J. Math. Soc. Japan
Vol. 66, No. 2 (2014) pp. 535–563
doi: 10.2969/jmsj/06620535

Relative stability and extremal metrics

By Toshiki Mabuchi

(Received July 2, 2010)
(Revised June 28, 2012)

Abstract. In this paper, by clarifying the concept of relative K-stability in [**28**], we shall solve the stability part of an extremal Kähler version of Donaldson-Tian-Yau's Conjecture. This extends the results in [**15**] and [**27**]. We then propose a program to solve the existence part of the conjecture.

1. Introduction.

In this paper, we shall study the relative K-stability in Székelyhidi [**28**] and the asymptotic relative Chow-stability in [**17**] (see also [**11**]) from the viewpoints of the existence problem of extremal Kähler metrics on a polarized algebraic manifold (M, L). In clarifying these concepts of relative stability, we are led to study piecewise bilinear forms associated to toric subvarieties of the Hilbert schemes (cf. Section 3, Theorem B). For a maximal compact connected subgroup K of the group $\mathrm{Aut}(M)$ of all holomorphic automorphisms of M, we here consider the extremal Kähler vector field $\mathcal{V} \in \mathfrak{k} := \mathrm{Lie}\, K$ for the class $c_1(L)_{\mathbb{R}}$. Let

$$T \in \mathcal{T}_{\mathrm{ex}}(M, L),$$

i.e., T is an algebraic torus in $\mathrm{Aut}(M)$ such that the maximal compact subgroup of T sits in K and that T contains the one-dimensional algebraic torus generated by $\mathcal{V}$. Then in terms of these concepts of relative stability, we propose in the last section a program to solve the following extremal Kähler version (cf. [**28**]) of Donaldson-Tian-Yau's Conjecture:

Conjecture A. *A polarized algebraic manifold (M, L) admits an extremal Kähler metric in the class $c_1(L)_{\mathbb{R}}$ if and only if (M, L) is K-stable relative to T above.*

The "only if" part of this conjecture will be proved affirmatively in Section 6, Theorem C, extending the results in [**15**] and [**27**]. In particular, our result solves the stability part of the original Donaldson-Tian-Yau's Conjecture, since by assuming the existence of constant scalar curvature Kähler metrics in $c_1(L)_{\mathbb{R}}$, we obtain $T = \{1\} \in \mathcal{T}_{\mathrm{ex}}(M, L)$.

2010 *Mathematics Subject Classification.* Primary 53C25; Secondary 32Q15, 32Q26.
Key Words and Phrases. K-stability, extremal Kähler metrics, relative stability, test configurations, Donaldson-Tian-Yau's Conjecture.
This research was supported by JSPS Grant-in-Aid for Scientific Research (A) (No. 20244005).

2. Background materials.

Here a *polarized algebraic manifold* (M, L) means a pair of a connected projective algebraic manifold M, defined over $\mathbb{C}$, and a very ample holomorphic line bundle L over M. Put $n := \dim_{\mathbb{C}} M$. For a maximal connected linear algebraic subgroup G of $\mathrm{Aut}(M)$, the Chevalley decomposition allows us to write G as a semidirect product

$$G = R_{\mathbb{C}} \ltimes U$$

of a reductive algebraic group $R_{\mathbb{C}}$ and the unipotent radical U of G. Let $\mathfrak{g} := \mathrm{Lie}\, G$ and $\mathfrak{r} := \mathrm{Lie}\, R_{\mathbb{C}}$ be the Lie algebras of G and $R_{\mathbb{C}}$, respectively. Then we may assume that $\mathfrak{r}$ is a complexification of $\mathfrak{k}$ in the introduction. As in [**5**], consider the Lie algebra characters

$$\mathcal{F}_p : \mathfrak{g} \to \mathbb{C}, \qquad p = 1, 2, \dots, n,$$

defined as obstructions to asymptotic Chow semistability of (M, L), where $\mathcal{F}_1$ is the classical Futaki character of M. For the center $\mathfrak{z}$ of $\mathfrak{r}$, define a subspace $\mathfrak{a}$ of $\mathfrak{z}$ consisting of all $A \in \mathfrak{z}$ such that

$$\mathcal{F}_p(A) = 0, \qquad \text{for all } p = 1, 2, \dots, n.$$

By setting $\mathfrak{z}_{\mathbb{Z}} := \{X \in \mathfrak{z}; \exp(2\pi\sqrt{-1}\, X) = \mathrm{id}_M\}$, we have an integral structure of $\mathfrak{z}$. Then by the nondegenerate symmetric bilinear form $\langle\ ,\ \rangle_0$ on $\mathfrak{g}$ as in [**6**], we define a complex Lie algebra

$$\mathfrak{b}_0 := \mathfrak{a}^{\perp 0}$$

to be the orthogonal complement, defined over $\mathbb{Q}$, of $\mathfrak{a}$ in $\mathfrak{z}$ consisting of all $B \in \mathfrak{z}$ such that $\langle A, B \rangle_0 = 0$ for all $A \in \mathfrak{a}$. Since $\mathrm{Ker}\, \mathcal{F}_1$ is perpendicular to $\mathfrak{t}_{\mathrm{ex}} := \mathbb{C}\mathcal{V}$ by $\langle\ ,\ \rangle_0$, we see that

$$\mathfrak{t}_{\mathrm{ex}} \subset \mathfrak{b}_0. \tag{2.1}$$

Let $\mathcal{T}_{\mathrm{ex}}(M, L)$ be the set of all algebraic tori T in G such that the maximal compact subgroup of T sits in K and that $\mathfrak{t} := \mathrm{Lie}\, T$ satisfies

$$\mathfrak{t}_{\mathrm{ex}} \subset \mathfrak{t}.$$

Now the infinitesimal action of the Lie algebra $\mathfrak{g}$ on M lifts to an infinitesimal bundle action of $\mathfrak{g}$ on L. Then by setting

$$V_m := H^0(M, \mathcal{O}(L^m)), \qquad m = 1, 2, \dots,$$

we view $\mathfrak{g}$ as a Lie subalgebra of $\mathfrak{sl}(V_m)$ by considering the traceless part. We now define a symmetric bilinear form $\langle\ ,\ \rangle_m$ on $\mathfrak{sl}(V_m)$ by

$$\langle X, Y\rangle_m = \mathrm{Tr}(XY)/m^{n+2}, \qquad X, Y \in \mathfrak{sl}(V_m),$$

whose asymptotic limit as $m \to \infty$ plays an important role (cf. [**28**]) as in Theorem B in Section 3. Since $\langle\ ,\ \rangle_m$ restricted to the Lie subalgebra $\mathfrak{z}$ of $\mathfrak{sl}(V_m)$ is nondegenerate for each positive integer m, we can define a complex Lie algebra

$$\mathfrak{b}_m := \mathfrak{a}^{\perp m}$$

as the orthogonal complement, defined over $\mathbb{Q}$, of $\mathfrak{a}$ in $\mathfrak{z}$ consisting of all $B \in \mathfrak{z}$ such that $\langle A, B\rangle_m = 0$ for all $A \in \mathfrak{a}$. Let $\mathfrak{t}_{\min}$ denote the complex Lie subalgebra, defined over $\mathbb{Q}$, of $\mathfrak{z}$ generated by all

$$\mathfrak{b}_m, \qquad m = 0, 1, \ldots,$$

in the center $\mathfrak{z}$. For instance, if the obstruction $Obstr(M, L)$ in [**5**] and [**10**] vanishes, then we have $\mathfrak{t}_{\min} = \{0\}$. Let $\mathcal{T}_{\min}(M, L)$ denote the nonempty set of all algebraic tori T in G such that the maximal compact subgroup of T sits in K and that $\mathfrak{t} := \mathrm{Lie}\, T$ satisfies

$$\mathfrak{t}_{\min} \subset \mathfrak{t},$$

where we need $\mathcal{T}_{\min}(M, L)$ only in the last section. For a maximal element $T_{\max}$ of $\mathcal{T}_{\min}(M, L)$, we see that $T_{\max}$ is a maximal algebraic torus in G satisfying $\mathfrak{t}_{\min} \subset \mathfrak{t}_{\max} := \mathrm{Lie}\, T_{\max}$. Let T_{ex} be the one-dimensional algebraic torus in G generated by $\mathcal{V}$, so that $\mathrm{Lie}\, T_{\mathrm{ex}} = \mathfrak{t}_{\mathrm{ex}}$. By (2.1), we have $\mathfrak{t}_{\mathrm{ex}} \subset \mathfrak{t}_{\min}$. Hence

$$\mathcal{T}_{\min}(M, L) \subset \mathcal{T}_{\mathrm{ex}}(M, L).$$

For each $T \in \mathcal{T}_{\mathrm{ex}}(M, L)$, let T_m denote the associated algebraic torus in $SL(V_m)$ such that $\mathfrak{t}_m := \mathrm{Lie}\, T_m$ is the Lie subalgebra of $\mathfrak{sl}(V_m)$ infinitesimally induced by $\mathfrak{t} = \mathrm{Lie}\, T$. Then by the T_m-action on V_m,

$$V_m = \bigoplus_{k=1}^{\nu_m} V(\chi_{m;k}),$$

where $V(\chi_{m;k}) := \{v \in V_m; g \cdot v = \chi_{m;k}(g)v \text{ for all } g \in T_m\}$ with mutually distinct multiplicative characters $\chi_{m;k} \in \mathrm{Hom}(T_m, \mathbb{C}^*)$, $k = 1, 2, \ldots, \nu_m$. Consider the algebraic subgroup S_m of $SL(V_m)$ defined by

$$S_m := \prod_{k=1}^{\nu_m} SL(V(\chi_{m;k})),$$

where each $SL(V(\chi_{m;k}))$ acts on V_m fixing $V(\chi_{m;i})$ if $i \neq k$. The centralizer H_m of S_m in $SL(V_m)$ consists of all diagonal matrices in $SL(V_m)$ acting on each $V(\chi_{m;k})$ by constant scalar multiplication. Hence the centralizer $Z(T_m)$ of T_m in $SL(V_m)$ is $H_m \cdot S_m$ with Lie algebra

$$\mathfrak{z}(\mathfrak{t}_m) = \mathfrak{h}_m + \mathfrak{s}_m,$$

where $\mathfrak{s}_m := \mathrm{Lie}\, S_m$ and $\mathfrak{h}_m := \mathrm{Lie}\, H_m$. In general, for a complex Lie subalgebra $\mathfrak{x}$ of $\mathfrak{sl}(V_m)$, we denote by $\mathfrak{x}_{\mathbb{Z}}$ the kernel of the map

$$\mathfrak{x} \ni X \mapsto \exp\left(2\pi\sqrt{-1}\, X\right) \in SL(V_m),$$

and if $\mathfrak{x}$ is abelian, we regard $\mathfrak{x}_{\mathbb{R}} := \mathfrak{x}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$ as a real Lie subalgebra of $\mathfrak{x}$. In particular, for $\mathfrak{x} = \mathfrak{h}_m$, we view $(\mathfrak{h}_m)_{\mathbb{R}} := (\mathfrak{h}_m)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$ as a real Lie subalgebra of $\mathfrak{h}_m$. For the orthogonal complement $\mathfrak{t}_m^{\perp}$ of $\mathfrak{t}_m$ $(= \mathfrak{t})$ in $\mathfrak{h}_m$ by the nondegenerate bilinear form $\langle\ ,\ \rangle_m$ above, let $T_m^{\perp}$ denote the corresponding algebraic torus sitting in H_m. We now define an algebraic subgroup G_m of $Z(T_m)$ by

$$G_m := T_m^{\perp} \cdot S_m.$$

3. Piecewise bilinear forms on $(\mathfrak{h}_m)_{\mathbb{R}}$.

In this section, let $T \in \mathcal{T}_{\mathrm{ex}}(M, L)$, and by fixing a positive integer m arbitrarily, we set $N_{\mathbb{R}} := (\mathfrak{h}_m)_{\mathbb{R}}/\mathfrak{g}^{\bullet}$ and $\tilde{N}_{\mathbb{R}} := (\mathfrak{h}_m)_{\mathbb{R}}/\mathfrak{t}^{\bullet}$, where $\mathfrak{g}^{\bullet} := \mathfrak{g} \cap (\mathfrak{h}_m)_{\mathbb{R}}$ and $\mathfrak{t}^{\bullet} := \mathfrak{t}_{\mathbb{R}} = \mathfrak{t} \cap (\mathfrak{h}_m)_{\mathbb{R}}$. We now consider the fan Δ in $N_{\mathbb{R}}$ associated to the toric variety $\mathcal{H}$ obtained as the closure of $H_m \cdot \gamma_M$ in the Hilbert scheme $\mathrm{Hilb}\, \mathbb{P}^*(V_m)$. Here γ_M denotes the point in $\mathrm{Hilb}\, \mathbb{P}^*(V_m)$ associated to the polarized subvariety (M, L^m) of $(\mathbb{P}^*(V_m), \mathcal{O}_{\mathbb{P}^*(V_m)}(1))$ in terms of the Kodaira embedding

$$\Phi_m : M \hookrightarrow \mathbb{P}^*(V_m)$$

by the complete linear system $|L^m|$. Note that the Lie algebra of the isotropy subgroup of H_m at γ_M is just the complexification in $\mathfrak{h}_m$ of the real Lie algebra $\mathfrak{g}^{\bullet}$. Let

$$\pi : (\mathfrak{h}_m)_{\mathbb{R}} \to N_{\mathbb{R}}, \quad \tilde{\pi} : (\mathfrak{h}_m)_{\mathbb{R}} \to \tilde{N}_{\mathbb{R}}, \quad \mathrm{pr} : \tilde{N}_{\mathbb{R}} \to N_{\mathbb{R}}$$

be the natural projections. Then Δ is a collection of strongly convex rational polyhedral cones C_j (cf. [**21**]), $j = 1, 2, \dots, r$, in $N_{\mathbb{R}}$ such that

$$N_{\mathbb{R}} = \bigcup_{j=1}^{r_1} C_j,$$

where $\{C_1, C_2, \dots, C_{r_1}\}$ denotes the set of all C_j's in Δ such that $\dim C_j = \dim N_{\mathbb{R}}$. For each $j = 1, 2, \dots, r$, by setting

$$\Sigma_j := \pi^{-1}(C_j) \quad \text{and} \quad \tilde{C}_j := \mathrm{pr}^{-1}(C_j),$$

we consider the open face Σ_j^0 of Σ_j. Let θ be a collection of continuous maps $\theta_j : \Sigma_j \times \Sigma_j \to \mathbb{R}$, $j = 1, 2, \dots, r_1$, which are symmetric, i.e., $\theta_j(X,Y) = \theta_j(Y,X)$ for all $(X,Y) \in \Sigma_j \times \Sigma_j$. Put $\Sigma_{ij} := \Sigma_i \cap \Sigma_j$.

Definition 3.1. θ is said to be a *piecewise bilinear form* if each θ_j extends to a symmetric bilinear form, denoted by the same θ_j by abuse of terminology, on $(\mathfrak{h}_m)_{\mathbb{R}}$ such that

$$\theta_{i|\Sigma_{ij}\times\Sigma_{ij}} = \theta_{j|\Sigma_{ij}\times\Sigma_{ij}}, \qquad i, j \in \{1, 2, \dots, r_1\}. \tag{3.2}$$

In view of the inclusion $\mathcal{H} \subset \operatorname{Hilb}\mathbb{P}^*(V_m)$, the universal family over the Hilbert scheme $\operatorname{Hilb}\mathbb{P}^*(V_m)$ restricts to a family

$$p : \mathcal{Z} \to \mathcal{H}$$

over $\mathcal{H}$ such that, via the H_m-actions on $\mathcal{H}$ and also on $\mathbb{P}^*(V_m)$, the subscheme $\mathcal{Z}$ of $\mathcal{H} \times \mathbb{P}^*(V_m)$ is preserved by the H_m-action with fibers

$$\mathcal{Z}_s \subset \{s\} \times \mathbb{P}^*(V_m) = \mathbb{P}^*(V_m), \qquad s \in \mathcal{H}, \tag{3.3}$$

regarded as the corresponding subschemes of $\mathbb{P}^*(V_m)$. Here for each $s \in \mathcal{H}$, we denote by $\mathcal{Z}_s := p^{-1}(s)$ the scheme-theoretic fiber of p over the point s. For simplicity, we put $\mathcal{L} := p_2^*\mathcal{O}_{\mathbb{P}^*(V_m)}(1)$, where $p_2 : \mathcal{Z} \to \mathbb{P}^*(V_m)$ is the restriction to $\mathcal{Z}$ of the projection of $\mathcal{H} \times \mathbb{P}^*(V_m)$ to the second factor $\mathbb{P}^*(V_m)$. For each $X \in \mathfrak{z}(\mathfrak{t}_m)_{\mathbb{Z}}$, by setting

$$\varphi_X(t) := \exp\{(\log t)X\}, \qquad t \in \mathbb{C}^*, \tag{3.4}$$

we have an algebraic group homomorphism $\varphi_X : \mathbb{C}^* \to Z(T_m)$. Hereafter until the end of this section, we assume that $X \in (\mathfrak{h}_m)_{\mathbb{Z}}$. We now observe that $(\mathfrak{h}_m)_{\mathbb{R}}$ is a disjoint union of all Σ_j^0, $j = 1, 2, \dots, r$, where for each such j, as long as $X \in \Sigma_j^0 \cap (\mathfrak{h}_m)_{\mathbb{Z}}$, the limit

$$\gamma_j := \lim_{t\to 0} \varphi_X(t) \cdot \gamma_M$$

depends only on j, and is independent of the choice of X in $\Sigma_j^0 \cap (\mathfrak{h}_m)_{\mathbb{Z}}$. In (3.3), by setting $s = \gamma_j$, we have the fiber $\mathcal{Z}_j := \mathcal{Z}_{\gamma_j}$ of $\mathcal{Z}$ over γ_j. For each $j = 1, 2, \dots, r$, we put $\mathcal{L}_j := \mathcal{L}_{|\mathcal{Z}_j}$ and let G_j be the algebraic torus in H_m generated by $\Sigma_j^0 \cap (\mathfrak{h}_m)_{\mathbb{Z}}$. Then the G_j-action on $(\mathcal{Z}, \mathcal{L})$ preserves the polarized subvariety $(\mathcal{Z}_j, \mathcal{L}_j)$, where (M, L^m) degenerates to $(\mathcal{Z}_j, \mathcal{L}_j)$ as $t \to 0$ for the action of the one-parameter group

$$\varphi_X : \mathbb{C}^* \to H_m, \qquad t \mapsto \varphi_X(t),$$

provided that $X \in \Sigma_j^0 \cap (\mathfrak{h}_m)_{\mathbb{Z}}$. On the other hand, the real subspace $\mathfrak{g}_{j_{\mathbb{R}}}$ of $(\mathfrak{h}_m)_{\mathbb{R}}$ generated by $\Sigma_j^0 \cap (\mathfrak{h}_m)_{\mathbb{Z}}$ is expressible as

$$\mathfrak{g}_{j\mathbb{R}} = (\mathfrak{h}_m)_{\mathbb{R}}, \qquad \text{if } 1 \le j \le r_1. \tag{3.5}$$

For positive integers ℓ, we consider the direct image sheaves $E_\ell := p_*\mathcal{L}^\ell$ over $\mathcal{H}$. In this paper, locally free sheaves and holomorphic vector bundles are used interchangeably. If $\ell \gg 1$, then E_ℓ is a vector bundle over $\mathcal{Z}$ and the fiber $(E_\ell)_{\gamma_j}$ over γ_j is identified with $H^0(\mathcal{Z}_j, \mathcal{L}_j^\ell)$. Put

$$d_\ell := \dim(E_\ell)_{\gamma_j} = \dim V_{\ell m}.$$

For each $X, Y \in \mathfrak{g}_{j\mathbb{R}}$, consider endomorphisms $X_{\ell;j}, Y_{\ell;j} \in \mathrm{End}(E_\ell)_{\gamma_j}$ induced by X, Y, respectively. For each $1 \le j \le r$ and $\ell \gg 1$, we have a symmetric bilinear forms $\theta_j^{(\ell)} : \mathfrak{g}_{j\mathbb{R}} \times \mathfrak{g}_{j\mathbb{R}} \to \mathbb{R}$, defined over $\mathbb{Q}$, by

$$\theta_j^{(\ell)}(X,Y) := \mathrm{Tr}(\hat{X}_{\ell;j}\hat{Y}_{\ell;j})/(\ell m)^{n+2}, \tag{3.6}$$

where $\hat{X}_{\ell;j}$, $\hat{Y}_{\ell;j} \in \mathfrak{sl}(E_\ell)_{\gamma_j}$ are traceless parts of $X_{\ell;j}$, $Y_{\ell;j}$ defined by

$$\hat{X}_{\ell;j} = X_{\ell;j} - \frac{\mathrm{Tr}(X_{\ell;j})}{d_\ell}\,\mathrm{id}_{(E_\ell)_{\gamma_j}}, \quad \hat{Y}_{\ell;j} = Y_{\ell;j} - \frac{\mathrm{Tr}(Y_{\ell;j})}{d_\ell}\,\mathrm{id}_{(E_\ell)_{\gamma_j}}.$$

For C_j, $C_k \in \Delta$, suppose that C_k is a face of C_j. Then by choosing an element X of $\Sigma_j^0 \cap (\mathfrak{h}_m)_{\mathbb{Z}}$, we see that $(\mathcal{Z}_k, \mathcal{L}_k)$ degenerates to $(\mathcal{Z}_j, \mathcal{L}_j)$ as $t \to 0$ for the action of the one-parameter group $\varphi_X(t)$, $t \in \mathbb{C}^*$, in H_m. Since E_ℓ can be G_j-equivariantly trivialized for degeneration along the one-parameter group, we hence obtain

$$\theta_j^{(\ell)}(X,Y) = \theta_k^{(\ell)}(X,Y), \qquad X, Y \in \mathfrak{g}_{k\mathbb{R}}. \tag{3.7}$$

Then by (3.5) and (3.7), $\theta^{(\ell)} = \{\theta_j^{(\ell)}; j = 1, 2, \dots, r_1\}$ is a piecewise symmetric bilinear form, since for $i, j \in \{1, 2, \dots, r_1\}$ with $\Sigma_{ij} \ne \emptyset$,

$$\theta_i^{(\ell)}(X,Y) = \theta_k^{(\ell)}(X,Y) = \theta_j^{(\ell)}(X,Y), \qquad X, Y \in \Sigma_{ij},$$

where $k \in \{1, 2, \dots, r\}$ is such that $C_k = C_i \cap C_j$. Now for $\ell = 1$, it is easy to check that the piecewise bilinear form $\theta^{(1)} = \{\theta_j^{(1)}\}$ coincides with $\langle\ ,\ \rangle_m$ on $(\mathfrak{h}_m)_{\mathbb{R}}$. On the other hand, for $\ell \to \infty$, we obtain

Theorem B. *The limit $\theta = \{\theta_j; j = 1, 2, \dots, r_1\}$ given by*

$$\theta_j(X,Y) := \lim_{\ell \to \infty} \theta_j^{(\ell)}(X,Y), \qquad X, Y \in \Sigma_j,$$

is a well-defined piecewise bilinear form such that each θ_j extends to a positive semidefinite bilinear form, defined over $\mathbb{Q}$, on $(\mathfrak{h}_m)_{\mathbb{R}}$.

PROOF. It suffices to show that, for each $j \in \{1, 2, \dots, r_1\}$, the bilinear form $\theta_j^{(\ell)}$ on $(\mathfrak{h}_m)_{\mathbb{R}}$ converges as $\ell \to \infty$ and also that the limit θ_j is a positive semidefinite bilinear form defined over $\mathbb{Q}$. Let us now define a quadratic form Q_ℓ on $\mathfrak{h}_m$ by

$$Q_\ell(X) := \theta_j^{(\ell)}(X, X), \qquad X \in (\mathfrak{h}_m)_{\mathbb{R}} \ (= \mathfrak{g}_{j\mathbb{R}}).$$

By the identity $2\theta_j^{(\ell)}(X, Y) = Q_\ell(X+Y) - Q_\ell(X) - Q_\ell(Y)$, the proof of the convergence of $\theta_j^{(\ell)}$ as $\ell \to \infty$ is reduced to showing the convergence of the sequence $\{Q_\ell(X); \ell = 1, 2, \dots\}$ for each fixed $X \in (\mathfrak{h}_m)_{\mathbb{R}}$. In view of [**28**] (see also [**4**]) and the definition (3.6) of $\theta_j^{(\ell)}$, the function $\ell^{n+2}Q_\ell(X)$ in $\ell \gg 1$ is a polynomial of degree $n+2$ with a leading coefficient α independent of the choice of $\ell \gg 1$, so that we can write

$$Q_\ell(X) = \alpha + O(\ell^{-1}),$$

where $\alpha = \int_{\mathcal{Z}_j} h_X^2 \omega_{\mathrm{FS}}^n$ for some real Hamiltonian function h_X on $\mathcal{Z}_j \hookrightarrow \mathbb{P}^*(V_m)$ associated to X. Hence $Q_\ell(X)$ converges to α as $\ell \to \infty$. Thus

$$\theta_j(X, X) = \alpha \geq 0.$$

Moreover if $X \in (\mathfrak{h}_m)_{\mathbb{Z}}$, then $\ell^{n+2}Q_\ell(X)$ is a polynomial in $\ell \gg 1$ with rational coefficients, so that its leading coefficient α sits in $\mathbb{Q}$. Hence the limit θ_j on $(\mathfrak{h}_m)_{\mathbb{R}}$ is a well-defined positive semidefinite bilinear form defined over $\mathbb{Q}$, as required. □

Since $\mathfrak{g}^\bullet \subset \Sigma_{ij} \subset \Sigma_j$ for all $i, j \in \{1, 2, \dots, r_1\}$, it follows from (3.2) that there exists a continuous map $u : \mathfrak{g}^\bullet \times (\mathfrak{h}_m)_{\mathbb{R}} \to \mathbb{R}$ such that

$$u_{|\mathfrak{g}^\bullet \times \Sigma_j} = \theta_j, \qquad j = 1, 2, \dots, r_1,$$

and that the restriction of u to $\mathfrak{g}^\bullet \times \mathfrak{g}^\bullet$ is the positive definite symmetric bilinear form $\langle\ ,\ \rangle_0$ as in [**6**] (see the remark in [**28**]). In view of $\mathfrak{t}^\bullet \subset \mathfrak{g}^\bullet$, the positive definiteness allows us to write $(\mathfrak{h}_m)_{\mathbb{R}}$ as a direct sum

$$(\mathfrak{h}_m)_{\mathbb{R}} = \mathfrak{t}^\bullet \oplus \mathfrak{t}^{\bullet \perp j}, \tag{3.8}$$

where $\mathfrak{t}^{\bullet\perp j}$ is the orthogonal complement of $\mathfrak{t}^\bullet$ in $(\mathfrak{h}_m)_{\mathbb{R}}$ by the symmetric bilinear form θ_j. In (3.8), let $\mathrm{pr}_j : (\mathfrak{h}_m)_{\mathbb{R}} \to \mathfrak{t}^{\bullet\perp j}$ be the projection to the second factor. On the other hand, by viewing the vector space $(\mathfrak{h}_m)_{\mathbb{R}}$ as a (not necessarily unique) direct sum $\tilde{N}_{\mathbb{R}} \oplus \mathfrak{t}^\bullet$, we see that

$$\mathfrak{t}_m^{\perp}{}' := \bigcup_{j=1}^{r_1} \mathrm{pr}_j(\Sigma_j)$$

sitting in $(\mathfrak{h}_m)_{\mathbb{R}}$ is a piecewise linear (and hence continuous) graph over $\tilde{N}_{\mathbb{R}}$. Thus the restriction of $\tilde{\pi} : (\mathfrak{h}_m)_{\mathbb{R}} \to \tilde{N}_{\mathbb{R}}$ to $\mathfrak{t}_m^{\perp}{}'$ is bijective, so that its inverse defines a continuous cross-section $\iota : \tilde{N}_{\mathbb{R}} \to (\mathfrak{h}_m)_{\mathbb{R}}$ to $\tilde{\pi}$. Now by setting $(\mathfrak{t}_m^{\perp}{}')_{\mathbb{Z}} := \mathfrak{t}_m^{\perp}{}' \cap (\mathfrak{h}_m)_{\mathbb{Z}}$, we define a subset $(\mathfrak{g}_m')_{\mathbb{Z}}$ of $\mathfrak{z}(\mathfrak{t}_m)_{\mathbb{Z}}$ by

$$(\mathfrak{g}_m')_{\mathbb{Z}} := (\mathfrak{t}_m^{\perp}{}')_{\mathbb{Z}} + (\mathfrak{s}_m)_{\mathbb{Z}} = \{X' + X''; X' \in (\mathfrak{t}_m^{\perp}{}')_{\mathbb{Z}}, X'' \in (\mathfrak{s}_m)_{\mathbb{Z}}\},$$

where $(\mathfrak{s}_m)_{\mathbb{Z}}$ denotes the set of all semisimple elements X'' in $\mathfrak{s}_m$ such that the equality $\exp(2\pi\sqrt{-1}\,X'') = \mathrm{id}_{V_m}$ holds.

Remark 3.9. The piecewise bilinear form θ above in Theorem B is essentially the same as the bilinear pairing by Székelyhidi [**28**] for $\mathbb{C}^*$-actions on a test configuration.

4. Relative K-stability.

In this section, we use test configurations introduced by Donaldson [**3**] (see also [**29**]). For a complex affine space $\mathbb{A}^1 := \{s \in \mathbb{C}\} \cong \mathbb{C}$, the algebraic torus $\mathbb{C}^*$ acts on $\mathbb{A}^1$ by multiplication of complex numbers,

$$\mathbb{C}^* \times \mathbb{A}^1 \to \mathbb{A}^1, \qquad (t, z) \mapsto tz.$$

Fix an element T of $\mathcal{T}_{\mathrm{ex}}(M, L)$, and let $X \in \mathfrak{z}(\mathfrak{t}_m)_{\mathbb{Z}}$. Then $\mathbb{C}^*$ acts on V_m and also on $\mathbb{P}^*(V_m)$ via the algebraic group homomorphism

$$\varphi_X : \mathbb{C}^* \to Z(T_m)$$

as in (3.4). Here for a positive integer α, if X is replaced by αX, then by the base change, the algebraic torus $\mathbb{C}^*$ is replaced by its unramified cover of order α. The DeContini Procesi family (cf. [**23**]) associated to X is the test configuration $(\mathcal{M}^X, \mathcal{L}^X)$ of (M, L^m) endowed with the $\mathbb{C}^*$-equivariant projective morphism of algebraic varieties,

$$\pi_X : \mathcal{M}^X \to \mathbb{A}^1,$$

where $\mathcal{M}^X$ is the subvariety of $\mathbb{A}^1 \times \mathbb{P}^*(V_m)$ obtained as the closure of the union $\bigcup_{z \in \mathbb{C}^*} \mathcal{M}_z^X$ of the fibers

$$\mathcal{M}_z^X = \pi_X^{-1}(z) = \{z\} \times \{\varphi_X(z) \cdot \Phi_m(M)\}.$$

Furthermore, we put $\mathcal{L}^X := p_2^*(\mathcal{O}_{\mathbb{P}^*(V_m)}(1))$ for the restriction p_2 to $\mathcal{M}^X$ of the projection of $\mathbb{A}^1 \times \mathbb{P}^*(V_m)$ to the second factor $\mathbb{P}^*(V_m)$. For the open subset $\mathbb{C}^*$ of $\mathbb{A}^1$, we see that the holomorphic map $\hbar : \mathbb{C}^* \to \mathrm{Hilb}\,\mathbb{P}^*(V_m)$ sending each $z \in \mathbb{C}^*$ to $\hbar(z) := p_2(\mathcal{M}_z^X) \in \mathrm{Hilb}\,\mathbb{P}^*(V_m)$ extends to a holomorphic map

$$\bar{\hbar} : \mathbb{A}^1 \to \mathrm{Hilb}\,\mathbb{P}^*(V_m),$$

and hence, we can view $\mathcal{M}^X$ as the pullback, by $\bar{\hbar}$, of the universal family over $\mathrm{Hilb}\,\mathbb{P}^*(V_m)$. For each positive integer ℓ, we have

$$\big(\mathcal{M}_z^X, (\mathcal{L}_z^X)^\ell\big) \cong (M, L^{\ell m}), \qquad z \in \mathbb{C}^*,$$

and hence $(\mathcal{M}^X, (\mathcal{L}^X)^\ell)$ is a test configuration of (M, L^m) of exponent ℓ. We first let $\ell = 1$. Since $\mathbb{A}^1 \times \mathcal{O}_{\mathbb{P}^*(V_m)}(-1)$ is viewed as the blow-up of $\mathbb{A}^1 \times V_m$ along $\mathbb{A}^1 \times \{0\}$, and since $\mathcal{M}_Z$ is an algebraic subvariety of $\mathbb{A}^1 \times \mathbb{P}^*(V_m)$, we have a $\mathbb{C}^*$-action on $(\mathcal{M}^X, \mathcal{L}^X)$ induced by

$$\mathbb{C}^* \times (\mathbb{A}^1 \times V_m) \to \mathbb{A}^1 \times V_m, \qquad (t, (z, v)) \mapsto (tz, \varphi_X(t)v).$$

Since T also acts on $\mathbb{A}^1 \times V_m$ by operating only on the second factor, the induced T-action on $\mathbb{A}^1 \times \mathbb{P}^*(V_m)$ preserves the subvariety $\mathcal{M}^X$, so that we have a natural T-action on $(\mathcal{M}^X, \mathcal{L}^X)$ commuting with the $\mathbb{C}^*$-action on $(\mathcal{M}^X, \mathcal{L}^X)$. For the scheme-theoretic fiber $\mathcal{M}_0^X$ of π_X over the origin $0 \in \mathbb{A}^1$, let $\mathcal{L}_0^X$ denote the restriction of $\mathcal{L}^X$ to $\mathcal{M}_0^X$. Let E_ℓ^X be the vector bundle over $\mathbb{A}^1$ associated to the direct image sheaf $(\pi_X)_*\{(\mathcal{L}^X)^\ell\}$. Then the fiber $(E_\ell^X)_0$ of E_ℓ^X over the origin is

$$(E_\ell^X)_0 \cong H^0(\mathcal{M}_0^X, (\mathcal{L}_0^X)^\ell),$$

for all integer $\ell \gg 1$. Note that $d_\ell = \dim V_{\ell m} = \dim(E_\ell^X)_0$. Consider the endomorphism $X_\ell \in \operatorname{End}(E_\ell^X)_0$ of $(E_\ell^X)_0$ induced by X. Let w_ℓ be the weight of the $\mathbb{C}^*$-action on $(E_\ell^X)_0$. Then for all $\ell \gg 1$,

$$\begin{cases} d_\ell = a_n \ell^n + a_{n-1}\ell^{n-1} + \cdots + a_1\ell + a_0, \\ w_\ell = \operatorname{Tr}(X_\ell) = b_{n+1}\ell^{n+1} + b_n\ell^n + \cdots + b_1\ell + b_0, \end{cases} \tag{4.1}$$

where rational numbers a_i, $b_j \in \mathbb{Q}$ are independent of the choice of ℓ. Note here that $a_n = m^n c_1(L)^n[M]/n! > 0$. Then for all ℓ as above,

$$w_\ell/\ell d_\ell = F_0 + F_1\ell^{-1} + F_2\ell^{-2} + \cdots \tag{4.2}$$

with coefficients $F_i = F_i(\mathcal{M}^X, \mathcal{L}^X) \in \mathbb{Q}$ independent of the choice of ℓ. In particular

$$F_1 = F_1(\mathcal{M}^X, \mathcal{L}^X) = \frac{a_n b_n - a_{n-1}b_{n+1}}{a_n^2}$$

is called the *Donaldson-Futaki invariant* (cf. [**3**]) for the test configuration $(\mathcal{M}^X, \mathcal{L}^X)$ of (M, L^m).

Let $\nu : \tilde{\mathcal{M}}^X \to \mathcal{M}^X$ be the normalization of $\mathcal{M}^X$, and we consider the pullback $\tilde{\mathcal{L}}^X := \nu^*\mathcal{L}^X$. Recall that $(\tilde{\mathcal{M}}^X, \tilde{\mathcal{L}}^X)$ is *trivial* if there exists a $\mathbb{C}^*$-equivariant isomorphism

$$(\tilde{\mathcal{M}}^X, \tilde{\mathcal{L}}^X) \cong (\mathbb{A}^1 \times M,\, \mathbb{A}^1 \times L^m),$$

where on the right-hand side, the group $\mathbb{C}^*$ acts on the second factors M and L^m trivially.

Now, the relative K-stability in [**28**] (see also [**8**], [**26**]) is formulated as follows:

Definition 4.3. (1) (M, L) is called *K-semistable relative to* T if $F_1(\mathcal{M}^X, \mathcal{L}^X) \le 0$ for all $X \in (\mathfrak{g}'_m)_{\mathbb{Z}}$ and all positive integers m.

(2) Let (M, L) be K-semistable relative to T. Then (M, L) is called *K-stable relative to* T, if $F_1(\mathcal{M}^X, \mathcal{L}^X) < 0$ for all $X \in (\mathfrak{g}'_m)_{\mathbb{Z}} \setminus \mathfrak{g}$, $m = 1, 2, \dots$, as long as $(\tilde{\mathcal{M}}^X, \tilde{\mathcal{L}}^X)$ is nontrivial.

5. Asymptotic relative Chow-stability.

In this section, let $T \in \mathcal{T}_{\mathrm{ex}}(M, L)$, and consider the T-equivariant Kodaira embedding $\Phi_m : M \hookrightarrow \mathbb{P}^*(V_m)$ associated to the complete linear system $|L^m|$ on M. Let $\delta(m)$ be the degree of the image $\Phi_m(M)$ in $\mathbb{P}^*(V_m)$. Take the $\delta(m)$-th symmetric tensor product $S^{\delta(m)}(V_m)$ of V_m. For the dual W_m^* of $W_m := S^{\delta(m)}(V_m)^{\otimes n+1}$, we have the Chow form

$$\hat{M}_m \in W_m^*$$

for the irreducible reduced algebraic cycle $\Phi_m(M)$ on $\mathbb{P}^*(V_m)$, so that the corresponding element $[\hat{M}_m]$ in $\mathbb{P}^*(W_m)$ is the Chow point for the cycle $\Phi_m(M)$. Consider the natural action of $SL(V_m)$ on W_m^* induced by the action of $SL(V_m)$ on V_m.

Definition 5.1. (1) (M, L^m) is said to be *Chow-stable relative to* T if the orbit $G_m \cdot \hat{M}_m$ is closed in W_m^*.

(2) (M, L) is said to be *asymptotically Chow-stable relative to* T if (M, L^m) is Chow-stable relative to T for all integers $m \gg 1$.

6. Extremal Kähler metrics.

For the "only if" part of Conjecture A, the algebraic torus T should be chosen as small as possible. For instance, the result of Stoppa and Székelyhidi [**27**] solves the case $T = T_{\max}$, which does not cover the stability part of the original Donaldson-Tian-Yau's Conjecture unless $\mathrm{Aut}(M)$ is discrete. In this section, by improving the arguments in [**15**], we shall prove the following theorem by showing relative stability for all $T \in \mathcal{T}_{\mathrm{ex}}(M, L)$ on a polarized algebraic manifold (M, L) with an extremal Kähler metric ω. Since we may assume that the compact group K in the introduction acts isometrically on ω (cf. [**1**]), the associated extremal Kähler vector field $\mathcal{V}$ belongs to $\mathfrak{k}$.

Theorem C. *A polarized algebraic manifold* (M, L) *with an extremal Kähler metric in* $c_1(L)_{\mathbb{R}}$ *is K-stable relative to every* $T \in \mathcal{T}_{\mathrm{ex}}(M, L)$.

Proof. Fix an element X in $(\mathfrak{g}'_m)_{\mathbb{Z}}$ and let ω be an extremal Kähler metric in the class $c_1(L)_{\mathbb{R}}$. Choose a Hermitian metric h for L such that $\omega = c_1(L; h)$. It then suffices to show the following:

i) $F_1(\mathcal{M}^X, \mathcal{L}^X) \le 0$;
ii) If $F_1(\mathcal{M}^X, \mathcal{L}^X) = 0$, then $X \in \mathfrak{g}$ as long as $(\tilde{\mathcal{M}}^X, \tilde{\mathcal{L}}^X)$ is nontrivial.

Hence by replacing the line bundle L^m by L, we may assume that $m = 1$ without loss of generality.

Step 1: In this step, following [**12**, Section 2], we study the asymptotic weighted Bergman kernel for the extremal Kähler polarized algebraic manifolds (M, L^ℓ) as $\ell \to +\infty$. Since the maximal compact subgroup of T sits in K, the corresponding Lie algebra $\mathfrak{t}$ satisfies $\sqrt{-1}\,\mathfrak{t}_{\mathbb{R}} \subset \mathfrak{k}$. We now define a Hermitian pairing $\langle\ ,\ \rangle_{L^2(h)}$ for V_ℓ by

$$\langle \sigma, \sigma' \rangle_{L^2(h)} := \int_M (\sigma, \sigma')_h \omega^n, \qquad \sigma, \sigma' \in V_\ell, \tag{6.1}$$

where $(\sigma, \sigma')_h$ is the pointwise Hermitian inner product of σ, σ' by the ℓ-multiple of h. Then by this Hermitian pairing $\langle\ ,\ \rangle_{L^2(h)}$, we have

$$V(\chi_{\ell;i}) \perp V(\chi_{\ell;j}), \qquad i \neq j,$$

where $V(\chi_{\ell;k})$ is as in Section 2. Put $n_{\ell;i} := \dim_{\mathbb{C}} V(\chi_{\ell;i})$. Let P_ℓ be the set of all pairs (i, α) of integers such that $1 \leq i \leq \nu_\ell$ and $1 \leq \alpha \leq n_{\ell;i}$. For the pairing (6.1), we say that an orthonormal basis $\{\sigma_{i,\alpha}; (i,\alpha) \in P_\ell\}$ for V_ℓ is *admissible*, if $\sigma_{i,\alpha} \in V(\chi_{\ell;i})$ for all $(i, \alpha) \in P_\ell$. Fix an admissible orthonormal basis $\{\sigma_{i,\alpha}; (i,\alpha) \in P_\ell\}$ for V_ℓ with $\langle\ ,\ \rangle_{L^2(h)}$. By setting $\beta_{\ell;i} := \exp\{-q^2(\chi_{\ell;i})_*(\sqrt{-1}\,\mathcal{V})\} - 1$, we define the asymptotic weighted Bergman kernel $Z_\ell(\omega)$, $\ell \gg 1$, by

$$Z_\ell(\omega) := n! q^n \sum_{i=1}^{\nu_\ell} \sum_{\alpha=1}^{n_{\ell;i}} (1 + \beta_{\ell;i})\, |\sigma_{i,\alpha}|_h^2, \tag{6.2}$$

where we put $q := \ell^{-1}$ and $|\sigma|_h^2 := (\sigma, \sigma)_h$ for all $\sigma \in V_\ell$. We write the sections $\tilde{\sigma}_{i,\alpha} := (1 + \beta_{\ell;i})^{1/2} \sigma_{i,\alpha}$ as $\tilde{\sigma}_{j(i,\alpha)}$ by introducing the notation

$$j(i, \alpha) := \alpha + \sum_{k=1}^{i-1} n_{\ell;i},$$

so that the basis $\{\tilde{\sigma}_{i,\alpha}; (i,\alpha) \in P_\ell\}$ for V_ℓ is written as $\tilde{\mathfrak{S}} := \{\tilde{\sigma}_j; j = 1, 2, \dots, d_\ell\}$, and the Kodaira embedding $\Phi_\ell : M \hookrightarrow \mathbb{P}^*(V_\ell)$ is given by

$$M \hookrightarrow \mathbb{P}^{d_\ell - 1}(\mathbb{C}), \quad p \mapsto \Phi_\ell(p) := (\tilde{\sigma}_1(p) : \tilde{\sigma}_2(p) : \cdots : \tilde{\sigma}_{d_\ell}(p)),$$

where $\mathbb{P}^*(V_\ell)$ and $\mathbb{P}^{d_\ell - 1}(\mathbb{C}) = \{(\zeta_1 : \zeta_2 : \cdots : \zeta_{d_\ell})\}$ are identified by the basis $\tilde{\mathfrak{S}}$. For later purposes, rewrite the homogeneous coordinates ζ_j, $1 \leq j \leq d_\ell$, as $\zeta_{i,\alpha}$, $1 \leq i \leq \nu_\ell$, $1 \leq \alpha \leq n_{\ell,i}$, by setting

$$\zeta_{i,\alpha} := \zeta_{j(i,\alpha)}.$$

Put $r_0 := \{2c_1(L)^n[M]\}^{-1}\{nc_1(L)^{n-1}c_1(M)[M] + \sqrt{-1}\int_M h^{-1}(\mathcal{V}h)\omega^n\}$. Then by Theorem B (see also p. 579) in [**11**], the asymptotic weighted Bergman kernel $Z_\ell(\omega)$, $\ell \gg 1$, for the extremal Kähler metric ω satisfies

$$Z_\ell(\omega) - (1 + r_0 q) = O(q^2). \tag{6.3}$$

Here (6.3) means that $|\text{L.H.S.}| \leq C_1 q^2$ for some positive constant C_1 independent of ℓ. For the Fubini-Study form

$$\omega_{\text{FS}} := \left(\sqrt{-1}/2\pi\right)\partial\bar{\partial}\log\left(\sum_{j=1}^{d_\ell}|\zeta_j|^2\right)$$

on $\mathbb{P}^*(V_\ell)$ $(= \mathbb{P}^{d_\ell - 1}(\mathbb{C}))$, the pullback $\Phi_\ell^*\omega_{\text{FS}}$ is $(\sqrt{-1}/2\pi)\partial\bar{\partial}\log Z_\ell(\omega)$, and hence by (6.3), we obtain

$$\Phi_\ell^*\omega_{\text{FS}} - \ell\omega = O(q^2). \tag{6.4}$$

Put $b_{\ell;i} := -q(\chi_{\ell;i})_*(\sqrt{-1}\,\mathcal{V}) \in \mathbb{R}$. Note also that, as in [**14**, Lemma 2.6], there exists a positive constant C_2 independent of the choice of $\ell \gg 1$ and i such that $|b_{\ell;i}| \leq C_2$. Hence

$$|\beta_{\ell;i}| = b_{\ell;i}q + O(q^2) = O(q) \qquad \text{for all } \ell \gg 1 \text{ and } i. \tag{6.5}$$

Step 2: Let $X \in (\mathfrak{g}_1')_{\mathbb{Z}}$, so that we consider the test configuration $(\mathcal{M}^X, \mathcal{L}^X)$ for (M, L) of exponent 1. Recall that the vector bundle E_ℓ^X over $\mathbb{A}^1$ associated to the direct image sheaf $(\pi_X)_*\{(\mathcal{L}^X)^\ell\}$ admits a $\mathbb{C}^*$-equivariant trivialization (cf. [**4**, Lemma 2])

$$E_\ell^X \cong \mathbb{A}^1 \times (E_\ell^X)_0. \tag{6.6}$$

For each $z \in \mathbb{A}^1$, let $(E_\ell^X)_z$ denote the fiber of the vector bundle E_ℓ^X over z. Then by (6.6), we may assume that the Hermitian metric $\rho_1 := \langle\ ,\ \rangle_{L^2(h)}$ on $V_\ell = (E_\ell^X)_1$ induces a Hermitian metric ρ_0 on the central fiber $(E_\ell^X)_0$ which is preserved by the action of $S^1 \subset \mathbb{C}^*$. Now,

$$W_\ell := S^{\delta(\ell)}((E_\ell^X)_0)^{\otimes n+1} \cong S^{\delta(\ell)}(V_\ell)^{\otimes n+1} \tag{6.7}$$

admits the Chow norm (cf. [**32**, 1.5]; see also Section 4 in [**11**])

$$W_\ell^* \ni w \mapsto \|w\|_{\text{CH}(\rho_0)} \in \mathbb{R}_{\geq 0}.$$

In view of the definition in Section 5, let $\hat{M}_\ell \in W_\ell^*$ denote the Chow form for the irreducible reduced algebraic cycle $\gamma := \Phi_\ell(M)$ on $\mathbb{P}^*(V_\ell)$, where $\mathbb{P}^*(V_\ell)$ is viewed as $\mathbb{P}^*((E_\ell^X)_0)$ by the identification

$$V_\ell = (E_\ell^X)_1 \cong (E_\ell^X)_0$$

induced by the trivialization (6.6). Since the $\mathbb{C}^*$-action on E_ℓ^X preserves $(E_\ell^X)_0$, we have a natural representation

$$\psi_\ell : \mathbb{C}^* \to GL\left((E_\ell^X)_0\right) \; (= GL(d_\ell; \mathbb{C}))$$

induced by the $\mathbb{C}^*$-action on E_ℓ^X. By the complete linear systems $|(\mathcal{L}_\ell^X)_z|$, $z \in \mathbb{A}^1$, we have the relative Kodaira embedding

$$\mathcal{M}^X \hookrightarrow \mathbb{P}^*(E_\ell^X)$$

over $\mathbb{A}^1$, where by (6.6) the projective bundle over $\mathbb{A}^1$ is regarded as the product bundle $\mathbb{A}^1 \times \mathbb{P}^*((E_\ell^X)_0)$. Then each fiber $\mathbb{P}^*((E_\ell^X)_z)$ over $z \in \mathbb{A}^1$ is naturally identified with $\mathbb{P}^*((E_\ell^X)_0)$, so that all $\mathcal{M}_z^X$, $z \in \mathbb{A}^1$, are regarded as subschemes of $\mathbb{P}^*((E_\ell^X)_0)$. Namely,

$$\mathcal{M}_t^X = \psi_\ell(t) \cdot \mathcal{M}_1^X, \qquad t \in \mathbb{C}^*,$$

where on the right-hand side, the element $\psi_\ell(t)$ in $GL((E_\ell^X)_0)$ acts naturally on $\mathbb{P}^*((E_\ell^X)_0)$ as the corresponding projective linear transformation. Note that $\mathcal{M}_1^X$ is nothing but γ as an algebraic cycle, and that $\mathcal{M}_0^X$ is preserved by the $\mathbb{C}^*$-action on $\mathbb{P}^*((E_\ell^X)_0)$. Consider the d_ℓ-fold covering $\hat{\mathbb{T}} := \{\hat{t} \in \mathbb{C}^*\}$ of the algebraic torus $\mathbb{T} := \{t \in \mathbb{C}^*\}$ by setting

$$t = \hat{t}^{d_\ell},$$

for the coordinates t and $\hat{t}$, where $d_\ell = \dim V_\ell$. Then the mapping $\psi_\ell^{SL} : \hat{\mathbb{T}} \to SL((E_\ell^X)_0)$ $(= SL(d_\ell; \mathbb{C}))$ defined by

$$\psi_\ell^{SL}(\hat{t}) := \frac{\psi_\ell(\hat{t}^{d_\ell})}{\det(\psi_\ell(\hat{t}))} = \frac{\psi_\ell(t)}{\det(\psi_\ell(\hat{t}))}, \qquad \hat{t} \in \hat{\mathbb{T}},$$

is also an algebraic group homomorphism. In view of (6.7), the group $SL((E_\ell^X)_0)$ acts naturally on W_ℓ^*. We then consider the function

$$f_\ell(s) := \log \|\psi_\ell^{SL}(\exp(\hat{s})) \cdot \hat{M}_\ell\|_{\mathrm{CH}(\rho_0)}, \qquad s \in \mathbb{R},$$

by setting $\hat{s} := s/d_\ell$. Note that $X = X' + X''$, where $X' \in (\mathfrak{t}_1^{\perp}{}')_{\mathbb{Z}}$ and $X'' \in (\mathfrak{s}_1)_{\mathbb{Z}}$. Let $\hat{X}_\ell'$, $\hat{X}_\ell''$, $\hat{\mathcal{V}}_\ell \in \mathfrak{sl}\,(E_\ell^X)_0$ be the endomorphisms of $(E_\ell^X)_0$ induced by X', X'', $\mathcal{V}$, respectively. Then for a suitable choice of an admissible orthonormal basis $\{\sigma_{i,\alpha}; (i,\alpha) \in P_\ell\}$ for V_ℓ, we obtain

$$\hat{X}_\ell'(\sigma_{i,\alpha}) = -e_{\ell;i}'\sigma_{i,\alpha}, \quad \hat{X}_\ell''(\sigma_{i,\alpha}) = -e_{\ell;i,\alpha}''\sigma_{i,\alpha}, \quad q\sqrt{-1}\,\hat{\mathcal{V}}_\ell(\sigma_{i,\alpha}) = -b_{\ell;i}\sigma_{i,\alpha}$$

for some positive integers $e'_{\ell;i}$ and $e''_{\ell;i,\alpha}$ satisfying $\Sigma_{i=1}^{\nu_\ell}\Sigma_{\alpha=1}^{n_{\ell;i}} e'_{\ell;i} = 0$ and $\Sigma_{\alpha=1}^{n_{\ell;i}} e''_{\ell;i,\alpha} = 0$ for all i. We now give an estimate of the first derivative $\dot{f}_m(0)$ at $s = 0$. In view of [**32**] (see also [**11**]),

$$\dot{f}_\ell(0) = (n+1)! \int_M \frac{\sum_{i=1}^{\nu_\ell}\sum_{\alpha=1}^{n_{\ell;i}} e_{\ell;i,\alpha}|\tilde{\sigma}_{i,\alpha}|_h^2}{\sum_{i=1}^{\nu_\ell}\sum_{\alpha=1}^{n_{\ell;i}} |\tilde{\sigma}_{i,\alpha}|_h^2} \Phi_\ell^* \omega_{\mathrm{FS}}^n \tag{6.8}$$

where $e_{\ell;i,\alpha} := e'_{\ell;i} + e''_{\ell;i,\alpha}$. Again by [**14**, Lemma 2.6], we obtain $|e'_{\ell;i}| = O(\ell)$ and $|e''_{\ell;i,\alpha}| = O(\ell)$, i.e., there exist positive constants C_3, C_4 independent of ℓ, i, α such that $|e'_{\ell;i}| \leq C_3\ell$ and $|e''_{\ell;i,\alpha}| \leq C_4\ell$. Now,

$$\sum_{i=1}^{\nu_\ell}\sum_{\alpha=1}^{n_{\ell;i}} e_{\ell;i,\alpha} b_{\ell;i} = \sum_{i=1}^{\nu_\ell} n_{\ell;i} e'_{\ell;i} b_{\ell;i} = q\,\mathrm{Tr}\left(\sqrt{-1}\,\hat{\mathcal{V}}_\ell \hat{X}'_\ell\right) = O(\ell^n), \tag{6.9}$$

where the last equality follows from the fact that $X' \in (\mathfrak{t}_1^{\perp\prime})_{\mathbb{Z}}$, since by $\theta(\sqrt{-1}\,\mathcal{V}, X') = 0$, we have (cf. [**28**])

$$\mathrm{Tr}\left(\hat{\mathcal{V}}_\ell \hat{X}'_\ell\right) = \theta\big(\sqrt{-1}\,\mathcal{V}, X'\big)\ell^{n+2} + O(\ell^{n+1}) = O(\ell^{n+1}).$$

Since $\sum_{i=1}^{\nu_\ell}\sum_{\alpha=1}^{n_{\ell;i}} |\tilde{\sigma}_{i,\alpha}|_h^2 = (\ell^n/n!)Z_\ell(\omega)$, by using $\sum_{i=1}^{\nu_\ell}\sum_{\alpha=1}^{n_{\ell;i}} e_{\ell;i,\alpha} = 0$ and $|e_{\ell;i,\alpha}| = O(\ell)$, we see from (6.3), (6.4), (6.5), (6.8) and (6.9) that

$$\begin{aligned}
\dot{f}_\ell(0) &= (n+1)! \int_M \frac{\sum_{i=1}^{\nu_\ell}\sum_{\alpha=1}^{n_{\ell;i}} e_{\ell;i,\alpha}(1+\beta_{\ell;i})|\sigma_{i,\alpha}|_h^2}{(\ell^n/n!)\{1 + r_0 q + O(q^2)\}} \{\ell\omega + O(q^2)\}^n \\
&= (n+1)! \int_M \frac{\sum_{i=1}^{\nu_\ell}\sum_{\alpha=1}^{n_{\ell;i}} e_{\ell;i,\alpha}\beta_{\ell;i}|\sigma_{i,\alpha}|_h^2}{(\ell^n/n!)\{1 + r_0 q + O(q^2)\}} \{\ell\omega + O(q^2)\}^n \\
&= \frac{(n+1)!}{1 + r_0 q} \sum_{i=1}^{\nu_\ell}\sum_{\alpha=1}^{n_{\ell;i}} e_{\ell;i,\alpha} b_{\ell;i} q + O(\ell^{n-1}) = O(\ell^{n-1}).
\end{aligned}$$

Recall the well-known fact (cf. [**32**]; see also [**11**, 4.5]) that f_ℓ is a convex function, i.e., $\ddot{f}_\ell(s) \geq 0$ for all $s \in \mathbb{R}$. Now by (8.8) in Appendix 1,

$$\lim_{s \to -\infty} \dot{f}_\ell(s) = (n+1)! a_n F_1 \ell^n + O(\ell^{n-1}). \tag{6.10}$$

Let $\ell \to \infty$. Then in view of $\dot{f}_\ell(0) = O(\ell^{n-1})$, the monotonicity of the function $\dot{f}_\ell(s)$ implies that

$$F_1(\mathcal{M}^X, \mathcal{L}^X) \leq 0.$$

Step 3: To complete the proof of Theorem C, by assuming that the invariant $F_1(\mathcal{M}^X, \mathcal{L}^X)$ vanishes, it suffices to show that $X \in \mathfrak{g}$ unless $(\tilde{\mathcal{M}}^X, \tilde{\mathcal{L}}^X)$ is trivial. Then by

$F_1(\mathcal{M}^X, \mathcal{L}^X) = 0$ and (6.10), we obtain

$$\lim_{s \to -\infty} \dot{f}_\ell(s) = O(\ell^{n-1}), \qquad \ell \gg 1. \tag{6.11}$$

For a sufficiently small positive real constant C_5 independent of ℓ, we put $\varepsilon := C_5(\log \ell)q$. Consider the local one-parameter group

$$g_{s,\ell} := \psi_\ell^{SL}(\exp(\hat{s})), \qquad -\varepsilon \leq s \leq 0.$$

In terms of the natural action of $SL(d_\ell, \mathbb{C})$ on $\mathbb{P}^{d_\ell - 1}(\mathbb{C})$, by setting $\omega_{s,\ell} := q(g_{s,\ell} \circ \Phi_\ell)^* \omega_{\mathrm{FS}}$, we see that the family of Kähler manifolds

$$(M, \omega_{s,\ell}), \qquad -\varepsilon \leq s \leq 0, \ \ell = 1, 2, \dots, \tag{6.12}$$

has bounded geometry as in Appendix 2. Let us now consider the holomorphic vector field $\mathcal{X}^{(\ell)}$ induced by $(\psi_\ell^{SL})_*(\partial/\partial s)$ on $\mathbb{P}^{d_\ell - 1}(\mathbb{C})$ which generates the local one-parameter group $g_{\ell,s}$, $-\varepsilon \leq s \leq 0$. For each $s \in [-\varepsilon, 0]$, we consider the holomorphic tangent bundle TM_s of $M_s := g_{s,\ell}(\Phi_\ell(M))$. For the Fubini-Study metric, let $TM_s^\perp$ denote the orthogonal complement of TM_s in $T\mathbb{P}^{d_\ell - 1}(\mathbb{C})_{|M_s}$, where $T\mathbb{P}^{d_\ell - 1}(\mathbb{C})$ is the holomorphic tangent bundle of $\mathbb{P}^{d_\ell - 1}(\mathbb{C})$. Hence $T\mathbb{P}^{d_\ell - 1}(\mathbb{C})_{|M_s}$ is differentiably a direct sum $TM_s \oplus TM_s^\perp$, and we can uniquely write

$$\mathcal{X}^{(\ell)}{}_{|M_s} = \mathcal{X}^{(\ell)}_{TM_s} + \mathcal{X}^{(\ell)}_{TM_s^\perp}, \tag{6.13}$$

where $\mathcal{X}^{(\ell)}_{TM_s}$ and $\mathcal{X}^{(\ell)}_{TM_s^\perp}$ are C^∞ sections of TM_s and $TM_s^\perp$, respectively. Note that $TM_s^\perp$ is regarded as the normal bundle of M_s in $\mathbb{P}^{d_\ell - 1}(\mathbb{C})$. Consider the exact sequence of holomorphic vector bundles

$$0 \to TM_s \to T\mathbb{P}^{d_\ell - 1}(\mathbb{C})_{|M_s} \to TM_s^\perp \to 0$$

over M_s. Then the pointwise estimate (cf. [**24**, (5.16)]) of the second fundamental form for this exact sequence is valid also in our case (cf. [**13**, Step 2]), and as in [**24**, (5.15)], we obtain the inequality

$$\int_{M_s} |\mathcal{X}^{(\ell)}_{TM_s^\perp}|^2_{\omega_{\mathrm{FS}}} \omega_{\mathrm{FS}}^n \geq C_6 \int_{M_s} |\bar{\partial} \mathcal{X}^{(\ell)}_{TM_s^\perp}|^2_{\omega_{\mathrm{FS}}} \omega_{\mathrm{FS}}^n, \tag{6.14}$$

where C_6 is a positive constant independent of the choice of s and ℓ. The space $\Theta := H^0(M, C^\infty(TM))$ of C^∞ sections of TM has the Hermitian L^2-pairing

$$\langle Y_1, Y_2 \rangle_{s,\ell} := \int_M (Y_1, Y_2)_{\omega_{s,\ell}} \omega_{s,\ell}^n, \qquad Y_1, Y_2 \in \Theta,$$

where $(Y_1, Y_2)_{\omega_{s,\ell}}$ denotes the pointwise Hermitian pairing of Y_1 and Y_2 by the Kähler metric $\omega_{s,\ell}$. For the subspace $\Gamma := H^0(M, \mathcal{O}(TM))$ of Θ, we consider its orthogonal complement $\Gamma^{\perp}_{s,\ell}$ in Θ by the pairing $\langle\, ,\, \rangle_{s,\ell}$. Then $\mathcal{X}^{(\ell)}_{TM_s}$ in (6.13) is expressible as

$$\mathcal{X}^{(\ell)}_{TM_s} = \mathcal{X}^{\circ}_{s,\ell} + \mathcal{X}^{\bullet}_{s,\ell},$$

where $\mathcal{X}^{\circ}_{s,\ell}$ and $\mathcal{X}^{\bullet}_{s,\ell}$ belong to $(g_{s,\ell} \circ \Phi_\ell)_* \Gamma$ and $(g_{s,\ell} \circ \Phi_\ell)_* \Gamma^{\perp}_{s,\ell}$, respectively. Recall that the second derivative $\ddot{f}_\ell(s)$ is given by

$$\ddot{f}_\ell(s) = \int_{M_s} |\mathcal{X}^{(\ell)}_{TM_s^{\perp}}|^2_{\omega_{\mathrm{FS}}} \omega_{\mathrm{FS}}^n \geq 0, \tag{6.15}$$

see for instance [**11**, Theorem 4.5]. Since $\dot{f}_\ell(0) - \dot{f}_\ell(-\varepsilon) = \int_{-\varepsilon}^0 \ddot{f}_\ell(s) ds \geq 0$, we see from $\dot{f}_\ell(0) = O(\ell^{n-1})$ and (6.10) that

$$\begin{aligned} O(\ell^{n-1}) = \dot{f}_\ell(0) - \lim_{s \to -\infty} \dot{f}_\ell(s) &\geq \dot{f}_\ell(0) - \dot{f}_\ell(-\varepsilon) \\ = \int_{-\varepsilon}^{0} \ddot{f}_\ell(s) ds &\geq \ddot{f}_\ell(s_\ell)\varepsilon, \end{aligned} \tag{6.16}$$

where s_ℓ, $\ell \gg 1$, are real numbers at which the functions $\ddot{f}_\ell(s)$, $-\varepsilon \leq s \leq 0$, attain their minima, i.e., $\ddot{f}_\ell(s_\ell) = \min_{-\varepsilon \leq s \leq 0} \ddot{f}_\ell(s)$. By

$$\ddot{f}_\ell(s_\ell) = \ell^{n+1} \int_{M_{s_\ell}} |\mathcal{X}^{(\ell)}_{TM_{s_\ell}^{\perp}}|^2_{q\omega_{\mathrm{FS}}} (q\omega_{\mathrm{FS}})^n,$$

it follows from (6.16) and $\varepsilon = O(q \log \ell)$ that

$$\int_{M_{s_\ell}} |\mathcal{X}^{(\ell)}_{TM_s^{\perp}}|^2_{q\omega_{\mathrm{FS}}} (q\omega_{\mathrm{FS}})^n = O(q/\log \ell), \qquad \ell \gg 1. \tag{6.17}$$

Since the left-hand side of (6.13) is holomorphic, by operating the $\bar{\partial}$-operator of the holomorphic vector bundle $T\mathbb{P}^{d_\ell - 1}(\mathbb{C})_{|M_s}$, we obtain

$$\bar{\partial} \mathcal{X}^{(\ell)}_{TM_s^{\perp}} = -\bar{\partial} \mathcal{X}^{(\ell)}_{TM_s} = -\bar{\partial} \mathcal{X}^{\bullet}_{s,\ell}. \tag{6.18}$$

Let $\Delta_{TM;s,\ell}$ denote the Laplacian on the space of C^∞ sections of the holomorphic tangent bundle TM of the Kähler manifold $(M, \omega_{s,\ell})$. Since the family (6.12) has bounded geometry, the first positive eigenvalue of the operator $-\Delta_{TM;s,\ell}$ on $\mathcal{A}^{0,0}(TM)$ is bounded from below by some positive constant C_7 independent of the choice of s and ℓ. Hence

$$\int_{M_{s_\ell}} |\bar{\partial} \mathcal{X}^{\bullet}_{s_\ell,\ell}|^2_{q\omega_{\mathrm{FS}}} (q\omega_{\mathrm{FS}})^n \geq C_7 \int_{M_{s_\ell}} |\mathcal{X}^{\bullet}_{s_\ell,\ell}|^2_{q\omega_{\mathrm{FS}}} (q\omega_{\mathrm{FS}})^n. \tag{6.19}$$

From (6.14), (6.18) and (6.19), we obtain

$$\int_{M_{s_\ell}} |\mathcal{X}^{(\ell)}_{TM_s^\perp}|^2_{q\omega_{\mathrm{FS}}} (q\omega_{\mathrm{FS}})^n \geq C_6 C_7\, q \int_{M_{s_\ell}} |\mathcal{X}^{\bullet}_{s_\ell,\ell}|^2_{q\omega_{\mathrm{FS}}} (q\omega_{\mathrm{FS}})^n. \tag{6.20}$$

Then from (6.17) and (6.20), it now follows that

$$\int_{M_{s_\ell}} |\mathcal{X}^{\bullet}_{s_\ell,\ell}|^2_{q\omega_{\mathrm{FS}}} (q\omega_{\mathrm{FS}})^n = O(1/\log \ell), \qquad \ell \gg 1. \tag{6.21}$$

Put $\tau_\ell := (\sum_{i=1}^{\nu_\ell}\sum_{\alpha=1}^{n_{\ell;i}} e_{\ell,i,\alpha}|\zeta_{i,\alpha}|^2)/(\ell \sum_{i=1}^{\nu_\ell}\sum_{\alpha=1}^{n_{\ell;i}} |\zeta_{i,\alpha}|^2)$ on $\mathbb{P}^{d_\ell -1}(\mathbb{C})$. Then by setting $c(\tau_\ell) := \{\int_{M_{s_\ell}} (q\omega_{\mathrm{FS}})^n\}^{-1}\int_{M_{s_\ell}} \tau_\ell (q\omega_{\mathrm{FS}})^n$, we define uniformly bounded real-valued C^∞ functions η_ℓ, $\ell \gg 1$, on M by

$$\eta_\ell := \{(g_{s_\ell,\ell}\circ \Phi_\ell)^*\tau_\ell\}_{|M} - c(\tau_\ell), \qquad \ell \gg 1,$$

which are uniformly bounded on M by $|e_{\ell,i,\alpha}| = O(\ell)$ (cf. Step 2). Hereafter, replace the sequence s_ℓ, $\ell \gg 1$, by its suitable subsequence s_{ℓ_j}, $j = 1, 2, \dots$, if necessary. We write ℓ_j, ℓ_j^{-1}, s_{ℓ_j}, $\langle\, ,\,\rangle_{s_{\ell_j},\ell_j}$, $g_{s_{\ell_j},\ell_j}$, $\omega_{s_{\ell_j},\ell_j}$, Φ_{ℓ_j}, η_{ℓ_j} as $\ell(j)$, $q(j)$, $s(j)$, $\langle\, ,\,\rangle_{(j)}$, $g(j)$, $\omega(j)$, $\Phi(j)$, $\eta(j)$, respectively. Since the family (6.12) has bounded geometry, we may assume that $\omega(j)$ converges to the extremal Kähler metric ω in C^∞, as $j \to \infty$ (see Appendix 2). For simplicity, we further put

$$\begin{cases} \mathcal{X}_{TM}(j) := (\Phi(j)^{-1})_*(g(j)^{-1})_*\mathcal{X}^{(\ell_j)}_{TM_{s_{\ell_j}}}, \\ \mathcal{X}^\circ(j) := (\Phi(j)^{-1})_*(g(j)^{-1})_*\mathcal{X}^\circ_{s_{\ell_j},\ell_j}, \\ \mathcal{X}^\bullet(j) := (\Phi(j)^{-1})_*(g(j)^{-1})_*\mathcal{X}^\bullet_{s_{\ell_j},\ell_j}. \end{cases}$$

Then the following cases 1 and 2 are possible:

Case 1: $I_j^\circ := \int_M |\mathcal{X}^\circ(j)|^2_{\omega(j)}\omega(j)^n$, $j = 1, 2, \dots$, are bounded. In this case, since $|\mathcal{X}_{TM}(j)|^2_{\omega(j)} = |\mathcal{X}^\circ(j)|^2_{\omega(j)} + |\mathcal{X}^\bullet(j)|^2_{\omega(j)}$, (6.21) together with the boundedness of I_j° implies that

$$\int_M |\mathcal{X}_{TM}(j)|^2_{\omega(j)}\omega(j)^n, \ j = 1, 2, \dots, \text{ are bounded.} \tag{6.22}$$

Since $\omega(j) \to \omega$ in C^∞, in view of (6.22) and $|\mathcal{X}_{TM}(j)|^2_{\omega(j)} = |\bar\partial\eta(j)|^2_{\omega(j)}$, we see that $\int_M |\bar\partial \eta(j)|^2_\omega \omega^n$, $j = 1, 2, \dots$, form a bounded sequence. Hence $\eta(j)$, $j = 1, 2, \dots$, are bounded in the Sobolev space $L^{1,2}(M, \omega^n)$. Then replacing $\eta(j)$, $j = 1, 2, \dots$, by its subsequence if necessary, we may further assume that, for some real-valued function $\eta_\infty \in L^2(M, \omega^n)$,

$$\eta(j) \to \eta_\infty \text{ strongly in } L^2(M, \omega^n), \quad \text{as } j \to \infty. \tag{6.23}$$

Put $\omega(\infty) := \omega$. Then for $j = 1, 2, \dots$, and also for $j = \infty$, the Lichnerowich operator Λ_j :

$C^\infty(M)_{\mathbb{C}} \to C^\infty(M)_{\mathbb{C}}$ for the Kähler manifold $(M, \omega(j))$ is an elliptic operator, of order 4, with kernel consisting of all Hamiltonian functions for the holomorphic Hamiltonian vector fields on $(M, \omega(j))$. Let $\Lambda_j^\# : C^\infty(M)_{\mathbb{C}} \to C^\infty(M)_{\mathbb{C}}$ be the formal adjoint of the operator Λ_j on the Kähler manifold $(M, \omega(j))$. Now, to each smooth function $f \in C^\infty(M)_{\mathbb{C}}$, we associate a complex vector field $\mathcal{V}_{f,j}$ of type $(1,0)$ on M such that

$$i(\mathcal{V}_{f,j})\omega(j) = \sqrt{-1}\,\bar\partial f, \qquad j = 1, 2, \dots,$$

where we can easily check that $\mathcal{V}_{\eta(j),j}$ coincides with $2\pi\mathcal{X}_{TM}(j)$. Hence for all $f \in C^\infty(M)_{\mathbb{C}}$, we can write $\int_M (\Lambda_j^\# f)\eta(j)\omega(j)^n$ as

$$\begin{aligned}\big(\Lambda_j^\# f, \eta(j)\big)_{L^2(M,\omega(j)^n)} &= (f, \Lambda_j \eta(j))_{L^2(M,\omega(j)^n)} = \big\langle \bar\partial \mathcal{V}_{f,j}, \bar\partial \mathcal{V}_{\eta(j),j}\big\rangle_{(j)} \\ &= 2\pi\big\langle \bar\partial\mathcal{V}_{f,j}, \bar\partial\{\mathcal{X}_{TM}(j)\}\big\rangle_{(j)} = 2\pi\big\langle \bar\partial\mathcal{V}_{f,j}, \bar\partial\{\mathcal{X}^\bullet(j)\}\big\rangle_{(j)}.\end{aligned}$$

Here the last equality follows from the identities $\mathcal{X}_{TM}(j) = \mathcal{X}^\circ(j) + \mathcal{X}^\bullet(j)$ and $\bar\partial\mathcal{X}^\circ(j) = 0$. Hence, for each fixed f in $C^\infty(M)_{\mathbb{C}}$, we obtain

$$\begin{cases} \left| \displaystyle\int_M (\Lambda_j^\# f)\eta(j)\omega(j)^n \right| = 2\pi \big| \langle \Delta_j \mathcal{V}_{f,j}, \mathcal{X}^\bullet(j)\rangle_{(j)} \big| \\ \qquad \le 2\pi \left\{ \displaystyle\int_M |\Delta_j \mathcal{V}_{f,j}|^2_{\omega(j)}\, \omega(j)^n \right\}^{1/2} \sqrt{I_j^\bullet}, \end{cases} \tag{6.24}$$

where $I_j^\bullet := \{\int_M |\mathcal{X}^\bullet(j)|^2_{\omega(j)}\omega(j)^n\}^{1/2}$ and $\Delta_j := \Delta_{TM;s(j),\ell_j}$. In (6.24), let $j \to \infty$. Since $I_j^\bullet \to 0$ by (6.21), and since $\omega(j) \to \omega$ in C^∞, by passing to the limit as $j \to \infty$, we see from (6.23) and (6.24) that

$$\int_M (\Lambda_\infty^\# f)\eta_\infty \omega^n = 0$$

for all $f \in C^\infty(M)_{\mathbb{C}}$. This shows that $\eta = \eta_\infty$ is a weak solution for the elliptic equation

$$\Lambda_\infty \eta = 0,$$

and hence is a strong solution. Thus we have a holomorphic vector field W on M such that $i(2\pi W)\omega = \bar\partial\eta_\infty$. Then by Appendix 3, under the assumption that $(\tilde{\mathcal{M}}^X, \tilde{\mathcal{L}}^X)$ is nontrivial, we obtain $X \in \mathfrak{g}$ as required.

Case 2: $I_j^\circ \to +\infty$ as $j \to \infty$. Here we replace I_j°, $j = 1, 2, \dots$, by its subsequence if necessary. In this case, $\hat{\mathcal{X}}^\circ(j) := \mathcal{X}^\circ(j)/\sqrt{I_j^\circ}$ satisfies

$$\int_M |\hat{\mathcal{X}}^\circ(j)|^2_{\omega(j)}\omega(j)^n = 1, \qquad j = 1, 2, \dots,$$

so that in view of the convergence $\omega(j) \to \omega$ in C^∞, as $j \to \infty$, we may assume that

$$\hat{\mathcal{X}}^\circ(j) \to \hat{\mathcal{X}}^\circ_\infty \ (\neq 0) \text{ in } \mathfrak{g}, \qquad \text{as } j \to \infty, \tag{6.25}$$

for some $\hat{\mathcal{X}}^\circ_\infty \in \mathfrak{g}$. Put $\hat{\eta}(j) := \eta(j)/\sqrt{I^\circ_j}$ and $\hat{\mathcal{X}}^\bullet(j) := \mathcal{X}^\bullet(j)/\sqrt{I^\circ_j}$. Since $\eta(j)$, $j = 1, 2, \dots$, are uniformly bounded on M, we see that

$$\hat{\eta}(j) \to 0 \text{ in } C^0(M), \qquad \text{as } j \to \infty. \tag{6.26}$$

Let $\hat{\eta}^\circ(j)$ and $\hat{\eta}^\bullet(j)$ be the Hamiltonian functions associated to the vector fields $\hat{\mathcal{X}}^\circ(j)$ and $\hat{\mathcal{X}}^\bullet(j)$, respetively, on the Kähler manifold $(M, \omega(j))$, so that

$$\begin{cases} i(2\pi\hat{\mathcal{X}}^\circ(j))\omega(j) = \sqrt{-1}\,\bar{\partial}(\hat{\eta}^\circ(j)), \\ i(2\pi\hat{\mathcal{X}}^\bullet(j))\omega(j) = \sqrt{-1}\,\bar{\partial}(\hat{\eta}^\bullet(j)), \end{cases}$$

where the functions $\hat{\eta}^\circ(j)$ and $\hat{\eta}^\bullet(j)$ are normalized by the vanishing of the integrals $\int_M \hat{\eta}^\circ(j)\omega(j)^n$ and $\int_M \hat{\eta}^\bullet(j)\omega(j)^n$, respectively. Then

$$\hat{\eta}(j) = \hat{\eta}^\circ(j) + \hat{\eta}^\bullet(j). \tag{6.27}$$

Now by (6.25), there exists a non-constant C^∞ function $\hat{\rho}$ on M such that $i(2\pi\hat{\mathcal{X}}^\circ_\infty)\omega = \sqrt{-1}\,\bar{\partial}\hat{\rho}$ and that

$$\hat{\eta}^\circ(j) \to \hat{\rho} \text{ in } C^\infty(M), \quad \text{as } j \to \infty.$$

Hence by (6.26) and (6.27), we see that

$$\hat{\eta}^\bullet(j) \to -\hat{\rho} \text{ in } C^0(M), \quad \text{as } j \to \infty.$$

On the other hand, by (6.21), we see that $\int_M |\bar{\partial}\hat{\eta}^\bullet(j)|^2_{\omega(j)}\omega(j)^n \to 0$ as $j \to \infty$, and hence for each fixed smooth $(0,1)$-form θ on M, we have

$$\begin{aligned} \left|(\hat{\eta}^\bullet(j), \bar{\partial}(j)^*\theta)_{L^2(M,\omega(j)^n)}\right| &= \left| \int_M (\bar{\partial}\hat{\eta}^\bullet(j), \theta)_{\omega(j)}\omega(j)^n \right| \\ \leq \left\{ \int_M |\bar{\partial}\hat{\eta}^\bullet(j)|^2_{\omega(j)}\omega(j)^n \right\}^{1/2} &\left\{ \int_M |\theta|^2_{\omega(j)}\omega(j)^n \right\}^{1/2} \to 0, \end{aligned}$$

where for $j \in \mathbb{Z}_+ \cup \{\infty\}$, we denote by $\bar{\partial}(j)^*$ the formal adjoint of the operator $\bar{\partial}$ on functions for the Kähler manifold $(M, \omega(j))$. Then by letting $j \to \infty$, we obtain the vanishing for the Hermitian L^2-inner product of functions $\hat{\rho}$ and $\bar{\partial}(\infty)^*\theta$,

$$(\hat{\rho}, \bar{\partial}(\infty)^*\theta)_{L^2(M,\omega^n)} = 0,$$

for every smooth $(0,1)$-form θ on M, i.e., $\bar{\partial}\hat{\rho} = 0$ in a weak sense, and hence in a strong sense. Thus we conclude that $\hat{\rho}$ is constant on M in contradiction to $\hat{\mathcal{X}}^\circ_\infty \neq 0$. This completes the proof of Theorem C. □

7. A program to solve Conjecture A.

As far as the K-stability of (M, L) relative to $T \in \mathcal{T}_{\mathrm{ex}}(M, L)$ is concerned, the stability condition is weakest in the case $T = T_{\max}$. Hence by Theorem C, it suffices to show the existence of an extremal Kähler metric in $c_1(L)_{\mathbb{R}}$ under the assumption that (M, L) is K-stable relative to $T_{\max}$, or more generally relative to $T \in \mathcal{T}_{\min}(M, L)$. Thus in this section, by assuming $T \in \mathcal{T}_{\min}(M, L)$, we discuss Conjecture A by dividing it into the following three parts:

Part 1. *If (M, L) is K-stable relative to T, then (M, L) is asymptotically Chow-stable relative to T.*

Part 2 (cf. [**17**]). *If (M, L) is asymptotically Chow-stable relative to T, then for all $m \gg 1$ there exist a series of weighted balanced metrics ω_m, $m \gg 1$, such that the m-th asymptotic Bergman kernel $B_m(\omega_m)$ is*

$$(m^n/n!) + f_m m^{n-1} + O(m^{n-2}), \qquad m \gg 1, \tag{7.1}$$

for some uniformly bounded real Hamiltonian function f_m on the Kähler manifold (M, ω_m) associated to a holomorphic vector field in $\mathfrak{t}$.

Part 3. *The Kähler metric ω_m in Part 2 converges to a Kähler metric ω_∞ on M in C^∞, as $m \to \infty$.*

Here Part 1 will be treated in [**19**], while Part 2 is proved in [**17**]. Note that Part 3 is studied by many authors, say, by Chen and Donaldson in the case $\dim M \leq 3$. For Part 3, we have some idea, though it will be discussed elsewhere (cf. [**18**]). If these three parts are done, then by $\dim \mathfrak{t} < +\infty$ and also by the uniform boundedness (cf. [**17**, Theorem A]) of f_m in (7.1), replacing f_m, $m = 1, 2, \ldots$, by its suitable subsequence if necessary, we may assume that f_m converges to some real Hamiltonian function f_∞ on the Kähler manifold (M, ω_∞) associated to a holomorphic vector field in $\mathfrak{t}$. Now by a theorem of Catlin-Lu-Tian-Yau-Zelditch ([**2**], [**9**], [**30**], [**31**]), we see from (7.1) that

$$f_m = \sigma(\omega_m)/2, \tag{7.2}$$

where for every Kähler metric ω in $c_1(L)_{\mathbb{R}}$, we denote by $\sigma(\omega)$ the scalar curvature of ω. In (7.2), let $m \to \infty$. Then we obtain $f_\infty = \sigma(\omega_\infty)/2$, and hence ω_∞ is an extremal Kähler metric in $c_1(L)_{\mathbb{R}}$, as required.

Since the statement of Conjecture A is supposed to be valid for all $T \in \mathcal{T}_{\mathrm{ex}}(M, L)$, it suggests the following:

Conjecture D. *A polarized algebraic manifold (M, L) is K-stable relative to T_{ex} if and only if (M, L) is K-stable relative to $T_{\max}$.*

Finally we observe that Conjecture A includes, as a special case, Donaldson-Tian-

Yau's conjecture on the existence of constant scalar curvature metrics. This is seen from the fact that, if (M, L) is K-stable, then the classical Futaki invariant (cf. [**7**]) of (M, L) vanishes so that any extremal Kähler metric on (M, L) has constant scalar curvature.

8. Appendix 1.

In this Appendix 1, we shall give another interpretation of the invariants F_j, $j = 1, 2, \dots,$ for test configurations by discussing the unpublished result (4.9) in [**15**]. Let $(\mathcal{M}, \mathcal{L})$ be a test configuration for (M, L) of exponent m in Donaldson's sense, so that there exists a $\mathbb{C}^*$-equivariant projective morphism of algebraic varieties,

$$\pi : \mathcal{M} \to \mathbb{A}^1,$$

with a relatively very ample line bundle $\mathcal{L}$ on the fiber space $\mathcal{M}$ over $\mathbb{A}^1 = \{s \in \mathbb{C}\}$ such that the $\mathbb{C}^*$-action on $\mathcal{M}$ lifts to a $\mathbb{C}^*$-linearization of $\mathcal{L}$ with isomorphisms of polarized algebraic manifolds,

$$(\mathcal{M}_s, \mathcal{L}_s) \cong (M, L^m), \qquad s \neq 0.$$

Here $\mathbb{C}^*$ acts on $\mathbb{A}^1$ by multiplication of complex numbers as in Section 4. Let E_ℓ, $\ell = 1, 2, \dots,$ be the holomorphic vector bundle over $\mathbb{A}^1$ associated to the direct image sheaves $\pi_* \mathcal{L}^\ell$. Then as in (6.6), we have a $\mathbb{C}^*$-equivariant trivialization

$$E_\ell \cong \mathbb{A}^1 \times (E_\ell)_0 \tag{8.1}$$

such that a Hermitian metric ρ_1 for $(E_\ell)_1 = V_{\ell m} = H^0(M, L^{\ell m})$ induces a Hermitian metric ρ_0 on the central fiber $(E_\ell)_0$ which is preserved by the action of $S^1 \subset \mathbb{C}^*$. Now for $\delta(\ell)$ in Section 5, the vector space $W_\ell := \{S^{\delta(\ell)}((E_\ell)_0)\}^{\otimes n+1}$ admits the Chow norm

$$W_\ell^* \ni w \mapsto \|w\|_{\mathrm{CH}(\kappa_0)} \in \mathbb{R}_{\geq 0},$$

as in Section 6. Let $\hat{M}_\ell \in W_\ell^*$ be such that the associated element $[\hat{M}_\ell]$ in $\mathbb{P}^*(W_\ell)$ is the Chow point for the reduced effective algebraic cycle

$$\gamma_1 := \Phi_{\ell m}(M)$$

on $\mathbb{P}^*((E_\ell)_0)$ for the Kodaira embedding $\Phi_{\ell m} : M \hookrightarrow \mathbb{P}^*(V_{\ell m})$ associated to the complete linear system $|L^{\ell m}|$ on M. Here each $(E_\ell)_s$, $s \neq 0$, is identified with $(E_\ell)_0$ via the trivialization (8.1), and by letting $s = 1$, we regard $\Phi_{\ell m}(M)$ on $\mathbb{P}^*(V_\ell)$ as the algebraic cycle γ_1 on $\mathbb{P}^*((E_\ell)_0)$. Since the T-action on E_ℓ preserves $(E_\ell)_0$, we have a representation

$$\psi_\ell : \mathbb{C}^* \to GL((E_\ell)_0) \tag{8.2}$$

induced by the $\mathbb{C}^*$-action on E_ℓ. Note that this $\mathbb{C}^*$-action on $(E_\ell)_0$ naturally induces a $\mathbb{C}^*$-action on $\mathbb{P}^*((E_\ell)_0)$. By the complete linear systems $|\mathcal{L}_s^\ell|$, $s \in \mathbb{A}^1$, we have the relative Kodaira embedding

$$\mathcal{M} \hookrightarrow \mathbb{P}^*(E_\ell),$$

over $\mathbb{A}^1$, where by (8.1) the projective bundle $\mathbb{P}^*(E_\ell)$ over $\mathbb{A}^1$ is viewed as product bundle $\mathbb{A}^1 \times \mathbb{P}^*((E_\ell)_0)$. Then each fiber $\mathbb{P}^*(E_\ell)_s$ of $\mathbb{P}^*(E_m)$ over $s \in \mathbb{A}^1$ is naturally identified with $\mathbb{P}^*((E_\ell)_0)$, so that all $\mathcal{M}_s$, $s \in \mathbb{A}^1$, are regarded as subschemes of $\mathbb{P}^*((E_\ell)_0)$. Then

$$\mathcal{M}_t = \psi_\ell(t) \cdot \mathcal{M}_1, \qquad t \in \mathbb{C}^*, \tag{8.3}$$

where on the right-hand side, the element $\psi_\ell(s)$ in $GL((E_\ell)_0)$ acts naturally on $\mathbb{P}^*((E_\ell)_0)$ as a projective linear transformation. Note that $\mathcal{M}_1$ is nothing but γ_1 as an algebraic cycle, and that $\mathcal{M}_0$ is preserved by the T-action on $\mathbb{P}^*((E_\ell)_0)$. Let $d_\ell := \dim(E_\ell)_0$ be as in (4.1), and we consider the d_ℓ-fold unramified covering $\hat{\mathbb{T}} := \{\hat{t} \in \mathbb{C}^*\}$ of the algebraic torus $\mathbb{T} := \{t \in \mathbb{C}^*\}$ by setting

$$t = \hat{t}^{d_\ell}$$

for t and $\hat{t}$. Then the mapping $\psi_\ell^{SL} : \hat{\mathbb{T}} \to SL((E_\ell)_0)$ defined by

$$\psi_\ell^{SL}(\hat{t}) := \frac{\psi_\ell(\hat{t}^{d_\ell})}{\det(\psi_\ell(\hat{t}))} = \frac{\psi_\ell(t)}{\det(\psi_\ell(\hat{t}))}, \qquad \hat{t} \in \hat{\mathbb{T}}, \tag{8.4}$$

is also an algebraic group homomorphism. Both $\psi_\ell(t)$ and $\psi_\ell^{SL}(\hat{t})$ induce exactly the same projective linear transformation on $\mathbb{P}^*((E_\ell)_0)$. Let γ_t be the algebraic cycle on $\mathbb{P}^*((E_m)_0)$ obtained as the image of γ_1 by this projective linear transformation. Now by (8.3), the algebraic cycle γ_t is nothing but $\mathcal{M}_t$ viewed just as an algebraic cycle on $\mathbb{P}^*((E_\ell)_0)$. Then as $t \to 0$, we have a limit algebraic cycle

$$\gamma_0 := \lim_{t \to 0} \gamma_t \tag{8.5}$$

on $\mathbb{P}^*((E_\ell)_0)$. Here γ_0 is the $\mathbb{T}$-invariant algebraic cycle on $\mathbb{P}^*((E_\ell)_0)$ associated to the subscheme $\mathcal{M}_0$ counted with multiplicities. Then let $\hat{M}_\ell^{(0)}$ denote the element in W_ℓ^* such that $[\hat{M}_\ell^{(0)}] \in \mathbb{P}^*(W_\ell)$ is the Chow point for the cycle γ_0 on $\mathbb{P}^*((E_\ell)_0)$. Then (8.5) is interpreted as

$$\lim_{\hat{t} \to 0} \left[\psi_\ell^{SL}(\hat{t}) \cdot \hat{M}_\ell\right] = \left[\hat{M}_\ell^{(0)}\right] \tag{8.6}$$

in $\mathbb{P}^*(W_\ell)$. Here by (8.2), the group $SL((E_\ell)_0)$ acts naturally on W_ℓ^*, and hence acts also on $\mathbb{P}^*(W_\ell)$. As in Section 6, we consider the function

$$f_\ell(s) := \log \left\|\psi_\ell^{SL}(\exp(\hat{s})) \cdot \hat{M}_\ell\right\|_{\mathrm{CH}(\rho_0)}, \qquad s \in \mathbb{R}, \tag{8.7}$$

by setting $\hat{s} := s/d_\ell$. Consider the first derivative $\dot{f}_\ell(s) := (df_\ell/ds)(s)$. The purpose of this appendix is to show the following (see Phong and Sturm [**25**, equation 7.29] for the leading term; see also [**4**, pp. 464–467]):

Theorem E. *Let a_n and F_j be as in Section 4. Then the function $\dot{f}_\ell(s)$ has a limit, as $s \to -\infty$, written in the following form for $\ell \gg 1$:*

$$\lim_{s\to-\infty} \dot{f}_\ell(s) = (n+1)!\, a_n(F_1\ell^n + F_2\ell^{n-1} + F_3\ell^{n-2} + \dots)$$
$$= (n+1)!\, a_n\left(\frac{w_\ell}{\ell d_\ell} - F_0\right)\ell^{n+1}. \tag{8.8}$$

Proof. Since γ_0 is preserved by the $\hat{\mathbb{T}}$-action on $(E_\ell)_0$, the Chow point $[\hat{M}^{(0)}]$ for γ_0 is fixed by the $\hat{\mathbb{T}}$-action on $\mathbb{P}^*(W_\ell)$, i.e.,

$$\psi_\ell^{SL}(\hat{t})\cdot \hat{M}_\ell^{(0)} = \hat{t}^{\lambda_\ell}\hat{M}_\ell^{(0)}, \qquad t\in\mathbb{C}^*,$$

for some $\lambda_\ell \in \mathbb{Z}$. Since the $\hat{\mathbb{T}}$-action on W_ℓ^* is diagonalizable, we can write $\hat{M}_\ell$ in the form

$$\hat{M}_\ell = \Sigma_{\alpha=1}^{\nu} u_\alpha, \tag{8.9}$$

where $0 \neq u_\alpha \in W_\ell^*$, $\alpha = 1, 2, \dots, \nu$, are such that, for an increasing sequence of integers $r_1 < r_2 < \cdots < r_\nu$, the equality

$$\psi_\ell^{SL}(\hat{t})\cdot u_\alpha = \hat{t}^{r_\alpha} u_\alpha \tag{8.10}$$

holds for all $\alpha \in \{1, 2, \dots, \nu\}$ and $\hat{t} \in \hat{\mathbb{T}}$. In particular, in view of (8.6), we can find a complex number $c \neq 0$ such that

$$\hat{M}_\ell^{(0)} = c\,u_1,$$

and hence λ_ℓ coincides with r_1. Then we may assume $c = 1$ without loss of generality. In view of (8.9) and (8.10), we can write $f_\ell(s)$ as

$$\log\left\|\exp\left(\frac{\lambda_\ell}{d_\ell}s\right)\cdot(u_1 + O(\hat{t}))\right\|_{\mathrm{CH}(\rho_0)} = \frac{\lambda_\ell}{d_\ell}s + \log\left\|(u_1+O(\hat{t}))\right\|_{\mathrm{CH}(\rho_0)},$$

so that by $\hat{t} = \exp(s/d_\ell)$, letting $s \to -\infty$, we obtain

$$\lim_{s\to-\infty}\dot{f}_\ell(s)\left(=\frac{r_1}{d_\ell}\right) = \frac{\lambda_\ell}{d_\ell}. \tag{8.11}$$

Hence it suffices to show that λ_ℓ/d_ℓ admits the asymptotic expansion as in the right-hand side of (8.8) above. Consider the graded algebra

$$\bigoplus_{k=0}^{\infty}(E_{k\ell})_0,$$

where via ψ_ℓ^{SL}, the group $\hat{\mathbb{T}}$ acts on $(E_\ell)_0$ and hence on $(E_{k\ell})_0$. Then by Mumford [**20**, Proposition 2.11], the weight τ_k for the $\hat{\mathbb{T}}$-action on $\det(E_{k\ell})_0$ satisfies the following:

$$\tau_k + \frac{\lambda_\ell}{(n+1)!}k^{n+1} = O(k^n), \qquad k \gg 1, \tag{8.12}$$

i.e., there exists a constant $C > 0$ independent of k, possibly depending on ℓ, such that the left-hand side of (8.12) has absolute value bounded by Ck^n for positive integers k. Let w_ℓ be as in (4.1). Then by the expression of ψ_ℓ^{SL} in (8.4), the weight τ_k for $\det(E_{k\ell})_0$ induced by the $\hat{\mathbb{T}}$-action on $(E_\ell)_0$ via ψ_ℓ^{SL} is expressible as

$$\tau_k = d_\ell w_{k\ell} - k\, w_\ell d_{k\ell}. \tag{8.13}$$

Here the term $d_\ell w_{k\ell}$ on the right-hand side of (8.13) is the weight in $\hat{t}$ for $\det(E_{k\ell})_0$ induced from the action of the numerator $\psi_\ell(t)$ of (8.4) on $(E_\ell)_0$, since it is nothing but the weight in $\hat{t}$ for the action of $\psi_{k\ell}(t)$ on $\det(E_{k\ell})_0$, while in view of the natural surjective homomorphism

$$S^k((E_\ell)_0) \to (E_{k\ell})_0, \qquad \ell \gg 1,$$

the term $k\, w_\ell d_{k\ell}$ is just the weight in $\hat{t}$ induced from the scalar action on $(E_\ell)_0$ by the denominator of (8.4). Then for $k \gg 1$, by (8.13) and (4.2), we obtain

$$\begin{aligned}
\tau_k &= d_\ell w_{k\ell} - k\, w_\ell d_{k\ell} = (k\ell) d_\ell d_{k\ell}\left\{\frac{w_{k\ell}}{(k\ell)d_{k\ell}} - \frac{w_\ell}{\ell d_\ell}\right\}\\
&= (k\ell) d_\ell d_{k\ell}\left\{\sum_{j\ge 0} F_j (k\ell)^{-j} - \sum_{j\ge 0} F_j \ell^{-j}\right\}\\
&= -(k\ell) d_\ell d_{k\ell}\{(F_1\ell^{-1} + F_2\ell^{-2} + F_3\ell^{-3} + \cdots) + O(k^{-1})\}\\
&= -k^{n+1} a_n d_\ell \{(F_1\ell^{n} + F_2\ell^{n-1} + F_3\ell^{n-2} + \cdots) + O(k^{-1})\},
\end{aligned}$$

where the last equality follows from $d_{k\ell} = (k\ell)^n\{a_n + O(1/k)\}$ obtained from (4.1) applied to $k\ell$ in place of ℓ. Then by comparing this expression of τ_k with (8.12), and then by (4.2), we obtain

$$\begin{aligned}
\frac{\lambda_\ell}{d_\ell} &= (n+1)!\, a_n (F_1\ell^n + F_2\ell^{n-1} + F_3\ell^{n-2} + \cdots)\\
&= (n+1)!\, a_n \left(\frac{w_\ell}{\ell d_\ell} - F_0\right)\ell^{n+1}.
\end{aligned}$$

9. Appendix 2.

In this Appendix 2, we shall show that the family of Kähler manifolds

$$(M, \omega_{s,\ell}), \qquad -\varepsilon \le s \le 0,\ \ell = 1, 2, \dots,$$

in (6.12) has bounded geometry in the sense that there exists a positive real constant R satisfying (cf. [**24**, p. 702])

a) $\omega_{s,\ell} - R^{-1}\omega$ is positive definite on M;
b) $\|\omega_{s,\ell} - \omega\|_{C^4(\omega)} < R$,

where ω is as in the proof of Theorem C. By (6.6), we identify $\mathbb{P}^*(E_\ell^X)$ with $\mathbb{A}^1 \times \mathbb{P}^*((E_\ell^X)_0)$, and let $\mathrm{pr}_2 : \mathbb{P}^*(E_\ell^X) \to \mathbb{P}^*((E_\ell^X)_0)$ denote the projection to the second factor. Then for the relative Kodaira embedding $\mathcal{M}^X \hookrightarrow \mathbb{P}^*(E_\ell^X)$ as in Section 6, the pullback

$$\mathcal{H} := \mathrm{pr}_2^* \, \mathcal{O}_{\mathbb{P}^*((E_\ell^X)_0)}(1)$$

to $\mathbb{P}^*(E_\ell^X)$ of the the hyperplane bundle $\mathcal{O}_{\mathbb{P}^*((E_\ell^X)_0)}(1)$ on $\mathbb{P}^*((E_\ell^X)_0)$ has the restriction

$$\mathcal{H}_{|\mathcal{M}^X} = (\mathcal{L}^X)^\ell. \tag{9.1}$$

Recall that the action of $\mathbb{T} = \{t \in \mathbb{C}^*\}$ on $\mathcal{M}^X$ lifts to a $\mathbb{T}$-linearization of $\mathcal{L}^X$, and hence $\mathbb{T}$ acts on $E_\ell^X = \mathbb{A}^1 \times (E_\ell^X)_0$ by

$$\mathbb{T} \times (\mathbb{A}^1 \times (E_\ell^X)_0) \to \mathbb{A}^1 \times (E_\ell^X)_0, \qquad (t, (s, e)) \mapsto (ts, \psi_\ell(t) \cdot e),$$

where ψ_ℓ is as in Section 6. This induces a $\mathbb{T}$-action on $\mathbb{P}^*(E_\ell^X)$. Let $\bar{\mathcal{L}}^X$ denote the complex conjugate of $\mathcal{L}^X$. By

$$\mathbb{T} \times \mathcal{L}^X \to \mathcal{L}^X, \qquad (t, \lambda) \mapsto g_{\mathcal{L}}(t) \cdot \lambda,$$

we mean the $\mathbb{T}$-action on $\mathcal{L}^X$, and the associated $\mathbb{T}$-action on the real line bundle $|\mathcal{L}^X|^2 := \mathcal{L}^X \otimes \bar{\mathcal{L}}^X$ on $\mathcal{M}^X$ will be denoted by

$$\mathbb{T} \times |\mathcal{L}^X|^2 \to |\mathcal{L}^X|^2, \qquad (t, \xi) \mapsto g_{|\mathcal{L}|^2}(t) \cdot \xi.$$

This $\mathbb{T}$-action on $|\mathcal{L}^X|^2$, covering the $\mathbb{T}$-action on $\mathcal{M}^X$, is independent of the choice of ℓ. In view of the definition of $g_{s,\ell}$, both $\psi_\ell(\exp(s))$ and $g_{s,\ell}$ induce the same projective linear transformation on $(E_\ell^X)_0$. Note also that $\varepsilon = C_3(\log \ell)q$, $\ell \gg 1$, and $-\varepsilon \le s \le 0$. Then by setting $\theta := 1 - e^{-C_3(\log \ell)q}$, we obtain

$$1 - \theta \le \exp(s) \le 1, \tag{9.2}$$

where $0 < \theta \ll 1$. As in Section 6, let $\{\sigma_{i,\alpha}; (i, \alpha) \in P_\ell\}$ be an admissible orthonormal basis for V_ℓ $(= (E^X_\ell)_1)$, and by the identification

$$(E^X_\ell)_1 \cong (E^X_\ell)_0,$$

the corresponding orthonormal basis for $(E^X_\ell)_0$ will be denoted by $\{\underline{\sigma}_{i,\alpha}; (i,\alpha) \in P_\ell\}$. In terms of these bases, both $\mathbb{P}^*((E^X_\ell)_0)$ and $\mathbb{P}^*((E^X_\ell)_1)$ $(= \mathbb{P}^*(V_\ell))$ are identified with

$$\mathbb{P}^{d_\ell - 1}(\mathbb{C}) = \{(z_1 : z_2 : \cdots : z_{d_\ell})\}.$$

Then $(n!/\ell^n)\Sigma_{\alpha=1}^{d_\ell}|z_\alpha|^2$ is regarded as a section for $|\mathcal{H}|^2 := \mathcal{H} \otimes \bar{\mathcal{H}}$, while by (9.1), we can write on $\mathcal{M}^X$

$$q\omega_{\mathrm{FS}} = \big(\sqrt{-1}/2\pi\big)\partial\bar{\partial}\log\Omega_{\mathrm{FS},\ell}.$$

Here $\Omega_{\mathrm{FS},\ell}$ denotes the positive real smooth section of $|\mathcal{L}^X|^2$ obtained as the restriction of $\{(n!/\ell^n)\Sigma_{\alpha=1}^{d_\ell}|z_\alpha|^2\}^q$ to $\mathcal{M}^X$. Put $t := \exp(s)$ for simplicity. In view of (9.1), identifying M with $\mathcal{M}^X_1$, we easily see that

$$\omega_{s,\ell} = \big(\sqrt{-1}/2\pi\big)\partial\bar{\partial}\log\big\{g_{|\mathcal{L}|^2}(t)^*\Omega_{\mathrm{FS},\ell}\big\}, \tag{9.3}$$

when restricted to $\mathcal{M}^X_1 \hookrightarrow \mathbb{P}^{d_\ell-1}(\mathbb{C})$. Here $g_{|\mathcal{L}|^2}(t)^*\Omega_{\mathrm{FS},\ell}$ is regarded as a positive real section of $|g_{\mathcal{L}}(t)^*\mathcal{L}^X|^2$ on $\mathcal{M}^X_1 \hookrightarrow \mathbb{P}^{d_\ell-1}(\mathbb{C})$. Consider the dual h^* of the Hermitian metric h, where h is such that $\omega = c_1(L;h)$ is the original extremal Kähler metric on M. Now by a theorem of Catlin-Lu-Tian-Zeldich ([**2**], [**9**], [**30**], [**31**]), we obtain

$$\Omega_{\mathrm{FS},\ell} \to h^* \text{ in } C^\infty, \tag{9.4}$$

as $\ell \to \infty$. In view of $t = \exp(s)$, $-\varepsilon \le s \le 0$, and (9.2), when restricted to $\mathcal{M}^X_1 (= M) \hookrightarrow \mathbb{P}^{d_\ell-1}(\mathbb{C})$, the difference between $g_{|\mathcal{L}|^2}(t)^*\Omega_{\mathrm{FS},\ell}$ and $\Omega_{\mathrm{FS},\ell}$ is small enough in the sense that its C^∞-norm on M is uniformly bounded from above by a constant $C(\theta)$ depending only on θ such that $C(\theta) \to 0$ as $\theta \to 0$. Thus we conclude from (9.3) that the family of Kähler manifolds $(M, \omega_{s,\ell})$ in (6.12) has bounded geometry.

REMARK 9.5. By $\varepsilon = C_3(\log \ell)q$ and $-\varepsilon \le s_\ell \le 0$, we see that θ above satisfies $\theta \to 0$ as $\ell \to \infty$, and hence $\omega(j) \to \omega$ as $j \to \infty$ in Section 6.

10. Appendix 3.

In the Case 1 of Step 3 of Section 6 in the proof of Theorem C, we assume that $(\tilde{\mathcal{M}}^X, \tilde{\mathcal{L}}^X)$ is nontrivial. Then by using [**16**], we shall show $X \in \mathfrak{g}$ as follows. Let $\eta^\circ(j)$ and $\eta^\bullet(j)$ be the Hamiltonian functions associated to the vector fields $\mathcal{X}^\circ(j)$ and $\mathcal{X}^\bullet(j)$, respetively, on the Kähler manifold $(M, \omega(j))$. Then

$$\begin{cases} i(2\pi\mathcal{X}^{\circ}(j))\omega(j) = \sqrt{-1}\,\bar{\partial}(\eta^{\circ}(j)), \\ i(2\pi\mathcal{X}^{\bullet}(j))\omega(j) = \sqrt{-1}\,\bar{\partial}(\eta^{\bullet}(j)), \end{cases}$$

where the functions $\eta^{\circ}(j)$ and $\eta^{\bullet}(j)$ are normalized by the vanishing of the integrals $\int_M \eta^{\circ}(j)\omega(j)^n$ and $\int_M \eta^{\bullet}(j)\omega(j)^n$, respectively. Then $\eta(j) = \eta^{\circ}(j) + \eta^{\bullet}(j)$, where by (6.21) and the assumption of Case 1,

$$I_j^{\bullet} \to 0 \text{ as } j \to \infty; \tag{10.1}$$

$$\{I_j^{\circ}\}_{j=1,2,\dots} \text{ is a bounded sequence.} \tag{10.2}$$

In view of (10.2), replacing $\omega(j)$, $j = 1, 2, \dots$, by its suitable subsequence if necessary, we may assume that

$$\mathcal{X}^{\circ}(j) \to \mathcal{X}^{\circ}_{\infty} \text{ in } \mathfrak{g}, \qquad \text{as } j \to \infty,$$

for some $\mathcal{X}^{\circ}_{\infty} \in \mathfrak{g}$. Hence there exists a C^{∞} function ρ on M such that $i(\mathcal{X}^{\circ}_{\infty})\omega = \sqrt{-1}\,\bar{\partial}\rho$ and that

$$\eta^{\circ}(j) \to \rho \text{ in } C^{\infty}(M), \qquad \text{as } j \to \infty.$$

This together with (6.23) implies

$$\eta^{\bullet}(j) \to \eta^{\bullet}_{\infty} \text{ in } L^2(M, \omega^n), \qquad \text{as } j \to \infty,$$

where $\eta^{\bullet}_{\infty} := \eta_{\infty} - \rho$. Let θ be an arbitrary smooth $(0,1)$-form θ on M. Then from (10.1) and $I_j^{\bullet} = \int_M |\bar{\partial}\eta^{\bullet}(j)|^2_{\omega(j)}\omega(j)^n$, it follows that

$$\begin{aligned} \left|(\eta^{\bullet}(j), \bar{\partial}(j)^*\theta)_{L^2(M,\omega(j)^n)}\right| &= \left| \int_M (\bar{\partial}\eta^{\bullet}(j), \theta)_{\omega(j)}\, \omega(j)^n \right| \\ &\leq \left\{ \int_M |\bar{\partial}\eta^{\bullet}(j)|^2_{\omega(j)}\omega(j)^n \right\}^{1/2} \left\{ \int_M |\theta|^2_{\omega(j)}\omega(j)^n \right\}^{1/2} \to 0, \end{aligned}$$

as $j \to \infty$. Then by letting $j \to \infty$, we obtain

$$\left(\eta^{\bullet}_{\infty}, \bar{\partial}(\infty)^*\theta\right)_{L^2(M,\omega^n)} = 0,$$

for every smooth $(0,1)$-form θ on M, i.e., $\bar{\partial}\eta^{\bullet}_{\infty} = 0$ in a weak sense, and hence in a strong sense. Thus $\eta^{\bullet}_{\infty}$ is constant on M, so that

$$0 = \eta^{\bullet}_{\infty} = \eta_{\infty} - \rho.$$

By setting $\underline{\mathcal{X}}(j) := (g(j)^{-1})_* \mathcal{X}^{(\ell_j)}_{|M_{s_{\ell_j}}}$ and $\underline{\mathcal{X}}_{TM^{\perp}}(j) := (g(j)^{-1})_* \mathcal{X}^{(\ell_j)}_{TM^{\perp}_{s_{\ell_j}}}$, we now have the expression

$$\mathcal{X}^{(\ell_j)}_{|\Phi(j)(M)} = \underline{\mathcal{X}}(j) = \underline{\mathcal{X}}_{TM^{\perp}}(j) + \Phi(j)_*\mathcal{X}^{\circ}(j) + \Phi(j)_*\mathcal{X}^{\bullet}(j).$$

Let $j \to \infty$. Then by [**16**], we conclude from (6.17) and (10.1) that

$$X = W \in \mathfrak{g}$$

in the Lie algebra $\mathfrak{sl}(V_1)$, as required.

Remark 10.3. The essential point of [**16**] is Appendix in Section 5, in which by using the normality of $\mathcal{M}$ implicitly, we observed that the nontriviality of $\Psi^{SL}_{1,X'}$ induces a nontrivial birational $\mathbb{C}^*$-action of an n-dimendsional irreducible component of $\mathcal{F}$ of $\mathcal{M}_0$ (see [**16**, pp. 22–23]). However, since $\mathcal{M}$ is not necessarily normal, it can occur that the induced birational $\mathbb{C}^*$-action on each n-dimendsional irreducible component of $\mathcal{F}$ of $\mathcal{M}_0$ is trivial, in which case the test configuration is trivial up to codimension ≥ 2 subvarieties of $\mathcal{M}$. Now by [**26**], our argument in [**16**] is still valid even if the revised version (cf. Definition 4.3) of K-stability due to [**8**] is used.

References

[1] E. Calabi, Extremal Kähler metrics. II, In: Differential Geometry and Complex Analysis, (eds. I. Chavel and H. M. Farkas), Springer-Verlag, Heidelberg, 1985, pp. 95–114.

[2] D. Catlin, The Bergman kernel and a theorem of Tian, In: Analysis and Geometry in Several Complex Variables, Katata, 1997, (eds. G. Komatsu and M. Kuranishi), Trends Math., Birkhäuser, Boston, 1999, pp. 1–23.

[3] S. K. Donaldson, Scalar curvature and stability of toric varieties, J. Differential Geom., **62** (2002), 289–349.

[4] S. K. Donaldson, Lower bounds on the Calabi functional, J. Differential Geom., **70** (2005), 453–472.

[5] A. Futaki, Asymptotic Chow semi-stability and integral invariants, Internat. J. Math., **15** (2004), 967–979.

[6] A. Futaki and T. Mabuchi, Bilinear forms and extremal Kähler vector fields associated with Kähler classes, Math. Ann., **301** (1995), 199–210.

[7] A. Futaki, Kähler-Einstein metrics and integral invariants, Lecture Notes in Math., **1314**, Springer-Verlag, Berlin, 1988.

[8] C. Li and C. Xu, Special test configurations and K-stability of Fano varieties, arXiv:math.AG/1111.5398.

[9] Z. Lu, On the lower order terms of the asymptotic expansion of Tian-Yau-Zelditch, Amer. J. Math., **122** (2000), 235–273.

[10] T. Mabuchi, An obstruction to asymptotic semistability and approximate critical metrics, Osaka J. Math., **41** (2004), 463–472.

[11] T. Mabuchi, Stability of extremal Kähler manifolds, Osaka J. Math., **41** (2004), 563–582.

[12] T. Mabuchi, Uniqueness of extremal Kähler metrics for an integral Kähler class, Internat. J. Math., **15** (2004), 531–546.

[13] T. Mabuchi, An energy-theoretic approach to the Hitchin-Kobayashi correspondence for manifolds. I, Invent. Math., **159** (2005), 225–243.

[14] T. Mabuchi, An energy-theoretic approach to the Hitchin-Kobayashi correspondence for manifolds. II, Osaka J. Math., **46** (2009), 115–139.

[15] T. Mabuchi, K-stability of constant scalar curvature polarization, arXiv:math.DG/0812.4093.

[16] T. Mabuchi, A stronger concept of K-stability, arXiv:math.DG/0910.4617.

[17] T. Mabuchi, Asymptotics of polybalanced metrics under relative stability constraints, Osaka J. Math., **48** (2011), 845–856.

[18] T. Mabuchi, The Yau-Tian-Donaldson Conjecture for general polarization, arXiv:math.DG/1307.3623.
[19] T. Mabuchi and Y. Nitta, K-stability and asymptotic Chow stability, arXiv:math.DG/1307.1959.
[20] D. Mumford, Stability of projective varieties, Enseignement Math. (2), **23** (1977), 39–110.
[21] T. Oda, Convex Bodies and Algebraic Geometry; An Introduction to the Theory of Toric Varieties, Ergeb. Math. Grenzgeb. (3), **15**, Springer-Verlag, Berlin, 1988.
[22] H. Ono, Y. Sano and N. Yotsutani, An example of an asymptotically Chow unstable manifold with constant scalar curvature, Ann. Inst. Fourier (Grenoble), **62** (2012), 1265–1287.
[23] S. T. Paul and G. Tian, Algebraic and analytic K-stability, arXiv:math.DG/0405530.
[24] D. H. Phong and J. Sturm, Scalar curvature, moment maps, and the Deligne pairing, Amer. J. Math., **126** (2004), 693–712.
[25] D. H. Phong and J. Sturm, Lectures on stability and constant scalar curvature, In: Handbook of Geometric Analysis. No. 3, (eds. L. Ji, P. Li, R. Schoen and L. Simon), Adv. Lect. Math. (ALM), **14**, Int. Press, Somerville, MA, 2010, pp. 357–436.
[26] J. Stoppa, A note on the definition of K-stability, arXiv:math.AG/1111.5826.
[27] J. Stoppa and G. Székelyhidi, Relative K-stability of extremal metrics, J. Eur. Math. Soc. (JEMS), **13** (2011), 899–909.
[28] G. Székelyhidi, Extremal metrics and K-stability, Bull. Lond. Math. Soc., **39** (2007), 76–84.
[29] G. Tian, Kähler-Einstein metrics with positive scalar curvature, Invent. Math., **130** (1997), 1–37.
[30] G. Tian, On a set of polarized Kähler metrics on algebraic manifolds, J. Differential Geom., **32** (1990), 99–130.
[31] S. Zelditch, Szegö kernels and a theorem of Tian, Internat. Math. Res. Notices, **1998** (1998), 317–331.
[32] S. Zhang, Heights and reductions of semi-stable varieties, Compositio Math., **104** (1996), 77–105.

Toshiki Mabuchi
Department of Mathematics
Osaka University
Toyonaka
Osaka 560-0043, Japan
E-mail: mabuchi@math.sci.osaka-u.ac.jp