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Jacobi inversion on strata of the Jacobian of the C_{rs} curve $y^r = f(x)$, II

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Abstract. Previous work by the authors (this journal, **60** (2008), 1009–1044) produced equations that hold on certain loci of the Jacobian of a cyclic C_{rs} curve. A curve of this type generalizes elliptic curves, and the equations in question are given in terms of (Klein's) generalization of Weierstrass' σ-function. The key tool is a matrix with entries that are polynomial in the coordinates of the affine plane model of the curve, thus can be expressed in terms of σ and its derivatives. The key geometric loci on the Jacobian of the curve give a stratification of Brill-Noether type. The results are of the type of Riemann-Kempf singularity theorem, the methods are germane to those used by J. D. Fay, who gave vanishing tables for Riemann's θ-function and its derivatives. The main objects we use were developed by several contemporary authors, aside from the classical definitions: meromorphic differentials were expressed in terms of the coordinates mainly by V. M. Buchstaber, J. C. Eilbeck, V. Z. Enolski, D. V. Leykin, and Taylor expansions for σ in terms of Schur polynomials also contributed by A. Nakayashiki, in terms of Sato's τ-function. Within this framework, following specific results for σ-derivatives given by Y. Ônishi, we arrive at our main results, namely statements on the vanishing on given strata of the partial derivatives of σ indexed by Young-diagrams subsets that can be worked out in terms of the Weierstrass semigroup of the curve at its point at infinity. The combinatorial statements hold not only for Jacobians but for the stratification of Sato's infinite-dimensional Grassmann manifold as well.

1. Introduction.

Vanishing theorems for Riemann's theta function have been investigated classically as well as in the current literature, and connected with problems as diverse as the Schottky problem and integrable nonlinear PDEs. The Riemann theta function has special algebraic properties when it comes from a Riemann surface; in fact we focus on a special type of curve X (reviewed in Section 2) which is a cyclic cover of $\mathbb{P}^1$, as in our previous paper [**MP1**]. We were able to express certain abelian functions in terms of the polynomial defining the affine part of the curve. We used the σ function, associated to the theta function, which Klein introduced [**K1**] for genus-2 curves to generalize the genus-1 Weierstrass σ; further work ensued in the 19th century, mainly for hyperelliptic curves (Klein, H. Burkhardt, O. Bolza); Klein gave a definition for general curves of genus 3 [**K2**, Section 27] and Korotkin with Shramchenko, inspired by Klein's constructions, recently

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produced both odd and (for generic curves) even σ-functions for all (smooth) curves, the latter invariant under modular transformations, and the former, invariant up to a root of unity [**KS**][1]. Baker brought the theory together in a monograph [**B3**] and extended the analysis (e.g., the aspect of power-series expansion and partial differential equations). More recently, σ was studied for all C_{rs} curves (cf., e.g., [**EEL**]). By taking suitable limits, we obtained the order of vanishing of σ on the stratification in the Jacobian given by the Abel image of the symmetric products $\mathcal{S}^k(X)$ of the curve [**MP1**, Remark 5.8]. We continue that analysis, using Schur polynomials and representation theory, to obtain a vanishing pattern using Young diagrams (Theorem 5.15) and moreover we express ratios of coefficients of the multivariable Taylor expansion of σ by algebraic functions (Theorem 5.24, as well as Proposition 5.26); we note also recent work [**EHKKLS**][1] that solves the Jacobi inversion problem on the singular strata of the Jacobian by explicit algebraic expressions for derivatives of σ in the hyperelliptic case, based on the embedding of the curve in the Jacobian and expressed in terms of the Weierstrass semigroup at the basepoint and attendant Schur-Weierstrasss polynomials. Moreover, equations for the periods of holomorphic and meromorphic differentials allow the authors of [**EHKKLS**] to perform numerical calculations and obtain qualitative information on the geodesics of black hole space-time. Our key technique consists in enabling partial derivatives on the Jacobian image of symmetric powers of the curve, cf. Section 4, just as Klein and Baker used the derivative along the curve. These techniques have produced new solutions of non-linear wave equations [**MP2**].

In [**Ô1**], Ônishi gave a non-vanishing theorem for σ over a hyperelliptic curve $X$of genus g given by affine equation $y^2 = x^{2g+1} + \lambda_{2g}x^{2g} + \cdots + \lambda_1 x + \lambda_0$ and a point ∞, as a special case of the Riemann Singularity Theorem (Section 5); specifically,

Theorem 1.1. *For $0 < k \le g-1$, let $D = P_1 + \cdots + P_k$ belong to $\mathcal{S}^k(X\backslash\infty) \setminus (\mathcal{S}^k_1(X) \cap \mathcal{S}^k(X\backslash\infty))$, where $\mathcal{S}^k_1(X)$ are divisors in $\mathcal{S}^k(X)$, the k-th symmetric product of the curve, whose linear series has projective dimension at least* 1; *let*

$$u^{[k]} := \sum_{i=1}^{k} \int_{\infty}^{P_i} \nu^I,$$

for a suitable basis ν^I of holomorphic differentials, and

$$\natural_k := \begin{cases} \{g, g-2, \dots, k+2, k\} & \textit{if } g-k \textit{ is even}, \\ \{g-1, g-3, \dots, k+3, k+1\} & \textit{otherwise}; \end{cases}$$

call $n_k := \#\natural_k$ the cardinality of the set $\natural_k$. The following holds:

1. *For every multiple index $(\alpha_1, \dots, \alpha_m)$ with $\alpha_i \in \{1, \dots, g\}$ (possibly repeated) and $m < n_k$,*

[1] We are very grateful to one Referee for this reference, which was not available at the time of our manuscript's submission.

$$\frac{\partial^m}{\partial u_{\alpha_1} \dots \partial u_{\alpha_m}} \sigma(u^{[k]}) = 0.$$

2. *For the multiple index* $\natural_k$,

$$\left(\prod_{\beta \in \natural_k} \frac{\partial}{\partial u_\beta} \right) \sigma(u^{[k]}) \neq 0. \tag{1.1}$$

We show some examples of $\natural_k$ in Table 1.1.

Table 1.1

(r,s)	g	$\natural_1$	$\natural_2$	$\natural_3$	$\natural_4$	$\natural_5$	$\natural_6$	$\natural_7$
$(2,3)$	1							
$(2,5)$	2	$\{2\}$						
$(2,7)$	3	$\{2\}$	$\{3\}$					
$(2,9)$	4	$\{2,4\}$	$\{3\}$	$\{4\}$				
$(2,11)$	5	$\{2,4\}$	$\{3,5\}$	$\{4\}$	$\{5\}$			
$(2,13)$	6	$\{2,4,6\}$	$\{3,5\}$	$\{4,6\}$	$\{5\}$	$\{6\}$		
$(2,15)$	7	$\{2,4,6\}$	$\{3,5,7\}$	$\{4,6\}$	$\{5,7\}$	$\{6\}$	$\{7\}$	
$(2,17)$	8	$\{2,4,6,8\}$	$\{3,5,7\}$	$\{4,6,8\}$	$\{5,7\}$	$\{6,8\}$	$\{7\}$	$\{8\}$

Ônishi generalized the theorem to cyclic C_{3s} and C_{5s} curves [**Ô2**], [**MÔ**]. For the analogous set $\natural_k$, similar (non-)vanishing results hold, exemplified in Table 1.2.

Table 1.2

(r,s)	g	$\natural_1$	$\natural_2$	$\natural_3$	$\natural_4$	$\natural_5$	$\natural_6$	$\natural_7$	$\natural_8$	$\natural_9$	$\natural_{10}$	$\natural_{11}$
$(3,4)$	3	$\{2\}$	$\{3\}$									
$(3,5)$	4	$\{2\}$	$\{3\}$	$\{4\}$								
$(3,7)$	6	$\{2,5\}$	$\{3,6\}$	$\{4\}$	$\{5\}$	$\{6\}$						
$(3,8)$	7	$\{2,5\}$	$\{3,6\}$	$\{4,7\}$	$\{5\}$	$\{6\}$	$\{7\}$					
$(3,10)$	9	$\{2,4,7\}$	$\{3,5,9\}$	$\{4,7\}$	$\{5,8\}$	$\{6,9\}$	$\{7\}$	$\{8\}$	$\{9\}$			
$(5,6)$	10	$\{2,5,8\}$	$\{3,7,9\}$	$\{4,8,10\}$	$\{5,9\}$	$\{6\}$	$\{7\}$	$\{8\}$	$\{9\}$	$\{10\}$		
$(5,7)$	12	$\{2,5,8,12\}$	$\{3,7,9\}$	$\{4,8,11\}$	$\{5,9,12\}$	$\{6,10\}$	$\{7,12\}$	$\{8\}$	$\{9\}$	$\{10\}$	$\{11\}$	$\{12\}$

In this article, we generalize these relations to the C_{rs} curve of Section 2, based on results of our previous paper [**MP1**] and on the theory of Young diagrams, which governs the σ-function as proved by Nakayashiki for general C_{rs} curves [**N**]; we note that such patterns for Schur-Weierstrass polynomials and their derivatives were introduced in [**BLE1**] and proven to satisfy the Riemann vanishing theorem. Nakayashiki's identification of the leading term in the expansion of σ in terms of Schur-Weierstrass polynomials is used below in a crucial way particularly in Section 5.

Theorem 5.15 below may be of independent interest in the theory of Schur functions and their derivatives on the stratification by partitions of Sato's Grassmannian (more general than our case, which is concerned with partitions related to the Weierstrass gaps of a C_{rs} curve). Precise knowledge of the order of vanishing of σ in terms of Weierstrass gaps is important not only for the intersection theory of the Jacobian, cf. [**BV**], but also for the intersection theory of the moduli space of Jacobians, in terms of Brill-Noether strata.

The paper can be read independently of the previous part [**MP1**]: below, we briefly set up the notation, cite the statements we need and give precise references to proofs in [**MP1**]. To illustrate the significance of patterns, we give two detailed examples, which go beyond the ones displayed above, being pentagonal and heptagonal. The Weierstrass-gap and Young-diagrams numerology is provided in Section 2, the Θ stratification and Schur function notation are found in Section 3; this notation is used in Section 5, which contains our main results.

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Guide to Symbols.

$N(n)$	order of pole at n-th Weierstrass non-gap ($N(0) = 0$, $N(g) = 2g$) p. 651
ϕ_i	monomial associated to the n-th Weierstrass non-gap p. 651
$d_>(t^\ell), d_<(t^\ell), d_\geq(t^\ell), d_\leq(t^\ell)$	truncated expansions in t p. 651
ν^I	holomorphic one form (2.2)
ν^{II}	meromorphic one form p. 657
w	Abel map ($w : \mathcal{S}^k(X) \to \mathbb{C}^g$) p. 652
$\deg_{\mathrm{w}}, \deg_{\mathrm{w}^{-1}}, \deg_\lambda$	degrees ($\deg_{\mathrm{w}}(\phi_n) = N(n)$) p. 651
$\mathcal{S}^k(X), \mathcal{S}^k_m(X)$	symmetric product of the curve and its singular strata p. 651
$\mathcal{J}, \Pi, \kappa$	Jacobian $\mathcal{J} = \mathbb{C}^g/\Pi$, $\kappa : \mathbb{C}^g \to \mathcal{J}$
$\mathcal{W}^k, \mathcal{W}^k_m$	strata in the Jacobian ($w(\mathcal{S}^k(X)) = \mathcal{W}^k$, $w(\mathcal{S}^k_m(X)) = \mathcal{W}^k_m$) (2.4) and p. 652
Λ	Young diagram p. 653
$\Psi_n, \Psi^{(\ell)}_n, \psi_n, \psi^{(\ell)}_n$	Frobenius-Stickelberger (FS) matrices and determinants p. 655
μ_n	Definition 2.5
$\omega', \omega'', \eta', \eta''$	complete integrals of the first kind and the second kind (3.1)
σ	σ function (3.5)
$L(u, v), \chi(\ell)$	quasi-periodicity of σ (3.6)
Θ^k, Θ^k_1	strata in the Jacobian (3.7), (3.8) and (3.9)
$h_n, h^{\langle \ell_1, \ell_2 \rangle}_n$	homogeneous symmetric functions p. 660
$T_n, T^{\langle \ell_1, \ell_2 \rangle}_n$	power symmetric functions, Proposition 3.6
$(a_1, \dots, a_r; b_1, \dots, b_r)$	characteristics of a partition p. 662
s_Λ, S_Λ	Schur function (3.11)
$u^\alpha, \lvert\alpha\rvert, d_>(u^\ell), \mathrm{w}_g(\alpha)$	multi-index convention (2.6) p. 654, p. 662

$T_n^{(g;k)}, T_n^{(k)}, h_n^{(g;k)}$	$T_n^{(k)} := T_n^{\langle 1,k\rangle}$, $T_n^{(g;k)} := T_n^{\langle k+1,g\rangle}$, $h_n^{(g;k)} := h_n^{\langle k+1,g\rangle}$ p. 672
$\Lambda^{(k)}, \Lambda^{[k]}$	truncated Young diagrams p. 671
$L^{[k]}(a_i, b_i)$	Proposition 5.12
H^i, h^i	cohomology and its dimension p. 667
H_{ℓ_1,ℓ_2}, H_i	matrices, Lemma 3.9, proof of Lemma 5.11
$\natural_k, \natural_k^{(i)}$	sequences, Definition 5.14
$u^{[k]}, u^{[g;k]}$,	$u^{[k]} := T^{(k)}_{\Lambda_i+g-i}, u^{[g;k]} := T^{(g;k)}_{\Lambda_i+g-i}$, Lemmas 5.11 and 5.25
$u^{[k]}, u^{[g;k]}, v^{(i)}$,	$v^{(i)} := w(P_i), u^{[k]} = u^{[k-1]} + v^{(k)}, u^{[g;k]} = u^{[g]} - u^{[k]}$

2. The C_{rs} curve and the pre-image of its Wirtinger varieties.

We recall from [**MP1**] notations and basic properties for the Riemann surface

$$X := \{(x, y) \mid y^r = f(x)\} \cup \infty$$

whose finite part is given by an affine equation

$$y^r = f(x), \quad f(x) := x^s + \lambda_{s-1}x^{s-1} + \cdots + \lambda_1 x + \lambda_0. \tag{2.1}$$

The integers r and s are such that $(r, s) = 1$ and $r < s$; the complex numbers $\lambda_0, \dots, \lambda_{s-1}$ are such that the finite part of X is smooth. This is a particular C_{rs} curve (a nomenclature introduced in the 1990s to generalize elliptic curves in Weierstrass form). It has genus $g = ((r-1)(s-1))/2$. In Section 5 we will consider the degeneration of X to a singular curve X_0 for which all λ's vanish.

Let $R := \mathbb{C}[x, y]/(y^r - f(x))$, $\mathcal{O}_X$ be the sheaf of holomorphic functions over X and $\mathcal{J}$ the Jacobian of X. We note that $R = \mathcal{O}_X(*\infty)$ is the ring of meromorphic functions on X regular outside ∞, where * stands for any order of pole.

For a non-negative integer n, we denote by $\phi_n \in \mathbb{C}[x, y]$ the (monic) monomial that at ∞ has a pole of order $N(n)$, the n-th integer in the (increasing) sequence complementary to the Weierstrass gaps: $\phi_0 = 1$, $\phi_1 = x$, etc.; by letting t_∞ be a local parameter at ∞, the leading term of ϕ_n is proportional to $t_\infty^{-N(n)}$. We routinely abuse notation slightly, to think of polynomials in $\mathbb{C}[x, y]$ as functions in R. A direct calculation gives $N(0) = 0$, $N(g-1) = 2g-2$, $N(g) = 2g$ for a $C_{r,s}$ curve. We define the w-degree, $\deg_{\mathrm{w}} : R \to \mathbb{Z}$, which assigns to an element of R its order of pole at ∞, $\deg_{\mathrm{w}}(x) = r$, $\deg_{\mathrm{w}}(y) = s$, $\deg_{\mathrm{w}}(\phi_n(P)) = N(n)$. We also consider the ring $R_\lambda := \mathbb{Q}[x, y, \lambda_0, \dots, \lambda_{s-1}]/(y^r - f(x))$ by regarding λ's as indeterminates, and define a λ-degree, $\deg_\lambda : R_\lambda \to \mathbb{Z}$ as an extension of the w-degree by assigning the degree $(s-i)r$ to each λ_i. This makes the polynomial defining the curve, $y^r - f(x)$, homogeneous of degree rs with respect to the λ-degree.

We denote a point $P \in X\backslash\infty$ by its affine coordinates (x, y); we also loosely denote by a k-tuple $(P_1, \dots, P_k)$, or by a divisor $D = \sum_{i=1}^k P_i$, an element of $\mathcal{S}^k(X)$, the k-th symmetric product of the curve. For a given local parameter t at some P in X, by $d_>(t^\ell)$ (resp. $d_<(t^\ell)$) we denote the terms of a function on X in its t-expansion whose orders of zero at P are greater (resp. less) than ℓ; $d_\geq(t^\ell)$ (resp. $d_\leq(t^\ell)$) includes terms of order ℓ. For the local parameter t_∞ at ∞, we have

$$x = \frac{1}{t_\infty^r}, \quad y = \frac{1}{t_\infty^s}(1 + d_>(t_\infty)), \quad \phi_n(P) = \frac{1}{t_\infty^{N(n)}}(1 + d_>(t_\infty)).$$

A basis $\{{\nu^I}_1, \dots, {\nu^I}_g\}$ of $H^0(X, K_X)$, where K_X as customary denotes the canonical bundle (and with a slight abuse of notation we do not distinguish between the bundle, given divisor, and sheaf that correspond to each other), is given in terms of the ϕ_i following [**B1**, Chapter VI, Section 91],

$$ {\nu^I}_i = \frac{\phi_{i-1}(P)dx}{ry^{r-1}}, \quad (i = 1, \dots, g). \tag{2.2}$$

We extend the w-degree to one-forms, by fixing the local parameter t_∞ at ∞, so that for a one-form $\nu = (t_\infty^n + d_>(t_\infty^n))dt_\infty$, $\deg_{\mathrm{w}}(\nu) = -n$. Since we have

$$ {\nu^I}_i = t_\infty^{2g-N(i-1)-2}(1 + d_{>0}(t_\infty))dt_\infty, \tag{2.3}$$

the degree is given by

$$\deg_{\mathrm{w}^{-1}}({\nu^I}_i) = 2g - N(i-1) - 2,$$

where $\deg_{\mathrm{w}^{-1}}(f) = -\deg_{\mathrm{w}}(f)$, and to entire functions on $\mathbb{C}^g$, by pulling them back to the curve via the Abel map defined next. Using the analytic as opposed to the algebraic nature of the curve, we consider the Abel images of the k-fold symmetric products of the curve:

$$\mathcal{W}^k := \kappa\left(\left\{\sum_{i=1}^k \int_\infty^{(x_i,y_i)} \begin{pmatrix} {\nu^I}_1 \\ \vdots \\ {\nu^I}_g \end{pmatrix} \middle| (x_i, y_i) \in X \right\}\right) \subset \mathcal{J}, \tag{2.4}$$

where κ is the projection $\mathbb{C}^g \to \mathcal{J} = \mathbb{C}^g/\Pi$, Π is the period lattice of the basis $\{{\nu^I}_1, \dots, {\nu^I}_g\}$, and $\mathcal{J}$ is the Jacobian of X. We denote by w the Abel map from $\mathcal{S}^k(X)$ of X to $\kappa^{-1}\mathcal{W}^k$ with base-point ∞, for any positive integer k. Note that there is a remaining Π-ambiguity due to the choice of path of integration: our results below will be independent of such ambiguity, but they require a g-tuple of complex numbers to be given explicitly, $w : (P_1, \dots, P_k) \mapsto w(P_1, \dots, P_k) = \sum_{i=1}^k \int_\infty^{P_i} \nu^I \in \mathbb{C}^g$, where we abbreviate by ν^{I} the g-vector of holomorphic differentials ${\nu^I}_i$. When an analytic function, say, of g complex variables is evaluated on $u := w(P_1, \dots, P_k)$, we view it as function of the coordinates $(u_1, \dots, u_g)$ of the (column) vector u, as the convention goes. We also introduce

$$\mathcal{S}_m^n(X) := \{D \in \mathcal{S}^n(X) \mid \dim |D| \ge m\},$$

where $|D|$ is the complete linear system $w^{-1}(w(D))$ [**ACGH**, IV.1]. If $n < g$, the singular locus of $\mathcal{S}^n(X)$ after moding out by linear equivalence, or projecting to the Picard group, is $\mathcal{S}_1^n(X)$ [**ACGH**, Chapter IV, Proposition 4.2, Corollary 4.5, where our $\mathcal{S}^n(X)$ is C_n^0]. We let $\mathcal{W}_m^n := w(\mathcal{S}_m^n(X))$.

The choice of basis $\{{\nu^I}_1, \dots, {\nu^I}_g\}$ allows us to connect the expansion of the σ function (cf. Section 3) in the attendant Abelian coordinates $\{u_1, \dots, u_g\}$ with the gap sequence at ∞. To do so, we introduce a Young diagram (cf., e.g., [**S**], [**BLE1**]) Λ relative to the Weierstrass-gap sequence: from the top down, $1 \le i \le g$, the rows have length:

$$\Lambda_i = N(g) - N(i-1) - g + i - 1 = g - N(i-1) + (i-1),$$

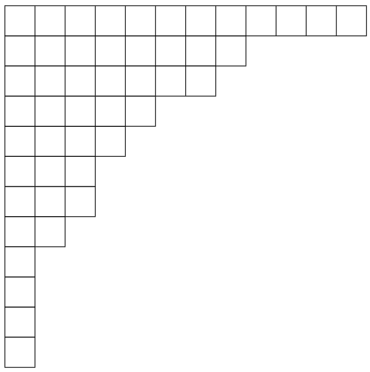
$$|\Lambda| = \sum_{i=1}^{g} \Lambda_i = \frac{1}{24}(r^2-1)(s^2-1) = g + \mathrm{w}(\infty),$$

where $\mathrm{w}(\infty)$ is the Weierstrass weight of the point ∞ (if we write Λ_j for $(j > g)$ we set it equal to zero).

We give two examples, the first in the pentagonal class, cf. [**MÔ**], and the second heptagonal (the fact that both r and s are prime numbers is not essential but more convenient): For the case $(r, s) = (5, 7)$ (Table 2.1), we have

Table 2.1

i	0	1	2	3	4	5	6	7	8	9	10	11	12
$\phi(i)$	1	x	y	x^2	xy	y^2	x^3	x^2y	xy^2	x^4	y^3	x^3y	x^2y^2
$N(i)$	0	5	7	10	12	14	15	17	19	20	21	22	24
Λ_i	–	12	8	7	5	4	3	3	2	1	1	1	1



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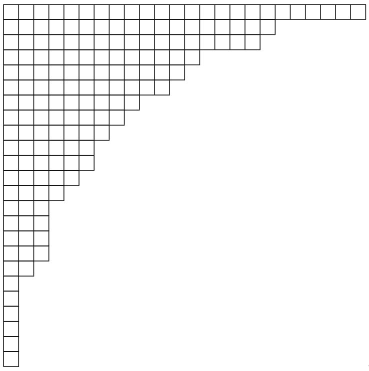
For the case $(r, s) = (7, 9)$ (Table 2.2), we have

Table 2.2

i	0	1	2	3	4	5	6	7	8	9	10	11	12
$\phi(i)$	1	x	y	x^2	xy	y^2	x^3	x^2y	xy^2	y^3	x^4	x^3y	x^2y^2
$N(i)$	0	7	9	14	16	18	21	23	25	27	28	30	32
Λ_i	–	24	18	17	13	12	11	9	8	7	6	6	5
i	13	14	15	16	17	18	19	20	21	22	23	24	
$\phi(i)$	xy^3	x^5	y^4	x^4y	x^3y^2	x^2y^3	x^6	xy^4	x^5y	y^5	x^4y^2	x^2y^4	
$N(i)$	34	35	36	37	39	41	42	43	44	45	46	48	
Λ_i	4	3	3	3	3	2	1	1	1	1	1	1	



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Lemma 2.1. *For $v \in w(P)$, $P \in X$,*

$$\deg_{\mathrm{w}^{-1}}(v_i) = N(g) - N(i-1) - 1 = 2g - N(i-1) - 1 = \Lambda_i + g - i,$$

and $\deg_{\mathrm{w}^{-1}}(v_g) = 1$.

Proof. From (2.3), around ∞, v_i is expressed by a local parameter t_∞:

$$v_i = \frac{1}{2g - N(i-1) - 1} t_\infty^{2g-N(i-1)-1}(1 + d_>(t_\infty)),$$

and thus we have $\deg_{\mathrm{w}^{-1}}(v_i) = 2g - N(i-1) - 1 = N(g-1) - N(i-1) + 1$. □

Remark 2.2. We extend the degree to a g-vector $u = w(P_1, \dots, P_k)$ in such a way that $\deg_{\mathrm{w}^{-1}}(u_i) = \deg_{\mathrm{w}^{-1}}(v_i)$ if v_i of $v = w(P_\ell)$ does not vanish for a certain $\ell \in \{1, 2, \dots, k\}$. Assuming that a point $(P_1, \dots, P_k)$ belongs to $\mathcal{S}^k(X) \setminus \mathcal{S}^k_1(X)$ and each P_j is near ∞, by using a local parameter $t_{\infty,j}$ for each P_j, we read this degree off the formal sum:

$$u_i = \frac{1}{2g - N(i-1) - 1}\left(t_{\infty,1}^{2g-N(i-1)-1} + \cdots + t_{\infty,k}^{2g-N(i-1)-1}\right)(1 + d_>(t_\infty)). \tag{2.5}$$

(it is important to note that we do not perform an actual sum, but just record formally the degree of each independent variable in the i-component of the abelian integral; for example, in the hyperelliptic case, if $k = 2$ and the points P_1, P_2 were hyperelliptic conjugates, the result of the sum would be zero). We introduce the multi-index convention: for non-negative integers α_i and $\alpha := (\alpha_1, \dots, \alpha_k)$, we write

$$t^\alpha := t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_k^{\alpha_k}, \quad \alpha = (\alpha_1, \alpha_2, \dots, \alpha_k), \quad |\alpha| := \sum_{i=1}^k \alpha_i, \tag{2.6}$$

and extend the definition $d_>(t_i^\ell)$ for variables, $t_1, \dots, t_k$: $d_>(t^\ell) \in \{\sum_{|\alpha|>\ell} a_\alpha t^\alpha\}$ and $d_\geq(t^\ell) \in \{\sum_{|\alpha|\geq\ell} a_\alpha t^\alpha\}$. For example,

$$u_g = (t_{\infty,1} + \cdots + t_{\infty,k})(1 + d_{>}(t_\infty)) \quad \text{thus} \quad \deg_{\mathrm{w}^{-1}}(u_g) = 1.$$

REMARK 2.3. By Serre duality, specifically $\dim H^0(X, D) = \deg D - g + 1 - \dim H^0(X, (2g-2)\infty - D)$, we have a symmetry in the Young diagram.

REMARK 2.4. Using Remark 2.3, $\deg_{\mathrm{w}^{-1}}(u_i) = \Lambda_i + g - i$ is the hooklength (cf. [**S**, Chapter 3]) of the node $(1, i)$ in the Young diagram Λ.

In [**MP1**], we introduced meromorphic functions on the curve, reviewed here in Definition 2.5, which generalize the polynomial U in Mumford's (U, V, W) parametrization of a hyperelliptic Jacobian (which he attributes to Jacobi) [**M**, Chapter IIIa].

To achieve an algebraic representation, e.g. in Section 4, of Abelian vector fields, we introduce the Frobenius-Stickelberger (FS) matrix and its determinant following [**MP1**]. Let n be a positive integer and $P_1, \ldots, P_n$ be in $X \backslash \infty$. We define the *ℓ-reduced Frobenius-Stickelberger* (FS) *matrix* by:

$$\Psi_n^{(\check{\ell})}(P_1, P_2, \ldots, P_n) := \begin{pmatrix} 1 & \phi_1(P_1) & \phi_2(P_1) & \cdots & \check{\phi}_\ell(P_1) & \cdots & \phi_n(P_1) \\ 1 & \phi_1(P_2) & \phi_2(P_2) & \cdots & \check{\phi}_\ell(P_2) & \cdots & \phi_n(P_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & \phi_1(P_n) & \phi_2(P_n) & \cdots & \check{\phi}_\ell(P_n) & \cdots & \phi_n(P_n) \end{pmatrix},$$

and $\psi_n^{(\check{\ell})}(P_1, P_2, \ldots, P_n) := |\Psi_n^{(\check{\ell})}(P_1, P_2, \ldots, P_n)|$ (a check on top of a letter signifies deletion). It is also convenient to introduce the simpler notation:

$$\psi_n(P_1, \ldots, P_n) := \left|\Psi_n^{(\check{n})}(P_1, \ldots, P_n)\right|, \quad \Psi_n(P_1, \ldots, P_n) := \Psi_n^{(\check{n})}(P_1, \ldots, P_n), \tag{2.7}$$

for the un-bordered matrix. We call this matrix *Frobenius-Stickelberger* (*FS*) *matrix* and its determinant *Frobenius-Stickelberger* (*FS*) *determinant*. These become undefined for some tuples in $(X \backslash \infty)^n$.

Meromorphic functions, viewed as divisors on the curve, allow us to express the addition structure of Pic$X$ in terms of FS-matrices. For n points $(P_i)_{i=1,\ldots,n} \in X \backslash \infty$, we find an element of R associated with any point $P = (x, y)$ in $(X \backslash \infty)$, $\alpha_n(P) := \alpha_n(P; P_1, \ldots, P_n) = \sum_{i=0}^n a_i \phi_i(P)$, $a_i \in \mathbb{C}$ and $a_n = 1$, which has a zero at each point P_i (with multiplicity, if the P_i are repeated) and has smallest possible order of pole at ∞ with this property. We obtain $\alpha_n(P)$ from the FS matrix as the μ_n defined herewith:

DEFINITION 2.5. For $P, P_1, \ldots, P_n \in (X \backslash \infty) \times \mathcal{S}^n(X \backslash \infty)$, we define $\mu_n(P)$ by

$$\mu_n(P) := \mu_n(P; P_1, \ldots, P_n) := \lim_{P_i' \to P_i} \frac{1}{\psi_n(P_1', \ldots, P_n')} \psi_{n+1}\big(P_1', \ldots, P_n', P\big),$$

where the P_i' are generic and the limit is taken (irrespective of the order) for each i; we define $\mu_{n,k}(P_1, \ldots, P_n)$ by

$$\mu_n(P) = \phi_n(P) + \sum_{k=0}^{n-1} (-1)^{n-k} \mu_{n,k}(P_1, \dots, P_n) \phi_k(P),$$

with the convention $\mu_{n,n}(P_1, \dots, P_n) = 1$.

Lemma 2.6. *Let n be a positive integer. For $(P_i)_{i=1,\dots,n} \in \mathcal{S}^n(X \backslash \infty)$, the function α_n over X induces the map (which we call by the same name)* :

$$\alpha_n : \mathcal{S}^n(X \backslash \infty) \to \mathcal{S}^{N(n)-n}(X),$$

i.e., to $(P_i)_{i=1,\dots,n} \in \mathcal{S}^n(X \backslash \infty)$ there corresponds an element $(Q_i)_{i=1,\dots,N(n)-n} \in \mathcal{S}^{N(n)-n}(X)$, such that

$$\sum_{i=1}^{n} P_i - n\infty \sim - \sum_{i=1}^{N(n)-n} Q_i + (N(n) - n)\infty.$$

For an effective divisor of degree n, $D \in \mathcal{S}^n(X)$, let D' be the maximal subdivisor of D which does not contain ∞, $D = D' + (n-m)\infty$ where $\deg D' = m (\leq n)$ and $D' \in \mathcal{S}^m(X \backslash \infty)$: we extend the map α_n to $\mathcal{S}^n(X)$ by defining $\overline{\alpha}_n(D) = \alpha_m(D') + [N(n) - n - (N(m) - m)]\infty$.

We see from the linear equivalence of Lemma 2.6:

Proposition 2.7. *For a positive integer, the Abel map composed with α_n induces*

$$\iota_n : \mathcal{W}^n \to \mathcal{W}^{N(n)-n}, \quad \kappa \circ w \mapsto -\kappa \circ w.$$

Let image(ι_n) be denoted by $[-1]\mathcal{W}^n$.

Remark 2.8. We recover the well-known result: The Serre involution on Pic^{g-1}, $\mathcal{L} \mapsto K_X \mathcal{L}^{-1}$, is given by ι_{g-1},

$$\iota_{g-1} : \mathcal{W}^{g-1} \to [-1]\mathcal{W}^{g-1}.$$

3. The σ-function.

In this section, we summarize the results of [**MP1**] that are needed below. As customary, we choose a basis α_i, β_j $(1 \leqq i, j \leqq g)$ of $H_1(X, \mathbb{Z})$ with intersection pairing $\alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = 0$, $\alpha_i \cdot \beta_j = \delta_{ij}$, and we denote the half-period matrices by

$$\begin{aligned} [\omega' \, \omega''] &= \frac{1}{2} \left[\int_{\alpha_i} \nu^I{}_j \ \int_{\beta_i} \nu^I{}_j \right]_{i,j=1,2,\dots,g}, \\ [\eta' \, \eta''] &= \frac{1}{2} \left[\int_{\alpha_i} \nu^{II}{}_j \ \int_{\beta_i} \nu^{II}{}_j \right]_{i,j=1,2,\dots,g}, \end{aligned} \tag{3.1}$$

where ${\nu^{II}}_j = {\nu^{II}}_j(x, y)$ $(j = 1, 2, \cdots, g)$ are differentials of the second kind, which we defined algebraically in [**MP1**] after [**EEL**].

The following Proposition gives a *generalized Legendre relation* [**B1**], [**BLE2**], [**EEL**].

PROPOSITION 3.1. *The matrix*

$$M := \begin{bmatrix} 2\omega' & 2\omega'' \\ 2\eta' & 2\eta'' \end{bmatrix}, \tag{3.2}$$

satisfies

$$M \begin{bmatrix} & -1 \\ 1 & \end{bmatrix} {}^t M = 2\pi\sqrt{-1} \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}. \tag{3.3}$$

By the Riemann relations [**F1**], it is known that $\mathrm{Im}\,({\omega'}^{-1}\omega'')$ is positive definite. Referring to Theorem 1.1 in [**F1**], let

$$\delta := \begin{bmatrix} \delta'' \\ \delta' \end{bmatrix} \in \left(\frac{1}{2}\mathbb{Z}\right)^{2g} \tag{3.4}$$

be the theta characteristic which gives the Riemann-constant vector $\omega_{\mathrm{R}} = 2\omega''\delta' + 2\omega'\delta''$ with respect to the base point ∞ and the period matrix $[2\omega'\ 2\omega'']$.

We define an entire function of (a column-vector) $u = {}^t(u_1, u_2, \ldots, u_g) \in \mathbb{C}^g$, associated (i.e., they differ by a multiplicative factor which is the exponential of a quadratic form in the variables, cf. [**L**, Chapter IV]) to a translate of the Riemann θ-function,

$$\begin{aligned} \sigma(u) &= \sigma(u; M) = \sigma(u_1, u_2, \ldots, u_g; M) \\ &= c \exp\left(-\frac{1}{2}\,{}^t u \eta' {\omega'}^{-1} u\right) \theta[\delta]\left(\frac{1}{2}{\omega'}^{-1} u;\ {\omega'}^{-1}\omega''\right) \\ &= c \exp\left(-\frac{1}{2}\,{}^t u \eta' {\omega'}^{-1} u\right) \\ &\quad \times \sum_{n \in \mathbb{Z}^g} \exp\left\{\sqrt{-1}\left[\pi\,{}^t(n+\delta''){\omega'}^{-1}\omega''(n+\delta'') + {}^t(n+\delta'')({\omega'}^{-1}u + \delta')\right]\right\}, \end{aligned} \tag{3.5}$$

where c is a constant that depends on the moduli of the curve. Since in this paper we deal only with ratios of σ-functions, or with the vanishing order of σ, we tacitly suppress the constant c.

For a given $u \in \mathbb{C}^g$, we introduce u' and u'' in $\mathbb{R}^g$ so that

$$u = 2\omega' u' + 2\omega'' u''.$$

PROPOSITION 3.2 ([**MP1**, Proposition 4.3]). *For u, $v \in \mathbb{C}^g$, and ℓ_{a} $(= 2\omega'{\ell_{\mathrm{a}}}' + 2\omega''{\ell_{\mathrm{a}}}'') \in \Pi$, we define*

$$L(u,v) := 2\,{}^t u(\eta' v' + \eta'' v''),$$
$$\chi(\ell_{\rm a}) := \exp\left[\pi\sqrt{-1}\big(2({}^t\ell_{\rm a}'\delta'' - {}^t\ell_{\rm a}''\delta') + {}^t\ell_{\rm a}'\ell_{\rm a}''\big)\right] \ (\in \{1,\,-1\}).$$

The following holds

$$\sigma(u+\ell_{\rm a}) = \sigma(u)\exp\left(L\left(u+\frac{1}{2}\ell_{\rm a},\ell_{\rm a}\right)\right)\chi(\ell_{\rm a}). \tag{3.6}$$

Remark 3.3. The above periodicity property of σ is essentially the same that holds for the normalized theta function in Chapter VI of [**L**]. The normalized theta function is based upon the Hodge structure of the Jacobian, whereas σ is related to the symplectic action appearing in the Legendre relation, cf. Proposition 3.1. We note here the specific periodicity because our vanishing results allow us to extend it to certain derivatives of σ, cf. Corollary 5.18.

We also note that, as in genus 1, σ is *modular invariant* (cf. [**N**] for any (r,s) curve), namely, for every $\gamma \in \mathrm{Sp}(2g,\boldsymbol{Z})$, we have

$$\sigma(u;\gamma M) = \sigma(u;M).$$

In the case of general curves, the definition of odd σ given in [**KS**] is modular invariant up to a given root of unity, and the even version is modular invariant.

The vanishing locus of σ is:

$$\Theta^{g-1} = (\mathcal{W}^{g-1} \cup [-1]\mathcal{W}^{g-1}) = \mathcal{W}^{g-1}. \tag{3.7}$$

The last equality is due to our choice of base point ∞ such that $(2g-2)\infty = K_X$; indeed, $-w(D) = w(-D)$ by definition for a divisor D, and when D has degree $g-1$, by Serre duality D is special if and only if $K_X - D$ is special. The reason for introducing $\mathcal{W}^{g-1}\cup[-1]\mathcal{W}^{g-1}$ is that the analogous loci when $g-1$ is replaced by k play an important role and $\mathcal{W}^k$ is not $[-1]$-invariant in general:

$$\Theta^k := \mathcal{W}^k \cup [-1]\mathcal{W}^k. \tag{3.8}$$

Similarly, we define

$$\Theta^k_1 := w\big(\mathcal{S}^k_1(X)\big) \cup [-1]w\big(\mathcal{S}^k_1(X)\big). \tag{3.9}$$

For $(r=2, s=2g+1)$ (hyperelliptic) curves and ∞ a branch point, Θ^k equals $\mathcal{W}^k$ for every positive integer k but in general it does not.

The main result in [**MP1**] is the following:

Theorem 3.4 (Jacobi inversion formulae over Θ^k). *The following relations hold*

1. Θ^g *case*: *for* $(P_1,\dots,P_g) \in \mathcal{S}^g(X)\setminus\mathcal{S}^g_1(X)$ *and* $u = \pm w(P_1,\dots,P_g) \in \kappa^{-1}(\Theta^g)$,

$$\frac{\sigma_i(u)\sigma_g(u) - \sigma_{gi}(u)\sigma(u)}{\sigma^2(u)} = (-1)^{g-i+1}\mu_{g,i-1}(P_1, \ldots, P_g), \quad \textit{for } 0 < i \leq g.$$

2. Θ^{g-1} *case*: *for* $(P_1, \ldots, P_{g-1}) \in \mathcal{S}^{g-1}(X) \setminus \mathcal{S}_1^{g-1}(X)$ *and* $u = \pm w(P_1, \ldots, P_{g-1}) \in \kappa^{-1}(\Theta^{g-1})$,

$$\frac{\sigma_i(u)}{\sigma_g(u)} = \begin{cases} (-1)^{g-i}\mu_{g-1,i-1}(P_1, \ldots, P_{g-1}) & \textit{for } 0 < i \leq g, \\ 1 & \textit{for } i = g. \end{cases}$$

3. Θ^k *case*: *for* $(P_1, \ldots, P_k) \in \mathcal{S}^k(X) \setminus \mathcal{S}_1^k(X)$ *and* $u = \pm w(P_1, \ldots, P_k) \in \kappa^{-1}(\Theta^k)$,

$$\frac{\sigma_i(u)}{\sigma_{k+1}(u)} = \begin{cases} (-1)^{k-i+1}\mu_{k,i-1}(P_1, \ldots, P_k) & \textit{for } 0 < i \leq k, \\ 1 & \textit{for } i = k+1, \\ 0 & \textit{for } k+1 < i \leq g. \end{cases}$$

REMARK 3.5. We could easily extend parts 2. and 3. of Theorem 3.4 to any non-singular (r, s) curve, with affine equation:

$$f(x, y) = y^r + x^s + \sum_{i,j: rs > si + rj \geq 0} \lambda_{ij} y^i x^j = 0, \tag{3.10}$$

where λ_{ij} are complex numbers. As stated in Section 2, we limit our study to the cyclic type to use the work in [**MP1**]. For the general C_{rs} curve, the definitions of Section 2 are naturally modified (cf. [**EEL**]). As mentioned in [**MP1**, Remark 5.10], Theorem 3.4 2. and 3. could be proved by Fay's and Jorgenson's results [**F1**], [**J**] which hold for every Riemann surface, including (3.10). A direct proof can also be given by the result of Nakayashiki [**N**, Theorem 1], where the sigma function is explicitly expressed in terms of the "prime form" and FS-matrices. When computing the vanishing order of both numerator and denominator of the left-hand side of 2. and 3. in essentially the same way as in [**MP1**, Section 5], the prime forms cancel and the ratio is reduced to a ratio of FS-matrices.

It should also be possible to prove statement 1. of Theorem 3.4 for the general curve (3.10), by using the relation in [**MP1**, Proposition 4.5]. To compare poles and zeros of both sides, as functions of $u = w(x, y)$ and (x, y), respectively, ((x, y) a point of (3.10)), one would use two facts: σ vanishes to order one on Θ^{g-1}, and the Weierstrass gap at $2g - 1$ is adjacent to non-gaps. While this would give the expected number of zeros, however, it is not easy to specify them because the numerator in the left-hand side of 1. is a difference.

In [**MP1**], we had arrived at these results by comparing abelian functions with meromorphic functions on the curve. In the present work, we give precise results on the order of vanishing of σ itself on the stratification Θ^k. To do so, we first give the expansion of the σ-function. We introduce the Schur function $\boldsymbol{s}_\Lambda(t)$,

$$\boldsymbol{s}_\Lambda(t) := \frac{|t_j^{\Lambda_i+g-i}|_{1\le i,j\le g}}{|t_j^{i-1}|_{1\le i,j\le g}}. \tag{3.11}$$

The complete homogeneous symmetric function $h_n^{\langle \ell_1,\ell_2\rangle} = h_n(t_{\ell_1},\dots,t_{\ell_2})$ for positive integers ℓ_1 and ℓ_2 ($\ell_1 < \ell_2$) is given by

$$\prod_{i=\ell_1}^{\ell_2} \frac{1}{(1-zt_i)} = \sum_{n\ge 0} h_n^{\langle \ell_1,\ell_2\rangle} z^n, \quad h_n^{\langle \ell_1,\ell_2\rangle} = 0 \text{ for } n<0.$$

Proposition 3.6 ([**S**, Theorem 4.5.1]). *Using the complete homogeneous symmetric functions* $h_n := h_n^{\langle 1,g\rangle}$, *we can express* $\boldsymbol{s}_\Lambda$ *by a* $(g\times g)$ *Jacobi-Trudi Determinant,* $|a_{ij}|_{1\le i,j\le g}$ *with* $a_{ij} = h_{\Lambda_i+j-i}$:

$$\boldsymbol{s}_\Lambda(t) := |h_{\Lambda_i+j-i}|, \quad h_n = \frac{1}{n!}\begin{vmatrix} T_1 & -1 & 0 & \cdots & \\ 2T_2 & T_1 & -2 & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (n-1)T_{n-1} & (n-2)T_{n-2} & (n-3)T_{n-3} & \cdots & 1-n \\ nT_n & (n-1)T_{n-1} & (n-2)T_{n-2} & \cdots & T_1 \end{vmatrix},$$

where $h_0 = 1$, $h_{i<0} = 0$ *and* $T_k := T_k^{\langle 1,g\rangle}$,

$$T_k^{\langle \ell_1,\ell_2\rangle} := \frac{1}{k}\sum_{j=\ell_1}^{\ell_2} t_j^k.$$

When regarded as a function of T, we rename $\boldsymbol{s}$, $S_\Lambda(T) := \boldsymbol{s}_\Lambda(t)$.

We now give an earlier version of the Jacobi-Trudi formula of Proposition 3.6; by connecting the two, we provide a proof of Proposition 3.6, as well as a modified formula which will be used in Section 5.

Lemma 3.7.

$$\boldsymbol{s}_\Lambda(t) := |h_{\Lambda_i+j-i}^{\langle j,g\rangle}|_{1\le i,j\le g}.$$

Proof. Using a Vandermonde determinant, (3.11) is reduced to

$$\frac{\begin{vmatrix} t_1^{\Lambda_1+g-1} & t_2^{\Lambda_1+g-1} & \cdots & t_{g-1}^{\Lambda_1+g-1} & t_g^{\Lambda_1+g-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t_1^{\Lambda_{g-1}+1} & t_2^{\Lambda_{g-1}+1} & \cdots & t_{g-1}^{\Lambda_{g-1}+1} & t_g^{\Lambda_{g-1}+1} \\ t_1^{\Lambda_g} & t_2^{\Lambda_g} & \cdots & t_{g-1}^{\Lambda_g} & t_g^{\Lambda_g} \end{vmatrix}}{\prod_{i<j}(t_i-t_j)}.$$

This equals

$$\frac{\begin{vmatrix} t_1^{\Lambda_1+g-1} - t_g^{\Lambda_1+g-1} & t_2^{\Lambda_1+g-1} - t_g^{\Lambda_1+g-1} & \cdots & t_{g-1}^{\Lambda_1+g-1} - t_g^{\Lambda_1+g-1} & t_g^{\Lambda_1+g-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t_1^{\Lambda_{g-1}+1} - t_g^{\Lambda_{g-1}+1} & t_2^{\Lambda_{g-1}+1} - t_g^{\Lambda_{g-1}+1} & \cdots & t_{g-1}^{\Lambda_{g-1}+1} - t_g^{\Lambda_{g-1}+1} & t_g^{\Lambda_{g-1}+1} \\ t_1^{\Lambda_g} - t_g^{\Lambda_g} & t_2^{\Lambda_g} - t_g^{\Lambda_g} & \cdots & t_{g-1}^{\Lambda_g} - t_g^{\Lambda_g} & t_g^{\Lambda_g} \end{vmatrix}}{\prod_{i<j}(t_i - t_j)}$$

and noting $t^\ell - t'^\ell = (t-t')h_\ell(t,t')$, this becomes

$$\frac{\begin{vmatrix} h_{\Lambda_1+g-2}(t_1,t_g) & h_{\Lambda_1+g-2}(t_2,t_g) & \cdots & h_{\Lambda_1+g-2}(t_{g-1},t_g) & t_g^{\Lambda_1+g-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{\Lambda_{g-1}}(t_1,t_g) & h_{\Lambda_{g-1}}(t_2,t_g) & \cdots & h_{\Lambda_{g-1}}(t_{g-1},t_g) & t_g^{\Lambda_{g-1}+1} \\ h_{\Lambda_g-1}(t_1,t_g) & h_{\Lambda_g-1}(t_2,t_g) & \cdots & h_{\Lambda_g-1}(t_{g-1},t_g) & t_g^{\Lambda_g} \end{vmatrix}}{\prod_{i<j<g}(t_i - t_j)}.$$

We go on similarly, to co-factorize by $\prod_i^{g-2}(t_{g-1} - t_i)$,

$$\frac{\begin{vmatrix} h_{\Lambda_1+g-3}(t_1,t_{g-1},t_g) & h_{\Lambda_1+g-3}(t_2,t_{g-1},t_g) & \cdots & h_{\Lambda_1+g-3}(t_{g-2},t_{g-1},t_g) & h_{\Lambda_1+g-2}(t_{g-1},t_g) & t_g^{\Lambda_1+g-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ h_{\Lambda_{g-1}-1}(t_1,t_{g-1},t_g) & h_{\Lambda_{g-1}-1}(t_2,t_{g-1},t_g) & \cdots & h_{\Lambda_{g-1}-1}(t_{g-2},t_{g-1},t_g) & h_{\Lambda_{g-1}}(t_{g-1},t_g) & t_g^{\Lambda_{g-1}+1} \\ h_{\Lambda_g-2}(t_1,t_{g-1},t_g) & h_{\Lambda_g-2}(t_2,t_{g-1},t_g) & \cdots & h_{\Lambda_g-2}(t_{g-2},t_{g-1},t_g) & h_{\Lambda_g-1}(t_{g-1},t_g) & t_g^{\Lambda_g} \end{vmatrix}}{\prod_{i<j<g-1}(t_i - t_j)}$$

and derive the statement. □

Lemma 3.7 will yield the Jacobi-Trudi determinant formula in Proposition 3.6. In order to relate the two expressions, we prove:

LEMMA 3.8. 1. $h_n^{\langle \ell_1,\ell_2\rangle} = h_n^{\langle \ell_1+1,\ell_2\rangle} + h_{n-1}^{\langle \ell_1,\ell_2\rangle} t_{\ell_1} = h_n^{\langle \ell_1,\ell_2-1\rangle} + h_{n-1}^{\langle \ell_1,\ell_2\rangle} t_{\ell_2}$.

2. *For positive integers ℓ_1, ℓ_2, n and m satisfying $\ell_2 > \ell_1$ and $m < \ell_2 - \ell_1$,* $h_n^{\langle \ell_1,\ell_2\rangle} = h_n^{\langle \ell_1+m,\ell_2\rangle} + h_{n-1}^{\langle \ell_1+m-1,\ell_2\rangle} h_1^{\langle \ell_1,\ell_1+m-1\rangle} + h_{n-2}^{\langle \ell_1+m-2,\ell_2\rangle} h_2^{\langle \ell_1,\ell_1+m-2\rangle} + \cdots + h_{n-m}^{\langle \ell_1,\ell_2\rangle} h_m^{\langle \ell_1,\ell_1\rangle}$.

3. *For positive integers ℓ_1, ℓ_2 and n satisfying $\ell_2 > \ell_1$,* $h_n^{\langle \ell_1,\ell_2\rangle} = h_n^{\langle \ell_2,\ell_2\rangle} + h_{n-1}^{\langle \ell_2-1,\ell_2\rangle} h_1^{\langle \ell_1,\ell_2-1\rangle} + h_{n-2}^{\langle \ell_2-2,\ell_2\rangle} h_2^{\langle \ell_1,\ell_2-2\rangle} + \cdots + h_{n-\ell_2+\ell_1}^{\langle \ell_1,\ell_2\rangle} h_{\ell_2-\ell_1}^{\langle \ell_1,\ell_1\rangle}$.

PROOF. The product $\prod_{i=\ell_1}^{\ell_2} 1/(1-zt_i) = 1/(1-zt_{\ell_1}) \prod_{i=\ell_1+1}^{\ell_2} 1/(1-zt_i)$ is expanded using

$$(1 - t_{\ell_1} z)\big(1 + h_1^{\langle \ell_1,\ell_2\rangle} z + h_2^{\langle \ell_1,\ell_2\rangle} z^2 + \cdots\big) = \big(1 + h_1^{\langle \ell_1+1,\ell_2\rangle} z + h_2^{\langle \ell_1+1,\ell_2\rangle} z^2 + \cdots\big),$$

which shows the first equality in the first statement; the second equality is similarly obtained. Statement 2. for the $m = 1$ case is reduced to the first statement for every n. We give a proof by induction: assume that 2. holds for $m \geq m_0$,

$$h_n^{\langle \ell_1,\ell_2\rangle} = h_n^{\langle \ell_1+m,\ell_2\rangle} + h_{n-1}^{\langle \ell_1+m-1,\ell_2\rangle} h_1^{\langle \ell_1+1,\ell_1+m-1\rangle} + h_{n-2}^{\langle \ell_1+m-2,\ell_2\rangle} h_2^{\langle \ell_1,\ell_1+m-2\rangle}$$
$$+ \cdots + h_{n-m}^{\langle \ell_1,\ell_2\rangle} h_m^{\langle \ell_1,\ell_1\rangle}.$$

We consider the $m+1$ case: part 1. implies that the individual terms in the right-hand side satisfy,

$$\begin{aligned} h_n^{\langle \ell_1,\ell_2\rangle} &= \big(h_n^{\langle \ell_1+m+1,\ell_2\rangle} + h_{n-1}^{\langle \ell_1+m,\ell_2\rangle} t_{\ell_1+m}\big) \\ &\quad + \big(h_{n-1}^{\langle \ell_1+m,\ell_2\rangle} + h_{n-2}^{\langle \ell_1+m-1,\ell_2\rangle} t_{\ell_2-m-1}\big) h_1^{\langle \ell_1+1,\ell_1+m+1\rangle} \\ &\quad + \big(h_{n-2}^{\langle \ell_1+m-1,\ell_2\rangle} + h_{n-3}^{\langle \ell_1+m-2,\ell_2\rangle} t_{\ell_2-m-2}\big) h_2^{\langle \ell_1+2,\ell_1+m-1\rangle} \\ &\quad + \cdots + \big(h_{n-m}^{\langle \ell_1+1,\ell_2\rangle} + h_{n-m-1}^{\langle \ell_1,\ell_2\rangle} t_{\ell_1}\big) h_m^{\langle \ell_1,\ell_1\rangle} \\ &= h_n^{\langle \ell_1+m+1,\ell_2\rangle} + h_{n-1}^{\langle \ell_1+m,\ell_2\rangle} \big(t_{\ell_1+m} + h_1^{\langle \ell_1+1,\ell_1+m+1\rangle}\big) \\ &\quad + h_{n-2}^{\langle \ell_1+m-1,\ell_2\rangle} \big(t_{\ell_1-m-1} h_1^{\langle \ell_1+1,\ell_1+m+1\rangle} + h_2^{\langle \ell_1+2,\ell_1+m-1\rangle}\big) \\ &\quad + \cdots + h_{n-m-1}^{\langle \ell_1,\ell_2\rangle} t_{\ell_1} h_m^{\langle \ell_1,\ell_1\rangle}, \end{aligned}$$

which gives the $m+1$ case. Statement 3. is obtained by setting $m = \ell_2 - \ell_1 - 1$. □

A modified version of the Jacobi-Trudi relation follows:

Lemma 3.9. *For any fixed k, $1 \le k \le g$, juxtaposing two matrices to obtain a $g \times g$ matrix,*

$$\boldsymbol{s}_\Lambda(t_1, \dots, t_g) = \big|H_{g,k}(t_1, \dots, t_g), H_{g,g-k}(t_{k+1}, \dots, t_g)\big|,$$

where $H_{a,b}(t_\ell, \dots, t_g)$'s are $a \times b$ matrices defined by

$$H_{g,k}(t_1, \dots, t_g) := (h_{\Lambda_i+j-i}(t_1, \dots, t_g))_{1 \le i \le g, 1 \le j \le k},$$
$$H_{g,g-k}(t_{k+1}, \dots, t_g) := (h_{\Lambda_i+j-i}(t_{k+1}, \dots, t_g))_{1 \le i \le g, k+1 \le j \le g}.$$

For a given Young diagram $\Lambda = (\Lambda_1, \Lambda_2, \dots, \Lambda_\ell)$, the length r of the diagonal is called the rank of the partition [**FH**, Section 4.1, p. 51]. Let a_i and b_i be the number of boxes below and to the right of the i-th box of the diagonal, reading from lower right to upper left. Frobenius named $(a_1, \dots, a_r; b_1, \dots, b_r)$ the characteristics of the partition [**FH**, Section 4.1 p. 51]. Here $a_i < a_j$ and $b_i < b_j$ for $i < j$.

We use the multi-index convention (cf. Guide to Symbols) and define a map for $\beta := (\beta_1, \dots, \beta_g)$,

$$\mathrm{w}_g(\beta) := ((2g - N(0) - 1)\beta_1, (2g - N(1) - 1)\beta_2, \dots, (2g - N(g-2) - 1)\beta_{g-1}, \beta_g) \in \mathbb{Z}^g$$

The following relation between Schur-Weierstrass polynomials and the σ-function was proved by Nakayashiki [**N**].

Proposition 3.10. *The expansion of $\sigma(u)$ at the origin takes the form*

$$\sigma(u) = S_\Lambda(T)|_{T_{\Lambda_i+g-i}=u_i} + \sum_{|\mathrm{w}_g(\alpha)|>|\Lambda|} c_\alpha u^\alpha$$

where $c_\alpha \in \mathbb{Q}[\lambda_j]$ and $S_\Lambda(T)$ is the lowest-order term in the w-degree of the u_i; $\sigma(u)$ is homogeneous of degree $|\Lambda|$ with respect to the λ-degrees.

We note that S_Λ is a function of $\{T_{\Lambda_i+g-i}\}_{i=1,\ldots,g}$, even though *a priori* it depends on $\{T_i\}_{i=1,\ldots,2g-1}$.

Proof. The result follows from [**N**, Theorem 3]. □

Remark 3.11. The symmetric functions have degree coming from the natural order of the multi-variables $(t_1, \ldots, t_g)$. In Proposition 3.10, the degree corresponds to the w-degree on $W^{k-1} \subset W^k$ and $S^{k-1}X \subset S^k X$, according to the convention of Remark 2.2.

4. Algebraic expression of the Jacobian of a coordinate change.

In order to detect the vanishing of the multiderivatives of σ, following Weierstrass, Klein, Baker and others [**B1**], [**B2**], [**B3**], [**K1**], [**W**], we consider certain vector fields on the symmetric products of the curve.

Let k be a positive integer $\leq g$. We fix a subset K_k of $\{1, 2, \ldots, g\}$ with k elements. We relabel the indices by the map $\iota : \{1, 2, \ldots, k\} \to K_k$ such that $\iota(i) < \iota(i+1)$ for $i = 1, \ldots, k-1$; we set $\iota'(i) := \iota(i) - 1$.

By inverting the Jacobian determinant for coordinate change where the truncated map (defined only locally around the points, lest the paths of integration differ by homotopy) is smooth,

$$\mathrm{proj}_{K_k} \circ w : \mathcal{S}^k(X) \to \mathbb{C}^g \to \mathbb{C}^k, \quad (P_1, \ldots, P_k) \mapsto \left(u_j = \sum_{i=1}^k \int_\infty^{P_i} \nu_j^{\mathrm{I}} \right)_{j \in K_k}, \tag{4.1}$$

we give an algebraic expression for vector fields that correspond to the 'partial' differentials. Assuming (4.1) invertible over an open set $\mathcal{U} \subset \mathcal{S}^k(X)$, and denoting, loosely, by

$$\partial_{u_j}, \text{ or, if typographically preferable, } \frac{\partial}{\partial u_j}, \ j \in K_k,$$

the coordinate vector field for projected coordinates $\mathbb{C}^g \to \mathbb{C}^k$, $(u_1, \ldots, u_g) \mapsto (u_{\iota(1)}, \ldots, u_{\iota(k)})$, which acts by holding constant the $u_i, i \in K_k, i \neq j$, we compute:

$$\begin{pmatrix} \partial_{u_{\iota(1)}} \\ \partial_{u_{\iota(2)}} \\ \vdots \\ \partial_{u_{\iota(k)}} \end{pmatrix} = r \begin{pmatrix} \phi_{\iota'(1)}(P_1) & \phi_{\iota'(2)}(P_1) & \cdots & \phi_{\iota'(k)}(P_1) \\ \phi_{\iota'(1)}(P_2) & \phi_{\iota'(2)}(P_2) & \cdots & \phi_{\iota'(k)}(P_2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{\iota'(1)}(P_k) & \phi_{\iota'(2)}(P_k) & \cdots & \phi_{\iota'(k)}(P_k) \end{pmatrix}^{-1} \begin{pmatrix} y_1^{r-1}\partial_{x_1} \\ y_2^{r-1}\partial_{x_2} \\ \vdots \\ y_k^{r-1}\partial_{x_k} \end{pmatrix}. \tag{4.2}$$

By letting $\partial_{v_g^{(i)}} := \partial/\partial v_g^{(i)} = (ry_i^{r-1}/\phi_{g-1}(x_i, y_i))(\partial/\partial x_i)$, this relation can be expressed by

$$\begin{pmatrix} \partial_{u_{\iota(1)}} \\ \partial_{u_{\iota(2)}} \\ \vdots \\ \partial_{u_{\iota(k)}} \end{pmatrix} = \begin{pmatrix} \phi_{\iota'(1)}(P_1)/\phi_{g-1}(P_1) & \cdots & \phi_{\iota'(k)}(P_1)/\phi_{g-1}(P_1) \\ \phi_{\iota'(1)}(P_2)/\phi_{g-1}(P_2) & \cdots & \phi_{\iota'(k)}(P_2)/\phi_{g-1}(P_2) \\ \vdots & \ddots & \vdots \\ \phi_{\iota'(1)}(P_k)/\phi_{g-1}(P_k) & \cdots & \phi_{\iota'(k)}(P_k)/\phi_{g-1}(P_k) \end{pmatrix}^{-1} \begin{pmatrix} \partial_{v_g^{(1)}} \\ \partial_{v_g^{(2)}} \\ \vdots \\ \partial_{v_g^{(k)}} \end{pmatrix}. \tag{4.3}$$

As a consequence:

LEMMA 4.1 ([**MP1**, Proposition 2.11]). *On the open set $\mathcal{U} \subset \mathcal{S}^k(X)$ and for $(P_1, \dots, P_k) \in \mathcal{U}$, $r\sum_{i=1}^k \epsilon_i(\partial/\partial u_{\iota(i)})$ is expressed by*

$$\begin{vmatrix} \phi_{\iota'(1)}(P_1) & \phi_{\iota'(2)}(P_1) & \cdots & \phi_{\iota'(k)}(P_1) \\ \phi_{\iota'(1)}(P_2) & \phi_{\iota'(2)}(P_2) & \cdots & \phi_{\iota'(k)}(P_2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{\iota'(1)}(P_k) & \phi_{\iota'(2)}(P_k) & \cdots & \phi_{\iota'(k)}(P_k) \end{vmatrix}^{-1} \begin{vmatrix} \phi_{\iota'(1)}(P_1) & \phi_{\iota'(2)}(P_1) & \cdots & \phi_{\iota'(k)}(P_1) & y_1^{r-1}\partial_{x_1} \\ \phi_{\iota'(1)}(P_2) & \phi_{\iota'(2)}(P_2) & \cdots & \phi_{\iota'(k)}(P_2) & y_2^{r-1}\partial_{x_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi_{\iota'(1)}(P_k) & \phi_{\iota'(2)}(P_k) & \cdots & \phi_{\iota'(k)}(P_k) & y_k^{r-1}\partial_{x_k} \\ \epsilon_1 & \epsilon_2 & \cdots & \epsilon_k & 0 \end{vmatrix},$$

where $(\epsilon_1, \dots, \epsilon_k)$ is any k-tuple of numbers.

LEMMA 4.2. *For the open set $\mathcal{U} \subset \mathcal{S}^k(X)$ and $(P_1, \dots, P_k) \in \mathcal{U}$, let $v^{(i)} := w(P_i)$. If $(P_1, \dots, P_{i-1}, \infty, P_{i+1}, \dots, P_k) \in \mathcal{U}$, we regard $v_j^{(i)}$ as a function of $v_g^{(i)}$ regardless of whether $g \in K_k$, $v_j^{(i)} = v_j^{(i)}(v_g^{(i)})$, and the following holds*:

$$\frac{\partial}{\partial v_j^{(i)}} = \big((v_g^{(i)})^{-N(g)+N(j-1)+2}(1 + d_>(v_g^{(i)})\big)\frac{\partial}{\partial v_g^{(i)}}.$$

PROOF. Using Lemma 2.1 and the chain rule,

$$\frac{\partial v_j^{(i)}}{\partial v_g^{(i)}} = \big(v_g^{(i)}\big)^{N(g)-N(j-1)-2}\big(1 + d_>(v_g^{(i)})\big). \qquad \square$$

In (4.1) and (4.3), we have seen differential operators with respect to the variables $u = \sum_{i=1}^k v^{(i)}$, the label $\{\iota(1), \dots, \iota(k)\}$, and the notation $u^{[k]} = \sum_{i=1}^k v^{(i)}$ and $(\partial/\partial u_{\iota(j)}^{[k]})_{j=1,\dots,k}$. Assume now that $\{\iota(1), \dots, \iota(\ell)\}$ and $\{g+\ell-k+1, \dots, g-1, g\}$ are disjoint. Then we also consider the differential operators $(\partial/\partial u_{\iota(j)}^{[\ell]})_{j=1,\dots,\ell}$ with respect to $u^{[\ell]} = \sum_{i=1}^{\ell} v^{(i)}$ for $\ell \le k$, and the differential operators $(\partial/\partial u_j^{[g;\ell]})_{j=g+\ell-k+1,\dots,g-1,g}$ with respect to $u^{[k;\ell]} = \sum_{i=\ell+1}^{k} v^{(i)}$. Now we give a transformation formula among them for a suitable open set $\mathcal{U} \subset \mathcal{S}^k(X)$, which is essentially the same as (5.11) in the proof of Lemma 5.11 and implies (5.18) in the proof of Proposition 5.26.

PROPOSITION 4.3. *For the open set $\mathcal{U} \subset \mathcal{S}^k(X)$ and $(P_1, \dots, P_k) \in \mathcal{U}$, let $v^{(i)} := w(P_i)$, $u^{[k]} := \sum_{i=1}^{k} v^{(i)}$, $u^{[\ell]} := \sum_{i=1}^{\ell} v^{(i)}$, $u^{[k;\ell]} := \sum_{i=\ell+1}^{k} v^{(i)}$, and $\iota(k-j) = g-j$ $(j = 0, \dots, k-\ell-1)$. The change of basis ${}^t\big(\partial/\partial u_{\iota(1)}^{[\ell]}, \dots, \partial/\partial u_{\iota(\ell)}^{[\ell]}, \partial/\partial u_{g+\ell-k+1}^{[g;\ell]}, \dots, \partial/\partial u_{g}^{[g;\ell]}\big)$ to ${}^t\big(\partial/\partial u_{\iota(1)}^{[k]}, \dots, \partial/\partial u_{\iota(\ell)}^{[k]}, \partial/\partial u_{\iota(\ell+1)}^{[k]}, \dots, \partial/\partial u_{\iota(k)}^{[k]}\big) = {}^t\big(\partial/\partial u_{\iota(1)}^{[k]}, \dots, \partial/\partial u_{\iota(\ell)}^{[k]}, \partial/\partial u_{g+\ell-k+1}^{[k]}, \dots, \partial/\partial u_{g}^{[k]}\big)$ is given by a matrix $\mathcal{M}_{u^{[k]},u^{[g;\ell]}}$,*

$$\begin{pmatrix} \phi_{\iota'(1)}(P_1) & \cdots & \phi_{\iota'(k)}(P_1) \\ \vdots & \ddots & \vdots \\ \phi_{\iota'(1)}(P_\ell) & \cdots & \phi_{\iota'(k)}(P_\ell) \\ \phi_{\iota'(1)}(P_{\ell+1}) & \cdots & \phi_{\iota'(k)}(P_{\ell+1}) \\ \vdots & \ddots & \vdots \\ \phi_{\iota'(1)}(P_k) & \cdots & \phi_{\iota'(k)}(P_k) \end{pmatrix}^{-1} \begin{pmatrix} \phi_{\iota'(1)}(P_1) \cdots \phi_{\iota'(\ell)}(P_1) & \\ \vdots \quad \ddots \quad \vdots & \\ \phi_{\iota'(1)}(P_\ell) \cdots \phi_{\iota'(\ell)}(P_\ell) & \\ & \phi_{g+\ell-k}(P_{\ell+1}) \cdots \phi_{g-1}(P_{\ell+1}) \\ & \vdots \quad \ddots \quad \vdots \\ & \phi_{g+\ell-k}(P_k) \cdots \phi_{g-1}(P_k) \end{pmatrix}.$$

PROOF. By considering the intermediate basis ${}^t\big(ry_1^{r-1}(\partial/\partial x_1), \dots, ry_k^{r-1}(\partial/\partial x_k)\big)$, (4.2) gives the result. □

From the expansion (2.5), we deduce the following result (we label it Lemma because it is used in the proof of Lemma 5.11):

LEMMA 4.4. *The transition matrix $\mathcal{M}_{u^{[k]},u^{[g;\ell]}}$ behaves like*

$$\mathcal{M}_{u^{[k]},u^{[g;\ell]}} = \begin{pmatrix} 1_\ell & \mathcal{C}_{k,\ell} \\ & 1_{k-\ell} \end{pmatrix} + \mathcal{M}_{>0}\big(v_g^{(\ell+1)}, \dots, v_g^{(k)}\big).$$

Here 1_ℓ is the $\ell \times \ell$ identity matrix, $\mathcal{C}_{k,\ell}$ is an $\ell \times (k-\ell)$ matrix which depends only on $u_{\iota(1)}^{[\ell]}, \dots, u_{\iota(\ell)}^{[\ell]}$, and $\mathcal{M}_{>0}(v_g^{(\ell+1)}, \dots, v_g^{(k)})$ is a $k \times k$ matrix whose entries are given by

$$\sum_{i_\ell, \dots, i_k, i_\ell + \cdots + i_k > 0} c_{i_\ell, \dots, i_k} {v_g^{(\ell+1)}}^{i_\ell} \cdots {v_g^{(k)}}^{i_k},$$

where i_j $(j = \ell, \dots, k)$ is a non-negative integer, and $c_{i_\ell, \dots, i_k}$ is a function of $u_{\iota(1)}^{[\ell]}, \dots, u_{\iota(\ell)}^{[\ell]}$.

PROOF. By letting

$$\Psi := \begin{pmatrix} \phi_{\iota'(1)}(P_1) & \cdots & \phi_{\iota'(k)}(P_1) \\ \vdots & \ddots & \vdots \\ \phi_{\iota'(1)}(P_k) & \cdots & \phi_{\iota'(k)}(P_k) \end{pmatrix},$$

$$\Psi_a := (\Psi_{i,j})_{i=1,\dots,\ell,j=1,\dots,\ell}, \quad \Psi_b := (\Psi_{i,j})_{i=1,\dots,\ell,j=\ell+1,\dots,k},$$
$$\Psi_c := (\Psi_{i,j})_{i=\ell+1,\dots,k,j=1,\dots,\ell}, \quad \Psi_d := (\Psi_{i,j})_{i=\ell+1,\dots,k,j=\ell+1,\dots,k},$$

$\mathcal{M}^{-1}_{u^{[k]},u^{[g;\ell]}}$ is given by

$$\begin{pmatrix} \Psi_a^{-1} & \\ & \Psi_d^{-1} \end{pmatrix} \begin{pmatrix} \Psi_a & \Psi_b \\ \Psi_c & \Psi_d \end{pmatrix}.$$

Since the w-degree of every entry of Ψ_d is greater than that of every entry of Ψ_c, $\Psi_d^{-1}\Psi_c$ vanishes if each P_j $(j = \ell+1,\dots,k)$ equals ∞. Indeed, if $P_j = \infty$ $(j = \ell+1,\dots,k)$ $(\mathcal{M}^{-1}_{u^{[k]},u^{[g;\ell]}})_{ij} = \delta_{ij}$ for $(i,j = 1,\dots,\ell)$ and $(i = \ell+1,\dots,k, j = 1,\dots,k)$ and thus $(\mathcal{M}_{u^{[k]},u^{[g;\ell]}})_{ij} = \delta_{ij}$ for $(i,j = 1,\dots,\ell)$ and $(i = \ell+1,\dots,k, j = 1,\dots,k)$. Further we have an expression,

$$\mathcal{M}_{u^{[k]},u^{[g;\ell]}} = \begin{pmatrix} \Psi_a & \Psi_b \\ \Psi_c & \Psi_d \end{pmatrix}^{-1} \begin{pmatrix} \Psi_a & \\ & \Psi_d \end{pmatrix}.$$

Let $\Psi^{(i,j)}$ be the minor of Ψ. By a natural extension of the w-degree with respect to P_i to $(P_{\ell+1},\dots,P_k)$ at $(\infty,\dots,\infty)$ as a multi-index (2.6), $|\Psi_a||\Psi_d|$ is the largest-degree term in the expansion of the determinant, $|\Psi| = |\Psi_a||\Psi_d| + \cdots$.

For the case $i = 1,\dots,\ell, j = \ell+1,\dots,k$, we argue as follows.

$$(\mathcal{M}_{u^{[k]},u^{[g;\ell]}})_{ij} = \sum_{i'=\ell+1}^{k} (-1)^{i+i'} \phi_{g-(k-j)-1}(P_{i'})\Psi^{(i',i)}/|\Psi|.$$

Since $\sum_{i'=1}^{k} \Psi^{-1}_{ii'}\Psi_{i'j} = \sum_{i'=1}^{k}(-1)^{i+i'}\phi_{g-(k-j)-1}(P_{i'})\Psi^{(i',i)}/|\Psi| = 0$, we have

$$(\mathcal{M}_{u^{[k]},u^{[g;\ell]}})_{ij} = -\sum_{i'=1}^{\ell}(-1)^{i+i'} \phi_{g-(k-j)-1}(P_{i'})\Psi^{(i',i)}/|\Psi|.$$

$\Psi^{(i',i)}$ is expanded as $\Psi^{(i',i)} = \Psi_a^{(i',i)}|\Psi_d| + \cdots$ and the first term has the largest degree with respect to the expansion at $(P_{\ell+1},\dots,P_k)$ at $(\infty,\dots,\infty)$ with the multi-index convention. When $P_{j'}$ approaches ∞ for every $j' = \ell+1,\dots,k$, $(\mathcal{M}_{u^{[k]},u^{[g;\ell]}})_{ij}$ becomes $-\sum_{i'=1}^{\ell}(-1)^{i+i'}\phi_{g-(k-j)-1}(P_{i'})\Psi_a^{(i',i)}/|\Psi_a|$, which is a function only of $(P_1,\dots,P_\ell)$. □

5. Vanishing of σ on Θ^k $(0 < k < g)$.

We finally give the vanishing order of σ, which we obtain directly from Proposition 3.10. Indeed, it is determined by the vanishing of the Schur function $\boldsymbol{s}_\Lambda$, which is the limit of the σ-function when X approaches the singular curve X_0. However, we can work with the θ-function of the nonsingular curve X, for which Theorems 3.4, 5.1, 5.3 hold, since we use properties of the Schur function only to obtain certain coefficients in the multivariable Taylor expansion of σ and check that they are different from zero.

We state Riemann's singularity theorem (cf. [**ACGH**, VI.1]), with the usual notation of h^i for the dimension of the cohomology space H^i:

THEOREM 5.1. *If D_k belongs to $\mathcal{S}^k(X\backslash\infty) \setminus (\mathcal{S}_1^k(X) \cap \mathcal{S}^k(X\backslash\infty))$, and we let*

$$u := \int_{k\infty}^{D_k} \nu^I,$$

$$n_k := h^0(X, D_k + (g-k-1)\infty) = \#\{\ell \mid 0 \le \ell, N(\ell) \le g-k-1\},$$

then:

1. *For every multi-index $(\alpha_1, \dots, \alpha_m)$ with $\alpha_i \in \{1, \dots, g\}$ and $m < n_k$,*

$$\frac{\partial^m}{\partial u_{\alpha_1} \dots \partial u_{\alpha_m}} \sigma(u) = 0.$$

2. *There exists a multi-index $I_\beta := (\beta_1, \dots, \beta_{n_k})$, which in general depends on D_k, such that*

$$\frac{\partial^{n_k}}{\partial u_{\beta_1} \dots \partial u_{\beta_{n_k}}} \sigma(u) \neq 0. \tag{5.1}$$

REMARK 5.2. Since the σ-function is either even or odd, in Theorem 5.1 we can replace the assumption for D_k with $D_k \in w(\mathcal{S}^k(X\backslash\infty) \setminus (\mathcal{S}_1^k(X) \cap \mathcal{S}^k(X\backslash\infty)) \cup [-1]w(\mathcal{S}^k(X\backslash\infty) \setminus (\mathcal{S}_1^k(X) \cap \mathcal{S}^k(X\backslash\infty))$, and we also extend via the operator $[-1]$ the defining set of u in the Theorem, $u \in \kappa^{-1}(\mathcal{W}^k) \subset \kappa^{-1}(\Theta^k)$, with the appropriate (extended) excluded subsets as in the Theorem.

For D_k and n_k as in Theorem 5.1, Fay [**F2**, Theorem 1.2] proved the following (cf. also [**BV**], [**SW**]):

THEOREM 5.3. *Let ν_i^+ $(0 \le \nu_1^+ < \nu_2^+ < \dots < \nu_{n_k}^+ \le g-1)$ be such that*

$$h^0(X, D_k + (g-k-\ell-1)\infty) = n_k - i + 1 \quad \text{for } \ell = \nu_i^+,$$
$$h^0(X, D_k + (g-k-\ell-1)\infty) \le n_k - i \qquad \text{for } \ell > \nu_i^+,$$

with ν_i^- defined for $[-1]D_k$ as ν_i^+ is for D_k and

$$N_k := n_k + \sum_{i=1}^{n_k} (\nu_i^+ + \nu_i^-).$$

Let $\hat{\nu^I}$ be the normalized basis of holomorphic one-forms

$$\hat{\nu^I} := {\omega'}^{-1} \nu^I.$$

For P and Q in X and $e-\omega_{\rm R}\in\Theta^k\setminus(\Theta^k_1\cup\Theta^{k-1})$ and for every $t\in\mathbb{C}$,

$$\theta\left(t\int_Q^P \hat{\nu^I}+e\right)=t^{n_k}\prod_{i=1}^{n_k}\left(\prod_{k=1}^{\nu_i^+}(t-k)\prod_{\ell=1}^{\nu_i^-}(t-\ell)\right)E(P,Q)^{N_k}\Phi(P,Q,t), \tag{5.2}$$

where $\theta(z)$ is the Riemann θ-function, $E(P,Q)$ is the prime form and $\Phi(P,Q,t)$ is an entire function of t for all $P\in X$ near Q, with

$$\Phi(P,P,t)=\frac{1}{N_k!}\sum_{i_1,\dots,i_{N_k}=1}^{g}\frac{\partial^{N_k}\theta(e)}{\partial z_{i_1}\dots\partial z_{i_{N_k}}}dz_{i_1}(P)\dots dz_{i_{N_k}}(P)\neq 0$$

where $e=(z_1,\dots,z_g)$.

Note that by $[-1]D_k$ here we mean simply a divisor linearly equivalent to $K-D_k$, so it has an order k' for which the overriding assumption of this section, $0<k'<g-1$, is not satisfied; however, the statement is ultimately concerned with the images under the Abel map. Note also that, unlike n_k, the ν's here depend *a priori* on the specific divisor D_k, but see Corollary 5.6. The following corollary follows from intersection theory as in [**BV**]. In this article, we prove it as a corollary of the above theorem.

COROLLARY 5.4. *For all $1\le k\le g-1$ (implicit in 5.1), $u^{[k]}\in\Theta^k\setminus(\Theta^k_1\cup\Theta^{k-1})$, $u^{[g]}\in\mathbb{C}^g$, $v\in\mathcal{W}^1$, and $t\in\mathbf{C}$ $(0<|t|<1)$, we have*

1. $\displaystyle\frac{\partial^\ell}{\partial {v_g}^\ell}\sigma(tv+u^{[k]})\Big|_{v=0}=0,\ \ell<N_k;\quad \frac{\partial^{N_k}}{\partial {v_g}^{N_k}}\sigma(tv+u^{[k]})\Big|_{v=0}\neq 0,$ *and*
2. $\displaystyle\frac{\partial^\ell}{\partial {u^{[g]}_g}^\ell}\sigma(u^{[g]})\Big|_{u^{[g]}=u^{[k]}}=0,\ \ell<N_k;\quad \frac{\partial^{N_k}}{\partial {u^{[g]}_g}^{N_k}}\sigma(u^{[g]})\Big|_{u^{[g]}=u^{[k]}}\neq 0.$

PROOF. We let $Q=\infty$, $e={\omega'}^{-1}u^{[k]}-\omega_{\rm R}$ in Theorem 5.3, and we let $v=\omega'\int_\infty^P\hat{\nu^I}$. We note that for a generic complex number t there exist $P_1,\dots,P_g\in X$ such that $tv=w(P_1,\dots,P_g)$, and thus $\theta(t{\omega'}^{-1}v+e)$ and $\theta({\omega'}^{-1}u^{[g]}-\omega_{\rm R})$ do not vanish in general where $u^{[g]}:=tv+u^{[k]}$. By differentiating the left-hand side of (5.2), using the chain rule and evaluating at $P=\infty$ where v is the zero vector, there exist $b_{\ell,i}\in\mathbb{C}$ satisfying

$$\frac{\partial^\ell}{\partial {v_g}^\ell}\theta(t{\omega'}^{-1}v+e)\Big|_{P=\infty}=\sum_{i=0}^{\ell}b_{\ell,i}t^i, \tag{5.3}$$

where

$$b_{\ell,\ell}:=\frac{\partial^\ell}{\partial (tv_g)^\ell}\theta(t{\omega'}^{-1}v+e)\Big|_{P=\infty}=\frac{\partial^\ell}{\partial {u^{[g]}_g}^\ell}\theta({\omega'}^{-1}u^{[g]}-\omega_{\rm R})\Big|_{u^{[g]}=u^{[k]}}.$$

This relation is obtained as follows: Since $\theta(t{\omega'}^{-1}v+e)$ is an entire function of the vector tv (ω' is an invertible coordinate change), it has an expansion $\theta(t{\omega'}^{-1}v + e) = \sum_{\beta\geq 0} a_\beta t^{|\beta|}v^\beta$, where β is a non-negative g-tuple, $\beta = (\beta_1, \dots, \beta_g)$ with the conventions $|\beta| = \beta_1 + \cdots + \beta_g$, $v^\beta = v_1^{\beta_1} \dots v_g^{\beta_g}$, and $\beta \geq 0$ for every $\beta_i \geq 0$, and each a_β is a complex number depending upon e, ω' and ω''. Now we compute up to lower-order terms in v_g. Recall that the orders of zero at ∞ of the chosen basis are decreasing (Section 2): $\deg_{\mathrm{w}^{-1}}(\nu_i^I) = 2g - N(i-1) - 2$ and $v_i = 1/(2g - N(i-1) - 1)v_g^{2g-N(i-1)-1}(1 + d_>(v_g))$ around $v = 0$. There exist coefficients $\hat{a}_\beta(e)$ and $\tilde{a}_{\beta,i}$ satisfying

$$\begin{aligned}\frac{\partial^\ell}{\partial {v_g}^\ell}\theta(t{\omega'}^{-1}v + e)\bigg|_{P=\infty} &= \frac{\partial^\ell}{\partial {v_g}^\ell}\sum_{\beta\geq 0} a_\beta t^{|\beta|}v^\beta\bigg|_{v=0} \\ &= \frac{\partial^\ell}{\partial {v_g}^\ell}\sum_{\beta\geq 0} \hat{a}_\beta t^{|\beta|} v_g^{|\mathrm{w}_g(\beta)|}\left(1 + \sum_{i=1} \tilde{a}_{\beta,i}v_g^i\right)\bigg|_{v_g=0}\end{aligned}$$

which is equal to the right-hand side of (5.3), and $(\partial^\ell/\partial(tv_g)^\ell)\theta(t{\omega'}^{-1}v + e)\Big|_{v=0} = b_{\ell,\ell} = \ell!\hat{a}_{(0,\dots,0,\ell)}$, since the conditions $|\mathrm{w}_g(\beta)| = \ell$ and $|\beta| = \ell$ mean $\beta = (0, \dots, 0, \ell)$.

We now consider the derivative of the right-hand side of (5.2) with respect to v_g, noting that $E(P, \infty) = t_\infty(1 + d_>(t_\infty))$ and $t_\infty = v_g(1 + d_>(v_g))$. Then for $\ell < N_k$, the derivative of the right-hand side of (5.2) vanishes when $P \to \infty$ or v_g vanishes. In the case that ℓ agrees with N_k, it contains a term consisting of $(\partial_{v_g} E(P, \infty))^{N_k} = (1 + d_>(t_\infty))$ times a non-zero number (use Theorem 5.3). Hence for $\ell < N_k$, we have

$$\frac{\partial^\ell}{\partial {v_g}^\ell}\theta({\omega'}^{-1}(tv + u^{[k]}) - \omega_{\mathrm{R}})\bigg|_{v=0} = 0, \quad \frac{\partial^{N_k}}{\partial {v_g}^{N_k}}\theta({\omega'}^{-1}(tv + u^{[k]}) - \omega_{\mathrm{R}})\bigg|_{v=0} \neq 0.$$

Since σ is associated to θ through multiplication by an exponential quadratic in the variable, they have the same vanishing order (the derivatives up to the order differ through multiplication by an invertible matrix), and the first statement of the Corollary is proved. Statement 2. is proved by comparing the coefficients t^ℓ in the derivative of both sides of (5.2); $b_{\ell,\ell}$ vanishes for $\ell < N_k$ and does not vanish for $\ell = N_k$. □

Remark 5.5. When computing a given number of derivatives of σ, as opposed to the order of a singular point of the theta divisor as in [**ACGH**, *loc. cit.*], we need to stay away from lower strata (recall that the derivatives of σ depend on deforming a given point along the curve). Thus, the exclusion of the point at ∞ from the divisors in $\mathcal{S}^k(X)$, and of the set Θ^{k-1}. For example, in genus 2 the locus $\mathcal{S}_1^2(X)$ is empty, whereas $\mathcal{S}^{g-1=1}(X)$ is the curve. In [**MP1**], as of Section 2 we extended the functions to be $\mathbb{P}^1$-valued, but beginning with Proposition 4.4, we excluded the special divisors $\mathcal{S}_1^k(X)$ (for which the right-hand side of [**MP1**, Proposition 4.4] could be infinite, if say $w(P) - w(P_1', \dots, P_g')$ is a zero of σ, a condition on the speciality of $\sum_{i=1}^g P_i' - P$, but $w(P) - w(P_1, \dots, P_g)$ is not, yet the points P, Q are distinct from the given poles P_i, P_j' so that the left-hand side is a finite number); here, we need (cf. Section 4, e.g.) a domain capable for a given number of points to move along the curve, so we may need positive divisors of degree less than g, thus special.

We can rephrase Theorem 5.3 as follows:

Corollary 5.6. *Let*

$$M_k := \{g - N(\ell) - k - 1 \mid g - N(\ell) - k - 1 \geq 0,\ \ell = 0, 1, \dots\},$$
$$\overline{M}_k := \{g - N(\ell + k) + k - 1 \mid g - N(\ell + k) + k - 1 \geq 0,\ \ell = 0, 1, \dots\}.$$

Then the quantities in Theorem 5.3 *are given by*

$$\{\nu_i^+\}_{1 \leq k \leq n_k} = M_k, \quad \{\nu_i^-\}_{1 \leq k \leq n_k} = \overline{M}_k,$$

and

$$N_k = \sum_{\ell=0}^{n_k - 1} (2g - N(\ell) - N(k + \ell) - 1). \tag{5.4}$$

Proof. M_k is obtained in a straightforward way. When we consider $\overline{M}_k$, we need to know the dimension of $H^0(X, [-1]D_k + (g - N(k) + k - 1)\infty)$. Let $D_k = P_1 + \cdots + P_k$ where each $P_i \in (X \backslash \infty)$. Using the notation in Definition 2.5, we have

$$H^0(X, [-1]D_k + (g - N(k) + k - 1)\infty) \ni \frac{\mu_{\ell+k}(P; P_1, \dots, P_{\ell+k})}{\mu_k(P; P_1, \dots, P_k)}, \quad \ell = 0, \dots, n_k - 1,$$

which gives $\overline{M}_k$ and N_k explicitly. □

For the expression (5.4), cf. also [**BV**], [**SW**].

Proposition 5.7. *The following numerical identity holds*:

$$N_k = \sum_{i=k}^{g-1} (N(g) - N(i) + i - g) = \sum_{i=k}^{g-1} (g - N(i) + i) = \sum_{i=k+1}^{g} \Lambda_i. \tag{5.5}$$

Proof. For a given $k > 1$, we have two cases: $n_{k-1} = n_k$, or $n_{k-1} + 1 = n_k$. The former case means that $g - k$ is the $(g - k - n_k + 1)$-th gap whereas the latter case implies that $g - k$ is in the n_k-th semigroup interval. As the dimensions of the linear systems $\mathcal{L}$ and $K_X \mathcal{L}^{-1}$ differ by the degree of $\mathcal{L}$ for any line bundle $\mathcal{L}$ (cf. Remark 2.3), in the latter case $2g - 2 - (g - k)$ is in the $(g - 1) - (g - k - n_k + 1)$-th semigroup interval. For each case we then have the relation:

$$\begin{aligned} n_{k-1} = n_k : \quad & N(k + n_k - 1) = g + k - 1, \\ n_{k-1} = n_k + 1 : \quad & N(n_k) = g - k. \end{aligned}$$

When $k = g - 1$, $n_{g-1} = 1$ and $N(0) = 1$, so (5.4) and (5.5) agree. Next, assume that for a given k the right-hand sides of the two expressions are equal. If $n_{k-1} = n_k$ the right-hand side of (5.4) for the $k-1$ case is given by

$$N_{k-1} = N_k - N(k-1) + N(k+n_k-1),$$

which is written using the above relation,

$$N_{k-1} = N_k - N(k-1) + g + k - 1.$$

When $n_{k-1} = n_k + 1$, the right-hand side of (5.4) for the $k-1$ case is

$$N_{k-1} = N_k - N(k-1) - N(n_k) + 2g - 1$$

which is also written using the above relation,

$$N_{k-1} = N_k - N(k-1) + g + k - 1.$$

We conclude that (5.4) and (5.5) are equal. □

Corollary 5.8. 1. *The number n_k is the first component of the node (n_k, m_k), $m_k := n_k + k$, encountered on the rim hook* [**S**, Definition 4.10.1] *of the diagram, in a right-to-left, up-to-down path, starting with n_{g-1} in row* 1. *For example, in the diagram of the* $(5,7)$ *curve* (*Table* 2.1) *the nodes that correspond to (n_k, m_k) are*: (1,12), (1,11), (1,10), (1,9), (1,8), (2,8), (2,7), (3,7), (3,6), (3,5), (4,5), (4,4).

2. *N_k is the number of cells in the rows of the diagram from row $k+1$ to g.*
3. *For $k = g-r, \dots, g-1$, $N_k = \deg_{w^{-1}}(u_{k+1})$ and $n_k = 1$*

Remark 5.9. Note that for the hyperelliptic case

$$N_k = (g-k)(g-k+1)/2.$$

One of the authors (S.M.) learned this relation from Victor Enolskii who discovered it by numerical computations in 2005. This turns out to be a corollary of Theorem 5.3 [**BV**], [**F2**] but the present study originated with Enolskii's communication of his discovery. Birkenhake and Vanhaecke [**BV**] showed that this number is a sum of hyperosculation degrees for embeddings of the curve into Grassmannians, defined by linear subseries of the complete linear series that defines the Weierstrass gaps at a point.

We introduce 'truncated Young diagrams' $\Lambda^{(k)} := (\Lambda_1, \dots, \Lambda_k)$ and $\Lambda^{[k]} := (\Lambda_{k+1}, \dots, \Lambda_g)$. We note that this truncated Young diagram was considered in [**BLE1**] for the $k = g-1$ case.

Corollary 5.8 gives:

Corollary 5.10. 1. $N_k = |\Lambda^{[k]}|$.

2. *n_k is the rank of the partition of $\Lambda^{[k]}$. Thus, we can read n_k in the diagram dually to corollary* 5.8 3., *by numbering k the boxes on the rim, starting with $k = g-1$ in row g; n_k is the number of boxes at the left of, and including, the one containing the*

number k.

3. *For the characteristics of the partition of $\Lambda^{[k]}$, $(a_1,\dots,a_{n_k};b_1,\dots,b_{n_k})$, $N_k = \sum_{i=1}^{n_k}(a_i+b_i+1)$.*

Bukhshtaber, Leĭkin and Ènol'skiĭ [**BLE1**] showed that the σ function over the singular curve X_0 given by $y^r = x^s$ is identified with a Schur function, cf. Proposition 3.10. Although we make use of formulas that hold for the σ-function, we implicitly go through the singular curve X_0 to pick up combinatorial results from the theory of Schur functions using Taylor expansions.

Using the Schur polynomials,

$$\boldsymbol{s}_{\Lambda^{(k)}}(t) := \frac{|t_j^{\Lambda_i+k-i}|_{1\le i,j\le k}}{|t_j^{i-1}|_{1\le i,j\le k}}, \quad \tilde{\boldsymbol{s}}_{\Lambda^{[k]}}(t) := \frac{|t_{k+j}^{\Lambda_{k+i}+g-k-i}|_{1\le i,j\le g-k}}{|t_{k+j}^{i-1}|_{1\le i,j\le g-k}},$$

and letting $T_j^{(k)} := (1/j)(t_1^j+\cdots+t_k^j) = T_j^{\langle 1,k\rangle}$, and $T_j^{(g;k)} := (1/j)(t_{k+1}^j+\cdots+t_g^j) = T_j^{\langle k+1,g\rangle}$, we define the following quantities,

$$S_{\Lambda^{(k)}}(T^{(k)}) := \boldsymbol{s}_{\Lambda^{(k)}}(t), \quad \tilde{S}_{\Lambda^{[k]}}(T^{(g;k)}) := \tilde{\boldsymbol{s}}_{\Lambda^{[k]}}(t).$$

The notation above for the left-hand sides $S_{\Lambda^{(k)}}(T^{(k)})$ and $\tilde{S}_{\Lambda^{[k]}}(T^{(k)})$ is consistent with Proposition 3.6, because they are functions of the $T^{(k)}$. For a given $(t_\ell)_{\ell=1,2,\dots,k}\in\mathbb{C}^k$, letting $u^{[k]} := (u_i^{[k]})_{i=1,\dots,g}\in\mathbb{C}^g$ be defined by $u_i^{[k]} := T^{(k)}_{\Lambda_i+g-i}$, we can also define consistently, in view of Lemma 5.11 proven below:

$$\boldsymbol{s}_{\Lambda^{(k)}}(u^{[k]}) = S_{\Lambda^{(k)}}(T^{(k)})\big|_{T^{(k)}_{\Lambda_i+g-i}=u_i^{[k]}}.$$

We also introduce the complete symmetric polynomial $h_n^{(g;k)} = h^{\langle k+1,g\rangle}$ such that $h_{n\ge1}^{(g;k)} = (-1)^{n-1}(n-1)!T_n^{(g;k)}+\cdots$, $h_0^{(g;k)}=1$ and $h_{n<0}^{(g;k)}=0$ (in particular, $h_n^{(g;k)}$ depends only on $t_{k+1},\dots,t_g$).

LEMMA 5.11. *For a given $(t_\ell)_{\ell=1,2,\dots,k}\in\mathbb{C}^k$, we let $u^{[k]} := (u_i^{[k]})_{i=1,\dots,g}\in\mathbb{C}^g$ be defined by $u_i^{[k]} := T^{(k)}_{\Lambda_i+g-i}$, and $u^{[g;k]} := (u_i^{[g;k]})_{i=1,\dots,g}\in\mathbb{C}^g$ be defined by $u_i^{[g;k]} := T^{(g;k)}_{\Lambda_i+g-i}$. For brevity, we denote by I a sequence of indices (among $\{k+1,\dots,g\}$) which may be repeated, and the notation $i\in I$ means that i runs through the sequence with given repetitions, if any; the order of the indices in the sequence is irrelevant. One such sequence for which I has the smallest number of elements n_k and the sum of the degrees in t (each variable t_ℓ having degree 1), $\sum_{i\in I}deg(u_i) = \sum_{i\in I}(\Lambda_i+g-i) = N_k$ is also minimum, is given in the proof. We use the notation: $\varepsilon_{\Lambda,I} := \varepsilon'(\prod_{i\in I}(\Lambda_i+g-i)!)^{-1}$, where ε' is a plus or minus sign depending on $\Lambda^{(g)}$ and k.*

1. *There exists some (possibly non-unique) finite sequence I (whose length may not be unique but is at least n_k) such that, by using the decomposition $u^{[g]} = u^{[k]}+u^{[g;k]}\in\mathbb{C}^g$,*

$$\boldsymbol{s}_{\Lambda^{(k)}}(u^{[k]}) = \varepsilon_{\Lambda,I}\left(\prod_{i\in I}\frac{\partial}{\partial u_i^{[g;k]}}\right)\boldsymbol{s}_{\Lambda^{(g)}}(u^{[g]})\bigg|_{u^{[g]}=u^{[k]}}.$$

In performing this partial derivative, we restrict the independent variables to a subspace $\mathbb{C}^{g-k}\subset\mathbb{C}^g$ *spanned by* $\{\partial/\partial u_i^{[g;k]}\}_{i=k+1,\ldots,g}$.

2. *There exists some (possibly non-unique) finite sequence I (whose length may not be unique but is at least n_k) such that,*

$$\boldsymbol{s}_{\Lambda^{(k)}}(u^{[k]}) = \varepsilon_{\Lambda,I}\left(\prod_{i\in I}\frac{\partial}{\partial u_i^{[g]}}\right)\boldsymbol{s}_{\Lambda^{(g)}}(u^{[g]})\bigg|_{u^{[g]}=u^{[k]}}.$$

PROOF. We treat both statements 1. and 2. together.

Using the modified Jacobi-Trudi determinant formula in Lemma 3.9,

$$\boldsymbol{s}_{\Lambda}(t_1,\ldots,t_g) := \begin{vmatrix} H_{k,k} & H_{k,g-k}\\ H_{g-k,k} & H_{g-k,g-k}\end{vmatrix}$$

where

$$\begin{aligned} H_{k,k}(t_1,\ldots,t_g) &:= (h_{\Lambda_i+j-i}(t_1,\ldots,t_g))_{1\le i,j\le k},\\ H_{k,g-k}(t_1,\ldots,t_g) &:= (h_{\Lambda_i+j-i}(t_1,\ldots,t_g))_{1\le i\le k,k+1\le j\le g},\\ H_{g-k,k}(t_{k+1},\ldots,t_g) &:= (h_{\Lambda_i+j-i}(t_{k+1},\ldots,t_g))_{k+1\le i\le g,1\le j\le k},\\ H_{g-k,g-k}(t_{k+1},\ldots,t_g) &:= (h_{\Lambda_i+j-i}(t_{k+1},\ldots,t_g))_{k+1\le i\le g,k+1\le j\le g}, \end{aligned}$$

the right-hand side of 1., without the constant $\varepsilon_{\Lambda,I}$, is given by

$$\left(\prod_{i\in I}\frac{\partial}{\partial u_i^{[g;k]}}\right)\begin{vmatrix} H_{k,k} & H_{k,g-k}\\ H_{g-k,k} & H_{g-k,g-k}\end{vmatrix}\Bigg|_{t_j=0:j=k+1,\ldots,g}. \tag{5.6}$$

Since we have $\boldsymbol{s}_{\Lambda^{(k)}}(u^{[k]}) = |H_{k,k}(t_1,\ldots,t_k)|$, we are to consider the derivative of

$$\tilde{\boldsymbol{s}}_{\Lambda^{[k]}}(t) = |H_{g-k,g-k}(t_{k+1},\ldots,t_g)|;$$

we will settle one case, in which I has the smallest number of elements and least degree in t as given in the statement.

Indeed, by producing a sequence I such that I satisfies

$$\left(\prod_{i\in I}\frac{\partial}{\partial u_i^{[g;k]}}\right)|H_{g-k,g-k}|\bigg|_{t_j=0:j=k+1,\ldots,g} = \varepsilon\prod_{i\in I}(\Lambda_i+g-i)! \tag{5.7}$$

and for any proper subsequence $J\subset I$,

$$\left(\prod_{i\in J}\frac{\partial}{\partial u_i^{[g;k]}}\right)|H_{g-k,g-k}|\Big|_{t_j=0:j=k+1,\dots,g}=0, \tag{5.8}$$

then

$$\left(\prod_{i\in I}\frac{\partial}{\partial u_i^{[g;k]}}\right)\boldsymbol{s}_\Lambda(t_1,\dots,t_g)\Big|_{t_j=0:j=k+1,\dots,g}=\varepsilon\prod_{i\in I}(\Lambda_i+g-i)!\boldsymbol{s}_\Lambda^{(k)}(u^{[k]})+\text{lower-order}. \tag{5.9}$$

The lower-order (in t) term vanishes for the following reasons: firstly, $H_{g-k,k}(t_{k+1},\dots,t_g)$ becomes the zero matrix and $H_{g-k,g-k}$ becomes a matrix whose every entry is zero except $h_0(\equiv 1)$ when $t_j=0:j=k+1,\dots,g$. The derivative lowers the order, so we have (5.7) and (5.8). The vanishing order of the entries in each column in $H_{g-k,k}$ is larger than that of the entries in $H_{g-k,g-k}$, thus the lowest-degree property of I means that any term in the lower-order part of (5.9) which comes from the derivative of an entry in $H_{g-k,k}$ in (5.9) vanishes. Secondly, due to (5.8), any term which comes from the derivative of an entry in $H_{k,k}$ or $H_{g-k,k}$ gives no contribution to the right-hand side in (5.9).

To find I as in (5.7) and (5.8), we consider the pattern:

$$s_{\Lambda^{[k]}}(t)=\begin{vmatrix} h^{(g;k)}_{\Lambda_{k+1}} & h^{(g;k)}_{\Lambda_{k+1}+1} & \cdots & h^{(g;k)}_{\Lambda_{k+1}+g-k-2} & h^{(g;k)}_{\Lambda_{k+1}+g-k-1}\\ h^{(g;k)}_{\Lambda_{k+2}-1} & h^{(g;k)}_{\Lambda_{k+2}} & \cdots & h^{(g;k)}_{\Lambda_{k+2}+g-k-3} & h^{(g;k)}_{\Lambda_{k+2}+g-k-2}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ h^{(g;k)}_{\Lambda_{g-1}-g+k+2} & h^{(g;k)}_{\Lambda_{g-1}-g+k+3} & \cdots & h^{(g;k)}_{\Lambda_{g-1}} & h^{(g;k)}_{\Lambda_{g-1}+1}\\ h^{(g;k)}_{\Lambda_{g}-g+k+1} & h^{(g;k)}_{\Lambda_{g}-g+k+2} & \cdots & h^{(g;k)}_{\Lambda_{g}-1} & h^{(g;k)}_{\Lambda_{g}} \end{vmatrix}$$

$$=\begin{vmatrix} \cdots & & \cdots & & & & & h^{(g;k)}_{\Lambda_{k+1}+g-k-1}\\ \cdots & & \cdots & & h^{(g;k)}_{\Lambda_{k+2}+g-k-\ell'} & & & \\ \cdots & & \cdots & & & & \cdots & \\ \cdots & h^{(g;k)}_0 & & & & & & \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \\ & & \cdots & h^{(g;k)}_0 & h^{(g;k)}_1 & & & \\ & & & \cdots & & h^{(g;k)}_0 & & \\ & & & \cdots & & & \ddots & \\ & & & \cdots & & & h^{(g;k)}_0 & h^{(g;k)}_1 \end{vmatrix}.$$

Since the elements in the lower-left part vanish, we can reduce the determinant to a combination of ones of smaller size. The pattern of $h_0^{(g;k)}$'s has the property that there is one situated Λ_i entries to the left of the diagonal elements in each i-th row for $i>n_k$ because of the configuration of the lower boundary of the diagram $\Lambda^{[k]}$. In the determinant calculation, a series of $h_0^{(g;k)}$ in the diagonal direction occurs in a unique position and contributes to the determinant a factor of 1. In order to omit that factor, we observe the following properties about the pattern of the matrix.

We define a sequence $(i_\ell,d_\ell)_{\ell=1,\dots,n'_k\le n_k}$ by the following conditions: $i_1=g$, $d_1=1$, $i_\ell>i_{\ell+1}$, $d_\ell>0$, and i_ℓ is the largest number satisfying $\Lambda_{i_\ell+\sum_{\ell'<\ell}d_{\ell'}}=$

$\Lambda_{i_\ell+\sum_{\ell'<\ell} d_{\ell'}+1} + d_\ell$. By using the sequence $(i_\ell, d_\ell)_{\ell=1,\dots,n_k'\le n_k}$, and denoting the elements of the matrix by a_{ij}, we define a hierarchy of submatrices starting from the above matrix whose determinant is $\boldsymbol{s}_{\Lambda^{[k]}}(t)$,

$$\begin{aligned}
H_1 &:= (a_{ij})_{1\le i,j\le g-k} = H_{g-k,g-k}(t_{k+1},\dots,t_g),\\
H_{\sum_{\ell'<\ell} d_{\ell'}+1} &:= (a_{ij})_{\sum_{\ell'<\ell} d_{\ell'}+1\le i\le \sum_{\ell'<\ell} d_{\ell'}+i_\ell-k,\ 1\le j\le i_\ell-k},\\
H_{\sum_{\ell'<\ell} d_{\ell'}+2} &:= (a_{ij})_{\sum_{\ell'<\ell} d_{\ell'}+2\le i\le \sum_{\ell'<\ell} d_{\ell'}+i_\ell-k,\ 1\le j\le i_\ell-k-1},\\
\cdots\quad &\cdots,\\
H_{\sum_{\ell'\le\ell} d_{\ell'}} &:= (a_{ij})_{\sum_{\ell'\le\ell} d_{\ell'}\le i\le \sum_{\ell'\le\ell} d_{\ell'}+i_\ell-k,\ 1\le j\le i_\ell-k-d_\ell+1}.
\end{aligned}$$

The above sequence of matrices stops when the number of entries becomes negative. As will be explained in the proof of Lemma 5.25, these matrices are directly related to the Schur functions of the Young diagrams associated with $\Lambda^{[k]}$.

For the example of the $(5,7)$ curve, when $k=4$, we have

$$H_1 = \begin{pmatrix}
h_4^{(12;4)} & h_5^{(12;4)} & h_6^{(12;4)} & h_7^{(12;4)} & h_8^{(12;4)} & h_9^{(12;4)} & h_{10}^{(12;4)} & h_{11}^{(12;4)}\\
h_2^{(12;4)} & h_3^{(12;4)} & h_4^{(12;4)} & h_5^{(12;4)} & h_6^{(12;4)} & h_7^{(12;4)} & h_8^{(12;4)} & h_9^{(12;4)}\\
h_1^{(12;4)} & h_2^{(12;4)} & h_3^{(12;4)} & h_4^{(12;4)} & h_5^{(12;4)} & h_6^{(12;4)} & h_7^{(12;4)} & h_8^{(12;4)}\\
 & h_0^{(12;4)} & h_1^{(12;4)} & h_2^{(12;4)} & h_3^{(12;4)} & h_4^{(12;4)} & h_5^{(12;4)} & h_6^{(12;4)}\\
 & & & h_0^{(12;4)} & h_1^{(12;4)} & h_2^{(12;4)} & h_3^{(12;4)} & h_4^{(12;4)}\\
 & & & & h_0^{(12;4)} & h_1^{(12;4)} & h_2^{(12;4)} & h_3^{(12;4)}\\
 & & & & & h_0^{(12;4)} & h_1^{(12;4)} & h_2^{(12;4)}\\
 & & & & & & h_0^{(12;4)} & h_1^{(12;4)}
\end{pmatrix},$$

$$H_2 = \begin{pmatrix}
h_2^{(12;4)} & h_3^{(12;4)} & h_4^{(12;4)}\\
h_1^{(12;4)} & h_2^{(12;4)} & h_3^{(12;4)}\\
 & h_0^{(12;4)} & h_1^{(12;4)}
\end{pmatrix}, \quad H_3 = \begin{pmatrix} h_1^{(12;4)} \end{pmatrix}, \quad H_4 = 0.$$

For the example of the $(7,9)$ curve, when $g=24$, $k=13$, we have

$$H_1 = \begin{pmatrix}
h_3^{(g;k)} & h_4^{(g;k)} & h_5^{(g;k)} & h_6^{(g;k)} & h_7^{(g;k)} & h_8^{(g;k)} & h_9^{(g;k)} & h_{10}^{(g;k)} & h_{11}^{(g;k)} & h_{12}^{(g;k)} & h_{13}^{(g;k)}\\
h_2^{(g;k)} & h_3^{(g;k)} & h_4^{(g;k)} & h_5^{(g;k)} & h_6^{(g;k)} & h_7^{(g;k)} & h_8^{(g;k)} & h_9^{(g;k)} & h_{10}^{(g;k)} & h_{11}^{(g;k)} & h_{12}^{(g;k)}\\
h_1^{(g;k)} & h_2^{(g;k)} & h_3^{(g;k)} & h_4^{(g;k)} & h_5^{(g;k)} & h_6^{(g;k)} & h_7^{(g;k)} & h_8^{(g;k)} & h_9^{(g;k)} & h_{10}^{(g;k)} & h_{11}^{(g;k)}\\
h_0^{(g;k)} & h_1^{(g;k)} & h_2^{(g;k)} & h_3^{(g;k)} & h_4^{(g;k)} & h_5^{(g;k)} & h_6^{(g;k)} & h_7^{(g;k)} & h_8^{(g;k)} & h_9^{(g;k)} & h_{10}^{(g;k)}\\
 & & h_0^{(g;k)} & h_1^{(g;k)} & h_2^{(g;k)} & h_3^{(g;k)} & h_4^{(g;k)} & h_5^{(g;k)} & h_6^{(g;k)} & h_7^{(g;k)} & h_8^{(g;k)}\\
 & & & & h_0^{(g;k)} & h_1^{(g;k)} & h_2^{(g;k)} & h_3^{(g;k)} & h_4^{(g;k)} & h_5^{(g;k)} & h_6^{(g;k)}\\
 & & & & & h_0^{(g;k)} & h_1^{(g;k)} & h_2^{(g;k)} & h_3^{(g;k)} & h_4^{(g;k)} & h_5^{(g;k)}\\
 & & & & & & h_0^{(g;k)} & h_1^{(g;k)} & h_2^{(g;k)} & h_3^{(g;k)} & h_4^{(g;k)}\\
 & & & & & & & h_0^{(g;k)} & h_1^{(g;k)} & h_2^{(g;k)} & h_3^{(g;k)}\\
 & & & & & & & & h_0^{(g;k)} & h_1^{(g;k)} & h_2^{(g;k)}\\
 & & & & & & & & & h_0^{(g;k)} & h_1^{(g;k)}
\end{pmatrix},$$

$$H_2 = \begin{pmatrix} h_2^{(g;k)} & h_3^{(g;k)} & h_4^{(g;k)} & h_5^{(g;k)} \\ h_1^{(g;k)} & h_2^{(g;k)} & h_3^{(g;k)} & h_4^{(g;k)} \\ h_0^{(g;k)} & h_1^{(g;k)} & h_2^{(g;k)} & h_3^{(g;k)} \\ & & h_0^{(g;k)} & h_1^{(g;k)} \end{pmatrix}, \quad H_3 = \begin{pmatrix} h_1^{(g;k)} & h_2^{(g;k)} \\ h_0^{(g;k)} & h_1^{(g;k)} \end{pmatrix}, \quad H_4 = 0.$$

It is clear that the number of H_i's is n_k and H_i contains H_j as a submatrix if $i < j$, with the convention $H_{n_k+1} = 0$. Every element $b_{d,e}$ $(d > c+1, e < c)$ of $H_i := (b_{d,e})$ vanishes unless it is in H_{i+1}. Every $(c+1, c)$ entry of H_i is equal to $h_0^{(g;k)} = 1$ if it is not in H_{i+1} and the number of $h_0^{(g;k)}$'s in H_1 is $g-k-n_k$. These facts imply that the term containing $(h_0^{(g;k)})^{g-k-n_k}$ in the expansion of the determinant given by products of all the entries with indices permuted, is given by the determinant of an $(n_k \times n_k)$ matrix, as shown:

$$|H_1| = \varepsilon''(h_0^{(g;k)})^{g-k-n_k} \begin{vmatrix} h_{\Lambda_{k+1}}^{(g;k)} & \cdots & h_{\Lambda_{k+1}+g-k-1}^{(g;k)} \\ \vdots & \cdots & \vdots \\ h_{\Lambda_{k+n_k}-n_k+1}^{(g;k)} & \cdots & h_{\Lambda_{k+n_k}+g-k-n_k}^{(g;k)} \end{vmatrix} + \cdots,$$

where ε'' is a plus or minus sign. Noting Corollary 5.10, let $(a_1, \dots, a_{n_k}; b_1, \dots, b_{n_k})$ be the partition characteristics of $\Lambda^{[k]}$ [**FH**, Section 4.1, p. 51]. We analyze the subscript j of $h_j^{(g;k)}$. The subscript of the upper-right corner H_i is given by $a_{n_k-i+1} + b_{n_k-i+1} + 1$. The subscripts of the elements on the straight line from the upper-right corner to the lower-left corner, $\Lambda_{k+1}+g-k-1, \dots, \Lambda_{k+n_k}-n_k+1$, are given by $a_{n_k}+b_{n_k}+1, \dots, a_3 + b_3 + 1, a_2 + b_2 + 1, a_1 + b_1 + 1$. The determinant is given by

$$\varepsilon' \prod_{i=1}^{n_k} h_{a_i+b_i+1}^{(g;k)} + \dots \tag{5.10}$$

where ε' is a plus or minus sign. We note that each term in the determinant has the same multidegree in the t variables, with each t_i weighing one, $\sum_{i=1}^{n_k}(a_i + b_i + 1) = N_k$. Moreover, $h_{a_i+b_i+1}^{(g;k)} = T_{a_i+b_i+1}^{(g;k)} + \cdots$. The subscript of the upper-right corner of H_1 is characterized by $\Lambda_k + g - k = a_{n_k} + b_{n_k} + 1$ and is the largest among the subscripts of the elements in H_1. In other words the first term in (5.10) is the only term that contains $T_{\Lambda_k+g-k}^{(k,g)}$ and cannot be cancelled. Hence we have

$$\left(\prod_{i=1}^{n_k} \frac{\partial}{\partial T_{a_i+b_i+1}^{(g;k)}}\right) \tilde{s}_{\Lambda^{[k]}}(t_{k+1}, \dots, t_g) = \varepsilon' \prod_{i=1}^{n_k} (a_i + b_i + 1)!.$$

Moreover, for every proper subsequence J of $\{1, 2, \dots, n_k\}$, we have

$$\left(\prod_{i\in J} \frac{\partial}{\partial T_{a_i+b_i+1}^{(g;k)}}\right) \tilde{\boldsymbol{s}}_{\Lambda^{[k]}}(t_{k+1}, \dots, t_g)\Big|_{t_j=0: j=k+1,\dots,g} = 0.$$

Since the element $h^{(g;k)}_{\Lambda_i+j-i}$ $(k+1 \le i \le g, 1 \le j \le k)$ in $H_{k,g-k}(t_{k+1},\dots,t_g)$ has sufficiently large degree in t, the contribution from $\big(\prod_{i=1}^{n_k}(\partial/\partial T^{(g;k)}_{a_i+b_i+1})\big)H_{k,g-k}(t_{k+1},\dots,t_g)$ vanishes when we compute (5.6). In other words, we have

$$\left(\prod_{i=1}^{n_k}\frac{\partial}{\partial T^{(g;k)}_{a_i+b_i+1}}\right)\boldsymbol{s}_{\Lambda^{(g)}}(t_1,\dots,t_g)\Bigg|_{t_j=0:j=k+1,\dots,g} = \varepsilon'\left(\prod_{i=1}^{n_k}(a_i+b_i+1)!\right)\boldsymbol{s}_{\Lambda^{(k)}}(t_1,\dots,t_k).$$

However, from Proposition 3.10, $\boldsymbol{s}_\Lambda(t) = \boldsymbol{s}_{\Lambda^{(g)}}(t)$ is a function only of $T_{\Lambda_j+g-j} = T^{(g)}_{\Lambda_j+g-j} = T^{(k)}_{\Lambda_j+g-j} + T^{(g;k)}_{\Lambda_j+g-j}$ for $j = 1,\dots,g$. There exists an integer ℓ_i such that $\Lambda_{\ell_i}+g-\ell_i = a_i+b_i+1$ for every i. By naming I the sequence $(\ell_1,\ell_2,\dots,\ell_{n_k})$, we obtain this way the smallest degree in t and the least number of derivatives, because, given the configuration of the H_i's, in the determinant (5.10) we have the largest number of h_0's (which equal 1 hence have degree zero). Since $\Lambda_{k+1}+g-k-1 = a_{n_k}+b_{n_k}+1$ and $a_i+b_i+1 \le a_{n_k}+b_{n_k}+1$, each element $j \in I$ belongs to $\{k+1,\dots,g\}$.

For this I, we obtain

$$\left(\prod_{i\in I}\frac{\partial}{\partial u_i^{[g;k]}}\right)\boldsymbol{s}_{\Lambda^{(g)}}(t_1,\dots,t_g)\Bigg|_{t_j=0:j=k+1,\dots,g} = \varepsilon'\left(\prod_{i\in I}(\Lambda_i+g-i)!\right)\boldsymbol{s}_{\Lambda^{(k)}}(t_1,\dots,t_k).$$

Similarly for every proper subsequence J of I,

$$\left(\prod_{i\in J}\frac{\partial}{\partial u_i^{[g;k]}}\right)\boldsymbol{s}_{\Lambda^{(g)}}(t_1,\dots,t_g)\Bigg|_{t_j=0:j=k+1,\dots,g} = 0.$$

The proof of 1. is complete.

For statement 2., we use the same sequence I.

Noting that $j \in I$ belongs to $\{k+1,\dots,g\}$, we have

$$\begin{pmatrix} du_1^{[g]} \\ du_2^{[g]} \\ \vdots \\ du_g^{[g]} \end{pmatrix} = M_T^{(g)}\begin{pmatrix} dt_1 \\ dt_2 \\ \vdots \\ dt_g \end{pmatrix}, \quad \begin{pmatrix} du_{k+1}^{[g;k]} \\ du_{k+2}^{[g;k]} \\ \vdots \\ du_g^{[g;k]} \end{pmatrix} = M_T^{(g;k)}\begin{pmatrix} dt_{k+1} \\ dt_{k+2} \\ \vdots \\ dt_g \end{pmatrix},$$

where

$$M_T^{(g)} := \begin{pmatrix} t_1^{\Lambda_1+g-2} & t_2^{\Lambda_1+g-2} & \cdots & t_g^{\Lambda_1+g-2} \\ t_1^{\Lambda_2+g-3} & t_2^{\Lambda_2+g-3} & \cdots & t_g^{\Lambda_2+g-3} \\ \vdots & \vdots & \ddots & \vdots \\ t_1^{\Lambda_{g-1}} & t_2^{\Lambda_{g-1}} & \cdots & t_g^{\Lambda_{g-1}} \\ t_1^{\Lambda_g-1} & t_2^{\Lambda_g-1} & \cdots & t_g^{\Lambda_g-1} \end{pmatrix},$$

$$M_T^{(g;k)} := \begin{pmatrix} t_{k+1}^{\Lambda_{k+1}+g-2} & t_{k+2}^{\Lambda_{k+1}+g-2} & \cdots & t_g^{\Lambda_{k+1}+g-2} \\ t_{k+1}^{\Lambda_{k+1}+g-3} & t_{k+2}^{\Lambda_{k+1}+g-3} & \cdots & t_g^{\Lambda_2+g-3} \\ \vdots & \vdots & \ddots & \vdots \\ t_{k+1}^{\Lambda_{g-1}} & t_{k+2}^{\Lambda_{g-1}} & \cdots & t_g^{\Lambda_{g-1}} \\ t_{k+1}^{\Lambda_g-1} & t_{k+2}^{\Lambda_g-1} & \cdots & t_g^{\Lambda_g-1} \end{pmatrix}.$$

Since the one-forms are given by

$$dt_j = \sum_{i=1}^{g} \left[\frac{\partial t_j}{\partial u_i^{[g]}}\right] du_i^{[g]} = \sum_{i=1}^{g} \left((M_T^{(g)})^{-1}\right)_{ji} du_i^{[g]}, \quad dt_j = \sum_{i=k+1}^{g} \left((M_T^{(g;k)})^{-1}\right)_{ji} du_i^{[g]},$$

we have

$$\frac{\partial}{\partial u_i^{[g]}} = \sum_{j=1}^{g} \left[\frac{\partial t_j}{\partial u_i^{[g]}}\right] \frac{\partial}{\partial t_j} = \sum_{j=1}^{g} \left((M_T^{(g)})^{-1}\right)_{ji} \frac{\partial}{\partial t_j}, \quad \frac{\partial}{\partial u_i^{[g;k]}} = \sum_{j=k+1}^{g} \left((M_T^{(g;k)}\right)^{-1})_{ji} \frac{\partial}{\partial t_j}.$$

We note that the $\partial/\partial u_i^{[g;k]}$ span a $\mathbb{C}^{g-k}$. $((M_T^{(g)})^{-1})_{ji}$ is given by

$$\left((M_T^{(g)})^{-1}\right)_{ji} = (-1)^{i+j} \frac{\begin{vmatrix} t_1^{\Lambda_1+g-2} & \cdots & \check{t}_j^{\Lambda_1+g-2} & \cdots & t_g^{\Lambda_1+g-2} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ t_1^{\Lambda_{i-1}+g-i} & \cdots & \check{t}_j^{\Lambda_{i-1}+g-i} & \cdots & t_g^{\Lambda_{i-1}+g-i} \\ t_1^{\Lambda_{i+1}+g-i-2} & \cdots & \check{t}_j^{\Lambda_{i+1}+g-i-2} & \cdots & t_g^{\Lambda_{i+1}+g-i-2} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ t_1^{\Lambda_g-1} & \cdots & \check{t}_j^{\Lambda_g-1} & \cdots & t_g^{\Lambda_g-1} \end{vmatrix}}{\begin{vmatrix} t_1^{\Lambda_1+g-2} & t_2^{\Lambda_1+g-2} & \cdots & t_g^{\Lambda_1+g-2} \\ t_1^{\Lambda_2+g-3} & t_2^{\Lambda_2+g-3} & \cdots & t_g^{\Lambda_2+g-3} \\ \vdots & \vdots & \ddots & \vdots \\ t_1^{\Lambda_{g-1}} & t_2^{\Lambda_{g-1}} & \cdots & t_g^{\Lambda_{g-1}} \\ t_1^{\Lambda_g-1} & t_2^{\Lambda_g-1} & \cdots & t_g^{\Lambda_g-1} \end{vmatrix}}.$$

We claim that for a symmetric function $h(u^{[g]})$, and subsequence $J \subset I$, we have

$$\left(\prod_{i\in J} \frac{\partial}{\partial u_i^{[g]}}\right) h(u^{[g]})\Big|_{t_{k+1}=0,\ldots,t_g=0} = \left(\prod_{i\in J} \frac{\partial}{\partial u_i^{[g;k]}}\right) h(u^{[g]})\Big|_{t_{k+1}=0,\ldots,t_g=0}. \tag{5.11}$$

This is essentially the same as Lemma 4.4 but we prove (5.11) directly as follows: Let $t_{k+1}, \ldots, t_g$ have the same order ϵ, written as $\epsilon_i = t_i$, $i = k+1, \ldots, g$.

We use the property of the chosen sequence I, namely that every $i \in I$ satisfies $k < i \leq g$. Then for $1 \leq j \leq k$, let $\Xi_{k,g,i} := \sum_{\ell=k,\ell\neq i}^{g} (\Lambda_\ell + g - \ell - 1)$ and $\Xi_{k+1,g} := \sum_{\ell=k+1}^{g} (\Lambda_\ell + g - \ell - 1)$. For $1 \leq j \leq k$, noting that $\Xi_{k,g,i} > \Xi_{k+1,g}$, we have

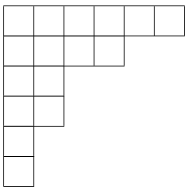
$$\left((M_T^{(g)})^{-1}\right)_{ji} = (-1)^{i+j} \frac{\begin{vmatrix} t_1^{\Lambda_1+g-2} & \cdots & \check{t}_j^{\Lambda_1+g-2} & \cdots & t_k^{\Lambda_1+g-2} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ t_1^{\Lambda_{k-1}+g-k} & \cdots & \check{t}_j^{\Lambda_{k-1}+g-k} & \cdots & t_g^{\Lambda_{k-1}+g-k} \end{vmatrix} d_{\geq}(\epsilon^{\Xi_{k,i,g}}) + d_{>}(\epsilon^{\Xi_{k,i,g}})}{\begin{vmatrix} t_1^{\Lambda_1+g-2} & \cdots & t_k^{\Lambda_1+g-2} \\ \vdots & \ddots & \vdots \\ t_1^{\Lambda_k+g-k-1} & \cdots & t_k^{\Lambda_k+g-k-1} \end{vmatrix} d_{\geq}(\epsilon^{\Xi_{k+1,g}}) + d_{>}(\epsilon^{\Xi_{k+1,g}})}$$

which vanishes when ϵ vanishes. Here $d_>(z^\ell) \in \{\sum_{|\alpha|>\ell} a_\alpha z^\alpha\} d_\geq(z^\ell) \in \{\sum_{|\alpha|\geq\ell} a_\alpha z^\alpha\}$ for $\alpha = (\alpha_1, \dots, \alpha_m)$, $z^\alpha = z_1^{\alpha_1} \cdots z_g^{\alpha_m}$, and $|\alpha| = \alpha_1 + \cdots + \alpha_m$. Then for $k+1 \leq j \leq g$, let $\Xi_{k+1,g,i} := \sum_{\ell=k+1, \ell\neq i}^{g} (\Lambda_\ell + g - \ell - 1)$. For $k+1 \leq j \leq g$, noting $\Xi_{k+1,g,i} < \Xi_{k+1,g}$, we have

$$\left((M_T^{(g)})^{-1}\right)_{ji} = (-1)^{i+j} \frac{\begin{vmatrix} t_1^{\Lambda_1+g-2} & \cdots & t_k^{\Lambda_1+g-2} \\ \vdots & \ddots & \vdots \\ t_1^{\Lambda_k+g-k-1} & \cdots & t_k^{\Lambda_k+g-k-1} \end{vmatrix} d_{\geq}(\epsilon^{\Xi_{k+1,i,g}}) + d_{>}(\epsilon^{\Xi_{k+1,i,g}})}{\begin{vmatrix} t_1^{\Lambda_1+g-2} & \cdots & t_k^{\Lambda_1+g-2} \\ \vdots & \ddots & \vdots \\ t_1^{\Lambda_k+g-k-1} & \cdots & t_k^{\Lambda_k+g-k-1} \end{vmatrix} d_{\geq}(\epsilon^{\Xi_{k+1,g}}) + d_{>}(\epsilon^{\Xi_{k+1,g}})}$$

which is singular for small ϵ. Its leading term becomes $((M_T^{(g;k)})^{-1})_{ji}$. This means that we have (5.11) and we proved the second statement.

In our last example, to give a visual description of the general pattern, rather than the (5,7) or (7,9) cases given above which would occupy several pages, we treat the trigonal (3,7) case for $k = 2$:



$$, \quad H_1 = \begin{pmatrix} h_2^{(6,2)} & h_3^{(6,2)} & h_4^{(6,2)} & h_5^{(6,2)} \\ h_1^{(6,2)} & h_2^{(6,2)} & h_3^{(6,2)} & h_4^{(6,2)} \\ & h_0^{(6,2)} & h_1^{(6,2)} & h_2^{(6,2)} \\ & & h_0^{(6,2)} & h_1^{(6,2)} \end{pmatrix}, \quad H_2 = \left(h_1^{(6,2)}\right), \quad H_3 = \emptyset.$$

Then the transition matrices expand as follows:

$$M_T^{(6)} = \begin{pmatrix} t_1^{10} & t_2^{10} & \epsilon_3^{10} & \epsilon_4^{10} & \epsilon_5^{10} & \epsilon_6^{10} \\ t_1^{7} & t_2^{7} & \epsilon_3^{7} & \epsilon_4^{7} & \epsilon_5^{7} & \epsilon_6^{7} \\ t_1^{4} & t_2^{4} & \epsilon_3^{4} & \epsilon_4^{4} & \epsilon_5^{4} & \epsilon_6^{4} \\ t_1^{3} & t_2^{3} & \epsilon_3^{3} & \epsilon_4^{3} & \epsilon_5^{3} & \epsilon_6^{3} \\ t_1 & t_2 & \epsilon_3 & \epsilon_4 & \epsilon_5 & \epsilon_6 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$(M_T^{(6)})^{-1})_{1,6} = -\frac{|t_2^{10}| d_{\geq}(\epsilon^{7+4+3+1}) + d_{>}(\epsilon^{15})}{\begin{vmatrix} t_1^{10} & t_2^{10} \\ t_1^{7} & t_2^{7} \end{vmatrix} d_{\geq}(\epsilon^{4+3+1+0}) + d_{>}(\epsilon^{8})},$$

$$(M_T^{(6)})^{-1})_{2,6} = \frac{|t_1^{10}| d_{\geq}(\epsilon^{7+4+3+1}) + d_{>}(\epsilon^{15})}{\begin{vmatrix} t_1^{10} & t_2^{10} \\ t_1^{7} & t_2^{7} \end{vmatrix} d_{\geq}(\epsilon^{4+3+1+0}) + d_{>}(\epsilon^{8})},$$

$$(M_T^{(6)})^{-1})_{3,6} = -\frac{\begin{vmatrix} t_1^{10} & t_2^{10} \\ t_1^{7} & t_2^{7} \end{vmatrix} \begin{vmatrix} \epsilon_4^4 & \epsilon_5^4 & \epsilon_6^4 \\ \epsilon_4^3 & \epsilon_5^3 & \epsilon_6^3 \\ \epsilon_4^1 & \epsilon_5^1 & \epsilon_6^1 \end{vmatrix} + d_{>}(\epsilon^{8})}{\begin{vmatrix} t_1^{10} & t_2^{10} \\ t_1^{7} & t_2^{7} \end{vmatrix} \begin{vmatrix} \epsilon_3^4 & \epsilon_4^4 & \epsilon_5^4 & \epsilon_6^4 \\ \epsilon_3^3 & \epsilon_4^3 & \epsilon_5^3 & \epsilon_6^3 \\ \epsilon_3^1 & \epsilon_4^1 & \epsilon_5^1 & \epsilon_6^1 \\ 1 & 1 & 1 & 1 \end{vmatrix} + d_{>}(\epsilon^{8})},$$

$$(M_T^{(6)})^{-1})_{4,6} = \frac{\begin{vmatrix} t_1^{10} & t_2^{10} \\ t_1^{7} & t_2^{7} \end{vmatrix} \begin{vmatrix} \epsilon_3^4 & \epsilon_5^4 & \epsilon_6^4 \\ \epsilon_3^3 & \epsilon_5^3 & \epsilon_6^3 \\ \epsilon_3^1 & \epsilon_5^1 & \epsilon_6^1 \end{vmatrix} + d_{>}(\epsilon^{8})}{\begin{vmatrix} t_1^{10} & t_2^{10} \\ t_1^{7} & t_2^{7} \end{vmatrix} \begin{vmatrix} \epsilon_3^4 & \epsilon_4^4 & \epsilon_5^4 & \epsilon_6^4 \\ \epsilon_3^3 & \epsilon_4^3 & \epsilon_5^3 & \epsilon_6^3 \\ \epsilon_3^1 & \epsilon_4^1 & \epsilon_5^1 & \epsilon_6^1 \\ 1 & 1 & 1 & 1 \end{vmatrix} + d_{>}(\epsilon^{8})}, \quad \cdots .$$

$$(M_T^{(6)})^{-1})_{1,3} = \frac{|t_2^{10}| d_{\geq}(\epsilon^{7+3+1+0}) + d_{>}(\epsilon^{11})}{\begin{vmatrix} t_1^{10} & t_2^{10} \\ t_1^{7} & t_2^{7} \end{vmatrix} d_{\geq}(\epsilon^{4+3+1+0}) + d_{>}(\epsilon^{8})},$$

$$(M_T^{(6)})^{-1})_{2,3} = -\frac{|t_1^{10}| d_{\geq}(\epsilon^{7+3+1+0}) + d_{>}(\epsilon^{11})}{\begin{vmatrix} t_1^{10} & t_2^{10} \\ t_1^{7} & t_2^{7} \end{vmatrix} d_{\geq}(\epsilon^{4+3+1+0}) + d_{>}(\epsilon^{8})},$$

$$(M_T^{(6)})^{-1})_{3,3} = \frac{\begin{vmatrix} t_1^{10} & t_2^{10} \\ t_1^{7} & t_2^{7} \end{vmatrix} \begin{vmatrix} \epsilon_4^3 & \epsilon_5^3 & \epsilon_6^3 \\ \epsilon_4^1 & \epsilon_5^1 & \epsilon_6^1 \\ 1 & 1 & 1 \end{vmatrix} + d_{>}(\epsilon^{4})}{\begin{vmatrix} t_1^{10} & t_2^{10} \\ t_1^{7} & t_2^{7} \end{vmatrix} \begin{vmatrix} \epsilon_3^4 & \epsilon_4^4 & \epsilon_5^4 & \epsilon_6^4 \\ \epsilon_3^3 & \epsilon_4^3 & \epsilon_5^3 & \epsilon_6^3 \\ \epsilon_3^1 & \epsilon_4^1 & \epsilon_5^1 & \epsilon_6^1 \\ 1 & 1 & 1 & 1 \end{vmatrix} + d_{>}(\epsilon^{8})},$$

$$(M_T^{(6)})^{-1})_{4,3} = -\frac{\begin{vmatrix} t_1^{10} & t_2^{10} \\ t_1^{7} & t_2^{7} \end{vmatrix} \begin{vmatrix} \epsilon_3^3 & \epsilon_5^3 & \epsilon_6^3 \\ \epsilon_3^1 & \epsilon_5^1 & \epsilon_6^1 \\ 1 & 1 & 1 \end{vmatrix} + d_{>}(\epsilon^{4})}{\begin{vmatrix} t_1^{10} & t_2^{10} \\ t_1^{7} & t_2^{7} \end{vmatrix} \begin{vmatrix} \epsilon_3^4 & \epsilon_4^4 & \epsilon_5^4 & \epsilon_6^4 \\ \epsilon_3^3 & \epsilon_4^3 & \epsilon_5^3 & \epsilon_6^3 \\ \epsilon_3^1 & \epsilon_4^1 & \epsilon_5^1 & \epsilon_6^1 \\ 1 & 1 & 1 & 1 \end{vmatrix} + d_{>}(\epsilon^{8})}, \quad \cdots . \qquad \square$$

The above proof yields the following Proposition:

PROPOSITION 5.12. *For the Young diagram Λ associated with the $C_{r,s}$ curve X of genus g, an integer k $(0 \le k < g)$, and the characteristics of the partition of $\Lambda^{[k]}$,*

$$(a_1, a_2, \dots, a_{n_k}; b_1, b_2, \dots, b_{n_k}),$$

the following holds:

1. *There exists an integer ℓ_i such that*

$$\Lambda_{\ell_i} + g - \ell_i = a_i + b_i + 1$$

   for every $i = 0, 1, \dots, n_k$;
2. *When the correspondence is denoted by*

$$L^{[k]}(a_i, b_i) := \ell_i,$$

   an example of I appearing in Lemma 5.11 2. is given by

$$I = \{L^{[k]}(a_1, b_1), L^{[k]}(a_2, b_2), \dots, L^{[k]}(a_{n_k}, b_{n_k})\};$$

3. *$L^{[k]}(a_{n_k}, b_{n_k}) = k + 1$, and*
4. *When the C_{rs} curve is hyperelliptic of genus g, i.e., $(r, s) = (2, 2g + 1)$, the set of indices I is equal to*

$$\begin{cases} \{g, g-2, \dots, k+2, k\} & \text{if } g-k \text{ is even,} \\ \{g-1, g-3, \dots, k+3, k+1\} & \text{otherwise.} \end{cases}$$

PROOF. The proof of Lemma 5.11 gives 1. and 2.; 3. is proved using the definition in 2. and the equality $a_{n_k} + b_{n_k} + 1 = \Lambda_{k+1} + g - k - 1$. 4. is obtained by straightforward computation. □

Note that since I in Proposition 5.12 4. corresponds to $\natural_k$ in Theorem 1.1, such I's are shown in Table 1.1.

For use as in Lemma 5.11, we define a family of sequences which we name Index.

DEFINITION 5.13. Let Index be the family of all finite sequences made up with numbers between 1 and g (some numbers may be repeated), though changing the order of the elements in a sequence would not change the values defined herewith for a given element of Index. For an element I_k of Index and $u \in \mathbb{C}^g$, define:

$$\sigma_{I_k} := \left(\prod_{i \in I_k} \frac{\partial}{\partial u_i} \right) \sigma,$$

$$\deg_{w^{-1}}(I_k) := \sum_{i \in I_k} \deg_{w^{-1}}(u_i).$$

In view of Proposition 5.12, we construct a set of indices as a natural extension of those in [**Ô1**], [**Ô2**], [**MÔ**].

Definition 5.14. For $k = 1, 2, \dots, g-1$, and the characteristics of the partition of $\Lambda^{[k]}$, $(a_1, \dots, a_r; b_1, \dots, b_r)$, we define

$$\natural_k := \left\{ L^{[k]}(a_1, b_1), L^{[k]}(a_2, b_2), \dots, L^{[k]}(a_{n_k}, b_{n_k}) \right\},$$

and

$$\natural_k^{(i)} := (\natural_k \setminus \{k+1\}) \bigcup \{i\}, \quad \text{for} \quad i = 1, 2, \dots, k.$$

Further, $\natural_g := \emptyset$ and $\natural_g^{(i)} := i$ for $i = 1, 2, \dots, g$.

We continue the examples of Tables 2.1, 2.2 (with (n_k, m_k) corresponding to k in Corollary 5.8 1.) in Table 5.1 for the case $(r, s) = (5, 7)$ and in Table 5.2 for the case $(r, s) = (7, 9)$.

Table 5.1a

i	0	1	2	3	4	5	6	7	8	9	10	11	12
$\phi(i)$	1	x	y	x^2	xy	y^2	x^3	x^2y	xy^2	x^4	y^3	x^3y	x^2y^2
$N(i)$	0	5	7	10	12	14	15	17	19	20	21	22	24
Λ_i	-	12	8	7	5	4	3	3	2	1	1	1	1
$\Lambda_i + g - i$	-	23	18	16	13	11	9	8	6	4	3	2	1
n_i	4	4	3	3	3	2	2	1	1	1	1	1	-
N_i	48	36	28	21	16	12	9	6	4	3	2	1	-

Table 5.1b

k	$(a_1, \dots, a_{n_k}; b_0, \dots, b_{n_k})$	$(a_i + b_i + 1)_{1 \le i \le n_k}$	$\sum(a_i + b_i + 1)$	$\natural_k$
0	$(1,4,6,11;1,4,6,11)$	$(3,9,13,23)$	48	$(10,6,4,1)$
1	$(0,3,5,10;0,2,5,7)$	$(1,6,11,18)$	36	$(12,8,5,2)$
2	$(2,4,9;1,3,6)$	$(4,8,16)$	28	$(9,7,3)$
3	$(1,3,8;0,2,4)$	$(2,6,13)$	21	$(11,8,4)$
4	$(0,2,7;0,1,3)$	$(1,4,11)$	16	$(12,9,5)$
5	$(1,6;1,2)$	$(3,9)$	12	$(10,6)$
6	$(0,5;0,2)$	$(1,8)$	9	$(12,7)$
7	$(4;1)$	(6)	6	(8)
8	$(3;0)$	(4)	4	(9)
9	$(2;0)$	(3)	3	(10)
10	$(1;0)$	(2)	2	(11)
11	$(0;0)$	(1)	1	(12)

Table 5.2a

i	0	1	2	3	4	5	6	7	8	9	10	11	12
$\phi(i)$	1	x	y	x^2	xy	y^2	x^3	x^2y	xy^2	y^3	x^4	x^3y	x^2y^2
$N(i)$	0	7	9	14	16	18	21	23	25	27	28	30	32
Λ_i	-	24	18	17	13	12	11	9	8	7	6	6	5
$\Lambda_i + g - i$	-	47	40	38	33	31	29	26	24	22	20	19	18
n_i	8	7	7	6	6	6	5	5	4	4	3	3	3
N_i	160	136	118	101	88	76	65	56	48	41	35	29	24
i	13	14	15	16	17	18	19	20	21	22	23	24	
$\phi(i)$	xy^3	x^5	y^4	x^4y	x^3y^2	x^2y^3	x^6	xy^4	x^5y	y^5	x^4y^2	x^2y^4	
$N(i)$	34	35	36	37	39	41	42	43	44	45	46	48	
Λ_i	4	3	3	3	3	2	1	1	1	1	1	1	
$\Lambda_i + g - i$	15	13	12	11	10	8	6	5	4	3	2	1	
n_i	3	3	2	2	1	1	1	1	1	1	1	-	
N_i	20	17	14	11	8	6	5	4	3	2	1	-	

Table 5.2b

k	$(a_1, \dots, a_{n_k}; b_0, \dots, b_{n_k})$	$(a_i + b_i + 1)_{1 \le i \le n_k}$	$\sum(a_i + b_i + 1)$	$\natural_k$
0	(0, 2, 5, 7, 9, 14, 16, 23; 0, 2, 5, 7, 9, 14, 16, 23)	(1, 5, 11, 15, 19, 29, 33, 47)	160	(24, 20, 16, 12, 11, 6, 4, 1)
1	(1, 4, 6, 8, 13, 15, 22; 1, 3, 6, 8, 10, 15, 17)	(3, 8, 13, 17, 24, 31, 40)	136	(22, 18, 14, 12, 8, 5, 2)
2	(0, 3, 5, 7, 12, 14, 21; 0, 2, 4, 7, 9, 11, 16)	(1, 6, 10, 15, 22, 26, 38)	118	(24, 19, 17, 13, 9, 7, 3)
3	(2, 4, 6, 11, 13, 20; 1, 3, 5, 8, 10, 12)	(4, 8, 12, 20, 24, 33)	101	(21, 18, 15, 10, 8, 4)
4	(1, 3, 5, 10, 12, 19; 0, 2, 4, 6, 9, 11)	(2, 6, 10, 17, 22, 31)	88	(23, 19, 17, 12, 9, 5)
5	(0, 2, 4, 9, 11, 18; 0, 1, 3, 5, 7, 10)	(1, 4, 8, 15, 19, 29)	76	(24, 21, 18, 13, 11, 6)
6	(1, 3, 8, 10, 17; 1, 2, 4, 6, 8)	(2, 6, 14, 17, 26)	65	(22, 19, 14, 12, 7)
7	(0, 2, 7, 9, 16; 0, 2, 3, 5, 7)	(1, 5, 11, 15, 24)	56	(24, 20, 16, 15, 8)
8	(1, 6, 8, 15; 1, 3, 4, 6)	(3, 10, 13, 22)	48	(22, 17, 14, 9)
9	(0, 5, 7, 14; 0, 2, 4, 5)	(1, 8, 12, 20)	41	(24, 18, 15, 10)
10	(4, 6, 13; 1, 3, 5)	(6, 10, 19)	35	(19, 17, 11)
11	(3, 5, 12; 0, 2, 4)	(17, 8, 4)	29	(21, 18, 12)
12	(2, 4, 11; 0, 1, 3)	(3, 6, 15)	24	(22, 19, 13)
13	(1, 3, 10; 0, 1, 2)	(2, 5, 13)	20	(23, 20, 14)
14	(0, 2, 9; 0, 1, 2)	(1, 4, 12)	17	(24, 21, 15)
15	(1, 8; 1, 2)	(3, 11)	14	(22, 16)
16	(0, 7; 0, 2)	(1, 10)	11	(24, 17)
17	(6; 1)	(8)	8	(18)
18	(5; 0)	(6)	6	(19)
19	(4; 0)	(5)	5	(20)
20	(3; 0)	(4)	4	(21)
21	(2; 0)	(3)	3	(22)
22	(1; 0)	(2)	2	(23)
23	(0; 0)	(1)	1	(24)

We can now state the main theorem (cf. Theorem 1.1 and Table 1.1):

THEOREM 5.15. *Let $\mathcal{I}_g = \{\emptyset\}$. For each $k = 1, 2, \dots, g$, there exists a subfamily of* Index, *$\mathcal{I}_k$, of cardinality n_k, whose element I_k is such that* $\deg_{\mathrm{w}^{-1}}(I_k) \ge N_k$, *and as a function over $\kappa^{-1}(\Theta^k \setminus (\Theta^k_1 \cup \Theta^{k-1}))$,*

$$\sigma_{J_k} = \begin{cases} \neq 0 & \text{for } J_k = I_k \\ = 0 & \text{for } J_k \subsetneqq I_k. \end{cases} \tag{5.12}$$

Moreover, $\{\natural_k, \natural_k^{(k)}, \natural_k^{(k-1)}, \dots, \natural_k^{(2)}, \natural_k^{(1)}\} \subset \mathcal{I}_k$.

REMARK 5.16. The property $\sigma_{\natural_k} \neq 0$ over $\kappa^{-1}(\Theta^k \setminus (\Theta^k_1 \cup \Theta^{k-1}))$ is the generalization of Ônishi's results in [**Ô1**], [**Ô2**], [**MÔ**]. Further we note that there exists $J_k \in$ Index such that $\#J_k = n_k$ but $\sigma_{J_k}(u) = 0$ for $u \in \kappa^{-1}(\Theta^k \setminus (\Theta^k_1 \cup \Theta^{k-1}))$.

Remark 5.17. Using Proposition 3.2, we have the following corollary, which shows $\sigma_{J_{k,i}}$ is also a normalized theta function over $\kappa^{-1}(\Theta^k \setminus (\Theta^k_1 \cup \Theta^{k-1}))$, cf. Remark 3.3:

Corollary 5.18. *For* $u \in \kappa^{-1}(\Theta^k \setminus (\Theta^k_1 \cup \Theta^{k-1}))$, ℓ_{a} $(= 2\omega'\ell_{\mathrm{a}}{}' + 2\omega''\ell_{\mathrm{a}}{}'') \in \Pi$, *and* $J_k \in \mathcal{I}_k$, *we have*

$$\sigma_{J_k}(u + \ell_{\mathrm{a}}) = \sigma_{J_k}(u) \exp\left(L\left(u + \frac{1}{2}\ell_{\mathrm{a}}, \ell_{\mathrm{a}}\right)\right)\chi(\ell_{\mathrm{a}}). \tag{5.13}$$

Proof. After we apply the differential operators $\prod_{i\in I_k}(\partial/\partial u_i)$ on both sides of the equality in Proposition 3.2, we restrict the domain to $\kappa^{-1}(\Theta^k \setminus (\Theta^k_1 \cup \Theta^{k-1}))$. Then by Theorem 5.15, the terms containing the lower-order derivatives of σ vanish and the equality follows. □

The former part of Theorem 5.15 is the same as Riemann's singularity theorem in Theorem 5.1. The latter part, which gives a specific subset of $\mathcal{I}_k$, is new and we will show it as follows.

Lemma 5.19. *For* $g - r - 1 \le k \le g - 1$, $\mathcal{I}_k = \{\{1\}, \{2\}, \dots, \{k + 1\}\}$.

Proof. Given that σ is even or odd, the analysis of $u^{[k]} \in \kappa^{-1}(\Theta^k\setminus(\Theta^k_1\cup\Theta^{k-1}))$ is essentially reduced to that of $u^{[k]} \in \kappa^{-1}(\mathcal{W}^k\setminus(\mathcal{W}^k_1\cup\mathcal{W}^{k-1}))$. We consider $u = u^{[g-1]}+v \in \kappa^{-1}(\Theta^g \setminus (\Theta^g_1 \cup \Theta^{g-1}))$ where $u^{[g-1]} \in \kappa^{-1}(\Theta^{g-1} \setminus (\Theta^{g-1}_1 \cup \Theta^{g-2}))$. By Theorem 5.1 and Corollary 5.8, $n_k = 1$ and there exists j such that $\sigma_j(u^{[g-1]})$ is not identically zero. From Theorem 3.4, $\sigma_i(u^{[g-1]}) = (-1)^{g-i}\mu_{g-1,i-1}(u^{[g-1]})\sigma_g(u^{[g-1]})$ and thus $\sigma_j(u^{[g-1]})$ does not vanish identically for $j = 1, \dots, g$. Similarly for $g-r-1 \le k \le g-1$, $N_{k-1} = \deg_{\mathrm{w}^{-1}}(u_k)$ for $k = g - r - 1, \dots, g - 1$. Thus for $u = u^{[k-1]} + v \in \kappa^{-1}(\Theta^k \setminus (\Theta^k_1 \cup \Theta^{k-1}))$ and $u^{[k-1]} \in \kappa^{-1}(\Theta^{k-1} \setminus (\Theta^{k-1}_1 \cup \Theta^{k-2}))$, we conclude that $\sigma_i(u^{[k-1]})$ does not identically equal zero $(i = 1, \dots, k)$ because $\sigma_i(u^{[k-1]}) = (-1)^{k-i+1}\mu_{k-1,i-1}(u^{[k-1]})\sigma_k(u^{[k-1]})$. □

Lemma 5.20. *For* $k < g$, $\mathcal{I}_k$ *contains an element* I_k *for which* $\ell \in I_k$, $\ell = 1, 2, \dots, k + 1$. *If a finite sequence* I_k *consists only of elements of* $\{k + 2, \dots, g\}$ *and* $\#I_k = n_k$, *then it does not belong to* $\mathcal{I}_k$.

Proof. The statement is obvious for $g - r \le k$. We thus consider $k < g - r$ and $n_k \ge 2$. Let us assume that every $I_k \in \mathcal{I}_k$ doesn't contain $k+1$. Let $u^{[k]} = u^{[k-1]}+v^{(k)} \in \kappa^{-1}$ $(\Theta^k \setminus (\Theta^k_1 \cup \Theta^{k-1}))$. The assumption means that for every $J_k \in$ Index such that $\#J_k = n_k - 1$, $\sigma_{\{k+1\}\cup J_k}(u^{[k-1]})$ vanishes. Since L'Hospital's theorem and Theorem 3.4 show

$$\begin{aligned} &\sigma_{\{i\}\cup J_k}(u^{[k]}) = (-1)^{k-i+1}\mu_{k,i-1}(u^{[k]})\sigma_{\{k+1\}\cup J_k}(u^{[k-1]}), \quad \text{for } i \le k \\ &\sigma_{\{i\}\cup J_k}(u^{[k]}) = 0 \times \sigma_{\{k+1\}\cup J_k}(u^{[k]}), \quad \text{for } i > k, \end{aligned} \tag{5.14}$$

every $\sigma_{\{i\}\cup J_k}(u^{[k]})$ vanishes for every $i = 1, 2, \dots, g$. This contradicts Theorem 5.1. Thus $\sigma_{\{k+1\}\cup J_k}(u^{[k-1]})$ cannot vanish identically and the statements are proved. □

Lemma 5.21. *For $k < g$, the sets $\natural_k$ and $\natural_k^{(i)}$ for $i = 1, 2, \ldots, k$ belong to $\mathcal{I}_k$.*

Proof. Corollary 5.10 shows that $\#\natural_k = n_k$ and $\deg_{\mathrm{w}^{-1}} \natural_k = N_k$. L'Hospital's theorem and Theorem 3.4 show that $\natural_k^{(i)}$ for $i = 1, 2, \cdots, k$ belongs to $\mathcal{I}_k$ if $\natural_k$ does. Proposition 3.10, the expansion (2.5), and Lemma 5.11 (2) imply that, as a function over $\kappa^{-1}(\Theta^k \setminus (\Theta_1^k \cup \Theta^{k-1})$,

$$\sigma_{\natural_k} \neq 0, \tag{5.15}$$

hence $\natural_k$ is an element of $\mathcal{I}_k$. Clearly $\deg_{\mathrm{w}^{-1}}(\natural_k^{(i)}) \geq N_k$. □

As a consequence of Proposition 3.10,

Corollary 5.22. *For $u^{[k]} \in \kappa^{-1}(\mathcal{W}^k \setminus (\mathcal{W}_1^k \cup \mathcal{W}^{k-1}))$, the expansion of $\sigma_{\natural_k}(u^{[k]})$ at the origin takes the form*

$$\sigma_{\natural_k}(u^{[k]}) = S_{\Lambda^{(k)}}(T) \mid_{T_{\Lambda_i+g-i} = u_i^{[k]}} + \sum_{|\mathrm{w}_g(\alpha)| > |\Lambda|} c_\alpha^{[k]} \cdot (u^{[k]})^\alpha$$

where $c_\alpha \in \mathbb{Q}[\lambda_j]$ and $S_{\Lambda^{(a)}}(T)$ is the lowest-order term in the w-degree of the $u_i^{[k]}$; $\sigma_{\natural_k}(u^{[k]})$ is homogeneous of degree $|\Lambda^{(a)}|$ with respect to the λ-degrees.

Remark 5.23. For example, by letting $J_i = \natural_k \setminus \{k+1\} \bigcup \{i\}$ for $i = k+2, k+3, \ldots, g$ and $u \in \kappa^{-1}(\Theta^k \setminus \Theta_1^k)$,

$$\sigma_{J_i}(u) = 0, \tag{5.16}$$

due to Theorem 3.4.

Here we note that there is an element in $\mathcal{I}_k \setminus \{\natural_k, \natural_k^{(k)}, \natural_k^{(k-1)}, \ldots, \natural_k^{(2)}, \natural_k^{(1)}\}$. Some examples are reported in [**MÔ**], where there is given an element I_k ($\#I_k = n_k$, $\deg_{\mathrm{w}^{-1}} I_k = N_k$) of Index which differs from $\natural_k$ and satisfies $\sigma_{I_k} \neq 0$ as a function over $\kappa^{-1}(\Theta^k \setminus (\Theta_1^k \cup \Theta^{k-1}))$.

Theorem 5.15 follows from the Lemmas above.

We can state a stronger version of Theorem 3.4 3.

Theorem 5.24. *For $k < g$, $(P_1, \ldots, P_k) \in \mathcal{S}^k(X \backslash \infty) \setminus (\mathcal{S}_1^k(X) \cap \mathcal{S}^k(X \backslash \infty))$ and $u = \pm w(P_1, \ldots, P_k) \in \kappa^{-1}(\Theta^k)$,*

$$\frac{\sigma_{\natural_k^{(i)}}(u)}{\sigma_{\natural_k}(u)} = (-1)^{k-i+1} \mu_{k,i-1}(P_1, \ldots, P_k).$$

Note that neither denominator nor numerator in the left-hand side vanish.

We also state another version of Theorem 3.4 3. besides Theorem 5.24, as follows.

In the proof of Lemma 5.11, we note that $h_{a_i+b_i+1}^{(g;k)} = \cdots + 1/(a_i + b_i + 1)! \times (T_1^{(g;k)})^{a_i+b_i+1} + \cdots$ which appears in (5.10) for the partition characteristics

$(a_1, \dots, a_{n_k}; b_1, \dots, b_{n_k})$ of $\Lambda^{[k]}$. We have the following Lemma:

Lemma 5.25. *By using the notations* ($u_i^{[k]} := T^{(k)}_{\Lambda_i+g-i}, u_i^{[g]} := T^{(g)}_{\Lambda_i+g-i}$) *in Lemma* 5.11, *for a subsequence* $J_\ell = \{L^{[k]}_{(a_\ell,b_\ell)}, L^{[k]}_{(a_{\ell+1},b_{\ell+1})}, \dots, L^{[k]}_{(a_{n_k},b_{n_k})}\} \subset \natural_k$ ($\ell \le n_k+1$) *using the characteristics of the partition* $(a_1, a_2, \dots, a_{n_k}; b_1, b_2, \dots, b_{n_k})$ *of* $\Lambda^{[k]}$, *we have*

$$\boldsymbol{s}_{\Lambda^{(k)}}(u^{[k]}) = \varepsilon'_{\Lambda,J_\ell,\natural_k}\left(\frac{\partial}{\partial u_g^{[g]}}\right)^{\deg_{\mathrm{w}^{-1}}(\natural_k\setminus J_\ell)}\left(\prod_{i\in J_\ell}\frac{\partial}{\partial u_i^{[g]}}\right)\boldsymbol{s}_{\Lambda^{(g)}}(u^{[g]})\Bigg|_{u^{[g]}=u^{[k]}}, \tag{5.17}$$

where $\varepsilon'_{\Lambda,J_\ell,\natural_k}$ *is a certain non-vanishing rational number.* (*Note that* $J_{n_k+1} = \emptyset$.)

Proof. With notation as in the proof of Lemma 5.11, using the characteristics of the partition, we introduce the Young diagrams $\Lambda^{[k,i]}$, ($i = 1, \dots, n_k$), given by $(a_1, \dots, a_i; b_1, \dots, b_i)$ such that $\Lambda^{[k,n_k]} = \Lambda^{[k]}$. Then by letting $t^{(g;k)} := (t_{k+1}, \dots, t_g)$, the determinant of the matrix in the proof of Lemma 5.11, H_j, equals the Schur function of $\Lambda^{[k,i]}$,

$$s_{\Lambda^{[k,i]}}(t^{(g;k)}) = |H_{n_k-i+1}|.$$

Since we are concerned only with the term of $((1/n!)T_1^{(g;k)})^n$ in $h_n^{(g;k)} = \cdots + (1/n!)(T_1^{(g;k)})^n + \cdots$, we analyze the behavior of T_1.

From the Jacobi-Trudi determinant expression in Proposition 3.6, for a Young diagram Λ, s_Λ contains the term

$$S_{\Lambda,T_1} := \left|\frac{1}{(\Lambda_i+j-i)!}T_1^{\Lambda_i+j-i}\delta_{(\Lambda_i+j-i)}\right|,$$

where $\delta_i = 1$ for $i \ge 0$ and, $\delta_i = 0$ for $i < 0$. Then the term S_{Λ,T_1}, i.e., the determinant of the matrix $((1/(\Lambda_i+j-i)!)T_1^{\Lambda_i+j-i})$, does not vanish as a polynomial of T_1 because the vectors $U_i := ((1/(\Lambda_i+j-i)!)T_1^{\Lambda_i+j-i})$, viewed as columns, are independent as their coordinates show. By considering their weight, S_{Λ,T_1} is a monomial $T_1^{|\Lambda|}$ with a non-vanishing rational factor.

On the other hand, (5.10) shows that $\tilde{\boldsymbol{s}}_{\Lambda^{[k]}}(t^{(g;k)})$ has a term $(u_g^{[g;k]})^{\deg_{\mathrm{w}^{-1}}(\natural_k\setminus J_\ell)} \times \prod_{i\in J_\ell} u_i^{[g;k]}$ up to a non-vanishing rational factor, where $u^{[g;k]} := u^{[g]} - u^{[k]}$. The claim follows. □

For $u^{[k]} \in \kappa^{-1}(\Theta^k)$ and $u^{[g]} \in \mathbb{C}^g$, we introduce the following notation as an extension of Definition 5.13,

$$\sigma_{J,g^N}(u^{[g]}) := \left(\frac{\partial}{\partial u_g^{[g]}}\right)^N\left(\prod_{i\in J}\frac{\partial}{\partial u_i^{[g]}}\right)\sigma(u^{[g]}),$$

and $\sigma_{J,g^N}(u^{[k]}) := \sigma_{J,g^N}(u^{[g]})|_{u^{[g]}=u^{[k]}}$.

We have now the variant proposition:

Proposition 5.26. *For $k < g$, $(P_1, \dots, P_k) \in \mathcal{S}^k(X\backslash\infty) \setminus (\mathcal{S}_1^k(X) \cap \mathcal{S}^k(X\backslash\infty))$, $u = \pm w(P_1, \dots, P_k) \in \kappa^{-1}(\Theta^k)$, subsequences $J_\ell = \{L^{[k]}_{(a_\ell, b_\ell)}, L^{[k]}_{(a_{\ell+1}, b_{\ell+1})}, \dots, L^{[k]}_{(a_{n_k}, b_{n_k})}\} \subset \natural_k$ $(\ell \le n_k+1)$ using the characteristics of the partition $(a_1, a_2, \dots, a_{n_k}; b_1, b_2, \dots, b_{n_k})$ of $\Lambda^{[k]}$, and $J_\ell^{(i)} := J_\ell \setminus \{k+1\} \cup \{i\}$ $(i = 1, 2, \dots, k)$, the following relations hold:*

1. *For $\ell \le n_k$,*

$$\frac{\sigma_{J_\ell^{(i)}, g^{\deg_{\mathrm{w}^{-1}}(\natural_k \setminus J_\ell)}}(u)}{\sigma_{J_\ell, g^{\deg_{\mathrm{w}^{-1}}(\natural_k \setminus J_\ell)}}(u)} = (-1)^{k-i+1} \mu_{k,i-1}(P_1, \dots, P_k),$$

especially,

$$\frac{\sigma_{i+1, g^{N_k - \deg_{\mathrm{w}^{-1}}(k+1)}}(u)}{\sigma_{k+1, g^{N_k - \deg_{\mathrm{w}^{-1}}(k+1)}}(u)} = (-1)^{k-i+1} \mu_{k,i-1}(P_1, \dots, P_k).$$

2. *For $\ell \le n_k$, we have as a function over $\kappa^{-1}(\Theta^k \setminus (\Theta_1^k \cup \Theta^{k-1}))$,*

$$\sigma_{J_\ell^{(i)}, g^{\deg_{\mathrm{w}^{-1}}(\natural_k \setminus J_\ell)}} \ne 0, \quad \sigma_{J, g^{N'}} = 0,$$

where $0 \le N' \le \deg_{\mathrm{w}^{-1}}(\natural_k \setminus J_\ell)$ and $J \subset J_\ell^{(i)}$ such that $\#J + N' < \#J_\ell^{(i)} + \deg_{\mathrm{w}^{-1}}(\natural_k \setminus J_\ell)$, and $\ell_{\mathrm{a}} \in \Pi$,

$$\sigma_{J_\ell^{(i)}, g^{\deg_{\mathrm{w}^{-1}}(\natural_k \setminus J_\ell)}}(u + \ell_{\mathrm{a}}) = \sigma_{J_\ell^{(i)}, g^{\deg_{\mathrm{w}^{-1}}(\natural_k \setminus J_\ell)}}(u) \exp\left(L\left(u + \frac{1}{2}\ell_{\mathrm{a}}, \ell_{\mathrm{a}}\right)\right)\chi(\ell_{\mathrm{a}}).$$

3. *For every $\ell = 1, 2, \dots, n_k$,*

$$\sigma_{J_\ell^{(i)}, g^{\deg_{\mathrm{w}^{-1}}(\natural_k \setminus J_\ell)}}(u) = \epsilon_{k, J_\ell} \sigma_{\natural_k^{(i)}}(u), \quad i = 1, \dots, k+1,$$
$$\sigma_{g^{N_k}}(u) = \epsilon_k \sigma_{\natural_k}(u),$$

where ϵ_{k, J_ℓ} and ϵ_k are non-vanishing rational numbers.

Proposition 5.26 3. in the hyperelliptic case is given in the work [**EHKKLS**].

Proof. We introduce the following objects and notation. As usual, $u^{[k]} \in \kappa^{-1}(w(\mathcal{S}^k(X\backslash\infty) \setminus (\mathcal{S}_1^k(X) \cap \mathcal{S}^k(X\backslash\infty)))$ is a g-vector; for $(P_1, \dots, P_g) \in \mathcal{S}^g(X)$, $v^{(i)} := w(P_i)$ for $(i = 1, 2, \dots, g)$, $u^{[\ell']} := \sum_{i=1}^{\ell'} v^{(i)}$, and $u^{[g;\ell']} := \sum_{i=\ell'+1}^{g} v^{(i)}$, $(\ell' = 1, \dots, g)$. Further we introduce the non-negative integer $\hat{N}_{k, J_\ell} := \deg_{\mathrm{w}^{-1}}(\natural_k \setminus J_\ell)$.

For a sequence J consisting of $\{k+1, k+2, \dots, g\}$, Lemma 4.4, which is essentially the same as (5.11), gives

$$\left(\frac{\partial}{\partial u_g^{[g]}}\right)^N \left(\prod_{i\in J}\frac{\partial}{\partial u_i^{[g]}}\right)\sigma(u^{[g]})\Big|_{u^{[g]}=u^{[k]}}$$
$$= \left(\frac{\partial}{\partial u_g^{[g;k]}}\right)^N \left(\prod_{i\in J}\frac{\partial}{\partial u_i^{[g;k]}}\right)\sigma(u^{[g]})\Big|_{u^{[g;k]}=0} + \text{lower order differentials of } \sigma(u^{[k]}). \tag{5.18}$$

From the proofs of Lemma 5.11 and Lemma 5.25, at $u^{[k]} \in \Theta^{[k]}$, $\sigma(u^{[g]})$ behaves like

$$\sigma(u^{[g]}) = \left(\prod_{i=1}^{n_k} h^{(g;k)}_{a_i+b_i+1}\right)_{T^{(g;k)}_{\Lambda_i+g-i}=u^{[g;k]}} \left(S_{\Lambda^{(k)}}(T^{(k)})|_{T^{(k)}_{\Lambda_i+g-i}=u^{[k]}} + \sum_{|\alpha|>\Lambda^{(k)}} a_\alpha \cdot (u^{[k]})^\alpha\right)$$
$$\times \big(1 + d_>(v_g^{(k+1)}, \dots, v_g^{(g)})\big) + \xi_k(u^{[g]}), \tag{5.19}$$

where $\xi_k(u^{[g]})$ represents the terms which do not contain $\big(\prod_{i=1}^{n_k} h^{(g;k)}_{a_i+b_i+1}\big)$, and $a_\alpha \in \mathbb{Q}[\lambda_i]$. Then we obviously have

$$\sigma_{J_\ell, g^{\hat{N}_{k,J_\ell}}}(u^{[k]}) = \epsilon' S_{\Lambda^{(k)}}(T^{(k)})\big|_{T^{(k)}_{\Lambda_i+g-i}=u^{[k]}} + \sum_{|\alpha|>\Lambda^{(k)}} a_\alpha \cdot (u^{[k]})^\alpha, \tag{5.20}$$

where the weight of the first term is $|\Lambda^{(k)}|$.

Since the $J_\ell = \natural_k$ $(k = 1, \dots, g-2)$ case is the same as Theorem 5.15, we consider only the $J_\ell \neq \natural_k$ $(k = 1, \dots, g-2)$ case.

We assume that $\ell \le n_k$. Since $k \ge g-r$ is also obvious, we consider only $k < g-r$. Due to Riemann's singularity theorem (Theorem 5.1), for $J_\ell \neq \natural_k$, $\sigma_{J_\ell}(u^{[k]}) = 0$ for $k < g-r$. On the other hand, since the relation (5.20) is not identically zero, there may exist a positive integer $N_{k,J_\ell} \le \hat{N}_{k,J_\ell}$ and a subsequence $J'_\ell \subset J_\ell$ such that we have, as a function over $\kappa^{-1}(\Theta^k \setminus (\Theta^k_1 \cup \Theta^{k-1}))$,

$$\sigma_{J'_\ell, g^{N_{k,J_\ell}}} \neq 0, \quad \sigma_{J', g^{N'}} = 0, \tag{5.21}$$

where $0 \le N' \le N_{k,J_\ell}$, $J' \subset J'_\ell$ such that $\#J' + N' < \#J'_\ell + N_{k,J_\ell}$. We show that such N_{k,J_ℓ} and J'_ℓ are identical to $\hat{N}_{k,J_\ell}$ and J_ℓ as follows.

Assume that $\sigma_{J'_\ell, g^{N_{k,J_\ell}}} \neq 0$ for some $N_{k,J_\ell} \le \hat{N}_{k,J_\ell}$, $J'_\ell \subset J_\ell$ such that $\#J'_\ell + N_{k,J_\ell} \le \#J_\ell + \hat{N}_{k,J_\ell}$. From (5.18), we have

$$\sigma(u^{[g]}) = \tilde{\epsilon}\big(u_g^{[g;k]}\big)^{N_{k,J_\ell}} \left(\prod_{i\in J'_\ell} u_i^{[g;k]}\right)\sigma_{J'_\ell, g^{N_{k,J_\ell}}}(u^{[k]})\big(1 + d_>(v_g^{(k+1)}, \dots, v_g^{(g)})\big) + \text{remainder},$$

where $\tilde{\epsilon}$ is a rational number. Thus $\sigma(u^{[k]} + tv^{(k+1)})$ is given by

$$\tilde{\epsilon}\big(tv_g^{(k+1)}\big)^{N_{k,J_\ell}} \left(\prod_{i\in J'_\ell} tv_i^{(k+1)}\right)\sigma_{J'_\ell, g^{N_{k,J_\ell}}}(u^{[k]})\big(1 + d_>(tv_g^{(k+1)}, 0, \dots, 0)\big) + \text{remainder}.$$

From Lemma 2.1, $\big(\partial^{N_{k,J_\ell}+\deg_{\mathrm{w}^{-1}}(J'_\ell)}/\partial {v_g^{(k+1)}}^{N_{k,J_\ell}+\deg_{\mathrm{w}^{-1}}(J'_\ell)}\big)\sigma(u^{[k]}+tv^{(k+1)})|_{v^{(k+1)}=0}$ is not identically zero. If we assume that $N_{k,J_\ell}+\deg_{\mathrm{w}^{-1}}(J'_\ell) < N_k$, it contradicts Corollary 5.4 1.; since the vanishing order N_k of σ in Corollary 5.4 agrees with the weight of the inflection at Θ^k [**BV**], we conclude that N_{k,J_ℓ} and J'_ℓ must be $\hat{N}_{k,J_\ell}$ and J_ℓ respectively and (5.21) must hold.

From Proposition 3.2, we have the translation formula for $\ell_{\mathrm{a}} \in \Pi$,

$$\sigma_{J_\ell,g^{\hat{N}_{k,J_\ell}}_{\mathrm{a}}}(u^{[k]}+\ell_{\mathrm{a}}) = \sigma_{J_\ell,g^{\hat{N}_{k,J_\ell}}_{\mathrm{a}}}(u^{[k]})\exp\left(L\left(u^{[k]}+\frac{1}{2}\ell_{\mathrm{a}},\ell_{\mathrm{a}}\right)\right)\chi(\ell_{\mathrm{a}}), \tag{5.22}$$

as in Corollary 5.18.

In the $J_\ell \neq \emptyset$ case, or $\ell \le n_k$ for $i=1,\dots,k$, we have

$$\begin{aligned}\left(\frac{\partial}{\partial u_g^{[g]}}\right)^{\hat{N}_{k,J_\ell}}&\left(\prod_{j=J_\ell\setminus\{k+1\}}\frac{\partial}{\partial u_j^{[g]}}\right)\big((\sigma_{k+1}(u^{[g]})\cdot\mu_{k,i-1}(u^{[g]})\big)\Big|_{u^{[g]}=u^{[k]}}\\ &=\sigma_{J_\ell,g^{\hat{N}_{k,J_\ell}}}(u^{[g]})\cdot\mu_{k,i-1}(u^{[g]}))\big|_{u^{[g]}=u^{[k]}},\end{aligned}$$

because $\mu_{k,i-1}$ does not vanish for $(P_1,\dots,P_k)\in\mathcal{S}^k(X\backslash\infty)\setminus(\mathcal{S}^k_1(X)\cap\mathcal{S}^k(X\backslash\infty))$ and does not diverge due to the assumption, whereas we have the vanishing property (5.21) when $u^{[g;k]}=0$. As a consequence, for every $i=1,\dots,k+1$,

$$\sigma_{J_\ell^{(i)},g^{\hat{N}_{k,J_\ell}}}(u^{[k]}) = (-1)^{k-1+1}\sigma_{J_\ell,g^{\hat{N}_{k,J_\ell}}}(u^{[k]})\cdot\mu_{k,i-1}(u^{[k]}).$$

As a function over $\kappa^{-1}(\Theta^k\setminus(\Theta^k_1\cup\Theta^{k-1}))$, we have

$$\sigma_{J_\ell^{(i)},g^{\hat{N}_{k,J_\ell}}}\neq 0,\quad \sigma_{J,g^{N'}}=0,$$

where $0\le N'\le\hat{N}_{k,J_\ell}$, $J\subset J_\ell^{(i)}$ such that $\#J+N'<\#J_\ell+\hat{N}_{k,J_\ell}$, and for $\ell_a\in\Pi$,

$$\sigma_{J_\ell^{(i)},g^{\hat{N}_{k,J_\ell}}}(u^{[k]}+\ell_{\mathrm{a}}) = \sigma_{J_\ell^{(i)},g^{\hat{N}_{k,J_\ell}}}(u^{[k]})\exp\left(L\left(u+\frac{1}{2}\ell_{\mathrm{a}},\ell_{\mathrm{a}}\right)\right)\chi(\ell_{\mathrm{a}}).$$

From Theorem 5.24 for every $i=1,\dots,k$, $\sigma_{J_\ell,g^{\hat{N}_{k,J_\ell}}}(u^{[k]})/\sigma_{\natural_k}(u^{[k]}) = \sigma_{J_\ell^{(i)},g^{\hat{N}_{k,J_\ell}}}(u^{[k]})/\sigma_{\natural_k^{(i)}}(u^{[k]})$ is a meromorphic function over Θ^k; we denote it by $q_\ell(u^{[k]})$.

Let the divisor of the meromorphic function $q_\ell(u^{[k]})$ as a function of $w(P_k)$ be $\sum_j p_j^+ - \sum_j p_j^-$, where $\sum_j p_j^+$ and $\sum_j p_j^-$ are effective divisors. We have $\sigma_{J_\ell^{(i)},g^{\hat{N}_{k,J_\ell}}}(u^{[k]}) = q_\ell(u^{[k]})\sigma_{\natural_k^{(i)}}(u^{[k]})$ for every $i=1,2,\cdots,k+1$, and every $\sigma_{J_\ell^{(i)},g^{\hat{N}_{k,J_\ell}}}(u^{[k]})$ is an entire function over $\kappa^{-1}\Theta^{(k)}$ as a function of $v^{(k)}=w(P_k)$ for $u^{[k]}=v^{(k)}+u^{[k-1]}$. Hence $\{p_j^-\}$ must be the preimage under the Abel map of a subset of the common divisors of $\sigma_{J_\ell^{(i)},g^{\hat{N}_{k,J_\ell}}}(u^{[k]})$ for every $i=1,2,\dots,k+1$, so that there exists $N_q\ge 0$ and $p_j^-=\infty$

$(j = 1, \dots, N_q)$. Hence q_ℓ is expanded as

$$q_\ell(u^{[k]}) = q_\ell(v_g^{(k)}, u^{[k-1]}) = (v_g^{(k)})^{-N_q}\big(q_{\ell,0} + q_{\ell,1}v_g^{(k)} + q_{\ell,2}(v_g^{(k)})^2 + \cdots\big),$$

where every coefficient $q_{\ell,i}$ is a function of $u^{[k-1]} \in \Theta^{k-1}$.

Corollary 5.22 gives the expansion of $\sigma_{\natural_k}$ and $\sigma_{J_\ell, g^{\hat{N}_{k,J_\ell}}}$ is given by (5.20), whereas we have $\sigma_{J_\ell, g^{\hat{N}_{k,J_\ell}}}(u^{[k]}) = q_\ell(u^{[k]})\sigma_{\natural_k}(u^{[k]})$. Hence q_ℓ is a constant function and $N_q = 0$. In other words, there is a non-vanishing rational number ϵ_{k,J_ℓ} such that

$$\sigma_{J_\ell^{(i)}, g^{\hat{N}_{k,J_\ell}}}(u^{[k]}) = \epsilon_{k,J_\ell}\sigma_{\natural_k^{(i)}}(u^{[k]}), \quad i = 1, 2, \dots, k+1. \tag{5.23}$$

Now we consider the $J_{n_k+1} = \emptyset$ case. Corollary 5.4 2. means that

$$\sigma_{g^{N_k}} \neq 0, \quad \sigma_{g^{N'}} = 0, \quad N' \le N_k. \tag{5.24}$$

Due to (5.20), we have $\sigma_{g^{N_k}}(u^{[k]}) = \epsilon_k \sigma_{\natural_k}(u^{[k]})$ for a suitable ϵ_k. □

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Note: We possibly list different transliterations of the name of the same author, following *MathSciNet* style.

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Errata for Part I [MP1].

1. On p. 1015 of [**MP1**], the wording "singular locus of $\mathcal{S}^n(X)$" is incorrect: Since any symmetric product of a smooth curve is smooth, what was indicated by $\mathcal{S}^n_1(X)$ in [**MP1**] is the singular locus of $\mathcal{S}^n(X)$ modulo linear equivalence.
2. The equation in [**MP1**, p. 1023],

$$\begin{pmatrix} \partial_{u_1} \\ \partial_{u_2} \\ \vdots \\ \partial_{u_k} \end{pmatrix} = r \begin{pmatrix} 1 & \phi_1(P_1) & \phi_1(P_2) & \cdots & \phi_1(P_k) \\ 1 & \phi_2(P_1) & \phi_2(P_2) & \cdots & \phi_2(P_k) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \phi_{k-1}(P_1) & \phi_{k-1}(P_2) & \cdots & \phi_{k-1}(P_k) \end{pmatrix}^{-1} \begin{pmatrix} y_1^{r-1}\partial_{x_1} \\ y_2^{r-1}\partial_{x_2} \\ \vdots \\ y_k^{r-1}\partial_{x_k} \end{pmatrix},$$

should read

$$\begin{pmatrix} \partial_{u_1} \\ \partial_{u_2} \\ \vdots \\ \partial_{u_k} \end{pmatrix} = r \begin{pmatrix} 1 & \phi_1(P_1) & \phi_2(P_1) & \cdots & \phi_{k-1}(P_1) \\ 1 & \phi_1(P_2) & \phi_2(P_2) & \cdots & \phi_{k-1}(P_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \phi_1(P_k) & \phi_2(P_k) & \cdots & \phi_{k-1}(P_k) \end{pmatrix}^{-1} \begin{pmatrix} y_1^{r-1}\partial_{x_1} \\ y_2^{r-1}\partial_{x_2} \\ \vdots \\ y_k^{r-1}\partial_{x_k} \end{pmatrix}.$$