J. Math. Soc. Japan
Vol. 66, No. 2 (2014) pp. 641–646
doi: 10.2969/jmsj/06620641

Wang's theorem for one-dimensional local rings

By Jun HORIUCHI and Hideto SAKURAI

(Received July 4, 2012)

Abstract. In this article, we show that, $Q :_A \mathfrak{m}^t \subseteq \mathfrak{m}^t$ for all integers $t > 0$, and for all parameter ideals $Q \subseteq \mathfrak{m}^{2t-1}$ in a one-dimensional Cohen-Macaulay local ring $(A, \mathfrak{m})$ provided that A is not a regular local ring. The assertion obtained by Wang can be extended to one-dimensional (hence, arbitrary dimensional) local rings after some mild modifications. We refer to these quotient ideals $I = Q :_A \mathfrak{m}^t$, t-th quasi-socle ideals of Q. Examples are explored.

1. Introduction.

Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$, and $\dim A > 0$. Let Q be a parameter ideal in A. Let $t > 0$ be a positive integer. With these notation, we set ideals $I = Q :_A \mathfrak{m}^t$, and call them t-th quasi-socle ideals of Q. This article studies t-th quasi-socle ideals in one-dimensional Cohen-Macaulay local rings. The purpose of this article is to extend Wang's theorem (see [**W**]) to one-dimensional Cohen-Macaulay local rings. We want to review the background of our researches briefly. When $t = 1$, the ideal $Q :_A \mathfrak{m}$ is called the socle ideal of Q. Let us recall one fundamental result on socle ideals given by Corso and Polini.

THEOREM 1.1 ([**CP**, Theorem 2.2]). *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring, which is not a regular local ring. Let $I = Q : \mathfrak{m}$ where Q is a parameter ideal in A. Then $I^2 = QI$.*

It seems natural to ask, "What will happen in the case when $t \geq 2$?" Bearing in our mind the case where $t = 1$, Polini and Ulrich conjectured that, by setting some conditions on the choice of parameter ideals Q, analogues of Theorem 1.1 might hold true for t-th quasi-socle ideals $I = Q :_A \mathfrak{m}^t$, $(t \geq 2)$. Namely, they posed the following profound conjecture. It is in the case when $t \geq 2$, A is a Cohen-Macaulay local ring, and $\dim A \geq 2$. Their conjecture is originally rooted in linkage theory.

CONJECTURE 1.2 ([**PU**]). *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring with $\dim A \geq 2$. Assume that $\dim A \geq 3$ when A is regular. Let $t \geq 2$ be an integer and Q a parameter ideal for A such that $Q \subseteq \mathfrak{m}^t$. Then $I = Q :_A \mathfrak{m}^t \subseteq \mathfrak{m}^t$.*

In 2007, Wang proved Cojecture 1.2 affirmatively in his remarkable paper [**W**]. We set $\mathrm{G}(\mathfrak{m}) = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ to be the associated graded ring of $\mathfrak{m}$.

2010 *Mathematics Subject Classification.* Primary 13H10; Secondary 13C13, 13C40.

Key Words and Phrases. parameter ideal, quasi-socle ideal, Cohen-Macaulay local ring, Buchsbaum local ring.

Theorem 1.3 ([**W**]). *Let A be a Cohen–Macaulay local ring and let $t \geq 2$ be an integer.*

(1) *The conjecture of Polini and Ulrich is true. Hence,* $\dim A \geq 2$*, and assume that* $\dim A \geq 3$ *when A is regular. Let $t \geq 2$ be an integer and let Q be a parameter ideal such that $Q \subseteq \mathfrak{m}^t$. Then $I = Q :_A \mathfrak{m}^t \subseteq \mathfrak{m}^t$, $\mathfrak{m}^t I = \mathfrak{m}^t Q$ and $I^2 = QI$.*
(2) *Assume that* $\operatorname{depth} \mathrm{G}(\mathfrak{m}) \geq 2$ *and let Q be a parameter ideal in A such that $Q \subseteq \mathfrak{m}^{t+1}$. Put $I = Q : \mathfrak{m}^t$. Then furthermore we have, $I \subseteq \mathfrak{m}^{t+1}$, $\mathfrak{m}^t I = \mathfrak{m}^t Q$ and $I^2 = QI$.*

The assumption that $\operatorname{depth} \mathrm{G}(\mathfrak{m}) \geq 2$ is satisfied if the ring A is a regular local ring of $\dim A \geq 2$. Wang's Theorem 1.3 deals with all Cohen–Macaulay local rings of $\dim A \geq 2$. It is natural to ask, "What will happen in the case when $\dim A = 1$?" Goto, Kimura, Matsuoka and Takahashi studied t-th quasi-socle ideals in one-dimensional local rings, and they have shown that the one-dimensional cases are different from higher-dimensional cases ($\dim A \geq 2$). It is difficult to control the t-th socle ideals $Q :_A \mathfrak{m}^t$ in one-dimensional local rings, even though A is a Cohen-Macaulay local ring and a Gorenstein local ring (see [**GKM**], [**GMT**]). We give an example of a one-dimensional Cohen-Macaulay local ring which shows that ideals $I = Q :_A \mathfrak{m}^t$, ($t \geq 2$) are not contained in $\mathfrak{m}^t$ when parameter ideal $Q \subseteq \mathfrak{m}^t$.

Example 1.4. Let $A = k[[X, Y]]/(X^2)$, where $k[[X, Y]]$ is the formal power series ring with two indeterminates X and Y over a field k. Put $\mathfrak{m} = (x, y) \subset A$, where x and y are the images of X and Y in A respectively. Then $\mathfrak{m}^n = (xy^{n-1}, y^n)$ for all positive integers $n > 0$. Let $t \geq 2$ be an integer and put $Q = (y^{2t-2})$. Then $Q \subseteq \mathfrak{m}^{2t-2} \subseteq \mathfrak{m}^t$ and $I = Q :_A \mathfrak{m}^t = (xy^{t-2}, y^{t-1}) = \mathfrak{m}^{t-1} \not\subseteq \mathfrak{m}^t$.

With these notation and terminology, we state the main result of this article.

Theorem 1.5. *Let $(A, \mathfrak{m})$ be a one-dimensional Cohen-Macaulay local ring and $t > 0$ a positive integer. Then, $Q :_A \mathfrak{m}^t \subseteq \mathfrak{m}^t$ for all parameter ideals $Q \subseteq \mathfrak{m}^{2t}$. Moreover if A is not a regular local ring, then $Q :_A \mathfrak{m}^t \subseteq \mathfrak{m}^t$ for all parameter ideals $Q \subseteq \mathfrak{m}^{2t-1}$.*

We shall remark that Example 1.4 shows that the value $2t-1$ of an order of parameter ideals $Q \subseteq \mathfrak{m}^{2t-1}$ in Theorem 1.5 is the best possible.

Goto, Kimura, Phuong and Truong explored quasi-socle ideals $I = Q :_A \mathfrak{m}^t$ in numerical semigroup rings, and they have shown some interesting results. Among them, they have shown some conditions for the associated graded ring $\mathrm{G}(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ to be Cohen–Macaulay [**GKPT**, Theorem 3.1]. However, their results do not cover Theorem 1.5. Because Theorem 1.5 and our discussion deal with all Cohen–Macaulay local rings of $d = 1$, and also for $d \geq 1$ (see Corollary 2.5, for the assertion $d \geq 1$).

2. Proof of Theorem 1.5.

In this section, we give a proof of Theorem 1.5. Firstly, we prove Theorem 2.3, and we derive Theorem 1.5 as a corollary. Let us begin with the following.

Lemma 2.1. *Let A be a commutative ring, and let $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{c}$ be ideals of A.*

Suppose that $\mathfrak{a}$ contains a non-zero divisor and $\mathfrak{a} \subseteq \mathfrak{b}$. Furthermore assume that there exists a subset F of $\mathfrak{a}$ such that $\mathfrak{a}$ is generated by F, and we assume that $(f) :_A \mathfrak{b} \subseteq \mathfrak{c}$ for all elements $f \in F$. Then we have $(a) :_A \mathfrak{b} \subseteq \mathfrak{c}$, for all non-zero divisors $a \in \mathfrak{a}$.

Proof. Let $a \in \mathfrak{a}$ be any non-zero divisor in A. Choose any element $x \in (a) :_A \mathfrak{b}$ and $f \in F$. Since $F \subseteq \mathfrak{a} \subseteq \mathfrak{b}$, we have $fx = ay$ for some $y \in A$. On the other hand, take any element $b \in \mathfrak{b}$, then we can express $bx = az$ for some $z \in A$. Thereby, we have

$$bay = b(fx) = f(bx) = faz.$$

Since a is a non-zero divisor, we get $by = fz$. Thus we see that $y \in (f) :_A \mathfrak{b}$. By our assumption $(f) :_A \mathfrak{b} \subseteq \mathfrak{c}$, we get $y \in \mathfrak{c}$. Therefore, $fx = ay \in a\mathfrak{c}$, and thus, $x \in a\mathfrak{c} :_A \mathfrak{a}$ because $\mathfrak{a}$ is generated by the set F. Now, because $a \in \mathfrak{a}$, we have $ax \in a\mathfrak{c}$. It is easy to see that $x \in \mathfrak{c}$, since a is a non-zero divisor. We get $(a) :_A \mathfrak{b} \subseteq \mathfrak{c}$ as claimed. □

Next Lemma is the key in our discussion.

Lemma 2.2. *Let $(A, \mathfrak{m})$ be a commutative local ring and assume that $\mathfrak{m}$ contains a non-zero divisor. Let $t > 0$ be a positive integer and let $s \geq 0$ be an integer. Let $a_1, a_2, \ldots, a_{t+s} \in \mathfrak{m}$ be non-zero divisors of A and we assume that $(a_1) \neq \mathfrak{m}$. Then $(a_1 a_2 \cdots a_{t+s}) :_A \mathfrak{m}^t \subseteq \mathfrak{m}^{s+1}$.*

Proof. Firstly, we prove the assertion in the case when $s > 0$. It is easy to see that

$$(a_1 a_2 \cdots a_{t+s}) :_A \mathfrak{m}^t \subseteq (a_1 a_2 \cdots a_{t+s}) :_A a_1 a_2 \cdots a_t \subseteq [(0) :_A a_1 \cdots a_t] + (a_{t+1} \cdots a_{t+s}).$$

Thereby, we have $(a_1 a_2 \cdots a_{t+s}) :_A \mathfrak{m}^t \subseteq (a_{t+1} \cdots a_{t+s})$, since $a_1 \cdots a_t$ is a non-zero divisor. We choose any element $x \in (a_1 a_2 \cdots a_{t+s}) :_A \mathfrak{m}^t$ and express it as $x = a_{t+1} \cdots a_{t+s} y$ where $y \in A$. It is enough to prove the following claim.

Claim 1. $y \in (a_1) :_A \mathfrak{m}$.

Proof of Claim 1. We choose any element $\alpha \in \mathfrak{m}$. Then we can express $\alpha a_2 \cdots a_t x = \alpha a_2 \cdots a_t a_{t+1} \cdots a_{t+s} y$, because $x = a_{t+1} \cdots a_{t+s} y$. On the other hand, we have that $\alpha a_2 \cdots a_t \in \mathfrak{m}^t$, hence, we have $\alpha a_2 \cdots a_t x = a_1 \cdots a_{t+s} z$, for some $z \in A$. Thus from these equations (recall that $a_2 \cdots a_{t+s}$ is a non-zero divisor), we have $\alpha y = a_1 z$. Therefore, $y \in (a_1) :_A \mathfrak{m}$ as claimed. □

Thanks to the assumption $(a_1) \neq \mathfrak{m}$, we have $(a_1) :_A \mathfrak{m} \subseteq \mathfrak{m}$. Hence, we have $y \in (a_1) :_A \mathfrak{m} \subseteq \mathfrak{m}$. Therefore, $x = a_{t+1} \cdots a_{t+s} y \in \mathfrak{m}^{s+1}$, thus, we get $(a_1 a_2 \cdots a_{t+s}) :_A \mathfrak{m}^t \subseteq \mathfrak{m}^{s+1}$ as claimed. The proof also works in the case when $s = 0$. We consider an element x itself instead of y, that is, the above proof of Claim 1 shows that $(a_1 a_2 \cdots a_t) :_A \mathfrak{m}^t \subseteq (a_1) :_A \mathfrak{m}$. □

We are ready to prove the key theorem.

Theorem 2.3. *Let $(A, \mathfrak{m})$ be a Noetherian local ring with* depth $A > 0$. *Let $t > 0$*

be a positive integer and let $s \geq 0$ *be an integer. Assume that* $\mathfrak{m}$ *is not principal. Then we have* $(a) :_A \mathfrak{m}^t \subseteq \mathfrak{m}^{s+1}$ *for all non-zero divisors* $a \in \mathfrak{m}^{t+s}$.

Proof. First of all, it is easy to see that $\mathfrak{m}^{t+s}$ is generated by the following set F:

$$F = \{a_1 a_2 \cdots a_{t+s} \mid a_1, a_2, \ldots, a_{t+s} \in \mathfrak{m} \text{ are non-zero divisors}\}.$$

Since $\mathfrak{m}$ is not principal, we have $(a_1 a_2 \cdots a_{t+s}) :_A \mathfrak{m}^t \subseteq \mathfrak{m}^{s+1}$ for all elements $a_1 a_2 \cdots a_{t+s} \in F$, by Lemma 2.2. Therefore, we get $(a) :_A \mathfrak{m}^t \subseteq \mathfrak{m}^{s+1}$ for all non-zero divisors $a \in \mathfrak{m}^{t+s}$, by Lemma 2.1. □

Applying Theorem 2.3 to one-dimensional Cohen-Macaulay local rings, we get the following.

Corollary 2.4. *Let* $(A, \mathfrak{m})$ *be a one-dimensional Cohen-Macaulay local ring. Let* $t > 0$ *be a positive integer, and let* $s \geq 0$ *be an integer. Then we have,* $Q :_A \mathfrak{m}^t \subseteq \mathfrak{m}^s$ *for all parameter ideals* $Q \subseteq \mathfrak{m}^{t+s}$. *Moreover, if* A *is not a regular local ring, we have* $Q :_A \mathfrak{m}^t \subseteq \mathfrak{m}^{s+1}$ *for all parameter ideals* $Q \subseteq \mathfrak{m}^{t+s}$.

Proof. If A is a regular local ring, then A is a DVR. Thereby, it is clear that $Q :_A \mathfrak{m}^t \subseteq \mathfrak{m}^s$ for all parameter ideals $Q \subseteq \mathfrak{m}^{t+s}$. Hence, we may assume that A is not a regular local ring. The assertion readily follows from Theorem 2.3, since $\mathfrak{m}$ is not principal. □

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. Set $s = t$ (resp. $s = t - 1$) in Corollary 2.4, we have the first (resp. second) assertion in Theorem 1.5. □

Finally the authors would like to give the following, which settles Polini-Ulrich Conjecture 1.2 of arbitrary dimension, although the assertion is almost covered by Wang's theorem (see [**W**]) in the case when $\dim A \geq 2$.

Corollary 2.5. *Let* $(A, \mathfrak{m})$ *be a Cohen-Macaulay local ring with* $d = \dim A > 0$. *Let* $t > 0$ *be a positive integer and let* $s \geq 0$ *be an integer, and assume that* $t + s \geq 2$. *Suppose that* $\mathfrak{m}$ *is not principal. Then we have,* $Q :_A \mathfrak{m}^t \subseteq \mathfrak{m}^{s+1}$ *for all parameter ideals* $Q \subseteq \mathfrak{m}^{t+s}$.

Proof. We prove the assertion by induction on $d = \dim A$. When $d = 1$, the assertion readily follows from Theorem 2.3. Suppose that $d \geq 2$ and assertion holds for $d - 1$. Let $Q = (a_1, a_2, \ldots, a_d) \subseteq \mathfrak{m}^{t+s}$ be a parameter ideal in A. We see that $A/(a_1)$ is not a regular local ring, since $a_1 \in \mathfrak{m}^{t+s} \subseteq \mathfrak{m}^2$. Thus, by passing to $A/(a_1)$, and thanks to the hypothesis of induction on d, we have,

$$Q :_A \mathfrak{m}^t \subseteq \mathfrak{m}^{s+1} + (a_1) \subseteq \mathfrak{m}^{s+1}.$$ □

3. One dimensional local rings.

In this section, we apply Theorem 1.5 to one-dimensional local rings $(A, \mathfrak{m})$. To do this, we give an application of Theorem 1.5 (see Corollary 3.1). We denote $H^0_{\mathfrak{m}}(A)$ the 0-th local cohomology module of A with respect to the maximal ideal $\mathfrak{m}$. First of all, we notice that if A is not a Cohen-Macaulay local ring, the assertion $Q :_A \mathfrak{m}^t \subseteq \mathfrak{m}^t$ does not hold for any parameter ideal $Q \subseteq \mathfrak{m}^t$, provided that the integer $t \gg 0$. In fact, suppose that A is not a Cohen-Macaulay local ring, hence $H^0_{\mathfrak{m}}(A) \neq (0)$. Then, there exists an integer $n > 0$ such that $H^0_{\mathfrak{m}}(A) \nsubseteq \mathfrak{m}^n$. On the other hand, there exists an integer $\ell > 0$ such that $H^0_{\mathfrak{m}}(A) = (0) :_A \mathfrak{m}^\ell$. We set an integer $t \geq \max\{n, \ell\}$, thus, $H^0_{\mathfrak{m}}(A) = (0) :_A \mathfrak{m}^\ell \subseteq Q :_A \mathfrak{m}^t$ for every parameter ideal Q in A. Then, since $H^0_{\mathfrak{m}}(A) \nsubseteq \mathfrak{m}^t$, we have $Q :_A \mathfrak{m}^t \nsubseteq \mathfrak{m}^t$. What will happen in case A is not necessarily a Cohen-Macaulay local ring? We give a following consequence.

COROLLARY 3.1. *Let $(A, \mathfrak{m})$ be a one-dimensional Noetherian local ring and $t > 0$ be a positive integer. Then, $Q :_A \mathfrak{m}^t \subseteq \mathfrak{m}^t + H^0_{\mathfrak{m}}(A)$ for all parameter ideals $Q \subseteq \mathfrak{m}^{2t}$. Moreover if $A/H^0_{\mathfrak{m}}(A)$ is not a regular local ring, then $Q :_A \mathfrak{m}^t \subseteq \mathfrak{m}^t + H^0_{\mathfrak{m}}(A)$ for all parameter ideals $Q \subseteq \mathfrak{m}^{2t-1}$.*

PROOF. Apply Theorem 1.5 to a Cohen-Macaulay local ring $A/H^0_{\mathfrak{m}}(A)$. □

Goto and the authors explored quasi-socle ideals in Buchsbaum local rings [**GHS**]. They have shown that quasi-socle ideals behave very well inside Buchsbaum local rings provided that $d = \dim A \geq 2$. Our interest for the application of Corollary 3.1 is especially Buchsbaum local rings. We refer to [**SV**] for basic properties of Buchsbaum local ring. It is known, among them, that, if A is a Buchsbaum local ring, then $H^0_{\mathfrak{m}}(A) = (0) :_A \mathfrak{m}$ (see [**SV**]). In the Example 3.2, we keep the same notation as in Example 1.4.

EXAMPLE 3.2. Let $A = k[[X, Y, Z]]/(X^2, XY, XZ, YZ)$, then A is a one-dimensional Buchsbaum local ring which is not a Cohen-Macaulay local ring. Put $\mathfrak{m} = (x, y, z)$, then we have $H^0_{\mathfrak{m}}(A) = (0) :_A \mathfrak{m} = (x)$. Hence, we have, $A/H^0_{\mathfrak{m}}(A) \simeq k[[Y, Z]]/(YZ)$. It is easy to check, $\mathfrak{m}^n = (y^n, z^n)$ for all integers $n > 1$. Let t be a positive integer and put $Q = (y^{2t-1} + z^{2t-1})$. Then, we have $Q : \mathfrak{m}^t = (y^t, z^t) + (x) = \mathfrak{m}^t + H^0_{\mathfrak{m}}(A)$.

Let A be a one-dimensional Cohen-Macaulay local ring (resp. Buchsbaum local ring). Thanks to Theorem 1.5 (resp. Corollary 3.1), we have $I = Q :_A \mathfrak{m}^t \subseteq \mathfrak{m}^t$ (resp. $I = Q :_A \mathfrak{m}^t \subseteq \mathfrak{m}^t + H^0_{\mathfrak{m}}(A)$), whence $I^2 \subseteq Q$. It is natural to expect that the equality $I^2 = QI$ holds true, but it is not true. In [**GKPT**], Goto and et al. explored the quasi-socle ideals $I = Q :_A \mathfrak{m}^t$ in numerical semigroup rings and they gave an example which shows that the reduction number of I with respect to parameter ideal Q is not equal to one. Thus, the equality $I^2 = QI$ does not hold true in general.

EXAMPLE 3.3 ([**GKPT**, Example 3.7]). Let k be a field and $R = k[[t^5, t^8, t^{12}]] \subseteq k[[t]]$ be a numerical semigroup ring. Then $(R, \mathfrak{m})$ is a one-dimensional Gorenstein local ring, where $\mathfrak{m} = (t^5, t^8, t^{12})$. Let $0 < \alpha \in \langle 5, 8, 12 \rangle$ be an integer, and suppose that $\alpha \geq 20$. Let $Q = (t^\alpha)$ be a parameter ideal in R, and let $I = Q : \mathfrak{m}^3$. We can check that $\mathfrak{m}^3 I \neq \mathfrak{m}^3 Q$ and $I^2 \neq QI$, hence the reduction number of I with respect to Q is not equal to one.

Question 3.4. Can we describe the reduction number of I with respect to Q in one-dimensional Cohen-Macaulay (Buchsbaum) local rings?

Acknowledgements. The authors would like to thank all members of Commutative Algebra Seminar at Meiji University for their suggestions and comments. Our thanks especially go to Dr. Satoru Kimura for his support and discussion.

References

[CP] A. Corso and C. Polini, Links of prime ideals and their Rees algebras, J. Algebra, **178** (1995), 224–238.

[GHS] S. Goto, J. Horiuchi and H. Sakurai, Quasi-socle ideals in Buchsbaum rings, Nagoya Math. J., **200** (2010), 93–106.

[GKM] S. Goto, S. Kimura and N. Matsuoka, Quasi-socle ideals in Gorenstein numerical semigroup rings, J. Algebra, **320** (2008), 276–293.

[GKPT] S. Goto, S. Kimura, T. T. Phuong and H. L. Truong, Quasi-socle ideals and Goto numbers of parameters, J. Pure Appl. Algebra, **214** (2010), 501–511.

[GMT] S. Goto, N. Matsuoka and R. Takahashi, Quasi-socle ideals in a Gorenstein local ring, J. Pure Appl. Algebra, **212** (2008), 969–980.

[PU] C. Polini and B. Ulrich, Linkage and reduction numbers, Math. Ann., **310** (1998), 631–651.

[SV] J. Stückrad and W. Vogel, Buchsbaum Rings and Applications, Springer-Verlag, 1986.

[W] H.-J. Wang, Links of symbolic powers of prime ideals, Math. Z., **256** (2007), 749–756.

Jun Horiuchi
Department of Mathematics
Nippon Institute of Technology
Miyashiro
Saitama 345-8501, Japan
E-mail: jhoriuchi.math@gmail.com

Hideto Sakurai
Department of Liberal Arts
Toyama National College of Technology
1-2 Ebieneriya, Imizu-shi
Toyama 933-0293, Japan
E-mail: sakurai-h@nc-toyama.ac.jp