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J. Math. Soc. Japan
Vol. 66, No. 2 (2014) pp. 565–580
doi: 10.2969/jmsj/06620565

Construction of Shiba behavior spaces on an open Riemann surface of infinite genus

Dedicated to Emeritus Professor Yukio Kusunoki on his 88th birthday

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(Received Nov. 15, 2010)
(Revised June 28, 2012)

Abstract. The concept of behavior spaces introduced by Shiba plays an important role of systematic investigation of abelian differentials on an open Riemann surface. A Shiba behavior space consists of harmonic differentials which satisfy a certain period condition and boundary behavior. In this paper, for any open Riemann surface of infinite genus we construct Shiba behavior spaces with arbitrarily prescribed period condition and with specific boundary behavior.

1. Introduction.

Let R be an open Riemann surface of genus g, $0 \leq g \leq \infty$. Let Λ be the usual real Hilbert space of all square integrable complex differentials on R equipped with inner product

$$\langle \omega, \sigma \rangle = \Re\biggl(\iint_R \omega \wedge {}^*\overline{\sigma} \biggr) \quad \text{for } \omega, \sigma \in \Lambda,$$

where $\Re(z)$ is the real part of $z \in \mathbb{C}$, $\overline{\sigma}$ is the complex conjugate differential of σ and ${}^*\overline{\sigma}$ is the conjugate differential of $\overline{\sigma}$. By Λ_x we denote a subspace of Λ, where 'x' designates the property of the subspace. For example we set

$$\Lambda_h = \{\lambda \in \Lambda \mid \lambda \text{ is harmonic}\},$$

where 'h' stands for 'harmonic'. If Λ_x is a subspace of Λ_h, we denote by $\Lambda_x^\perp$ the orthogonal complement of Λ_x in Λ_h. For $z \in \mathbb{C}$ we set $z\Lambda_x = \{z\omega \mid \omega \in \Lambda_x\}$. Observe that $(z\Lambda_x)^\perp = z\Lambda_x^\perp$. Let ${}^*\Lambda_x = \{{}^*\omega \mid \omega \in \Lambda_x\}$. We use the following subspaces:

$$\Lambda_{hse} = \biggl\{ \lambda \in \Lambda_h \mid \lambda \text{ is semiexact, i.e., } \int_\gamma \lambda = 0 \text{ for every dividing cycle } \gamma \biggr\},$$

$$\Lambda_{he} = \biggl\{ \lambda \in \Lambda_h \mid \lambda \text{ is exact, i.e., } \int_\gamma \lambda = 0 \text{ for every cycle } \gamma \biggr\},$$

2010 *Mathematics Subject Classification.* Primary 30F30.
Key Words and Phrases. Riemann surface, harmonic differential, Shiba behavior space.

$$\Lambda_a = \{\lambda \in \Lambda_h \mid \lambda \text{ is analytic}\},$$

$$\Lambda_{\bar a} = \{\lambda \in \Lambda_h \mid \overline{\lambda} \in \Lambda_a\}.$$

Moreover, let $\Lambda_{h0} = {}^*\Lambda_{he}^{\perp}$ and $\Lambda_{hm} = {}^*\Lambda_{hse}^{\perp}$. We also use the following real subspaces:

$$\Gamma_h = \{\lambda \in \Lambda_h \mid \lambda \text{ is real}\},$$

$$\Gamma_{hse} = \Gamma_h \cap \Lambda_{hse}, \quad \Gamma_{he} = \Gamma_h \cap \Lambda_{he},$$

$$\Gamma_{h0} = {}^*\Gamma_{he}^{\perp}, \quad \Gamma_{hm} = {}^*\Gamma_{hse}^{\perp},$$

where $\Gamma_{he}^{\perp}$ and $\Gamma_{hse}^{\perp}$ are the orthogonal complements taken in Γ_h of Γ_{he} and Γ_{hse}, respectively. Note that the superscript '$\perp$' is used for two different orthogonal complements; one is taken in Λ_h and the other in Γ_h. We shall write explicitly the space in which orthogonal complement is taken, unless it can be clearly understood from the context.

Let $\{R_m\}_{m=1}^{\infty}$ be a canonical exhaustion of R and let $\{A_j, B_j\}_{j=1}^{g}$ be a canonical homology basis associated with $\{R_m\}_{m=1}^{\infty}$, i.e.,

i) the restriction $\{A_j, B_j\}_{j=1}^{p(m)}$ of $\{A_j, B_j\}_{j=1}^{g}$ to R_m is a canonical homology basis of R_m mod ∂R_m (cf. [**1**]), where $p(m)$ is the genus of R_m,
ii) for each j, B_j crosses A_j from left to right.

Definition 1.1 (Shiba behavior space). Let $\mathcal{L} = \{L_j\}_{j=1}^{g}$ be a family of lines $L_j = L(\theta_j) = \{re^{i\theta_j} \mid r \in \mathbb{R}\}$ passing through the origin in $\mathbb{C}$, where $\theta_j \in \mathbb{R}$. A subspace $\Lambda_x = \Lambda_x(\mathcal{L})$ of Λ_{hse} is said to be a *Shiba behavior space associated with* $\mathcal{L}$ if the following structure condition and period condition are satisfied:

1) (structure condition) $\Lambda_x = i^*\Lambda_x^{\perp}$,
2) (period condition) both $\int_{A_j} \lambda$ and $\int_{B_j} \lambda$ lie on L_j for every $j = 1, \ldots, g$ and $\lambda \in \Lambda_x$.

Shiba behavior spaces play a central role in the formulation of Riemann-Roch's and Abel's theorems for certain classes of abelian differentials with prescribed boundary behavior.

It is easy to see that if Γ_x is a subspace such that $\Gamma_{hm} \subset \Gamma_x \subset \Gamma_{he}$, then $\Gamma_x + i^*\Gamma_x^{\perp}$ is a Shiba behavior space associated with $\mathcal{L} = \{L_j = L(\pi/2)\}_{j=1}^{g}$. Historically, Kusunoki [**3**] gave the theory of abelian integrals on an open Riemann surface by using canonical semiexact differentials and applied it to the vertical slit mappings (generalization of Koebe's uniformization). The canonical semiexact differentials are represented by use of the space $\Gamma_{hm} + i\Gamma_{hse}$. The $\Gamma_{he} + i\Gamma_{ho}$ is also typical behavior space which is used to construct horizontal slit mappings of a planar domain. Yoshida [**7**] extended these typical cases by use of so called Γ_χ-behavior. Shiba [**6**] introduced the concept of behavior spaces and showed that the result of Yoshida reduced to the special case of $\Gamma_x + i^*\Gamma_x^{\perp}$ ($\Gamma_{hm} \subset \Gamma_x \subset \Gamma_{he}$). As an example of showing the significance of his extension, he gave a conformal mapping of a compact bordered Riemann surface onto a region on $\mathbb{C}$ with slits whose directions are arbitrarily prescribed. This result can not be represented by Yoshida's method. Further he gave an example of a behavior space associated with $\mathcal{L} = \{L_j = L(0) \text{ or } L(\pi/2)\}_{j=1}^{g}$.

Although our assertions are undoubtedly true for Riemann surfaces of finite genus, let the genus g be infinite for the simplicity of representations. Our main subject is to construct a Shiba behavior space associated with an arbitrary family of lines passing through the origin. More precisely we prove the following.

THEOREM 1.1. *Let $\mathcal{L} = \{L_j\}_{j=1}^g$ be a family of lines $L_j = \{re^{i\theta_j} \mid r \in \mathbb{R}\}$ passing through the origin in the complex plane. Then there exists a Shiba behavior space associated with $\mathcal{L}$.*

In fact, we shall construct Shiba behavior spaces which are regarded, roughly speaking, as the form $\Gamma_x + i^*\Gamma_x^\perp$ with $\Gamma_{hm} \subset \Gamma_x \subset \Gamma_{he}$ in a neighborhood of ideal boundary, and then realize our aim for Theorem 1.1 as the limit of a convergent sequence of thus constructed behavior spaces.

Further we will give more general Shiba behavior spaces. Let $K \geq 2$ be an integer, and suppose K distinct unimodular constants $e^{i\varphi_1}, e^{i\varphi_2}, \ldots, e^{i\varphi_K}$ are given. We will construct a Shiba behavior space of the form

$$Cl\left(\sum_{k=1}^{K} e^{i\varphi_k}\Gamma_k\right), \quad \text{where } \Gamma_k \subset \Gamma_{hse}, \tag{1}$$

which has more general boundary behavior than those constructed in the proof of Theorem 1.1. Observe that $\Gamma_x + i^*\Gamma_x^\perp$ is represented as above with $K = 2$, $\varphi_1 = 0$, $\varphi_2 = \pi/2$, $\Gamma_1 = \Gamma_x$ and $\Gamma_2 = {}^*\Gamma_x^\perp$.

THEOREM 1.2. *Let $K \geq 2$ be an integer and suppose K distinct unimodular constants $e^{i\varphi_1}, e^{i\varphi_2}, \ldots, e^{i\varphi_K}$ are given. Let $\mathcal{L} = \{L_j\}_{j=1}^g$ be a family of lines $L_j = \{re^{\theta_j} \mid r \in \mathbb{R}\}$ such that $\{\theta_j\}_{j=1}^g = \{\varphi_1, \ldots, \varphi_K\}$. Then there exists a Shiba behavior space of the form* (1) *associated with $\mathcal{L}$.*

In Section 5 we will show examples of the form (1). One of them is applicable to the above mentioned slit mapping with arbitrarily prescribed directions. These examples illustrate the difference between Theorem 1.1 and Theorem 1.2.

2. Pre-behavior space.

A subspace Λ_x of Λ_h is said to be a *pre-behavior space* if $\Lambda_x = i\,{}^*\Lambda_x^\perp$. For a sequence of pre-behavior spaces $\{\Lambda^{(n)}\}_{n=1}^\infty$, we consider the following subspace:

$$\Lambda_s = \left\{\lambda \in \Lambda_h \mid \text{there exists } \lambda_n \in \Lambda^{(n)} \text{ such that } \lim_{n\to\infty} \|\lambda_n - \lambda\| = 0\right\}.$$

We say that Λ_s is the strong limit of $\{\Lambda^{(n)}\}_{n=1}^\infty$. We give a sequence of Shiba behavior spaces $\{\Lambda^{(n)}\}_{n=1}^\infty$ whose strong limit is a Shiba behavior space associated with an infinite number of lines $\mathcal{L} = \{L_j\}_{j=1}^g$. In order to show the existence of above $\{\Lambda^{(n)}\}_{n=1}^\infty$, the following lemmas may be a key. Since the proof of the first lemma is simple, we omit the proof.

Lemma 2.1 (cf. [**1**]). *Let Λ_x, Λ_y be closed subspaces of Λ_h.*

(1) *If Λ_x is orthogonal to Λ_y, then $\Lambda_x + \Lambda_y$ is a closed subspace of Λ_h.*
(2) *If the dimension of Λ_x is finite, then $\Lambda_x + \Lambda_y$ is a closed subspace of Λ_h.*

Lemma 2.2. (1) *Let $\{U_n\}_{n=1}^{\infty}$ be an increasing sequence of closed subspaces of Λ_h. Set $U_0 = Cl(\bigcup_{n=1}^{\infty} U_n)$, where $Cl(X)$ denotes the closure of a set X. For each $\lambda \in \Lambda_h$, let $\lambda = \omega_n + \sigma_n$ be the orthogonal decomposition of λ, where $\omega_n \in U_n$ and $\sigma_n \in U_n^{\perp}$. Then $\{\omega_n\}_{n=1}^{\infty}$ (resp. $\{\sigma_n\}_{n=1}^{\infty}$) converges to $\omega_0 \in U_0$ (resp. $\sigma_0 \in U_0^{\perp}$).*
(2) *Let $\{V_n\}_{n=1}^{\infty}$ be a decreasing sequence of closed subspaces of Λ_h. Set $V_0 = \bigcap_{n=1}^{\infty} V_n$. For each $\lambda \in \Lambda_h$, let $\lambda = \omega_n + \sigma_n$ be the orthogonal decomposition of λ, where $\omega_n \in V_n$ and $\sigma_n \in V_n^{\perp}$. Then $\{\omega_n\}_{n=1}^{\infty}$ (resp. $\{\sigma_n\}_{n=1}^{\infty}$) converges to $\omega_0 \in V_0$ (resp. $\sigma_0 \in V_0^{\perp}$).*

Proof. (1) For $m > n$, the differential $\sigma_m \in U_m^{\perp}$ belongs to $U_n^{\perp}$. Note that

$$\sigma_n - \sigma_m = (\lambda - \omega_n) - (\lambda - \omega_m) = \omega_m - \omega_n.$$

Hence $\omega_m - \omega_n$ is orthogonal to ω_n. We have

$$\begin{aligned}
&\langle \omega_n, \omega_m \rangle = \langle \omega_n, \omega_m - \omega_n + \omega_n \rangle = \|\omega_n\|^2, \\
&0 \le \|\omega_n - \omega_m\|^2 = \|\omega_n\|^2 - 2\langle \omega_n, \omega_m \rangle + \|\omega_m\|^2 \\
&\quad = \|\omega_m\|^2 - \|\omega_n\|^2 \le \|\omega_m\|^2 \le \|\lambda\|^2.
\end{aligned}$$

Hence $\{\|\omega_n\|\}_{n=1}^{\infty}$ is a bounded increasing sequence. It follows that $\{\omega_n\}_{n=1}^{\infty}$ is a Cauchy sequence and converges to an $\omega \in U_0$. Also $\{\sigma_n\}_{n=1}^{\infty}$ converges to a $\sigma \in \bigcap_{n=1}^{\infty} U_n^{\perp} = U_0^{\perp}$. Since $\lambda = \omega_0 + \sigma_0 = \omega_n + \sigma_n = \omega + \sigma$, we have $\omega_0 = \omega$, $\sigma_0 = \sigma$.

(2) We can apply (1) to $U_n = V_n^{\perp}$. □

Theorem 2.1. *Let $\{\Lambda^{(n)}\}_{n=1}^{\infty}$ be a sequence of pre-behavior spaces which satisfy the following: $\Lambda^{(n)} = U_n + V_n$, where $\{U_n\}_{n=1}^{\infty}$ (resp. $\{V_n\}_{n=1}^{\infty}$) is an increasing (resp. a decreasing) sequence of closed subspaces of Λ_h. Then the strong limit Λ_s of $\{\Lambda^{(n)}\}_{n=1}^{\infty}$ is a pre-behavior space.*

Proof. Let W_n be the orthogonal projection of V_n to $U_n^{\perp}$. Since $\Lambda^{(n)} = U_n + V_n = U_n + W_n$ is a closed subspace of Λ_h, we obtain that W_n is also a closed subspace of Λ_h and $\{W_n\}_{n=1}^{\infty}$ is a decreasing sequence in Λ_h, because $\{V_n\}_{n=1}^{\infty}$ is decreasing. We can write each $\varphi \in \Lambda_a$ in the form

$$\varphi = \lambda_n + i\,{}^*\lambda_n, \text{ where } \lambda_n \in \Lambda^{(n)},$$

and further

$$\lambda_n = \omega_n + \sigma_n, \text{ where } \omega_n \in U_n \text{ and } \sigma_n \in V_n.$$

Take the orthogonal decomposition of $\sigma_n = \alpha_n + \beta_n$, where $\alpha_n \in U_n$ and $\beta_n \in U_n^\perp$. Then β_n belongs to W_n and $\omega_n + \alpha_n \in U_n$ is the orthogonal projection of φ to U_n. By Lemma 2.2 the sequence $\{\omega_n + \alpha_n\}_{n=1}^\infty$ converges to $\lambda_0 \in U_0 = Cl(\bigcup_{n=1}^\infty U_n)$ and $\lambda_0 \in \Lambda_s$. By Lemma 2.2 the sequence $\{\beta_n\}_{n=1}^\infty$ also converges to $\beta_0 \in W_0 = \bigcap_{n=1}^\infty W_n$. Since W_n is contained in $\Lambda^{(n)}$, the differential β_0 belongs to Λ_s. Therefore $\{\lambda_n = \omega_n + \alpha_n + \beta_n\}_{n=1}^\infty$ converges to $\lambda_0 + \beta_0 \in \Lambda_s$ in the sense of Dirichlet norm. We have

$$\varphi = \lambda_0 + \beta_0 + i\,{}^*(\lambda_0 + \beta_0) \in \Lambda_s + i\,{}^*\Lambda_s \text{ and } \Lambda_a \subset \Lambda_s + i\,{}^*\Lambda_s.$$

Similarly, we can show $\Lambda_{\bar{a}} \subset \Lambda_s + i\,{}^*\Lambda_s$. We get $\Lambda_h = \Lambda_a + \Lambda_{\bar{a}} \subset \Lambda_s + i\,{}^*\Lambda_s$. For $\lambda, \mu \in \Lambda_s$, there exist $\lambda_n, \mu_n \in \Lambda_n$ so that

$$\lim_{n\to\infty} \|\lambda_n - \lambda\| = 0, \quad \lim_{n\to\infty} \|\mu_n - \mu\| = 0.$$

Hence we have

$$\langle \lambda, i\,{}^*\mu \rangle = \lim_{n\to\infty} \langle \lambda_n, i\,{}^*\mu_n \rangle = 0.$$

It follows that $i\,{}^*\Lambda_s$ is orthogonal to Λ_s. As a result, Λ_s is a pre-behavior space. □

3. Proof of Theorem 1.1.

We give a generalization of the behavior space $\Gamma_x + i^*\Gamma_x^\perp$ ($\Gamma_{hm} \subset \Gamma_x \subset \Gamma_{he}$). On a Riemann surface of genus g, let $\sigma_\gamma \in \Gamma_{ho}$ denote the period reproducing differential for a cycle γ, i.e.,

$$\int_\gamma \omega = \langle \omega, {}^*\sigma_\gamma \rangle \text{ for } \omega \in \Gamma_h.$$

By our choice of a canonical homology basis $\{A_j, B_j\}$, we note that

$$\int_{B_k} \sigma_{A_j} = \langle \sigma_{A_j}, {}^*\sigma_{B_k} \rangle = \delta_{jk}, \quad \int_{A_k} \sigma_{B_j} = \langle \sigma_{B_j}, {}^*\sigma_{A_k} \rangle = -\delta_{jk},$$
$$\int_{A_k} \sigma_{A_j} = \langle \sigma_{A_j}, {}^*\sigma_{A_k} \rangle = 0, \qquad \int_{B_k} \sigma_{B_j} = \langle \sigma_{B_j}, {}^*\sigma_{B_k} \rangle = 0,$$

where δ_{jk} is the Kronecker delta.

Let $\mathcal{L}$ be a given family of lines $\{L_j = L(\theta_j)\}_{j=1}^g$ and

$$\mathcal{L}^m = \{L_j \mid L_j = L(\theta_j) \text{ for } j \le p(m) \text{ and } L_j = L(\pi/2) \text{ for } j > p(m)\}.$$

Take a closed subspace Γ_x which satisfies $\Gamma_{hm} \subset \Gamma_x \subset \Gamma_{he}$, and consider the following subspaces of Λ_h:

$$S_m = \left\{ \sum_{j=1}^{p(m)} (a_j \sigma_{A_j} + b_j \sigma_{B_j}) \,\middle|\, a_j \text{ and } b_j \text{ are real} \right\},$$

$$S^m = Cl\Bigg(\Bigg\{ \sum_{j=p(m)+1}^{g} (a_j \sigma_{A_j} + b_j \sigma_{B_j}) \,\Bigg|\, a_j \text{ and } b_j \text{ are real}$$

$$\text{and only a finite number of } \{a_k, b_\ell\} \text{ do not vanish} \Bigg\} \Bigg),$$

$$S_m(\mathcal{L}) = \left\{ \sum_{j=1}^{p(m)} \left(a_j e^{i\theta_j} \sigma_{A_j} + b_j e^{i\theta_j} \sigma_{B_j} \right) \,\middle|\, a_j \text{ and } b_j \text{ are real} \right\},$$

$$\underline{\Lambda}_x^{(m)} = \underline{\Lambda}_x^{(m)}(\mathcal{L}) = \Gamma_x + S_m(\mathcal{L}),$$

$$\Lambda_x^{(m)} = \Lambda_x^{(m)}(\mathcal{L}) = \underline{\Lambda}_x^{(m)}(\mathcal{L}) + i\left({}^*\Gamma_x^{\perp} \cap {}^*S_m^{\perp}\right),$$

where $S_m^{\perp}$ is the orthogonal complement of S_m in Γ_h.

We have the following.

Lemma 3.1. *The subspace $\Lambda_x^{(m)}$ is a Shiba behavior space associated with $\mathcal{L}^m$.*

Proof. First, we prove the period condition. Note that

$$\Gamma_{hse} \supset {}^*\Gamma_x^{\perp} \supset \Gamma_{ho} \supset S_m.$$

By using period reproducing differentials, we see that

$${}^*\Gamma_x^{\perp} \cap {}^*S_m^{\perp} \subset {}^*S_m^{\perp} = \left\{ \omega \in \Gamma_h \,\middle|\, \int_{A_j} \omega = 0, \int_{B_j} \omega = 0 \text{ for every } j \le p(m) \right\},$$

and each differential $\omega \in S_m(\mathcal{L})$ satisfies

$$\int_{A_j} \omega = \int_{B_j} \omega = 0 \text{ for } j > p(m).$$

Hence for $\lambda \in \Lambda_x^{(m)}$,

$$\int_{A_j} \lambda \in L(\theta_j) \text{ and } \int_{B_j} \lambda \in L(\theta_j) \text{ for } j \le p(m),$$

$$\int_{A_j} \lambda \in L(\pi/2) \text{ and } \int_{B_j} \lambda \in L(\pi/2) \text{ for } j > p(m).$$

Second, we prove $\Lambda_x^{(m)\perp} \supset i\,{}^*\Lambda_x^{(m)}$. Since the dimension of $S_m(\mathcal{L})$ is finite, by Lemma 2.1, $\underline{\Lambda}_x^{(m)}(\mathcal{L}) = \Gamma_x + S_m(\mathcal{L})$ is a closed subspace. Let $\omega \in \Gamma_x$, $\sigma \in S_m(\mathcal{L})$ and $\lambda \in \Gamma_x^{\perp} \cap S_m^{\perp}$. We have $\langle \omega + \sigma, \lambda \rangle = 0$, because the real part of σ belongs to S_m. It follows that $\underline{\Lambda}_x^{(m)}$ is orthogonal to $\Gamma_x^\perp \cap S_m^\perp$. The subspace Γ_x ($\subset \Gamma_{he}$) is orthogonal to $i\,{}^*\Gamma_x$ and $i\,{}^*S_m(\mathcal{L})$. By using period reproducing differentials we have

$$\begin{aligned}
\left\langle e^{i\theta_j}\sigma_{A_j}, ie^{i\theta_k}\,{}^*\sigma_{A_k}\right\rangle &= -\Re\left(ie^{i(\theta_j-\theta_k)}\right)\langle\sigma_{A_j}, {}^*\sigma_{A_k}\rangle = 0,\\
\left\langle e^{i\theta_j}\sigma_{B_j}, ie^{i\theta_k}\,{}^*\sigma_{A_k}\right\rangle &= -\Re\left(ie^{i(\theta_j-\theta_k)}\right)\langle\sigma_{B_j}, {}^*\sigma_{A_k}\rangle = 0,\\
\left\langle e^{i\theta_j}\sigma_{B_j}, ie^{i\theta_k}\,{}^*\sigma_{B_k}\right\rangle &= -\Re\left(ie^{i(\theta_j-\theta_k)}\right)\langle\sigma_{B_j}, {}^*\sigma_{B_k}\rangle = 0.
\end{aligned}$$

Hence $S_m(\mathcal{L})$ is orthogonal to $i\,{}^*S_m(\mathcal{L})$. It follows that $\underline{\Lambda}_x^{(m)}$ is orthogonal to $i\,{}^*\underline{\Lambda}_x^{(m)}$. The space $i({}^*\Gamma_x^\perp \cap {}^*S_m^\perp)$ is clearly orthogonal to $\Gamma_x^\perp \cap S_m^\perp$. Thus $\Lambda_x^{(m)}$ is orthogonal to $i\,{}^*\Lambda_x^{(m)}$, i.e., $\Lambda_x^{(m)\perp} \supset i\,{}^*\Lambda_x^{(m)}$.

Finally, we show $\Lambda_x^{(m)\perp} \subset i\,{}^*\Lambda_x^{(m)}$. Suppose $\lambda \in \Lambda_h$ is orthogonal to $i\,{}^*\Lambda_x^{(m)}$. Then for $1 \le j \le p(m)$

$$\begin{aligned}
0 &= \left\langle \lambda, ie^{i\theta_j}\,{}^*\sigma_{A_j}\right\rangle = \Re\left(-ie^{-i\theta_j}\int_{A_j}\lambda\right),\\
0 &= \left\langle \lambda, ie^{i\theta_j}\,{}^*\sigma_{B_j}\right\rangle = \Re\left(-ie^{-i\theta_j}\int_{B_j}\lambda\right).
\end{aligned}$$

Hence there exist real numbers a_j, b_j which satisfy

$$\int_{A_j}\lambda = a_je^{i\theta_j} \text{ and } \int_{B_j}\lambda = b_je^{i\theta_j}.$$

Set

$$\lambda_0 = \sum_{j=1}^{p(m)}\left(-a_je^{i\theta_j}\sigma_{B_j} + b_je^{i\theta_j}\sigma_{A_j}\right).$$

Then, by the property of reproducing differentials, we have

$$\int_{A_j}(\lambda-\lambda_0) = \int_{B_j}(\lambda-\lambda_0) = 0 \text{ for } j \le p(m).$$

We see that $\lambda-\lambda_0$ is orthogonal to ${}^*S_m$ and $i\,{}^*S_m$. By assumption, λ is orthogonal to $i\,{}^*\Gamma_x$ and so is $\lambda_0 \in S_m(\mathcal{L})$. Hence $\lambda-\lambda_0$ is orthogonal to $i\,{}^*\Gamma_x$. By $S^m \subset {}^*\Gamma_x^\perp \cap {}^*S_m^\perp$ we have $i\,{}^*\Lambda_x^{(m)} \supset \Gamma_x^\perp \cap S_m^\perp \supset {}^*S^m$. It follows that

$$0 = \langle\lambda, {}^*\sigma_{A_j}\rangle = \Re\left(\int_{A_j}\lambda\right),\quad 0 = \langle\lambda, {}^*\sigma_{B_j}\rangle = \Re\left(\int_{B_j}\lambda\right) \text{ for } j > p(m).$$

Since

$$0 = \langle \lambda_0, {}^*\sigma_{A_j} \rangle = \Re\biggl(\int_{A_j} \lambda_0 \biggr), \quad 0 = \langle \lambda_0, {}^*\sigma_{B_j} \rangle = \Re\biggl(\int_{B_j} \lambda_0 \biggr) \text{ for } j > p(m),$$

we get $\Re(\lambda - \lambda_0) = \sigma \in \Gamma_{he}$. The real part of λ_0 belongs to S_m. Since λ is orthogonal to $\Gamma_x^\perp \cap S_m^\perp = (\Gamma_x + S_m)^\perp$, the real part of λ belongs to $Cl(\Gamma_x + S_m)$. The subspace S_m is of finite dimension and, by Lemma 2.1, $\Gamma_x + S_m$ is a closed subspace. It follows that $\sigma \in (\Gamma_x + S_m) \cap \Gamma_{he} = \Gamma_x$. Let η be the imaginary part of $\lambda - \lambda_0$. Since $\lambda - \lambda_0$ and σ are orthogonal to $i\,{}^*(S_m + \Gamma_x)$, so is $i\eta$. We see $\eta \in {}^*\Gamma_x^\perp \cap {}^*S_m^\perp$. Therefore $\lambda = \sigma + \lambda_0 + i\eta \in \Gamma_x + S_m(\mathcal{L}) + i({}^*\Gamma_x^\perp \cap {}^*S_m^\perp) = \Lambda_x^{(m)}$. This shows that $(i\,{}^*\Lambda_x^{(m)})^\perp \subset \Lambda_x^{(m)}$, i.e., $\Lambda_x^{(m)\perp} \subset i\,{}^*\Lambda_x^{(m)}$. Thus we have $\Lambda_x^{(m)} = i\,{}^*\Lambda_x^{(m)\perp}$. □

Theorem 1.1 is represented as the following Theorem.

Theorem 3.1. *Let $\mathcal{L} = \{L_j\}_{j=1}^g$ be an arbitrarily given family of lines. The strong limit Λ_{xs} of $\{\Lambda_x^{(m)}\}_{m=1}^\infty$ is a Shiba behavior space associated with $\mathcal{L}$.*

Proof. We note that the sequence $\{\underline{\Lambda}_x^{(m)}\}_{m=1}^\infty$ is clearly an increasing sequence of closed subspaces of Λ_h. Let W_m be the orthogonal projection of $i({}^*\Gamma_x^\perp \cap {}^*S_m^\perp)$ to $\underline{\Lambda}_x^{(m)\perp}$. Then $\Lambda_x^{(m)} = \underline{\Lambda}_x^{(m)} + W_m$. Thus we see that W_m is a closed subspace of Λ_h and that $\{W_m\}$ is decreasing. By Theorem 2.1 we have the strong limit Λ_{xs} of $\{\Lambda_x^{(m)}\}_{m=1}^\infty$ is a pre-behavior space. For $\lambda \in \Lambda_{xs}$ it is clear that

$$\int_{A_j} \lambda \in L_j, \quad \int_{B_j} \lambda \in L_j.$$

Therefore Λ_{xs} is a Shiba behavior space associated with $\mathcal{L}$. □

4. Proof of Theorem 1.2.

On a Riemann surface of infinite genus, we already know that the period condition gives an influence to the boundary behavior of concerned differentials. In the case that the period condition contains at least three lines in a neighborhood of the ideal boundary, the Shiba behavior space has not been given except [**4**]. The boundary behavior of $\Lambda_x^{(m)}$ is, roughly speaking, stipulated by Γ_x.

In this section we construct Shiba behavior spaces under a suitable condition different from those in Section 3. For this purpose, we rearrange the period condition of a Shiba behavior space. We divide the set of numbers $J = \{1, 2, \dots, g\}$ to K divisions $\mathbb{J} = \{J_k\}_{k=1}^K$ ($K \leq \infty$), i.e., $J = \bigcup_{k=1}^K J_k$, $J_k \cap J_\ell = \emptyset$ for $k \neq \ell$. Let $\mathcal{L}(\mathbb{J}) = \{L_k\}_{k=1}^K$ be a family of lines passing through the origin in $\mathbb{C}$. We call a subspace $\Lambda_x = \Lambda_x(\mathcal{L}(\mathbb{J}))$ of Λ_{hse} a *Shiba behavior space associated with* $\mathcal{L}(\mathbb{J})$ if the following structure condition and period condition are satisfied:

1) (structure condition) $\Lambda_x = i\,{}^*\Lambda_x^\perp$,

2) (period condition) $\displaystyle\int_{A_j} \lambda \in L_k, \ \int_{B_j} \lambda \in L_k$, for $j \in J_k, \ \lambda \in \Lambda_x$.

For $K = 1$ we can easily construct a Shiba behavior space. Take a closed subspace Γ_x which satisfies $\Gamma_{hm} \subset \Gamma_x \subset \Gamma_{he}$, and set $\Lambda(1) = \Gamma_x + i\,{}^*\Gamma_x^{\perp}$. Then $i\,{}^*\Lambda(1)^{\perp} = \Lambda(1)$. By $\Gamma_{hm} \subset \Gamma_x$, we have $\Gamma_{hse} \supset {}^*\Gamma_x^{\perp}$ and $\Lambda(1) \subset \Lambda_{hse}$. From $\Gamma_x \subset \Gamma_{he}$, every $\lambda \in \Lambda(1)$ is able to have only imaginary period for an arbitrary cycle. Hence $\Lambda(1)$ is a Shiba behavior space associated with $\mathcal{L}(\mathbb{J}) = \{L_1 = L(\pi/2)\}$. This case corresponds to the result of Yoshida [**7**].

For a finite K, we construct a pre-behavior space. Let $\mathcal{L}(\mathbb{J}) = \{L_k = L(\theta_k)\}_{k=1}^{K}$, where $L_j \cap L_k = \{0\}$ if $j \neq k$. Take a set of closed subspaces $\{\Gamma^{(k)} \subset \Gamma_h\}_{k=1}^{K}$ and set $\Lambda(K) = \sum_{k=1}^{K} e^{i\theta_k}\Gamma^{(k)}$.

Proposition 4.1. *Suppose $\Gamma^{(j)}$ is orthogonal to ${}^*\Gamma^{(k)}$ $(k \neq j)$ and ${}^*\Gamma^{(1)\perp} = \sum_{k=2}^{K}\Gamma^{(k)}$. Then $Cl(\Lambda(K))$ is a pre-behavior space.*

Proof. When Λ_x is a pre-behavior space, $e^{i\theta}\Lambda_x$ is a pre-behavior space. By a simple argument we may assume that $\theta_1 = 0$.

First, we show that $\Lambda(K)^{\perp} \supset i\,{}^*\Lambda(K)$.

For $\sigma = \sum_{k=1}^{K} e^{i\theta_k}\sigma_k \in \Lambda(K)$ and $\omega = \sum_{\ell=1}^{K} e^{i\theta_\ell}\omega_\ell \in \Lambda(K)$, where $\sigma_k \in \Gamma^{(k)}$, $\omega_\ell \in \Gamma^{(\ell)}$, we have

$$\langle \sigma, i\,{}^*\omega\rangle = \left\langle \sum_{k=1}^{K} e^{i\theta_k}\sigma_k, i\,{}^*\left(\sum_{\ell=1}^{K} e^{i\theta_\ell}\omega_\ell\right)\right\rangle = -\sum_{k=1}^{K}\sum_{\ell=1}^{K} \Re\big(ie^{i(\theta_k-\theta_\ell)}\big)\langle \sigma_k, {}^*\omega_\ell\rangle = 0.$$

This shows $\Lambda(K)$ is orthogonal to $i\,{}^*\Lambda(K)$ and $Cl(\Lambda(K))$ is also orthogonal to $i\,{}^*Cl(\Lambda(K))$.

By Lemma 2.1, $Cl(\Lambda(K)) + i\,{}^*Cl(\Lambda(K))$ is closed. It follows that

$$Cl(\Lambda(K)) + i\,{}^*Cl(\Lambda(K)) = Cl(\Lambda(K) + i\,{}^*\Lambda(K)).$$

Next, we show that $(\Lambda(K) + i\,{}^*\Lambda(K))^{\perp} \cap \Lambda_a = \{0\}$.

For $\varphi \in (\Lambda(K) + i\,{}^*\Lambda(K))^{\perp} \cap \Lambda_a = \Lambda(K)^{\perp} \cap i\,{}^*\Lambda(K)^{\perp} \cap \Lambda_a$, we denote by ω the real part of φ. Since φ is orthogonal to $\Gamma^{(1)}$, ω is also orthogonal to $\Gamma^{(1)}$. By assumption, we have a representation

$$\omega = {}^*\left(\sum_{k=2}^{K}\alpha_k\right) \in {}^*\left(\sum_{k=2}^{K}\Gamma^{(k)}\right), \text{ where } \alpha_k \in \Gamma^{(k)}.$$

Since φ is orthogonal to $e^{i\theta_j}\Gamma^{(j)}$, we have

$$e^{-i\theta_j}\varphi = \cos\theta_j\omega + \sin\theta_j\,{}^*\omega + i(-\sin\theta_j\omega + \cos\theta_j\,{}^*\omega)$$

and $\cos\theta_j\omega + \sin\theta_j\,{}^*\omega$ is orthogonal to $\Gamma^{(j)}$. Hence

$$\mu_j = \cos\theta_j\omega + \sin\theta_j\,{}^*\omega = \cos\theta_j\,{}^*\left(\sum_{k=2}^{K}\alpha_k\right) - \sin\theta_j\left(\sum_{k=2}^{K}\alpha_k\right) \in \Gamma^{(j)\perp}.$$

We get

$$0 = \langle \mu_j, \alpha_j \rangle = \left\langle \cos\theta_j \, {}^*\Big(\sum_{k=2}^{K} \alpha_k \Big) - \sin\theta_j \Big(\sum_{k=2}^{K} \alpha_k \Big), \alpha_j \right\rangle,$$

$$0 = \cos\theta_j \langle {}^*\alpha_j, \alpha_j \rangle - \sin\theta_j \left\langle \sum_{k=2}^{K} \alpha_k, \alpha_j \right\rangle = \sin\theta_j \langle {}^*\omega, \alpha_j \rangle.$$

Since $\sin\theta_j \neq 0$ for $j \neq 1$, we have $\langle {}^*\omega, \alpha_j \rangle = 0$ and

$$\langle {}^*\omega, {}^*\omega \rangle = \left\langle {}^*\omega, -\sum_{j=2}^{K} \alpha_j \right\rangle = -\sum_{j=2}^{K} \langle {}^*\omega, \alpha_j \rangle = 0.$$

It follows that $\omega = 0 = \varphi$ and $\Lambda(K)^\perp \cap i\,{}^*\Lambda(K)^\perp \cap \Lambda_a = \{0\}$.

Analogously we can show that $\Lambda(K)^\perp \cap i\,{}^*\Lambda(K)^\perp \cap \Lambda_{\bar{a}} = \{0\}$. It follows that

$$\Lambda(K)^\perp \cap i\,{}^*\Lambda(K)^\perp \cap \Lambda_h = \{0\}$$

and

$$\Lambda_h = Cl(\Lambda(K) + i\,{}^*\Lambda(K)) = Cl(\Lambda(K)) + i\,{}^*Cl(\Lambda(K)).$$

Therefore $Cl(\Lambda(K))$ is a pre-behavior space. $\square$

REMARK 4.1. The subspace $\Lambda(2)$ is always a closed subspace so that $\Lambda(2) = i\,{}^*\Lambda(2)^\perp$. Let $\{e^{i\theta_1}\omega_n + e^{i\theta_2}\sigma_n\}_{n=1}^\infty$ be a convergent sequence in $\Lambda(2)$, where $\omega_n \in \Gamma^{(1)}$ and $\sigma_n \in \Gamma^{(2)} = {}^*\Gamma^{(1)\perp}$. Then $\{\omega_n + e^{i(\theta_2-\theta_1)}\sigma_n\}_{n=1}^\infty$ and its imaginary part $\{\sin(\theta_2 - \theta_1)\sigma_n\}_{n=1}^\infty$ are Cauchy sequences. This shows that $\{\sigma_n\}_{n=1}^\infty$ is a Cauchy sequence converging to $\sigma \in \Gamma^{(2)}$. Hence $\{\omega_n\}_{n=1}^\infty$ is a Cauchy sequence converging to $\omega \in \Gamma^{(1)}$. Therefore $\{e^{i\theta_1}\omega_n + e^{i\theta_2}\sigma_n\}_{n=1}^\infty$ converges to $e^{i\theta_1}\omega + e^{i\theta_2}\sigma \in \Lambda(2)$. We see that $\Lambda(2)$ is closed. Thus, by Proposition 4.1, the subspace $\Lambda(2)$ is a pre-behavior space.

We set, for $\mathbb{J} = \{J_k\}_{k=1}^K$,

$$S(J_k) = Cl\Bigg(\Bigg\{ \sum_{j \in J_k} (a_j \sigma_{A_j} + b_j \sigma_{B_j}) \,\Bigg|\, a_j \text{ and } b_j \text{ are real} \\ \text{and only a finite number of } \{a_k, b_\ell\} \text{ do not vanish} \Bigg\} \Bigg) \subset \Gamma_h.$$

In order to prove Theorem 1.2, it is sufficient to show the following Theorem.

THEOREM 4.1. *Suppose* $\Gamma^{(1)}, \Gamma^{(2)}, \ldots, \Gamma^{(K)} (K < \infty)$ *are closed subspaces of* Γ_h *such that*

(1) $\Gamma^{(j)} \subset {}^*\Gamma^{(k)\perp} \quad (j < k)$,

(2) ${}^*\Gamma^{(1)\perp} = \displaystyle\sum_{k=2}^{K} \Gamma^{(k)}$,

(3) $S(J_k) \subset \Gamma^{(k)} \subset \Gamma_{hse} \quad (k = 1, \ldots, K)$.

Set $\Lambda(K) = \sum_{k=1}^{K} e^{i\theta_k} \Gamma^{(k)}$. *Then* $Cl(\Lambda(K))$ *is a Shiba behavior space associated with* $\mathcal{L}(\mathbb{J})$.

Proof. We note that for $j < k$

$$\Gamma^{(k)} \subset {}^*\Gamma^{(j)\perp} \subset {}^*S(J_j)^{\perp}.$$

By Proposition 4.1 $Cl(\Lambda(K))$ is a pre-behavior space. Since $\Gamma^{(k)}$ is contained in Γ_{hse}, we see that $\Lambda(K) \subset \Lambda_{hse}$. For $\omega \in \Gamma^{(k)} \subset {}^*S(J_j)^{\perp}$ $(j \neq k)$ and $\ell \in J_j$, we have

$$\int_{A_\ell} \omega = \langle \omega, {}^*\sigma_{A_\ell} \rangle = 0, \text{ and } \int_{B_\ell} \omega = \langle \omega, {}^*\sigma_{B_\ell} \rangle = 0.$$

For $\omega \in \Gamma^{(k)}$ and $\ell \in J_k$, we have

$$\int_{A_\ell} \omega, \ \int_{B_\ell} \omega \in \mathbb{R}, \text{ and } \int_{A_\ell} e^{i\theta_k} \omega, \ \int_{B_\ell} e^{i\theta_k} \omega \in L_k.$$

For $\lambda = \sum_{j=1}^{K} e^{i\theta_j} \omega_j \in \Lambda_K$, where $\omega_j \in \Gamma^{(j)}$, we have

$$\int_{A_\ell} \lambda = \sum_{j=1}^{K} \int_{A_\ell} e^{i\theta_j} \omega_j \in L_k, \text{ and } \int_{B_\ell} \lambda = \sum_{j=1}^{K} \int_{B_\ell} e^{i\theta_j} \omega_j \in L_k \text{ for } \ell \in J_k.$$

The subspace $\Lambda(K)$ satisfies the period condition, so does $Cl(\Lambda(K))$. Hence $Cl(\Lambda(K))$ is a Shiba behavior space associated with $\mathcal{L}(\mathbb{J})$. □

Corollary 4.1 (cf. [**5**]). *For closed subspaces* Γ_x *and* $\Gamma^{(1)}$, *assume that*

$$\Gamma_{hm} \subset \Gamma_x \subset \Gamma_{he} \quad \textit{and} \quad \Gamma_x + S(J_1) \subset \Gamma^{(1)} \subset {}^*\Gamma_x^{\perp} \cap \left(\bigcap_{j=2}^{K} {}^*S(J_j)^{\perp} \right).$$

Set ${}_1\Gamma = {}^*\Gamma^{(1)\perp}$, $\Gamma^{(2)} = {}_1\Gamma \cap (\bigcap_{j=3}^{K} {}^*S(J_j)^{\perp})$, ${}_2\Gamma = {}^*\Gamma^{(2)\perp}$, *and* $\Gamma^{(k)} = (\bigcap_{j=1}^{k-1} {}_j\Gamma) \cap (\bigcap_{j=k+1}^{K} {}^*S(J_j)^{\perp})$, ${}_k\Gamma = {}^*\Gamma^{(k)\perp}$, $k = 3, \ldots, K$.

If $\sum_{j=1}^{K-1} \Gamma^{(j)}$ *is closed, then* $Cl(\Lambda(K))$ *is a Shiba behavior space associated with* $\mathcal{L}(\mathbb{J})$.

Proof. We show that the assumption in Theorem 4.1 is satisfied.

(1) For $j < k$, we have $\Gamma^{(k)} \subset {}_j\Gamma = {}^*\Gamma^{(j)\perp}$. Hence $\Gamma^{(j)}$ is orthogonal to ${}^*\Gamma^{(k)}$.

(2) From $\Gamma^{(K)} = (\bigcap_{j=1}^{K-1} {}_j\Gamma)$, we get ${}^*\Gamma^{(K)\perp} = Cl(\sum_{j=1}^{K-1} \Gamma^{(j)}) = \sum_{j=1}^{K-1} \Gamma^{(j)}$.

(3) By the assumption we have for each ℓ (≥ 2)

$$\Gamma^{(\ell)} \subset {}_1\Gamma \subset Cl\biggl(\Gamma_x + \sum_{j=2}^{K} S(J_j)\biggr) \subset \Gamma_{hse},$$

and so we get

$${}_\ell\Gamma = Cl\biggl(\sum_{j=1}^{\ell-1} \Gamma^{(j)} + \sum_{j=\ell+1}^{K} S(J_j)\biggr) \subset \Gamma_{hse}.$$

We have also

$$S(J_1) \subset \Gamma^{(1)} \subset {}^*\Gamma_x^{\perp} \subset \Gamma_{hse}, \quad {}_1\Gamma \supset \Gamma_x + \sum_{j=2}^{K} S(J_j).$$

It follows that

$$\sum_{j=k}^{K} S(J_j) \subset \bigcap_{\ell=1}^{k-1} {}_\ell\Gamma.$$

Hence

$$\Gamma^{(k)} = \biggl(\bigcap_{j=1}^{k-1} {}_j\Gamma\biggr) \cap \biggl(\bigcap_{j=k+1}^{K} {}^*S(J_j)^{\perp}\biggr) \supset S(J_k).$$

By Theorem 4.1 the conclusion follows. $\square$

5. Examples of Shiba behavior spaces in Section 4.

We show examples of Shiba behavior space different from those in Section 3.

Example 1. Let

$$\begin{aligned} J_{mk} &= \{j \mid (j \in J_k) \text{ and } (j \leq p(m))\}, \\ J_K^m &= \{j \mid (j \in J_K) \text{ or } (j > p(m) \text{ and } j \notin J_1)\}, \\ \mathbb{J}^m &= \{J_1, J_{m2}, \ldots, J_{m(K-1)}, J_K^m\}, \\ S(J_{mk}) &= \biggl\{ \sum_{j \in J_{mk}} (a_j \sigma_{A_j} + b_j \sigma_{B_j}) \biggm| a_j \text{ and } b_j \text{ are real} \biggr\}, \end{aligned}$$

$$S(J_K^m) = Cl\Bigg(\Bigg\{ \sum_{j \in J_K^m} (a_j \sigma_{A_j} + b_j \sigma_{B_j}) \Bigg| \, a_j \text{ and } b_j \text{ are real}$$
$$\text{and only a finite number of } \{a_k, b_\ell\} \text{ do not vanish} \Bigg\}\Bigg),$$

and $\{\Gamma_{hm,k}\}_{k=2}^{K}$ be closed subspaces of Γ_{hm} such that $\sum_{k=2}^{K} \Gamma_{hm,k} = \Gamma_{hm}$ and the dimension of each $\Gamma_{hm,k}$ $(k = 2, \ldots, K-1)$ is finite. Set

$$\Gamma_m^{(1)} = \Gamma_{hse} \cap \left(\bigcap_{k=2}^{K-1} {}^*S(J_{mk})^{\perp} \right) \cap {}^*S(J_K^m)^{\perp},$$
$$\Gamma_m^{(k)} = \Gamma_{hm,k} + S(J_{mk}) \quad (k = 2, 3, \ldots, K-1),$$
$$\Gamma_m^{(K)} = Cl(\Gamma_{hm,K} + S(J_K^m)).$$

Then we have the following:

(1) For k $(k = 2, \ldots, K-1)$

$${}^*\Gamma_m^{(k)\perp} = {}^*\Gamma_{hm,k}^{\perp} \cap {}^*S(J_{mk})^{\perp} \supset \Gamma_{hse} \cap {}^*S(J_{mk})^{\perp} \supset \Gamma_m^{(j)} \quad (j \neq k),$$

and

$${}^*\Gamma_m^{(K)\perp} = {}^*\Gamma_{hm,K}^{\perp} \cap {}^*S(J_K^m)^{\perp} \supset \Gamma_{hse} \cap {}^*S(J_K^m)^{\perp} \supset \Gamma_m^{(j)} \quad (j \neq K).$$

(2)
$$\begin{aligned} {}^*\Gamma_m^{(1)\perp} &= Cl\Bigg(\Gamma_{hm} + \sum_{k=2}^{K-1} S(J_{mk}) + S(J_K^m)\Bigg) \\ &= \sum_{k=2}^{K-1} (\Gamma_{hm,k} + S(J_{mk})) + Cl(\Gamma_{hm,K} + S(J_K^m)) = \sum_{k=2}^{K} \Gamma_m^{(k)}, \end{aligned}$$

because the dimension of $\sum_{k=2}^{K-1} (\Gamma_{hm,k} + S(J_{mk}))$ is finite.

(3) For k $(k = 2, \ldots, K-1)$

$$S(J_{mk}) \subset \Gamma_{hm,k} + S(J_{mk}) = \Gamma_m^{(k)} \subset \Gamma_{hse},$$
$$S(J_K^m) \subset \Gamma_{hm,K} + S(J_K^m) \subset \Gamma_m^{(K)} \subset \Gamma_{hse},$$

and

$$S(J_1) \subset \Gamma_m^{(1)} \subset \Gamma_{hse}.$$

Setting $\Lambda(K)^{(m)} = \sum_{k=1}^{K} e^{i\theta_k} \Gamma_m^{(k)}$, we have by Lemma 2.1,

$$Cl(\Lambda(K)^{(m)}) = Cl\Big(\sum_{k=1}^{K} e^{i\theta_k}\Gamma_m^{(k)}\Big) = \sum_{k=2}^{K-1} e^{i\theta_k}\Gamma_m^{(k)} + Cl\big(e^{i\theta_1}\Gamma_m^{(1)} + e^{i\theta_K}\Gamma_m^{(K)}\big),$$

and, by the same argument as in Remark 4.1, $Cl(e^{i\theta_1}\Gamma_m^{(1)} + e^{i\theta_K}\Gamma_m^{(K)}) = e^{i\theta_1}\Gamma_m^{(1)} + e^{i\theta_K}\Gamma_m^{(K)}$. Hence $Cl(\Lambda(K)^{(m)}) = \Lambda(K)^{(m)}$. By Theorem 4.1, we see that $\Lambda(K)^{(m)}$ is a Shiba behavior space associated with $\mathcal{L}(\mathbb{J}^m)$, where $\Lambda(K)^{(m)}$ has the period condition containing two lines in a neighborhood of the ideal boundary. Further the closed space $U_m = \sum_{k=2}^{K-1} e^{i\theta_k}\Gamma_m^{(k)}$ is increasing and $V_m = e^{i\theta_1}\Gamma_m^{(1)} + e^{i\theta_K}\Gamma_m^{(K)}$ is decreasing. Therefore, by Theorem 2.1, the strong limit $\Lambda_s(K)$ of $\{\Lambda(K)^{(m)}\}$ is a Shiba behavior space associated with $\mathcal{L}(\mathbb{J})$, which is different from those given in Section 3.

Example 2. Let $\Gamma_{hm}^{(1)} = Cl(\Gamma_{hm} + S(J_1))$ and $\{\Gamma_{hse,k}\}_{k=2}^K$ be closed subspaces of Γ_{hse} such that

$$S(J_{mk}) \subset \Gamma_{hse,k} \subset \Gamma_{hse} \cap {}^*S(J_1)^\perp \cap \Bigg(\bigcap_{j=2, j\neq k}^{K} {}^*\Gamma_{hse,j}^\perp\Bigg)$$

and

$$\sum_{k=2}^{K} \Gamma_{hse,k} = \Gamma_{hse} \cap {}^*S(J_1)^\perp = {}^*\Gamma_{hm}^{(1)\perp}.$$

Then

$${}^*S(J_{mk})^\perp \supset {}^*\Gamma_{hse,k}^\perp \supset \Gamma_{hm} + S(J_1) + \sum_{j=2, j\neq k}^{K} \Gamma_{hse,j}.$$

By Theorem 4.1 $Cl(e^{i\theta_1}\Gamma_{hm}^{(1)} + \sum_{k=2}^K e^{i\theta_k}\Gamma_{hse,k})$ is a Shiba behavior space associated with $\mathcal{L}(\mathbb{J})$. This type of behavior spaces may contain, for the interior of a compact bordered Riemann surface, the behavior spaces which give the slit mapping with arbitrarily prescribed directions, see [**4**] and [**6**].

Now, we give an example of $\{\Gamma_{hse,k}\}_{k=2}^K$. Let a Riemann surface R have a regular partition $\{\beta_k\}_{k=2}^K$ of the Kerékjártó-Stoïlow ideal boundary Δ and U_k be a neighborhood of β_k on $R^* = R \cup \Delta$ such that $U_j \cap U_k = \emptyset$ for $j \neq k$. Take an R_N whose complement is contained in $\bigcup_{k=2}^K U_k$. Set

$$J_1 = \{j \mid 1 \leq j \leq p(N)\}$$

and for $2 \leq k \leq K$

$$J_k = \{j \mid A_j, B_j \subset (R - R_N) \cap U_k\}.$$

Let Γ_k^1 be a subspace of C^1 differentials $\{\omega\}$ such that ω is exact on $(R - R_N) \cap U_k$

and for an $n > N$ depending on ω is 0 on $(R - R_n) \cap U_k$. Set $\Gamma_{hse,k} = \Gamma_h \cap {}^*\Gamma_k^{1\perp}$. For $\sigma \in S(J_k)$ and $\omega \in \Gamma_k^1$, we see that $\langle {}^*\sigma, \omega \rangle = 0$ and $S(J_k) \subset \Gamma_{hse,k}$. Since the closed subspace $Cl(\Gamma_k^1)$ contains $\Gamma_{hm} + \sum_{j \neq k} S(J_j)$, we have

$$\Gamma_{hse,k} \subset \Gamma_{hse} \cap \left(\bigcap_{j \neq k} {}^*S(J_j)^\perp \right).$$

Every differential $\omega \in \Gamma_{hse,k}$ is exact on $R - ((R - R_N) \cap U_k)$ and is orthogonal to ${}^*\Gamma_k^1$. The $\omega = df$ on $R - ((R - R_N) \cap U_k)$ is approximated by df_n which is 0 on a neighborhood of $\bigcup_{j \neq k} \beta_j$. Hence we see that $\Gamma_{hse,k} \subset Cl(\Gamma_j^1)$ $(j \neq k)$. It follows that

$$\Gamma_{hse,k} \subset \Gamma_{hse} \cap {}^*S(J_1)^\perp \cap \left(\bigcap_{j=2, j \neq k}^{K} {}^*\Gamma_{hse,j}^\perp \right).$$

Since any differential in $\Gamma_{hse} \cap {}^*S(J_1)^\perp$ restricted to $(R - R_N) \cap U_k$ is extended to a differential in $\Gamma_{hse,k} + \Gamma_{eo}$, it also can be showed that

$$\sum_{k=2}^{K} \Gamma_{hse,k} = \Gamma_{hse} \cap {}^*S(J_1)^\perp.$$

Every differential in the behavior space of this example is, roughly speaking, L_k-valued along β_k.

Finally, we decompose a Shiba behavior space. Under a suitable condition we can reconstruct it from the decomposition by the method in Section 4. Let $K < \infty$, $\mathbb{J} = \{J_k\}_{k=1}^K$ be a division of numbers $\{1, 2, \ldots, g\}$ and $\mathcal{L}(\mathbb{J}) = \{L_k = L(\theta_k)\}_{k=1}^K$, where $L_j \cap L_k = \{0\}$ for $j \neq k$. Let Λ_x be an arbitrary Shiba behavior space associated with $\mathcal{L}(\mathbb{J})$. We set

$${}_k\Gamma = \{\text{Imaginary part of } e^{-i\theta_k}\omega \mid \omega \in \Lambda_x\}, \quad \Gamma^{(k)} = {}^*({}_k\Gamma^\perp)$$

and

$$\tilde{\Lambda}_x = \sum_{k=1}^{K} e^{i\theta_k} \Gamma^{(k)}.$$

Theorem 5.1. *If* ${}^*\Gamma^{(1)\perp} \subset \sum_{k=2}^K \Gamma^{(k)}$, *then* $Cl(\tilde{\Lambda}_x) = \Lambda_x$.

Proof. For $\lambda \in \Lambda_x$ and $\sigma_k \in \Gamma^{(k)} (k = 1, 2, \ldots, K)$, we have

$$\langle \lambda, ie^{i\theta_k}\, {}^*\sigma_k \rangle = \langle e^{-i\theta_k}\lambda, i\, {}^*\sigma_k \rangle = \langle \text{Imaginary part of } e^{-i\theta_k}\lambda, {}^*\sigma_k \rangle = 0$$

and

$$\left\langle \lambda, i\,^*\left(\sum_{k=1}^{K} e^{i\theta_k}\sigma_k\right)\right\rangle = 0.$$

This shows that Λ_x is orthogonal to $i\,^*\tilde{\Lambda}_x$ i.e. $i\,^*\tilde{\Lambda}_x^{\perp} \supset \Lambda_x = i\,^*\Lambda_x^{\perp}$. Hence $\tilde{\Lambda}_x \subset \Lambda_x$. For $\sigma_j \in \Gamma^{(j)}$ and $\sigma_k \in \Gamma^{(k)}$ $(j \neq k)$, differentials $e^{i\theta_j}\sigma_j$ and $e^{i\theta_k}\sigma_k$ belong to $\tilde{\Lambda}_x$. We have

$$0 = \langle e^{i\theta_j}\sigma_j, ie^{i\theta_k}\,^*\sigma_k\rangle = \Re(-ie^{i(\theta_j-\theta_k)})\langle \sigma_j, {}^*\sigma_k\rangle.$$

Since $\Re(-ie^{i(\theta_j-\theta_k)}) \neq 0$, it holds $\langle \sigma_j, {}^*\sigma_k\rangle = 0$. Therefore $\Gamma^{(j)}$ is orthogonal to ${}^*\Gamma^{(k)}$. Particularly, $\Gamma^{(1)\perp} \supset \sum_{k=2}^{K} {}^*\Gamma^{(k)}$. By assumption we get ${}^*\Gamma^{(1)\perp} = \sum_{k=2}^{K} \Gamma^{(k)}$. It follows, by Proposition 4.1, that $Cl(\tilde{\Lambda}_x)$ is a pre-behavior space and $Cl(\tilde{\Lambda}_x) = i\,^*Cl(\tilde{\Lambda}_x)^{\perp} \supset \Lambda_x$. Therefore $Cl(\tilde{\Lambda}_x) = \Lambda_x$. $\square$

Acknowledgements. The authors are grateful to Professor Hiroaki Aikawa for helpful suggestions and kind advices. They also express their sincere thanks to the referees for their careful reading and valuable comments.

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