^{[s/2]}\cdot \prod_{j\in \boldsymbol{I}_m} \chi_{\pi_j}(\tau'_j) \quad \text{for } \tau'_j \in B'_j \ (j \in \boldsymbol{I}_m),$$

and $\chi_\pi(\sigma') = 0$ elsewhere. Accordingly,

$$\int_{S'} \left|\chi_\pi(\sigma')\right|^2 d\mu_{S'}(\sigma') = 2^{s} \prod_{j\in \boldsymbol{I}_m\setminus \boldsymbol{I}_r} \int_{B'_j} \left|\chi_{\pi_j}(\tau'_j)\right|^2 d\mu_{S'_j}(\tau'_j) = 1.$$

(iii) The irreducibility of π is equivalent to $\|\chi_\pi\|_{\ell^2(S')} = 1$. □

5.2. Proof of Proposition 4.7.

For this purpose, we need to come back to the general situation where, among π_j's given, π_i $(i \in \boldsymbol{I}_m^{\mathrm{sa}})$ are self-associate, and π_j $(j \in \boldsymbol{I}_m^{\mathrm{nsa}})$ are non-self-associate with $\boldsymbol{I}_m = \boldsymbol{I}_m^{\mathrm{sa}} \sqcup \boldsymbol{I}_m^{\mathrm{nsa}}$. Express these sets of indexes as

$$\boldsymbol{I}_m^{\mathrm{sa}} = \{i_1,\ldots,i_r\},\ i_1<\cdots<i_r, \quad \boldsymbol{I}_m^{\mathrm{nsa}} = \{j_1,\ldots,j_s\},\ j_1<\cdots<j_s, \tag{5.3}$$

where $r+s=m$. The character formula for $\pi = \pi_1\hat{*}\pi_2\hat{*}\cdots\hat{*}\pi_m$ of $S' = S'_1\hat{*}S'_2\hat{*}\cdots\hat{*}S'_m$ in this general situation is given as follows.

(i) In case s is odd, π is non-self-associate, and

$$\chi_\pi\big(\tau'_1\tau'_2\cdots\tau'_m\big) = 2^{[s/2]}\cdot \prod_{j\in \boldsymbol{I}_m} \chi_{\pi_j}(\tau'_j) \quad \text{for } \tau'_j \in B'_j \ (j \in \boldsymbol{I}_m), \tag{5.4}$$

$$\chi_\pi\big(\sigma'_1\sigma'_2\cdots\sigma'_m\big) = (2i)^{[s/2]}\cdot \prod_{j\in \boldsymbol{I}_m^{\mathrm{sa}}} \delta_{\pi_j}(\tau'_j)\cdot \prod_{j\in \boldsymbol{I}_m^{\mathrm{nsa}}} \chi_{\pi_j}(\kappa'_j), \tag{5.5}$$

$$\text{for } \sigma'_j = \tau'_j \in B'_j \ (j \in \boldsymbol{I}_m^{\mathrm{sa}}), \quad \sigma'_j = \kappa'_j \in C'_j \ (j \in \boldsymbol{I}_m^{\mathrm{nsa}}).$$

(ii) In case s is even, π is self-associate, and

$$\chi_\pi\big(\tau'_1\tau'_2\cdots\tau'_m\big) = 2^{[s/2]}\cdot \prod_{j\in \boldsymbol{I}_m} \chi_{\pi_j}(\tau'_j) \quad \text{for } \tau'_j \in B'_j \ (j \in \boldsymbol{I}_m).$$

(iii) In any case, $\chi_\pi(\sigma') = 0$ elsewhere.

Orthogonality for $\Omega^{\mathrm{odd}}(S')$: To prove the mutual orthogonality of $\{\chi_\pi,\ \mathrm{sgn}\cdot\chi_\pi\,;\, \pi \in \Omega^{\mathrm{odd}}(S')\}$, we apply character formula (i) and (iii) above. Take another $\pi' = \pi'_1\hat{*}\pi'_2\hat{*}\cdots\hat{*}\pi'_m$ from $\Omega^{\mathrm{odd}}(S')$ with $\pi'_j \in \Omega(S'_j)$, for which $\pi'_j \neq \pi_j$ for some $j \in \boldsymbol{I}_m$, and $s' = \sharp\{\pi'_j; \pi'_j \in \Omega^{\mathrm{nsa}}(S'_j)\}$.

(1) Assume that $\pi'_j \in \Omega^{\mathrm{sa}}(S'_j)$ for $j \in \boldsymbol{I}'^{\mathrm{sa}}_m$ and $\pi'_j \in \Omega^{\mathrm{nsa}}(S'_j)$ for $j \in \boldsymbol{I}'^{\mathrm{nsa}}_m := \boldsymbol{I}_m\setminus\boldsymbol{I}'^{\mathrm{sa}}_m$.

Then, we assert that $\chi_{\pi'} \perp (\mathrm{sgn})^{\alpha}\chi_{\pi}$ ($\alpha = 0, 1$) unless $\boldsymbol{I}_m^{\mathrm{sa}} = \boldsymbol{I}_m'^{\mathrm{sa}}$.

In fact, it is sufficient to prove that the integrals of $\chi_{\pi'}(\sigma')\overline{\chi_{\pi}(\sigma')}$ for the part (5.4) and for the part (5.5) are both equal to 0. For the part (5.4), the integral is equal to

$$2^{[s/2]+[s'/2]} \prod_{j \in \boldsymbol{I}_m} \int_{B'_j} \chi_{\pi'_j}(\tau'_j)\overline{\chi_{\pi_j}(\tau'_j)}d\mu_{S'_j}(\tau'_j) = 2^{[s/2]+[s'/2]-m} \prod_{j \in \boldsymbol{I}_m} \langle \chi_{\pi'_j}, \chi_{\pi_j} \rangle_{\ell^2(B'_j)}. \quad (*)$$

This is equal to 0 because of Lemma 4.5 (i) applied to S'_j for which $\pi'_j \neq \pi_j$.

For the part (5.5), the integral is trivially equal to 0 since the supports for $\chi_{\pi'}$ and for χ_{π} are mutually disjoint.

(2) Now assume that $\boldsymbol{I}_m^{\mathrm{sa}} = \boldsymbol{I}_m'^{\mathrm{sa}}$. In this case, the integral for the part (5.4) is again equal to the above integral $(*)$ and equal to 0 by the same reason. The integral for the part (5.5) is equal to

$$4^{[s/2]} \prod_{j \in \boldsymbol{I}_m^{\mathrm{sa}}} \int_{B'_j} \delta_{\pi'_j}(\tau'_j)\overline{\delta_{\pi_j}(\tau'_j)}d\mu_{S'_j}(\tau'_j) \times \prod_{j \in \boldsymbol{I}_m^{\mathrm{nsa}}} \int_{C'_j} \chi_{\pi'_j}(\kappa'_j)\overline{\chi_{\pi_j}(\kappa'_j)}d\mu_{S'_j}(\kappa'_j). \quad (**)$$

Then, for the integrals in the first factor, we apply Lemma 4.5 (i) to S'_j for which $\pi'_j \neq \pi_j$ (if exists). For the integrals in the second factor, we apply Lemma 4.5 (iii) to S'_j for which $\pi'_j \neq \pi_j$ (if exists). Thus we see that the integral $(**)$ is equal to 0.

Orthogonality for $\Omega^{\mathrm{even}}(S')$: The mutual orthogonality of the set of characters $\{\chi_\pi; \pi \in \Omega^{\mathrm{even}}(S')\}$ comes from the character formula (ii)–(iii) above and Lemma 4.5 (i) applied to each S'_j ($j \in \boldsymbol{I}_m$).

Now the proof of Proposition 4.7 is complete.

5.3. Proof of the completeness.

Let us prove the completeness of the set $\mho_{\mathrm{tw}}(S')$ of twisted central products $\pi_1 \hat{*} \cdots \hat{*} \pi_m$. For this, we apply the equality (2.1) in Proposition 2.1. Since $\dim(\pi_1 \hat{*} \cdots \hat{*} \pi_m) = 2^{[s/2]} \prod_{j \in \boldsymbol{I}_m} \dim \pi_j$, we have

$$\big(\dim(\pi_1 \hat{*} \cdots \hat{*} \pi_m)\big)^2 = \begin{cases} 2^{s-1} \prod_{j \in \boldsymbol{I}_m} (\dim \pi_j)^2 & \text{if } s \text{ is odd}, \\ 2^s \prod_{j \in \boldsymbol{I}_m} (\dim \pi_j)^2 & \text{if } s \text{ is even}. \end{cases}$$

(1). For the convenience of notations, we return to the case where, among π_j's, π_j ($j \in \boldsymbol{I}_r$) are self-associate, and π_j ($j \in \boldsymbol{I}_m \setminus \boldsymbol{I}_r$) are non-self-associate. (But the calculations have generalities.) When s is odd, by Proposition 4.3 (ii),

$$\big(\dim(\pi_1 \hat{*} \cdots \hat{*} \pi_m)\big)^2 = \frac{1}{2} \sum_{\substack{\alpha_j = 0,1 \\ (j \in \boldsymbol{I}_m \setminus \boldsymbol{I}_r)}} \prod_{j \in \boldsymbol{I}_r} (\dim \pi_j)^2 \prod_{j \in \boldsymbol{I}_m \setminus \boldsymbol{I}_r} \big(\dim(\mathrm{sgn}_j)^{\alpha_j}\pi_j\big)^2. \quad (5.6)$$

Here, for representations themselves, we have

$$\begin{aligned}&\pi_1\hat{*}\cdots\hat{*}\pi_r\hat{*}\big((\mathrm{sgn}_{r+1})^{\alpha_{r+1}}\pi_{r+1}\big)\hat{*}\cdots\hat{*}\big((\mathrm{sgn}_m)^{\alpha_m}\pi_m\big)\\ &\quad\cong \mathrm{sgn}^{\alpha_{r+1}+\cdots+\alpha_m}\cdot\pi_1\hat{*}\cdots\hat{*}\pi_m,\end{aligned}$$

and the parity of the sum $\sum_{j=r+1}^{j=m}\alpha_j = 0,1 \pmod 2$ determines the equivalence class of the twisted central product representation above. Hence, we get from (5.6) that

$$\begin{aligned}&\sum_{\alpha_m=0,1}\big(\dim(\pi_1\hat{*}\cdots\hat{*}\pi_r\hat{*}\pi_{r+1}\hat{*}\cdots\hat{*}\pi_{m-1}\hat{*}((\mathrm{sgn}_m)^{\alpha_m}\pi_m)\big)^2\\ &\quad=\sum_{\substack{\alpha_j=0,1\\(j\in\boldsymbol{I}_m\setminus\boldsymbol{I}_r)}}\prod_{j\in\boldsymbol{I}_r}(\dim\pi_j)^2\prod_{j\in\boldsymbol{I}_m\setminus\boldsymbol{I}_r}\big(\dim(\mathrm{sgn}_j)^{\alpha_j}\pi_j\big)^2. \qquad (5.7)\end{aligned}$$

(2). When s is even, by Proposition 4.4, we have similarly

$$\big(\dim(\pi_1\hat{*}\cdots\hat{*}\pi_m)\big)^2=\sum_{\substack{\alpha_j=0,1\\(j\in\boldsymbol{I}_m\setminus\boldsymbol{I}_r)}}\prod_{j\in\boldsymbol{I}_r}\big(\dim\pi_j\big)^2\prod_{j\in\boldsymbol{I}_m\setminus\boldsymbol{I}_r}\big(\dim(\mathrm{sgn}_j)^{\alpha_j}\pi_j\big)^2. \qquad (5.8)$$

(3). In the discussions until here, we assumed that, among spin IRs π_j, self-associate ones are π_j with $j\in\boldsymbol{I}_r$ and non-self-associate ones are π_j with $j\in\boldsymbol{I}_m\setminus\boldsymbol{I}_r$. Now, in the general case, when we fix the set A of suffixes j of self-associate π_j's, then that of non-self-associate π_j's is $B=\boldsymbol{I}_m\setminus A$, and we can obtain similarly as above the equalities corresponding to (5.7) and (5.8), with (A,B) in place of $(\boldsymbol{I}_r,\ \boldsymbol{I}_m\setminus\boldsymbol{I}_r)$.

Adding thus obtained equalities over all different (A,B), we arrive to

$$\sum_{[\pi_1\hat{*}\cdots\hat{*}\pi_m]}\big(\dim(\pi_1\hat{*}\cdots\hat{*}\pi_m)\big)^2=\prod_{j\in\boldsymbol{I}_m}\sum_{[\pi_j]\in\widehat{S'_j}^{\mathrm{spin}}}(\dim\pi_j)^2=\prod_{j\in\boldsymbol{I}_m}\frac{1}{2}|S'_j|=\frac{1}{2}|S'|,$$

where the sum on the left hand side is over different equivalence classes $[\pi_1\hat{*}\cdots\hat{*}\pi_m]$. This is exactly the equality (2.1) to be proved.

Thus we have proved the following completeness theorem:

Theorem 5.3. *For $(S'_j, z_j, \mathrm{sgn}_j)\in\mathscr{G}'$ $(j\in\boldsymbol{I}_m)$, let $\mho_{\mathrm{tw}}(S')$ be the set of spin IRs of the twisted central product $S'=S'_1\hat{*}S'_2\hat{*}\cdots\hat{*}S'_m$ obtained as twisted central products $\pi_1\hat{*}\pi_2\hat{*}\cdots\hat{*}\pi_m$ of spin IRs π_j of S'_j $(j\in\boldsymbol{I}_m)$. Then it is complete in the sense that any spin IR of S' is equivalent to someone in $\mho_{\mathrm{tw}}(S')$.*

Note that the same assertion is valid in more general case of $(S'_j, z_j, \mathrm{sgn}_j)$ $(j\in\boldsymbol{I}_m)$ taken from the bigger category $\mathscr{G}$.

(4). For an $S'\in\mathscr{G}$, let $\overset{\mathrm{ass}}{\sim}$ be the equivalence relation introduced in Section 2.1, that is, $\pi\overset{\mathrm{ass}}{\sim}\pi' \overset{\mathrm{def}}{\iff} \pi'\cong\pi$ or $\pi'\cong\mathrm{sgn}\cdot\pi$. The equivalence class of spin IR π under $\overset{\mathrm{ass}}{\sim}$ is denoted by $[\pi]_{\mathrm{ass}}$ and the set of equivalence classes is denoted by ${}^{\mathrm{ass}}\widehat{S'}^{\mathrm{spin}}$.

Now let $S'=S'_1\hat{*}S'_2\hat{*}\cdots\hat{*}S'_m$. Recall the notations introduced in Section 4.3. For each S'_j, take a complete set of representatives $\Omega(S'_j)=\Omega^{\mathrm{sa}}(S'_j)\sqcup\Omega^{\mathrm{nsa}}(S'_j)$ of ${}^{\mathrm{ass}}\widehat{S'_j}^{\mathrm{spin}}$, where $\{[\pi_j]_{\mathrm{ass}};\pi_j\in\Omega^{\mathrm{sa}}(S'_j)\}$ covers associate equivalence classes of self-associate spin IRs, and $\{[\pi_j]_{\mathrm{ass}};\pi_j\in\Omega^{\mathrm{nsa}}(S'_j)\}$ covers those of non-self-associate spin IRs. For a set $(\pi_j)_{j\in\boldsymbol{I}_m}$ of $\pi_j\in\Omega(S'_j)$, let $s:=\sharp\{j\in\boldsymbol{I}_m;\pi_j\in\Omega^{\mathrm{nsa}}(S'_j)\}$, and denote by $\Omega^{\mathrm{even}}(S')$ the set of $\pi=\pi_1\hat{*}\cdots\hat{*}\pi_m$ with s even, and by $\Omega^{\mathrm{odd}}(S')$ the one for s odd.

Theorem 5.4. *For $(S'_j,z_j,\mathrm{sgn}_j)\in\mathscr{G}'$ $(j\in\boldsymbol{I}_m)$, take their twisted central product $S'=S'_1\hat{*}S'_2\hat{*}\cdots\hat{*}S'_m$. Then the set $\Omega^{\mathrm{even}}(S')$ can be taken as a set of representatives $\Omega^{\mathrm{sa}}(S')$, and the one $\Omega^{\mathrm{odd}}(S')$ as such a set $\Omega^{\mathrm{nsa}}(S')$, that is to say, $\{[\pi];\pi\in\Omega^{\mathrm{even}}(S')\}$ is the totality of equivalence classes of self-associate spin IRs, and $\{[\pi],[\mathrm{sgn}\cdot\pi];\pi\in\Omega^{\mathrm{odd}}(S')\}$ is the totality of equivalence classes of non-self-associate spin IRs. In this sense, the union $\Omega^{\mathrm{ass}}(S'):=\Omega^{\mathrm{even}}(S')\sqcup\Omega^{\mathrm{odd}}(S')$ gives a parametrization of the spin dual $\widehat{S'}^{\mathrm{spin}}$ of S', modulo 'association', that is, a parametrization of ${}^{\mathrm{ass}}\widehat{S'}^{\mathrm{spin}}=\widehat{S'}^{\mathrm{spin}}/\overset{\mathrm{ass}}{\sim}$.*

Note 5.1. When s is odd, pick up one non-self-associate $\pi_i\in\Omega^{\mathrm{nsa}}(S'_i)$, then

$$\mathrm{sgn}\cdot(\pi_1\hat{*}\pi_2\hat{*}\cdots\hat{*}\pi_m)\cong\pi_1\hat{*}\cdots\hat{*}\pi_{i-1}\hat{*}(\mathrm{sgn}_i\cdot\pi_i)\hat{*}\pi_{i+1}\hat{*}\cdots\hat{*}\pi_m.$$

6. Spin IRs of normal subgroup $B'=\mathrm{Ker}(\mathrm{sgn})$ of $S'=S'_1\hat{*}S'_2\hat{*}\cdots\hat{*}S'_m$.

The kernel $B'=\mathrm{Ker}(\mathrm{sgn})$ of the character sgn of the twisted central product $S'=S'_1\hat{*}S'_2\hat{*}\cdots\hat{*}S'_m$ is normal and of index 2. Therefore Proposition 2.2 can be applied to the pair (S',B'). By this, we can obtain from Theorem 5.4 a parametrization of spin dual $\widehat{B'}^{\mathrm{spin}}$ of B' as follows:

Theorem 6.1. *Let $(S'_j,z_j,\mathrm{sgn}_j)\in\mathscr{G}$ for $j\in\boldsymbol{I}_m$, and suppose that at least one S'_j comes from the subcategory $\mathscr{G}'$. For the normal subgroup $B'=\mathrm{Ker}(\mathrm{sgn})$ of the twisted central product $S'=S'_1\hat{*}S'_2\hat{*}\cdots\hat{*}S'_m$, a complete set of representatives of its spin dual $\widehat{B'}^{\mathrm{spin}}$ is obtained from the complete set $\Omega^{\mathrm{ass}}(S')$ of representatives of spin dual $\widehat{S'}^{\mathrm{spin}}/\overset{\mathrm{ass}}{\sim}$ modulo 'association' of S', in Theorem 5.4, as follows.*

The restriction $\pi|_{B'}$ onto B' of each $\pi=\pi_1\hat{*}\cdots\hat{*}\pi_m\in\Omega^{\mathrm{even}}(S')$ is a direct sum of mutually non-equivalent two spin IRs $\rho'=\rho'(\pi_1\hat{*}\cdots\hat{*}\pi_m)$ and $\rho''=\rho''(\pi_1\hat{*}\cdots\hat{*}\pi_m)$, and with any element κ' in $C'=S'\setminus B'$, $\rho''\cong{}^{\kappa'}(\rho')$. Denote by $\Omega^{\mathrm{even}}(B')$ the totality of thus obtained spin IRs of B'.

The restriction $\pi|_{B'}$ onto B' of each $\pi=\pi_1\hat{*}\cdots\hat{*}\pi_m\in\Omega^{\mathrm{odd}}(S')$ is itself a spin IR $\rho(\pi_1\hat{*}\cdots\hat{*}\pi_m)$. Denote by $\Omega^{\mathrm{odd}}(B')$ the totality of these IRs of B'.

Then the union $\Omega^{\mathrm{spin}}(B'):=\Omega^{\mathrm{even}}(B')\sqcup\Omega^{\mathrm{odd}}(B')$ is a complete set of representatives of the spin dual $\widehat{B'}^{\mathrm{spin}}$ of B'.

7. Relation to projective representations of finite groups.

In his papers [**13**] and [**14**], Schur founded the theory of projective representations of finite groups. The motivation to our present study comes principally from general theory of such representations and also from our recent studies on irreducible projective (spin) representations of complex reflection groups. So we should explain the relations of the present paper to these subjects a little in detail, to justify the present study on spin representations of double covering groups. For an introduction to the theory of projective representations, we refer e.g. [**5**].

7.1. Stand point in the general theory of projective representations. The notion of a projective representation of a finite group G is first introduced by Schur [**13**] in 1904 and its general theory is founded by his work [**13**] and [**14**]. A *projective representation* T of G is a map from G into the group $GL(V)$ of invertible linear operators on a (finite-dimensional) vector space V which satisfies

$$T(e) = I, \quad T(g)T(h) = r_{g,h}T(gh), \tag{7.1}$$

where I is the identity operator on V and $r_{g,h} \in \boldsymbol{C}^\times := \{w \in \boldsymbol{C}; w \neq 0\}$. (We call T often *spin* representation of G.) The function $r_{g,h}$ on $G \times G$ is called the *factor set* of T. When $T(g)$ is replaced by $T'(g) := \lambda_g T(G)$ with $\lambda_g \in \boldsymbol{C}^\times$, the factor set $r'_{g,h}$ of T' is given by

$$r'_{g,h} = \frac{\lambda_g \lambda_h}{\lambda_{gh}} \cdot r_{g,h}.$$

Introducing equivalence relation $(r_{g,h}) \sim (r'_{g,h})$, we come to the cohomology group $H^2(G, \boldsymbol{C}^\times)$, which is called *Schur multiplier* of G.

On the other hand, consider a central extension $\widetilde{G}$ of G by an abelian group Z as $1 \to Z \to \widetilde{G} \to G \to 1$ (exact), and take a section $s : G \to \widetilde{G}$. Let $\widetilde{T}$ be an irreducible linear representation of $\widetilde{G}$ and put $T(g) := \widetilde{T}(s(g))$. Then we have $T(g)T(h) = r_{g,h}T(gh)$ with $r_{g,h} \in \boldsymbol{C}^\times$. In fact, $s(g)s(h) = z_{g,h}s(gh)$ with a $z_{g,h} \in Z$, and $\widetilde{T}(z_{g,h}) = r_{g,h}I$ since $\widetilde{T}$ is irreducible. Schur proved that, for any finite group G, there exists a finite central extension $\widetilde{G}$ such that

(∗) *any projective representation of G is obtained in this way from a linear representation of $\widetilde{G}$.*

A *representation group* of G is defined as a central extension of G with the property (∗) and with minimum degree among such coverings. In [**13**] the following are proved:

(1). *For any finite group G, there exist a finite number of non-isomorphic representation groups. However the central subgroup Z for extension is unique and isomorphic to the Schur multiplier $H^2(G, \boldsymbol{C}^\times)$.*

The representation theory for these representation groups are mutually equivalent, and so we take one of them and denote it by $R(G)$. Since the set of *linear* representations of $R(G)$ covers the set of projective representations of G, we can replace largely the study of projective representations of G to that of linear representations of $R(G)$.

In the case of a connected Lie group G, any projective representation of G can be linearized if we go up to its universal covering group. So that, even for a finite group G, we may call $R(G)$ a *universal covering group* of G, even though it is not unique.

A characterization of representation group is given as follows (cf. [**14**, Introduction]).

(2). *A group G^* is a representation group of a finite group G if and only if there exists a central subgroup Z of G^* such that*

(i) *Z is contained in $[G^*, G^*] \cap Z(G^*)$, where $Z(G^*)$ denotes the center of G^*,*
(ii) $1 \longrightarrow Z \longrightarrow G^* \longrightarrow G \longrightarrow 1$ (exact),
(iii) $|Z| = |H^2(G, \boldsymbol{C}^\times)|$.

For the study of linear representations of $R(G)$ of a specified group G, one-dimensional characters of the abelian group $Z = H^2(G, \boldsymbol{C}^\times)$ have important meaning as explained below, and this invite us to the study of *spin* representations of a *double* covering groups of G in many cases. This is one of our motivations to the present paper.

Now, take an IR π of $R(G)$, then every element $z \in Z \subset Z(G)$ is mapped to a scalar operator, that is, $\pi(z) = \chi_Z^\pi(z) I$, where χ_Z^π is a character of Z, which we call central type of π. Hence the set of IRs of $R(G)$ is divided into subsets

$$\mathrm{Irr}\big(R(G); \chi\big) := \{\pi\,;\, \chi_Z^\pi = \chi\}, \quad \chi \in \widehat{Z}.$$

For $\chi \in \widehat{Z}$, put $Z_\chi := \mathrm{Ker}(\chi)$ and $\widetilde{G}^\chi := R(G)/Z_\chi$. Then any IR π with $\chi_Z^\pi = \chi$ can be considered as a representation of the quotient group $\widetilde{G}^\chi$. This is, in turn, a central extension of G by central subgroup Z/Z_χ, and so a q_χ-times covering group of G, where $q_\chi := |Z/Z_\chi|$ is the order of χ. Thus we arrive naturally at the following fundamental problems for construction of spin irreducible representations.

Problem 7.1. *Let G be a finite group, and $\boldsymbol{Z}_q$ a cyclic group of order q. Let $\widetilde{G}$ be a central extension of G with central subgroup $\boldsymbol{Z}_q$ as $1 \to \boldsymbol{Z}_q \to \widetilde{G} \to G \to 1$ (exact). Give a general method to construct spin IRs of $\widetilde{G}$ with central type $\chi \in \widehat{\boldsymbol{Z}_q}$.*

Problem 7.2. *Let G_1' and G_2' be central extensions by $\boldsymbol{Z}_q$ of finite groups G_1 and G_2 respectively. Define a twisted central product $G_1' \hat{*}_q G_2'$ as a central extension of $G_1 \times G_2$ by $\boldsymbol{Z}_q$ which contains naturally both G_1' and G_2'.*

Moreover, for spin IR π_i of G_i' of the same central type $\chi \in \widehat{\boldsymbol{Z}_q}$ for $i = 1, 2$, give a general formula for constructing twisted central product $\pi_1 \hat{*}_q \pi_2$ as a spin IR of $G_1' \hat{*}_q G_2'$ extending π_i of $G_i' \subset G_1' \hat{*}_q G_2'$ for $i = 1, 2$.

Extend this to the case of G_i' $(i = 1, 2, \ldots, m)$ and also for π_i $(i = 1, 2, \ldots, m)$.

In the book [**10**], we see many explicit examples of Schur multipliers $Z = H^2(G, \boldsymbol{C}^\times)$, and find that prime factors of the order of Z are dominantly powers of 2. Hence the case of $q = 2$, or of double covering groups, is of great importance. This is the case which we treat here.

7.2. Studies on projective representations of complex reflection groups. Let $D_n(T) = T^n$ be the direct product of n copies of $T = \boldsymbol{Z}_m$ (understood as a multiplicative group), and make $\mathfrak{S}_n$ acts on it as permutations of coordinates. We consider the semidirect product $\mathfrak{S}_n(T) := D_n(T) \rtimes \mathfrak{S}_n$. Moreover, for a subgroup S of T, define a normal subgroup as $\mathfrak{S}_n(T)^S := \{(d, \sigma) \in \mathfrak{S}_n(T); P(d) \in S\}$, where $P(d) := t_1 t_2 \cdots t_n$ for $d = (t_j)_{j \in \boldsymbol{I}_n}$. Note that any subgroup $T = \boldsymbol{Z}_m$ is given as $S(p) := \{t^p; t \in T\}$ for a factor p of m. Then $\mathfrak{S}_n(\boldsymbol{Z}_m)$ is a realization of the complex reflection group

$G(m,1,n)$ called a *generalized symmetric group*, and for $p|m$, $\mathfrak{S}_n(\boldsymbol{Z}_m)^{S(p)}$ is a realization of complex reflection group $G(m,p,n)$. Note that $G(1,1,n)=\mathfrak{S}_n$, and $G(2,1,n)$ and $G(2,2,n)$ are respectively the Weyl groups of type BC_n and D_n.

Projective representations of Weyl groups and generalised symmetric groups have been studied by many mathematicians, in particular by A.O. Morris et al. We are now studying construction of spin IRs and calculation of their characters (called spin characters) by a quite different method (cf. [**3**] and [**4**]), and the limiting process as $n\to\infty$. In these studies, Schur multipliers are fundamental ingredients at the starting point. They are determined for Weyl groups in [**9**], for generalized symmetric groups $G(m,1,n)$ in [**1**], and for complex reflection groups $G(m,p,n)$ in [**12**]. For these groups G, they are all of the form of $\boldsymbol{Z}_2^k$. This means that studies on double covering groups of G and their spin representations are decisive.

The important part of our studies is on generalized symmetric groups, and the most interesting case is the case of m even. For $n\geq 4$, the Schur multiplier $H^2(G(m,1,n),\boldsymbol{C}^\times)$ is isomorphic to the abelian group $Z':=\langle z_1,z_2,z_3\rangle\cong\boldsymbol{Z}_2^3$ generated by 3 generators z_i ($i\in\boldsymbol{I}_3$) of order 2. A representation group $R(G(m,1,n))$ of $G(m,1,n)$ is 8-times covering group ($8=2^3$), and contains naturally the representation group $\widetilde{\mathfrak{S}}_n$ (double covering) of $\mathfrak{S}_n$, and

$$\{e\}\longrightarrow Z'\longrightarrow R\big(G(m,1,n)\big)\xrightarrow{\Phi}G(m,1,n)\longrightarrow\{e\}\ \text{(exact)},$$

where Φ is the natural homomorphism. The restriction $\Phi_{\mathfrak{S}}:=\Phi|_{\widetilde{\mathfrak{S}}_n}$ is the natural homomorphism $\widetilde{\mathfrak{S}}_n\to\mathfrak{S}_n$ (cf. [**3**, Section 3, Theorem 3.3]). A spin IR π of $G(m,1,n)$ is a linear representation of $R(G(m,1,n))$ with a certain central character $\chi^\pi_{Z'}$ as $\pi(z)=\chi^\pi_{Z'}(z)I$ $(z\in Z')$.

When we construct spin IRs of $G(m,1,n)$, with a fixed central character $\chi^\pi_{Z'}$, the study of spin IRs of subgroups of Schur-Young type $\widetilde{\mathfrak{S}}_{\boldsymbol{\nu}}$ of $\widetilde{\mathfrak{S}}_n$, defined in Example 1.3, becomes essential. For an *ordered* decomposition $\boldsymbol{\nu}=(\nu_j)_{j\in\boldsymbol{I}_m}$, $|\boldsymbol{\nu}|:=\nu_1+\nu_2+\cdots+\nu_m=n$, take a subgroup of Schur-Young type $\widetilde{\mathfrak{S}}_{\boldsymbol{\nu}}:=\Phi_{\mathfrak{S}}^{-1}(\mathfrak{S}_{\nu_1}\times\cdots\times\mathfrak{S}_{\nu_m})$ of $\widetilde{\mathfrak{S}}_n$, which is isomorphic to the twisted central product of $\widetilde{\mathfrak{S}}_{\nu_j}=\Phi_{\mathfrak{S}}^{-1}(\mathfrak{S}_{\nu_j})$ $(j\in\boldsymbol{I}_m)$: $\widetilde{\mathfrak{S}}_{\boldsymbol{\nu}}\cong\widetilde{\mathfrak{S}}_{\nu_1}\hat{*}\cdots\hat{*}\widetilde{\mathfrak{S}}_{\nu_m}$. Then we need to treat the twisted central product $\pi_1\hat{*}\pi_2\hat{*}\cdots\hat{*}\pi_m$ of spin IRs π_j of $\widetilde{\mathfrak{S}}_{\nu_j}$ $(j\in\boldsymbol{I}_m)$, and so on.

8. Spin representation of $\mathfrak{S}_n$ and its subgroups.

This section is devoted to review, with appropriate renewals, necessary informations from Schur's paper [**15**], thus preparing definitions and notations for the succeeding sections. In [**15**], spin representations of the representation groups $\mathfrak{T}_n$ of $\mathfrak{S}_n$ and $\mathfrak{B}_n$ of $\mathfrak{A}_n$ have been studied in detail.

8.1. Representation groups of $\mathfrak{S}_n$ and $\mathfrak{A}_n$.

As an application of the general theory developed in Sections 1–6, we treat spin IRs of Schur-Young subgroups of the n-th symmetric group $\mathfrak{S}_n$. As an abstract group, $\mathfrak{S}_n$ is given by a set of generators $\{s_1,s_2,\ldots,s_{n-1}\}$ and a set of fundamental relations: $s_i^2=e(i\in\boldsymbol{I}_{n-1})$, $(s_is_{i+1})^3=e$ $(i\in\boldsymbol{I}_{n-2})$, $s_is_j=s_js_i$ $(|i-j|\geq 2,\ i,j\in\boldsymbol{I}_{n-1})$. Here s_i

corresponds to a simple reflection $s_i = (i\ \ i{+}1)$.

Representation groups of $\mathfrak{S}_n$ are isomorphic to $\mathfrak{S}_n$ itself for $n = 2, 3$. For $n \geq 4$, we have two representation groups, given in [**15**, Section 3] as $\mathfrak{T}_n$ and $\mathfrak{T}'_n$, not mutually isomorphic for $n \neq 6$. Here we use the latter $\mathfrak{T}'_n$ and denote it by $\widetilde{\mathfrak{S}}_n$.

Theorem 8.1. *For $n \geq 2$, define a double covering group $\widetilde{\mathfrak{S}}_n$ by giving a set of generators $\{z, r_1, r_2, \ldots, r_{n-1}\}$ and a set of fundamental relations*

$$\begin{cases} z^2 = e,\ zr_i = r_i z & (i \in \boldsymbol{I}_{n-1}), \\ r_i^2 = e & (i \in \boldsymbol{I}_{n-1}), \\ (r_i r_{i+1})^3 = e & (i \in \boldsymbol{I}_{n-2}), \\ r_i r_j = z r_j r_i & (|i-j| \geq 2,\ i, j \in \boldsymbol{I}_{n-1}). \end{cases} \tag{8.1}$$

Put $Z := \{e, z\}$, then the chain $\{e\} \to Z = \{z, e\} \to \widetilde{\mathfrak{S}}_n \stackrel{\Phi_{\mathfrak{S}}}{\to} \mathfrak{S}_n \to \{e\}$ is exact, where the covering map $\Phi_{\mathfrak{S}} : \widetilde{\mathfrak{S}}_n \to \mathfrak{S}_n$ is given by $z \to e$ and $r_i \to s_i$ $(i \in \boldsymbol{I}_{n-1})$.

For $n \geq 4$, $\widetilde{\mathfrak{S}}_n$ is a representation group of $\mathfrak{S}_n$.

Note that $\widetilde{\mathfrak{S}}_n := \mathfrak{S}_n \times Z$ $(n = 2, 3)$, and we put $\widetilde{\mathfrak{S}}_1 := Z$ for convenience. For an element $\sigma \in \mathfrak{S}_n$, let $L(\sigma)$ be the length of σ with respect to simple reflections, and $\mathrm{sgn}(\sigma) := (-1)^{L(\sigma)}$ the sign of σ. For a cycle $\xi = (i_1\ i_2\ \ldots\ i_\ell)$, put $\ell(\xi) := \ell$ its length, then $L(\xi) \equiv \ell(\xi) - 1 \pmod 2$. For an element σ' of the covering group $\widetilde{\mathfrak{S}}_n$, take $\sigma = \Phi_{\mathfrak{S}}(\sigma') \in \mathfrak{S}_n$ and define $L(\sigma') := L(\sigma)$, $\mathrm{sgn}(\sigma') := \mathrm{sgn}(\sigma)$. According to $\mathrm{sgn}(\sigma') = 1$ or -1, we call σ' *even* or *odd*.

A representation group of the alternating group $\mathfrak{A}_n$ is given as follows. For $n \geq 3$, we have a double covering group of $\mathfrak{A}_n = \{\sigma \in \mathfrak{S}_n; \mathrm{sgn}(\sigma) = 1\}$ as

$$\mathfrak{B}_n := \left\{\sigma' \in \widetilde{\mathfrak{S}}_n; \mathrm{sgn}(\sigma') = 1\right\} = \mathrm{Ker}_{\widetilde{\mathfrak{S}}_n}(\mathrm{sgn}). \tag{8.2}$$

For $n = 1, 2$, we put $\mathfrak{B}_n := Z$. Representation groups of $\mathfrak{A}_n$ is unique up to isomorphism. For $n \geq 4$, $\neq 6, 7$, $\mathfrak{B}_n$ is a representation group of $\mathfrak{A}_n$. For $n = 6, 7$, representation group is 6-times covering of $\mathfrak{A}_n$, and $\mathfrak{B}_n$ is its quotient [**15**, Section 5]. In any case, $(\mathfrak{B}_n, z, \mathrm{sgn}|_{\mathfrak{B}_n})$, with the trivial $\mathrm{sgn}|_{\mathfrak{B}_n}$, is an element of $\mathscr{G} \setminus \mathscr{G}'$.

8.2. Schur's 'Hauptdarstellung' of $\widetilde{\mathfrak{S}}_n$.

In [**15**, Section 22], a fundamental spin IR Δ_n of $\mathfrak{S}_n$ or an IR of the representation group $\mathfrak{T}_n$, called Hauptdarstellung, is constructed. We transcribe it to our group $\widetilde{\mathfrak{S}}_n = \mathfrak{T}'_n$ and denote it by Δ'_n and call it again 'Hauptdarstellung'. Let ε, a, b, c be 2×2 matrices given in (2.12). For $n \geq 3$, put $N := [(n-1)/2]$ and consider square matrices X_j $(j \in \boldsymbol{I}_{2N+1})$ of degree $M = 2^N$ as

$$\begin{cases} X_{2k-1} := c^{\otimes(k-1)} \otimes a \otimes \varepsilon^{\otimes(N-k)} & (1 \leq k \leq N), \\ X_{2k} := c^{\otimes(k-1)} \otimes b \otimes \varepsilon^{\otimes(N-k)} & (1 \leq k \leq N), \\ X_{2N+1} := c^{\otimes N}. \end{cases} \tag{8.3}$$

Then they satisfy $X_j^2 = E$ $(j \in \boldsymbol{I}_{n-1})$, $X_jX_k = -X_kX_j$ $(j \neq k,\ j,k \in \boldsymbol{I}_{n-1})$, with $E = E_M$ the unit matrix. Putting $T_j := a_{j-1}X_{j-1} + b_jX_j$ $(X_0 := O)$ for $j \in \boldsymbol{I}_{n-1}$, we wish to have the following property, corresponding to (8.1):

$$\begin{cases} T_j^2 = E & (j \in \boldsymbol{I}_{n-1}), \\ (T_jT_{j+1})^3 = E & (j \in \boldsymbol{I}_{n-2}), \\ T_jT_k = -T_kT_j & (|j-k| \geq 2,\ j,k \in \boldsymbol{I}_{n-1}). \end{cases} \tag{8.4}$$

For the second equality, it is sufficient to have $T_jT_{j+1} + T_{j+1}T_j + E = O$ $(j \in \boldsymbol{I}_{n-2})$. In fact, multiply $(T_jT_{j+1} - E)$ from the left, then, under $T_j^2 = E$, $T_{j+1}^2 = E$,

$$\begin{aligned} O &= (T_jT_{j+1} - E)(T_jT_{j+1} + T_{j+1}T_j + E) \\ &= (T_jT_{j+1})^2 + E + T_jT_{j+1} - T_jT_{j+1} - T_{j+1}T_j - E, \end{aligned}$$

and so $(T_jT_{j+1})^2 = T_{j+1}T_j$. Multiply T_jT_{j+1} from the left, then $(T_jT_{j+1})^3 = E$.

Hence we get the following equations for the coefficients a_j, b_j:

$$\begin{cases} a_0 = 0, \quad b_1^2 = 1, \\ a_{j-1}^2 + b_j^2 = 1 & (j \in \boldsymbol{I}_{n-1}), \\ 2a_jb_j = -1 & (j \in \boldsymbol{I}_{n-2}). \end{cases} \tag{8.5}$$

Lemma 8.2. *A set of solutions of* (8.5) *is given by* $a_0 = 0$, $b_1 = 1$, *and*

$$\begin{cases} a_{2\nu} = -\dfrac{\sqrt{\nu}}{\sqrt{2\nu+1}}, & b_{2\nu+1} = \dfrac{\sqrt{\nu+1}}{\sqrt{2\nu+1}} \quad (2\nu+1 \leq n-1), \\ a_{2\nu+1} = -\dfrac{\sqrt{2\nu+1}}{2\sqrt{\nu+1}}, & b_{2\nu+2} = \dfrac{\sqrt{2\nu+3}}{2\sqrt{\nu+1}} \quad (2\nu+2 \leq n-1). \end{cases}$$

Theorem 8.3. *Let* $n \geq 4$. *For* $j \in \boldsymbol{I}_{n-1}$, *put* $T_j := a_{j-1}X_{j-1} + b_jX_j$. *Then,* $\Delta'_n(r_j) := T_j$ *gives a spin IR* Δ'_n *of* $\widetilde{\mathfrak{S}}_n$, *which is called 'Hauptdarstellung'.*

For the character $\chi_{\Delta'_n}$ of 'Hauptdarstellung' of $\widetilde{\mathfrak{S}}_n$ and the complement $\delta_{\Delta'_n}$ when Δ'_n is self-associate, we transcribe Schur's result. A cycle decomposition of $\sigma \in \mathfrak{S}_n$ is $\sigma = \sigma_1\sigma_2\cdots\sigma_t$ with disjoint cycles σ_j such that $\mathrm{supp}(\sigma_j) \cap \mathrm{supp}(\sigma_k) = \emptyset$ $(j \neq k)$. Admit cycles of length 1, and if $\ell_1 + \cdots + \ell_t = n$ with $\ell_j = \ell(\sigma_j)$, we call this decomposition *saturated*. For $\sigma' \in \widetilde{\mathfrak{S}}_n$, let $\sigma = \sigma_1\sigma_2\cdots\sigma_t$ be a saturated cycle decomposition of $\sigma = \Phi_{\mathfrak{S}}(\sigma')$, and take an inverse image $\sigma'_j \in \widetilde{\mathfrak{S}}_n$, $\sigma_j = \Phi_{\mathfrak{S}}(\sigma'_j)$, appropriately, then we have $\sigma' = \sigma'_1\sigma'_2\cdots\sigma'_t$ (call this a *cycle decomposition* of σ'). On the other hand, any element of $\widetilde{\mathfrak{S}}_n$ is conjugate to σ' or $z\sigma'$ for a standard element σ' of the form

$$\sigma' = \sigma'_1\sigma'_2\cdots\sigma'_t, \quad \sigma'_j = r_{L_j+1}r_{L_j+2}\cdots r_{L_{j+1}-1}\ (j \in \boldsymbol{I}_t), \tag{8.6}$$

with $L_0 = 0$, $L_i = \ell_1 + \cdots + \ell_i$ $(i \in \boldsymbol{I}_t)$.

Theorem 8.4. *Assume $n \geq 4$. For a $\sigma' \in \widetilde{\mathfrak{S}}_n$, let a saturated cycle decomposition be $\sigma' = \sigma'_1\sigma'_2\cdots\sigma'_t$.*

(i) *Let σ' be even. Then $\chi_{\Delta'_n}(\sigma') \neq 0$ only if all σ'_j are even, and with $N = [(n-1)/2]$,*

$$\chi_{\Delta'_n}(\sigma') = 2^N \cdot \prod_{j\in\boldsymbol{I}_t} \chi_{\Delta'_n}(\sigma'_j)/2^N = \pm(-1)^{(n-t)/2}\, 2^{[(t-1)/2]}.$$

If σ' is a standard element as in (8.6)*, the above top sign is $+$.*

(ii) *Let n be even. Then Δ'_n is non-self-associate. For $\kappa' \in \mathfrak{C}_n = \widetilde{\mathfrak{S}}_n \setminus \mathfrak{B}_n$, $\chi_{\Delta'_n}(\kappa') \neq 0$ only when $\kappa = \Phi_{\mathfrak{S}}(\kappa')$ is a cycle of the longest length n, and for the standard element $\kappa' = r_1r_2\cdots r_{n-1}$,*

$$\chi_{\Delta'_n}(\kappa') = i^N\sqrt{N+1} = i^{n/2-1}\sqrt{n/2}.$$

(iii) *Let n be odd. Then Δ'_n is self-associate. For the complement $\delta_{\Delta'_n}$, $\delta_{\Delta'_n}(\tau') \neq 0$ for $\tau' \in \mathfrak{B}_n$ only when $\tau = \Phi_{\mathfrak{S}}(\tau')$ is a cycle of the longest length n, and for the standard $\kappa' = r_1r_2\cdots r_{n-1}$,*

$$\delta_{\Delta'_n}(\kappa') = i^N\sqrt{2N+1} = i^{(n-1)/2}\sqrt{n}.$$

8.3. Spin irreducible representations of $\widetilde{\mathfrak{S}}_n$ and of $\mathfrak{B}_n$.

Take an ordered partition $\boldsymbol{\nu} = (\nu_i)_{i\in\boldsymbol{I}_m}$, $n = \nu_1 + \nu_2 + \cdots + \nu_m$, $\nu_j \geq 1$, of n, and define subintervals of $\boldsymbol{I}_n = \{1, 2, \ldots, n\} = [1, n]$ as $J_1 := [1, \nu_1]$, $J_i := [\nu_1 + \cdots + \nu_{i-1} + 1, \nu_1 + \cdots + \nu_i]$ $(2 \leq i \leq m)$. Consider a (Frobenius-)Young type subgroup of $\mathfrak{S}_n = \mathfrak{S}_{\boldsymbol{I}_n}$ as

$$\mathfrak{S}_{\boldsymbol{\nu}} := \mathfrak{S}_{J_1} \times \mathfrak{S}_{J_2} \times \cdots \times \mathfrak{S}_{J_m} \cong \mathfrak{S}_{\nu_1} \times \mathfrak{S}_{\nu_2} \times \cdots \times \mathfrak{S}_{\nu_m}, \tag{8.7}$$

denoted as $\mathfrak{S}_{\boldsymbol{\nu}} = \mathfrak{S}_{\nu_1} \times \mathfrak{S}_{\nu_2} \times \cdots \times \mathfrak{S}_{\nu_m}$ for brevity. For the double covering group $\widetilde{\mathfrak{S}}_n$, its Schur-Young type subgroup is by definition the full inverse image of $\mathfrak{S}_{\boldsymbol{\nu}}$ as

$$\widetilde{\mathfrak{S}}_{\boldsymbol{\nu}} := \Phi_{\mathfrak{S}}^{-1}(\mathfrak{S}_{\boldsymbol{\nu}}) = \Phi_{\mathfrak{S}}^{-1}(\mathfrak{S}_{\nu_1} \times \mathfrak{S}_{\nu_2} \times \cdots \times \mathfrak{S}_{\nu_m}). \tag{8.8}$$

Denote $\widetilde{\mathfrak{S}}_{J_i} := \Phi_{\mathfrak{S}}^{-1}(\mathfrak{S}_{J_i}) \cong \widetilde{\mathfrak{S}}_{\nu_i}$ simply by $\widetilde{\mathfrak{S}}_{\nu_i}$. Then, $\widetilde{\mathfrak{S}}_{\boldsymbol{\nu}}$ is a double covering of $\mathfrak{S}_{\boldsymbol{\nu}}$, and naturally isomorphic to the twisted central product $\widetilde{\mathfrak{S}}_{\nu_1} \hat{*} \widetilde{\mathfrak{S}}_{\nu_2} \hat{*} \cdots \hat{*} \widetilde{\mathfrak{S}}_{\nu_m}$.

Now, let us describe a complete set of representatives of spin IRs for the spin dual of the whole group $\widetilde{\mathfrak{S}}_n$. To do so, we change the symbol $\boldsymbol{\nu} = (\nu_j)_{j\in\boldsymbol{I}_m}$ with $\boldsymbol{\lambda} = (\lambda_j)_{j\in\boldsymbol{I}_m}$ and assume that

$$\lambda_1 > \lambda_2 > \cdots > \lambda_m > 0 \quad (n = \lambda_1 + \cdots + \lambda_m). \tag{8.9}$$

Let $\mathcal{P}_n^{\mathrm{str}}$ be the set of all such strict partitions of n where m varies, and introduce in it the inverse lexicographic order $\prec$, cf. [**15**, Section 37], so that $(n) \prec (n-1,1) \prec (n-2,2) \prec (n-3,2,1) \prec \cdots$, for $n \geq 6$. Put $s(\boldsymbol{\lambda}) := \sharp\{j \in \boldsymbol{I}_m; \lambda_j \text{ even}\}$ for $\boldsymbol{\lambda} \in \mathcal{P}_n^{\mathrm{str}}$, then $s(\boldsymbol{\lambda}) \equiv n - m \pmod 2$. Define subsets of $\mathcal{P}_n^{\mathrm{str}}$ as

$$\begin{cases} \mathcal{P}_{n,\mathrm{od}}^{\mathrm{str}} := \big\{\boldsymbol{\lambda} \in \mathcal{P}_n^{\mathrm{str}}; s(\boldsymbol{\lambda}) \text{ odd}\big\}, \\ \mathcal{P}_{n,\mathrm{ev}}^{\mathrm{str}} := \big\{\boldsymbol{\lambda} \in \mathcal{P}_n^{\mathrm{str}}; s(\boldsymbol{\lambda}) \text{ even}\big\}. \end{cases} \tag{8.10}$$

For $\boldsymbol{\lambda} = (\lambda_j)_{j \in \boldsymbol{I}_m} \in \mathcal{P}_n^{\mathrm{str}}$, take spin IR Δ'_{λ_j} of $\widetilde{\mathfrak{S}}_{\lambda_j}$ $(j \in \boldsymbol{I}_m)$ and then its twisted central product $\Delta'_{\boldsymbol{\lambda}} := \Delta'_{\lambda_1} \hat{*} \Delta'_{\lambda_2} \hat{*} \cdots \hat{*} \Delta'_{\lambda_m}$, and induce it up to $\widetilde{\mathfrak{S}}_n$ as

$$\Pi_{\boldsymbol{\lambda}} := \mathrm{Ind}_{\widetilde{\mathfrak{S}}_{\boldsymbol{\lambda}}}^{\widetilde{\mathfrak{S}}_n} \Delta'_{\boldsymbol{\lambda}}. \tag{8.11}$$

LEMMA 8.5. (i) *For $\boldsymbol{\lambda} \in \mathcal{P}_{n,\mathrm{od}}^{\mathrm{str}}$, $\Delta'_{\boldsymbol{\lambda}}$ and $\Pi_{\boldsymbol{\lambda}}$ are non-self-associate.*
(ii) *For $\boldsymbol{\lambda} \in \mathcal{P}_{n,\mathrm{ev}}^{\mathrm{str}}$, $\Delta'_{\boldsymbol{\lambda}}$ and $\Pi_{\boldsymbol{\lambda}}$ are self-associate.*

Apply Propositions 4.3 and 4.4 to the twisted central product $\Delta'_{\boldsymbol{\lambda}} = \Delta'_{\lambda_1} \hat{*} \Delta'_{\lambda_2} \hat{*} \cdots \hat{*} \Delta'_{\lambda_m}$, then we obtain explicitly the characters $\chi_{\Delta'_{\boldsymbol{\lambda}}}$ and the complement $\delta_{\Delta'_{\boldsymbol{\lambda}}}$ (if $s(\boldsymbol{\lambda})$ is even) from Theorem 8.4. Then the character $\chi_{\Pi_{\boldsymbol{\lambda}}}$ can be calculated.

There exists a unique irreducible component of $\Pi_{\boldsymbol{\lambda}}$, denoted by $\pi_{\boldsymbol{\lambda}}$ and determined inductively along $\prec$, such that $\Pi_{\boldsymbol{\lambda}}$ is a direct sum of $\pi_{\boldsymbol{\lambda}}$ and of multiples of $\pi_{\boldsymbol{\lambda}'}$, $\mathrm{sgn} \cdot \pi_{\boldsymbol{\lambda}'}$, $\boldsymbol{\lambda}' \prec \boldsymbol{\lambda}$. This means that $\pi_{\boldsymbol{\lambda}}$ is the top irreducible component of $\Pi_{\boldsymbol{\lambda}}$. More in detail, in the level of characters,

$$\chi_{\Pi_{\boldsymbol{\lambda}}} = \chi_{\pi_{\boldsymbol{\lambda}}} + \sum_{\substack{\boldsymbol{\lambda}' \prec \boldsymbol{\lambda} \\ \lambda' \in \mathcal{P}_{n,\mathrm{ev}}^{\mathrm{str}}}} m(\boldsymbol{\lambda}', \boldsymbol{\lambda}) \chi_{\pi_{\boldsymbol{\lambda}'}} + \sum_{\substack{\boldsymbol{\lambda}' \prec \boldsymbol{\lambda} \\ \lambda' \in \mathcal{P}_{n,\mathrm{od}}^{\mathrm{str}}}} m(\boldsymbol{\lambda}', \boldsymbol{\lambda}) \big(\chi_{\pi_{\boldsymbol{\lambda}'}} + \mathrm{sgn} \cdot \chi_{\pi_{\boldsymbol{\lambda}'}}\big),$$

where $m(\boldsymbol{\lambda}', \boldsymbol{\lambda})$ denotes the multiplicity of $\pi_{\boldsymbol{\lambda}'}$ in $\Pi_{\boldsymbol{\lambda}}$.

The matrix of multiplicities $(m(\boldsymbol{\lambda}', \boldsymbol{\lambda}))$, $\boldsymbol{\lambda}', \boldsymbol{\lambda} \in \mathcal{P}_n^{\mathrm{str}}$, is upper triangular with diagonal entries all equal to 1. Irreducible characters $\chi_{\pi_{\boldsymbol{\lambda}}}$, restricted on $\mathfrak{B}_n$, can be obtained by Gram-Schmidt orthogonalization process in $\ell^2(\mathfrak{B}_n)$ applied to the set of induced characters $\{\chi_{\Pi_{\boldsymbol{\lambda}}}|_{\mathfrak{B}_n}; \boldsymbol{\lambda} \in \mathcal{P}_n^{\mathrm{str}}\}$. In [**15**, Absch. IX–X], spin irreducible characters $\chi_{\pi_{\boldsymbol{\lambda}}}$ on $\mathfrak{B}_n$ are studied by another method in detail in relation to induced characters $\chi_{\Pi_{\boldsymbol{\lambda}'}}|_{\mathfrak{B}_n}$, and character formula for $\chi_{\pi_{\boldsymbol{\lambda}}}|_{\mathfrak{B}_n}$ is given. The character $\pi_{\boldsymbol{\lambda}}$ on $\mathfrak{C}_n = \widetilde{\mathfrak{S}}_n \setminus \mathfrak{B}_n$, and the complement $\delta_{\pi_{\boldsymbol{\lambda}}}$ (if $s(\boldsymbol{\lambda})$ is even) can be calculated more easily. Final result is given in Satz IX in [**15**, Section 41].

Classification of spin IRs of $\widetilde{\mathfrak{S}}_n$ and that of spin IRs of $\mathfrak{B}_n$ are given in [**15**, Section 42] by means of so-called shifted Young diagrams $\boldsymbol{\lambda} \in \mathcal{P}_n^{\mathrm{str}}$. They are summarized as follows.

THEOREM 8.6. (i) *For $n \geq 4$, a complete set of representatives for spin dual $\widehat{\widetilde{\mathfrak{S}}_n}^{\mathrm{spin}}$ is given by $\{\pi_{\boldsymbol{\lambda}},\ \mathrm{sgn} \cdot \pi_{\boldsymbol{\lambda}}\ (\boldsymbol{\lambda} \in \mathcal{P}_{n,\mathrm{od}}^{\mathrm{str}}),\ \pi_{\boldsymbol{\lambda}}\ (\boldsymbol{\lambda} \in \mathcal{P}_{n,\mathrm{ev}}^{\mathrm{str}})\}$.*
(ii) *For $n \geq 4$, a complete set of representatives for spin dual $\widehat{\mathfrak{B}_n}^{\mathrm{spin}}$ is given by*

$\{\rho_{\boldsymbol\lambda} := \pi_{\boldsymbol\lambda}|_{\mathfrak{B}_n}\ (\boldsymbol\lambda \in \mathcal{P}^{\rm str}_{n,{\rm od}}),\ \rho'_{\boldsymbol\lambda},\ \rho''_{\boldsymbol\lambda}\ (\boldsymbol\lambda \in \mathcal{P}^{\rm str}_{n,{\rm ev}})\}$, *where* $\pi_{\boldsymbol\lambda}|_{\mathfrak{B}_n} \cong \rho'_{\boldsymbol\lambda} \oplus \rho''_{\boldsymbol\lambda}$ *for* $\boldsymbol\lambda \in \mathcal{P}^{\rm str}_{n,{\rm ev}}$.

9. Spin IRs of Schur-Young type subgroups.

Let $\boldsymbol\nu = (\nu_j)_{j\in \boldsymbol{I}_m}$ be an ordered partition of $n \geq 4$ (here we do not assume the order of *large* or *small* among ν_j's), and take Schur-Young type subgroup $\widetilde{\mathfrak{S}}_{\boldsymbol\nu} = \Phi_{\mathfrak{S}}^{-1}(\mathfrak{S}_{\nu_1}\times\cdots\times\mathfrak{S}_{\nu_m})) \cong \widetilde{\mathfrak{S}}_{\nu_1}\hat{*}\,\widetilde{\mathfrak{S}}_{\nu_2}\hat{*}\cdots\hat{*}\,\widetilde{\mathfrak{S}}_{\nu_m}$ of $\widetilde{\mathfrak{S}}_n$. Then, applying Theorem 5.4 to $S' = S'_1\hat{*}\cdots\hat{*}\,S'_m$, $S'_j = \widetilde{\mathfrak{S}}_{\nu_j}$, and Theorem 6.1 to $B' = \mathrm{Ker}(\mathrm{sgn})$, we obtain complete sets of representatives for the spin dual of $\widetilde{\mathfrak{S}}_{\boldsymbol\nu}$ and that of the spin dual of its normal subgroup $\mathfrak{B}_n\cap\widetilde{\mathfrak{S}}_{\boldsymbol\nu}$ of index 2.

9.1. Spin IRs of $\widetilde{\mathfrak{S}}_{\boldsymbol\nu}$.

By Theorem 8.6, we prepare for each $S'_j = \widetilde{\mathfrak{S}}_{\nu_j}$, a complete set of representatives of its spin dual $\widehat{S'_j}^{\,\rm spin}$ as

$$\big\{\pi_{\boldsymbol\lambda^{(j)}},\ \mathrm{sgn}\cdot\pi_{\boldsymbol\lambda^{(j)}}\ (\boldsymbol\lambda^{(j)} \in \mathcal{P}^{\rm str}_{\nu_j,{\rm od}}),\ \pi_{\boldsymbol\lambda^{(j)}}\ (\boldsymbol\lambda^{(j)} \in \mathcal{P}^{\rm str}_{\nu_j,{\rm ev}})\big\},$$

where $\boldsymbol\lambda^{(j)} = (\lambda^{(j)}_i)_{i\in\boldsymbol{I}_{m_j}}$, $\nu_j = \lambda^{(j)}_1 + \cdots + \lambda^{(j)}_{m_j}$, $\lambda^{(j)}_1 > \lambda^{(j)}_2 > \cdots > \lambda^{(j)}_{m_j} > 0$, is a shifted Young diagram of size ν_j. Put $\boldsymbol\Lambda := (\boldsymbol\lambda^{(j)})_{j\in\boldsymbol{I}_m}$ and denote by $s(\boldsymbol\Lambda)$ the number of even $\lambda^{(j)}_i$'s: $s(\boldsymbol\Lambda) = s(\boldsymbol\lambda^{(1)})+\cdots+s(\boldsymbol\lambda^{(m)})$, then $s(\boldsymbol\Lambda) \equiv n-\sum_j m_j \pmod 2$. Take the twisted central product

$$\pi_{\boldsymbol\Lambda} := \pi_{\boldsymbol\lambda^{(1)}}\,\hat{*}\,\pi_{\boldsymbol\lambda^{(2)}}\,\hat{*}\,\cdots\,\hat{*}\,\pi_{\boldsymbol\lambda^{(m)}}. \tag{9.1}$$

Then, by Propositions 4.3 and 4.4, $\pi_{\boldsymbol\Lambda}$ is self-associate or not according as $s(\boldsymbol\Lambda)$ is even or odd. Moreover Theorem 5.4 gives us the following.

Theorem 9.1. *Let $\boldsymbol\nu = (\nu_j)_{j\in\boldsymbol{I}_m}$ be an ordered partition of $n \geq 4$, and $\widetilde{\mathfrak{S}}_{\boldsymbol\nu} = \widetilde{\mathfrak{S}}_{\nu_1}\hat{*}\,\widetilde{\mathfrak{S}}_{\nu_2}\hat{*}\cdots\hat{*}\,\widetilde{\mathfrak{S}}_{\nu_m}$ be the corresponding Schur-Young subgroup of $\widetilde{\mathfrak{S}}_n$. Then a complete set of representatives of the spin dual $\widehat{\widetilde{\mathfrak{S}}_{\boldsymbol\nu}}^{\,\rm spin}$ of $\widetilde{\mathfrak{S}}_{\boldsymbol\nu}$ is given by the union of two sets as*

$$\big\{\pi_{\boldsymbol\Lambda},\ \mathrm{sgn}\cdot\pi_{\boldsymbol\Lambda}\,;\ \boldsymbol\Lambda = (\boldsymbol\lambda^{(j)})_{j\in\boldsymbol{I}_m},\ \boldsymbol\lambda^{(j)} \in \mathcal{P}^{\rm str}_{\nu_j},\ s(\boldsymbol\Lambda)\ \text{odd}\big\}$$
$$\bigsqcup\big\{\pi_{\boldsymbol\Lambda}\,;\ \boldsymbol\Lambda = (\boldsymbol\lambda^{(j)})_{j\in\boldsymbol{I}_m},\ \boldsymbol\lambda^{(j)} \in \mathcal{P}^{\rm str}_{\nu_j},\ s(\boldsymbol\Lambda)\ \text{even}\big\}.$$

9.2. Spin IRs of $B' = \mathfrak{B}_n\cap\widetilde{\mathfrak{S}}_{\boldsymbol\nu}$.

Take a standard normal subgroup $B' = \mathfrak{B}_n\cap\widetilde{\mathfrak{S}}_{\boldsymbol\nu}$ of $\widetilde{\mathfrak{S}}_{\boldsymbol\nu}$ of index 2. Then, by Theorem 6.1, a complete set of representatives of its spin dual $\widehat{B'}^{\,\rm spin}$ is deduced from Theorem 9.1 as follows.

Theorem 9.2. *Let the notation be as in the preceding theorem. Then a complete set of representatives of the spin dual of $B' := \mathfrak{B}_n\cap\widetilde{\mathfrak{S}}_{\boldsymbol\nu}$ is given by the union of two sets as*

$$\left\{\rho_{\mathbf{\Lambda}} := \pi_{\mathbf{\Lambda}}\big|_{B'}\,;\; \mathbf{\Lambda} = (\boldsymbol{\lambda}^{(j)})_{j\in \boldsymbol{I}_m},\; \boldsymbol{\lambda}^{(j)} \in \mathcal{P}^{\mathrm{str}}_{\nu_j},\; s(\mathbf{\Lambda}) \text{ is odd}\right\}$$
$$\bigsqcup \left\{\rho'_{\mathbf{\Lambda}},\, \rho''_{\mathbf{\Lambda}}\,;\; \mathbf{\Lambda} = (\boldsymbol{\lambda}^{(j)})_{j\in \boldsymbol{I}_m},\; \boldsymbol{\lambda}^{(j)} \in \mathcal{P}^{\mathrm{str}}_{\nu_j},\; s(\mathbf{\Lambda}) \text{ is even}\right\},$$

where $\pi_{\mathbf{\Lambda}}|_{B'} \cong \rho'_{\mathbf{\Lambda}} \oplus \rho''_{\mathbf{\Lambda}}$ *for* $\mathbf{\Lambda}$*, with* $s(\mathbf{\Lambda})$ *even.*

10. Characters of spin representations.

We introduce certain subsets of $\widetilde{\mathfrak{S}}_n$. Let $\mathscr{B}_n$ be the set of $\sigma' \in \widetilde{\mathfrak{S}}_n$ such that $\sigma = \Phi_{\mathfrak{S}}(\sigma')$ has a cycle decomposition $\sigma = \sigma_1\sigma_2\cdots\sigma_t$ with $\ell_j = \ell(\sigma_j)$ all odd. Moreover we denote the set of $\sigma' \in \widetilde{\mathfrak{S}}_n$ such that $\sigma = \Phi_{\mathfrak{S}}(\sigma')$ are cycles of the maximum length n, by $\mathscr{C}_n$ if n is even, and by $\mathscr{D}_n$ if n is odd.

10.1. 'Hauptdarstellung Δ'_n of $\widetilde{\mathfrak{S}}_n$.

We see from Theorem 8.4 that

if n is odd, then $\mathrm{supp}(\chi_{\Delta'_n}) \subset \mathscr{B}_n$, $\mathrm{supp}(\delta_{\Delta'_n}) \subset \mathscr{D}_n \subset \mathfrak{B}_n$;
if n is even, then $\mathrm{supp}(\chi_{\Delta'_n}) \cap \mathfrak{B}_n \subset \mathscr{B}_n$, $\mathrm{supp}(\chi_{\Delta'_n}) \cap \mathfrak{C}_n \subset \mathscr{C}_n$.

10.2. Fundamental spin IR $\Delta'_{\boldsymbol{\lambda}}$ of $\widetilde{\mathfrak{S}}_{\boldsymbol{\lambda}}$.

For an ordered partition $\boldsymbol{\lambda} = (\lambda_j)_{j\in \boldsymbol{I}_m}$ of n, in Schur-Young type subgroup $\widetilde{\mathfrak{S}}_{\boldsymbol{\lambda}} = \widetilde{\mathfrak{S}}_{\lambda_1} \hat{*}\, \widetilde{\mathfrak{S}}_{\lambda_2} \hat{*} \cdots \hat{*}\, \widetilde{\mathfrak{S}}_{\lambda_m}$, we put

$$\mathscr{B}_{\boldsymbol{\lambda}} := \mathscr{B}_{\lambda_1}\cdot \mathscr{B}_{\lambda_2}\cdots \mathscr{B}_{\lambda_m} = \left\{b'_1 b'_2 \cdots b'_m; b'_j \in \mathscr{B}_{\lambda_j} \subset \widetilde{\mathfrak{S}}_{\lambda_j}\right\}.$$

Moreover, the subset $\mathscr{X}_{\lambda_1}\cdot \mathscr{X}_{\lambda_2}\cdots \mathscr{X}_{\lambda_m}$ with $\mathscr{X}_{\lambda_j} = \mathscr{D}_{\lambda_j}$ or $\mathscr{C}_{\lambda_j}$ (in $\widetilde{\mathfrak{S}}_{\lambda_j}$) according as λ_j is odd or even, is denoted by $\mathscr{C}_{\boldsymbol{\lambda}}$ if $s(\boldsymbol{\lambda})$ is odd, and by $\mathscr{D}_{\boldsymbol{\lambda}}$ if $s(\boldsymbol{\lambda})$ is even. Then we see from Table 4.1 that, for spin IR $\Delta'_{\boldsymbol{\lambda}} = \Delta'_{\lambda_1} \hat{*} \cdots \hat{*} \Delta'_{\lambda_m}$ of $\widetilde{\mathfrak{S}}_{\boldsymbol{\lambda}}$,

$$\text{if } s(\boldsymbol{\lambda}) \text{ is odd, then} \quad \begin{cases} \mathrm{supp}(\chi_{\Delta'_{\boldsymbol{\lambda}}}) \cap [\mathrm{sgn} = 1] \subset \mathscr{B}_{\boldsymbol{\lambda}} \subset \widetilde{\mathfrak{S}}_{\boldsymbol{\lambda}} \cap \mathfrak{B}_n, \\ \mathrm{supp}(\chi_{\Delta'_{\boldsymbol{\lambda}}}) \cap [\mathrm{sgn} = -1] \subset \mathscr{C}_{\boldsymbol{\lambda}} \subset \widetilde{\mathfrak{S}}_{\boldsymbol{\lambda}} \cap \mathfrak{C}_n; \end{cases}$$

$$\text{if } s(\boldsymbol{\lambda}) \text{ is even, then} \quad \begin{cases} \mathrm{supp}(\chi_{\Delta'_{\boldsymbol{\lambda}}}) \subset \mathscr{B}_{\boldsymbol{\lambda}} \subset \widetilde{\mathfrak{S}}_{\boldsymbol{\lambda}} \cap \mathfrak{B}_n, \\ \mathrm{supp}(\delta_{\Delta'_{\boldsymbol{\lambda}}}) \subset \mathscr{D}_{\boldsymbol{\lambda}} \subset \widetilde{\mathfrak{S}}_{\boldsymbol{\lambda}} \cap \mathfrak{B}_n; \end{cases}$$

where $[\mathrm{sgn} = \epsilon 1]$, $\epsilon = \pm$, denotes the subset defined by the condition $\mathrm{sgn}(\sigma') = \epsilon 1$.

10.3. Spin IR $\pi_{\boldsymbol{\lambda}}$ of $\widetilde{\mathfrak{S}}_n$.

For a spin IR $\pi_{\boldsymbol{\lambda}}$ of $\widetilde{\mathfrak{S}}_n$ with a shifted Young diagram $\boldsymbol{\lambda}$,

$$\text{if } s(\boldsymbol{\lambda}) \text{ is odd, then} \quad \begin{cases} \mathrm{supp}(\chi_{\pi_{\boldsymbol{\lambda}}}) \cap \mathfrak{B}_n \subset [\mathscr{B}_{\boldsymbol{\lambda}}], \\ \mathrm{supp}(\chi_{\pi_{\boldsymbol{\lambda}}}) \cap \mathfrak{C}_n \subset [\mathscr{C}_{\boldsymbol{\lambda}}]; \end{cases}$$

$$\text{if } s(\boldsymbol{\lambda}) \text{ is even, then} \quad \begin{cases} \mathrm{supp}(\chi_{\pi_{\boldsymbol{\lambda}}}) \subset [\mathscr{B}_{\boldsymbol{\lambda}}], \\ \mathrm{supp}(\delta_{\pi_{\boldsymbol{\lambda}}}) \subset [\mathscr{D}_{\boldsymbol{\lambda}}], \end{cases}$$

where, for a subset K of $\widetilde{\mathfrak{S}}_n$, $[K]$ denotes the union of $\sigma' K \sigma'^{-1}$ over $\sigma' \in \widetilde{\mathfrak{S}}_n$.

10.4. Spin IR $\pi_{\boldsymbol{\Lambda}}$ of Schur-Young type subgroup $\widetilde{\mathfrak{S}}_{\boldsymbol{\nu}}$.

For an ordered partition $\boldsymbol{\nu} = (\nu_j)_{j \in \boldsymbol{I}_m}$ of n, take an m-tuple $\boldsymbol{\Lambda} = (\boldsymbol{\lambda}^{(j)})_{j \in \boldsymbol{I}_m}$ of shifted Young diagrams with $\nu_j = |\boldsymbol{\lambda}^{(j)}|$ the size of $\boldsymbol{\lambda}^{(j)}$. Define subsets $\mathscr{B}_{\boldsymbol{\Lambda}}$, $\mathscr{D}_{\boldsymbol{\Lambda}}$ and $\mathscr{C}_{\boldsymbol{\Lambda}}$ of $\widetilde{\mathfrak{S}}_{\boldsymbol{\nu}}$ as follows: put

$$\mathscr{B}_{\boldsymbol{\Lambda}} := \mathscr{B}_{\boldsymbol{\lambda}^{(1)}} \cdot \mathscr{B}_{\boldsymbol{\lambda}^{(2)}} \cdots \mathscr{B}_{\boldsymbol{\lambda}^{(m)}}$$

and the subset $\mathscr{X}_{\boldsymbol{\lambda}^{(1)}} \cdot \mathscr{X}_{\boldsymbol{\lambda}^{(2)}} \cdots \mathscr{X}_{\boldsymbol{\lambda}^{(m)}}$ with $\mathscr{X}_{\boldsymbol{\lambda}^{(j)}} = \mathscr{D}_{\boldsymbol{\lambda}^{(j)}}$ or $\mathscr{C}_{\boldsymbol{\lambda}^{(j)}}$ (in $\widetilde{\mathfrak{S}}_{\nu_j}$) according as $s(\boldsymbol{\lambda}^{(j)})$ is even or odd, is denoted by $\mathscr{C}_{\boldsymbol{\Lambda}}$ if $s(\boldsymbol{\Lambda})$ is odd, and by $\mathscr{D}_{\boldsymbol{\Lambda}}$ if $s(\boldsymbol{\Lambda})$ is even.

Theorem 10.1. *For spin IR $\pi_{\boldsymbol{\Lambda}}$, $\boldsymbol{\Lambda} = (\boldsymbol{\lambda}^{(j)})_{j \in \boldsymbol{I}_m}$, $\boldsymbol{\lambda}^{(j)} = (\lambda_i^{(j)})_{i \in \boldsymbol{I}_{m_j}}$ of $\widetilde{\mathfrak{S}}_{\boldsymbol{\nu}}$, its character $\chi_{\pi_{\boldsymbol{\Lambda}}}$ and complement $\delta_{\pi_{\boldsymbol{\Lambda}}}$ (if $s(\boldsymbol{\Lambda})$ is odd) are given by Propositions* 4.3 *and* 4.4, *by means of characters and complements of $\pi_{\boldsymbol{\lambda}^{(j)}}$'s. Their supports are evaluated as follows*:

$$\textit{if } s(\boldsymbol{\Lambda}) \textit{ is odd, then} \quad \begin{cases} \mathrm{supp}(\chi_{\pi_{\boldsymbol{\Lambda}}}) \cap [\mathrm{sgn} = 1] \subset \mathscr{B}_{\boldsymbol{\Lambda}}, \\ \mathrm{supp}(\chi_{\pi_{\boldsymbol{\Lambda}}}) \cap [\mathrm{sgn} = -1] \subset \mathscr{C}_{\boldsymbol{\Lambda}}, \end{cases}$$

$$\textit{if } s(\boldsymbol{\Lambda}) \textit{ is even, then} \quad \begin{cases} \mathrm{supp}(\chi_{\pi_{\boldsymbol{\Lambda}}}) \subset \mathscr{B}_{\boldsymbol{\Lambda}}, \\ \mathrm{supp}(\delta_{\pi_{\boldsymbol{\Lambda}}}) \subset \mathscr{D}_{\boldsymbol{\Lambda}}. \end{cases}$$

10.5. Increasing sequence of groups.

The group $\mathfrak{S}_n$ is naturally imbedded into $\mathfrak{S}_{n+1}$ as a subgroup consisting elements leaving $n+1$ invariant, and the covering group $\widetilde{\mathfrak{S}}_n$ is imbedded into $\widetilde{\mathfrak{S}}_{n+1}$ as the full inverse image $\Phi_{\mathfrak{S}}^{-1}(\mathfrak{S}_n)$ for $\Phi_{\mathfrak{S}} : \widetilde{\mathfrak{S}}_{n+1} \to \mathfrak{S}_{n+1}$. Then we have an increasing sequence of groups as $\cdots \subset \widetilde{\mathfrak{S}}_n \subset \widetilde{\mathfrak{S}}_{n+1} \subset \cdots$, which goes up to a double covering $\widetilde{\mathfrak{S}}_\infty$ of the infinite symmetric group $\mathfrak{S}_\infty$. Take a spin IR π_n of $\widetilde{\mathfrak{S}}_n$ for each $\widetilde{\mathfrak{S}}_n$, and consider the series of their normalized characters $\widetilde{\chi}_{\pi_n}$. The study of the limit $\lim_{n \to \infty} \widetilde{\chi}_{\pi_n}$ in relation to $\widetilde{\mathfrak{S}}_\infty$, is started from Vershik-Kerov [**16**] and is continued by them and others (cf. [**11**]).

In the case of spin (projective) representations of generalized symmetric groups $G(m,1,n)$, in some types of spin IRs, the asymptotic theory of characters as above is reduced essentially to such problem for Schur-Young type subgroups $\widetilde{\mathfrak{S}}_{\boldsymbol{\nu}}$ as $n = |\boldsymbol{\nu}| \to \infty$ (cf. Example 1.2, and [**2**], [**3**] and [**4**]). In that case the following result has an important meaning.

Theorem 10.2. (i) *Let the notations be as in Theorems* 9.2 *and* 10.1. *For spin IR $\pi_{\boldsymbol{\Lambda}}$ of $\widetilde{\mathfrak{S}}_{\boldsymbol{\nu}}$ with $n = |\boldsymbol{\nu}|$, if it is restricted on the subgroup $\widetilde{\mathfrak{S}}_{n-1} \cap \widetilde{\mathfrak{S}}_{\boldsymbol{\nu}}$, then the support of its character is contained in $\mathscr{B}_{\boldsymbol{\Lambda}}$. In other words, for $\sigma' \in \widetilde{\mathfrak{S}}_{\boldsymbol{\nu}}$, suppose the order of the support of $\sigma = \Phi(\sigma')$ is $\leq n-1$, then $\chi_{\pi_{\boldsymbol{\Lambda}}}(\sigma') \neq 0$ only when $\sigma' \in \mathscr{B}_{\boldsymbol{\Lambda}}$.*

(ii) *On the subgroup $\widetilde{\mathfrak{S}}_{n-1} \cap \widetilde{\mathfrak{S}}_{\boldsymbol{\nu}}$, the normalized characters $\widetilde{\chi}_{\pi_{\boldsymbol{\Lambda}}}$ and $\widetilde{\chi}_{\pi'_{\boldsymbol{\Lambda}}}$ with $\pi'_{\boldsymbol{\Lambda}} =$* $\mathrm{sgn} \cdot \pi_{\boldsymbol{\Lambda}}$, *and also $\widetilde{\chi}_{\rho_{\boldsymbol{\Lambda}}}$ (if $s(\boldsymbol{\Lambda})$ is odd), $\widetilde{\chi}_{\rho'_{\boldsymbol{\Lambda}}}$, $\widetilde{\chi}_{\rho''_{\boldsymbol{\Lambda}}}$ (if $s(\boldsymbol{\Lambda})$ is even), are all essentially equal to each other and zero outside $\mathscr{B}_{\boldsymbol{\Lambda}}$.*

Proof. (i) This follows from Theorem 10.1. In fact, if $|\mathrm{supp}(\sigma)| \leq n-1$, then σ' can be an element of neither $\mathscr{C}_{\mathbf{\Lambda}}$ nor $\mathscr{D}_{\mathbf{\Lambda}}$. Then, (ii) is easy to prove. □

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References

[1] J. W. Davies and A. O. Morris, The Schur multiplier of the generalized symmetric group, J. London Math. Soc. (2), **8** (1974), 615–620.

[2] T. Hirai, E. Hirai and A. Hora, Towards projective representations and spin characters of finite and infinite complex reflection groups, In: Proceedings of the Fourth German-Japanese Symposium, Infinite Dimensional Harmonic Analysis IV, Tokyo, 2007, (eds. J. Hilgert, A. Hora, T. Kawazoe, K. Nishiyama and M. Voit), World Scientific, 2009, pp. 112–128.

[3] T. Hirai, E. Hirai and A. Hora, Projective representations and spin characters of complex reflection groups $G(m,p,n)$ and $G(m,p,\infty)$, I, In: Projective Representations and Spin Characters of Complex Reflection Groups $G(m,p,n)$ and $G(m,p,\infty)$, MSJ Mem., **29**, Math. Soc. Japan, Tokyo, 2013, pp. 49–122.

[4] T. Hirai, A. Hora and E. Hirai, Projective representations and spin characters of complex reflection groups $G(m,p,n)$ and $G(m,p,\infty)$, II, In: Projective Representations and Spin Characters of Complex Reflection Groups $G(m,p,n)$ and $G(m,p,\infty)$, MSJ Mem., **29**, Math. Soc. Japan, Tokyo, 2013, pp. 123–272.

[5] T. Hirai, A. Hora and E. Hirai, Introductory expositions on projective representations of groups, In: Projective Representations and Spin Characters of Complex Reflection Groups $G(m,p,n)$ and $G(m,p,\infty)$, MSJ Mem., **29**, Math. Soc. Japan, Tokyo, 2013, pp. 1–47.

[6] P. N. Hoffman and J. F. Humphreys, Twisted products and projective representations of monomial groups, Exposition. Math., **3** (1985), 91–95.

[7] P. N. Hoffman and J. F. Humphreys, Projective Representations of the Symmetric Groups, Q-Functions and Shifted Tableaux, Oxford Math. Monogr., Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1992.

[8] J. F. Humphreys and S. Tickner, Principal indecomposable modules for twisted central products, Comm. Algebra, **25** (1997), 357–368.

[9] S. Ihara and T. Yokonuma, On the second cohomology groups (Schur-multipliers) of finite reflection groups, J. Fac. Sci. Univ. Tokyo Sect. I, **11** (1965), 155–171.

[10] G. Karpilovski, The Schur Multiplier, London Math. Soc. Monogr. (N.S.), **2**, The Clarendon Press, Oxford University Press, New York, 1987.

[11] S. V. Kerov, Asymptotic Representation Theory of the Symmetric Group and its Applications in Analysis, Transl. Math. Monogr., **219**, Amer. Math. Soc., Providence, RI, 2003.

[12] E. W. Read, On the Schur multipliers of the finite imprimitive unitary reflection groups $G(m,p,n)$, J. London Math. Soc. (2), **13** (1976), 150–154.

[13] J. Schur, Über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen, J. Reine Angew. Math., **127** (1904), 20–50.

[14] J. Schur, Untersuchungen über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen, J. Reine Angew. Math., **132** (1907), 85–137.

[15] J. Schur, Über die Darstellung der symmetrischen und der alternierenden Gruppen durch gebrochene lineare Substitutionen, J. Reine Angew. Math., **139** (1911), 155–250.

[16] A. M. Vershik and S. V. Kerov, Asymptotic theory of characters of the symmetric group, Funct. Anal. Appl., **15** (1981), 246–255; Russian original, **15**, no. 4 (1981), 15–27.

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