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Riesz measures and Wishart laws associated to quadratic maps

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Abstract. We introduce a natural definition of Riesz measures and Wishart laws associated to an Ω-positive (virtual) quadratic map, where $\Omega \subset \boldsymbol{R}^n$ is a regular open convex cone. In this context we prove new general formulas for moments of the Wishart laws on non-symmetric cones. For homogeneous cases, all the quadratic maps are characterized and the associated Riesz measure and Wishart law with its moments are described explicitly. We apply the theory of relatively invariant distributions and a matrix realization of homogeneous cones obtained recently by the second author.

1. Introduction.

The objective and motivation of this paper is to present a natural approach to Wishart laws and Riesz measures on regular convex cones via quadratic maps, and to apply it to the computation of moments of Wishart laws, in particular on homogeneous cones. Note that such a "quadratic" approach and moment formulas are lacking in a series of recent papers ([**1**], [**3**], [**20**]) devoted to certain Wishart laws on non-symmetric cones.

Riesz measures and distributions on convex cones form one of fundamental tools of harmonic analysis and of the theory of the wave equation, cf. [**5**] in the case of symmetric cones and [**7**], [**8**], [**12**] for homogeneous cones. Moreover, exponential families generated by Riesz measures are composed of Wishart laws and are of great significance in random matrix theory and in statistics.

Wishart laws are probability distributions on symmetric or Hermitian matrices with very important applications in multivariate statistics. Their role in statistics is due to two reasons:

– they are probability distributions of the maximum likelihood estimator (MLE) of the covariance matrix in a multivariate normal sample ([**21**], [**1**]).
– in Bayesian statistics, Wishart laws form a Diaconis-Ylvisaker family ([**4**]) of prior distributions for the covariance parameter in a covariance selection model ([**20**]).

On the other hand, recent developments in random matrix theory of chiral Gaussian ensembles containing Wishart laws, are intense and motivated by applications in mathematical physics, cf. [**17**] and references therein.

These numerous modern applications of Wishart laws make it necessary to develop the theory of Wishart laws and Riesz measures on more general cones than in the classical

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case of the symmetric cones of real symmetric or complex Hermitian matrices. For example, in an r-dimensional Gaussian model X, if the marginal variables X_i and X_j are known to be conditionally independent given all the other variables, the statistical analysis of the covariance matrix of X must be done on the cone $\mathcal{P}$ of positive definite symmetric matrices Y with $Y_{ij} = Y_{ji} = 0$ and on its dual cone $\mathcal{Q}$ ([**20**]). The cones $\mathcal{P}$ and $\mathcal{Q}$ are usually no longer symmetric. This led to some important papers in recent statistical and probabilistic literature about Wishart laws on more general cones: homogeneous cones ([**1**]) or cones related to graphical models ([**20**]). In these papers, Wishart laws are introduced via their density functions (see Section 3.8).

A natural approach and definition of Wishart laws is by quadratic maps. If X is a standard normal random matrix, then the symmetric matrix

$$Y = X^{\mathrm{t}} X$$

has a Wishart law and this is the first step of a usual definition of all classical Wishart laws [**21**], [**5**]. However, the authors of [**1**], [**20**] never consider a quadratic construction of Wishart laws. For Riesz measures, a quadratic construction is presented for symmetric cones in [**5**], but not explicitly noticed in [**7**], [**12**] for homogeneous cones.

In this paper we construct and study Riesz measures and Wishart laws on regular convex cones via quadratic maps. For a regular open convex cone $\Omega \subset \boldsymbol{R}^n$ and an Ω-positive quadratic map $q : \boldsymbol{R}^m \to \boldsymbol{R}^n$, the Riesz measure associated to q is defined as the image of the Lebesgue measure dx on $\boldsymbol{R}^m$ by q. Wishart laws studied in this paper are obtained from $\boldsymbol{R}^m$-valued normal random vectors X as the law of $Y := q(X)/2$.

In Section 2 of the paper we explain the details of the quadratic construction of Riesz measures on regular convex cones and next we define the corresponding Wishart laws. We compute their Laplace transforms. More general Riesz measures and Wishart laws associated to *virtual* quadratic maps are introduced in Section 2.4.

In Sections 2.3 and 2.5 we get formulas for the expectation, covariance and higher moments of Wishart laws (Theorems 2.8, 2.9 and 2.12), following in a straightforward way from the Laplace transform formulas. Moments formulas are generalized in Theorem 2.13, which is not so obvious as it may seem. These results on moments are an essential contribution into the theory of Wishart laws on non-symmetric cones and they have important statistical applications. Without introducing of the associated linear map ϕ in Definition 2.2, the moment formulas were unavailable by the techniques of [**1**], [**20**]. Moreover, the notion of virtuality is indispensable here.

Group equivariance of Wishart laws is studied at the end of the Section 2.

Section 3 of the article is thoroughly devoted to the case when Ω is a homogeneous cone and the quadratic map q is homogeneous. A crucial role in the analysis of these maps and of related Riesz measures and Wishart laws is played by a matrix realization of any homogeneous cone, coming from [**15**] and explained in Section 3.2. It allows, among others, to define basic and standard quadratic maps in Sections 3.3 and 3.4. They play a role of generators for homogeneous quadratic maps q needed to construct all Riesz measures and Wishart laws on Ω. Next we apply the results of [**12**] on Gindikin-Riesz distributions on Ω and on the orbit decomposition of $\overline{\Omega}$, the closure of Ω. We explain the relation between Riesz measures related to homogeneous quadratic maps and the

Gindikin-Riesz distributions on Ω (Theorem 3.13). In Section 3.7, we prove the Bartlett decomposition for the Wishart laws on homogeneous cones (Theorems 3.15 and 3.17).

Here we summarize what we have done, compared with preceding works. Families of Wishart laws that we construct and study in Section 3 comprise Wishart distributions studied in papers [**1**] and [**20**] (homogeneous case) and are significantly bigger: we describe all singular Wishart laws and many more absolutely continuous Wishart laws than in papers [**1**] and [**20**]. For the symmetric cone case, our Wishart laws cover the ones studied in [**11**] as well. All the results of Section 2 apply to them, in particular the formula in Theorem 2.13 does for the moments. Let us underline the novelty and usefulness of the technique of matrix realization of homogeneous cones in the study of Wishart laws on such cones. The Bartlett decomposition via the standard quadratic maps is obtained on homogeneous cones thanks to this technique. Moment formulas are explicit thanks to basic quadratic maps construction.

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2. Riesz measure and Wishart law on a convex cone. Moments of Wishart laws.

2.1. Regular cones and quadratic maps.

In this paper, an open convex cone $\Omega \subset \boldsymbol{R}^n$ is always assumed to be *regular*, that is, $\overline{\Omega} \cap (-\overline{\Omega}) = \{0\}$, where $\overline{\Omega}$ denotes the closure of Ω. Then the dual cone $\Omega^* := \big\{\eta \in (\boldsymbol{R}^n)^* \,;\, \langle y, \eta\rangle > 0 \ (\forall y \in \overline{\Omega} \setminus \{0\})\big\}$ is a regular open convex cone again in the dual vector space $(\boldsymbol{R}^n)^*$, and we have $(\Omega^*)^* = \Omega$.

Definition 2.1. A *quadratic map* $q : \boldsymbol{R}^m \ni x \mapsto {}^{\mathsf{t}}(f_1(x), \dots, f_n(x)) \in \boldsymbol{R}^n$ is a map where each $f_k(x)$ $(k = 1, \dots, n)$ is a quadratic form of x. We say that q is Ω*-positive* if

(i) $q(x) \in \overline{\Omega}$ for all $x \in \boldsymbol{R}^m$, and
(ii) $q(x) = 0$ implies $x = 0$.

The conditions (i) and (ii) are restated in a single condition $q(x) \in \overline{\Omega} \setminus \{0\}$ $(\forall x \in \boldsymbol{R}^m \setminus \{0\})$.

Definition 2.2. For the quadratic map q, we define *the associated linear map* $\phi = \phi_q : (\boldsymbol{R}^n)^* \to \mathrm{Sym}(m, \boldsymbol{R})$ in such a way that

$$ {}^{\mathsf{t}}x\phi(\eta)x = \langle q(x), \eta\rangle \qquad (\eta \in (\boldsymbol{R}^n)^*,\ x \in \boldsymbol{R}^m). $$

Note that the associated linear map ϕ is a generalization of the coefficient matrix of a quadratic form on a linear space. The Ω-positivity of q is equivalent to the following property of ϕ:

$$ \eta \in \Omega^* \Rightarrow \phi(\eta) \text{ is positive definite.} \tag{2.1} $$

Example 1. Let Ω be the open convex cone in $\boldsymbol{R}^3$ defined by

$$\begin{aligned}\Omega &:= \left\{ t_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t_4 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} ; t_1, t_2, t_3, t_4 > 0 \right\} \\ &= \left\{ \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \boldsymbol{R}^3 ; y_1 > 0,\ y_2 > 0,\ -y_1 + y_3 > 0,\ -y_2 + y_3 > 0 \right\}. \qquad (2.2)\end{aligned}$$

If we identify $(\boldsymbol{R}^3)^*$ with $\boldsymbol{R}^3$ by $\langle y, \eta \rangle := y_1\eta_1 + y_2\eta_2 + y_3\eta_3$ $(y, \eta \in \boldsymbol{R}^3)$, we have

$$\begin{aligned}\Omega^* &= \left\{ \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} \in \boldsymbol{R}^3 ; \eta_3 > 0,\ \eta_1 + \eta_3 > 0,\ \eta_1 + \eta_2 + \eta_3 > 0,\ \eta_2 + \eta_3 > 0 \right\} \\ &= \left\{ t_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t_4 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} ; t_1, t_2, t_3, t_4 > 0 \right\},\end{aligned}$$

see [**14**]. Let $q : \boldsymbol{R}^4 \to \boldsymbol{R}^3$ be the quadratic map given by

$$q(x) := (x_1)^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + (x_2)^2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + (x_3)^2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (x_4)^2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad (x \in \boldsymbol{R}^4).$$

Clearly, this q is Ω-positive. By a simple calculation, we have

$$\phi(\eta) = \begin{pmatrix} \eta_3 & 0 & 0 & 0 \\ 0 & \eta_1 + \eta_3 & 0 & 0 \\ 0 & 0 & \eta_1 + \eta_2 + \eta_3 & 0 \\ 0 & 0 & 0 & \eta_2 + \eta_3 \end{pmatrix} \quad (\eta \in \boldsymbol{R}^3).$$

Example 2. Let Sym_r^+ be the set of positive definite real symmetric matrices of size r. Then Sym_r^+ is a regular open convex cone in the vector space $\mathrm{Sym}(r, \boldsymbol{R})$ of real symmetric matrices. If we identify the space $\mathrm{Sym}(r, \boldsymbol{R})$ with its dual vector space by the inner product $\langle y, \eta \rangle := \mathrm{tr}(y\eta)$ $(y, \eta \in \mathrm{Sym}(r, \boldsymbol{R}))$, then the dual cone $(\mathrm{Sym}_r^+)^*$ coincides with Sym_r^+. We define $q_{r,s} : \mathrm{Mat}(r, s; \boldsymbol{R}) \to \mathrm{Sym}(r, \boldsymbol{R})$ by

$$q_{r,s}(x) = x\,{}^{\mathrm{t}}x \quad (x \in \mathrm{Mat}(r, s; \boldsymbol{R})).$$

Then $q_{r,s}$ is Sym_r^+-positive. We denote the (i, j) component of $x \in \mathrm{Mat}(r, s; \boldsymbol{R})$ by $x_{r(j-1)+i}$, so that $\mathrm{Mat}(r, s; \boldsymbol{R})$ is identified with $\boldsymbol{R}^{rs}$. Then we have for $\eta \in \mathrm{Sym}(r, \boldsymbol{R})$

$$\phi(\eta) = \begin{pmatrix} \eta & & & \\ & \eta & & \\ & & \ddots & \\ & & & \eta \end{pmatrix} \in \mathrm{Sym}(rs, \boldsymbol{R}),$$

where η is put s times. In this case, the map $\phi : \mathrm{Sym}(r, \boldsymbol{R}) \to \mathrm{Sym}(rs, \boldsymbol{R})$ is a Jordan algebra representation, and q is exactly the quadratic map associated to the representation ([**5**, Chapter IV, Section 4]).

EXAMPLE 3. Let $\mathcal{Z}$ be a subspace of $\mathrm{Sym}(r, \boldsymbol{R})$, and put $\mathcal{P} := \mathcal{Z} \cap \mathrm{Sym}_r^+$. Then $\mathcal{P}$ is a regular open convex cone in $\mathcal{Z}$. Let $\mathcal{Q} \subset \mathcal{Z}^*$ be the dual cone of $\mathcal{P}$. We shall construct a $\mathcal{Q}$-positive quadratic map $q_{\mathcal{Z}} : \boldsymbol{R}^r \to \mathcal{Z}^*$ whose associated linear map $\phi_{\mathcal{Z}} : \mathcal{Z} \to \mathrm{Sym}(r, \boldsymbol{R})$ equals the inclusion map. Let us define the surjective linear map $\pi_{\mathcal{Z}^*} : \mathrm{Sym}(r, \boldsymbol{R}) \to \mathcal{Z}^*$ by

$$\langle y, \pi_{\mathcal{Z}^*}(S) \rangle := \mathrm{tr}\, yS \qquad (y \in \mathcal{Z},\ S \in \mathrm{Sym}(r, \boldsymbol{R})).$$

Then the quadratic map $q_{\mathcal{Z}} : \boldsymbol{R}^r \to \mathcal{Z}^*$ is given by $q_{\mathcal{Z}}(x) := \pi_{\mathcal{Z}^*}(x{}^{\mathsf{t}}x)$ $(x \in \boldsymbol{R}^r)$. In fact, for $x \in \boldsymbol{R}^r \setminus \{0\}$ and $y \in \mathcal{P}$ we have

$$\langle y, q_{\mathcal{Z}}(x) \rangle = \mathrm{tr}(yx{}^{\mathsf{t}}x) = {}^{\mathsf{t}}xyx > 0 \tag{2.3}$$

because y is positive definite. Therefore we get $q_{\mathcal{Z}}(x) \in \overline{\mathcal{Q}} \setminus \{0\}$, so that $q_{\mathcal{Z}}$ is $\mathcal{Q}$-positive. Keeping the natural isomorphism $(\mathcal{Z}^*)^* \simeq \mathcal{Z}$ in mind, we see from (2.3) that $\phi_{\mathcal{Z}}(y) = y$ $(y \in \mathcal{Z})$. Soon later, we shall consider the cases

$$\mathcal{Z} := \left\{ \begin{pmatrix} y_{11} & 0 & 0 \\ 0 & y_{22} & y_{32} \\ 0 & y_{32} & y_{33} \end{pmatrix} \in \mathrm{Sym}(3, \boldsymbol{R})\, ;\, y_{11},\, y_{22},\, y_{32},\, y_{33} \in \boldsymbol{R} \right\} \tag{2.4}$$

and

$$\mathcal{Z} := \left\{ \begin{pmatrix} y_{11} & 0 & y_{31} \\ 0 & y_{22} & y_{32} \\ y_{31} & y_{32} & y_{33} \end{pmatrix} \in \mathrm{Sym}(3, \boldsymbol{R})\, ;\, y_{11},\, y_{31},\, y_{32},\, y_{22},\, y_{33} \in \boldsymbol{R} \right\} \tag{2.5}$$

as concrete examples. Actually, in the latter case (2.5), the cones $\mathcal{Q}$ and $\mathcal{P}$ are called *the Vinberg cone* and *the dual Vinberg cone* respectively, which are the lowest dimensional non-symmetric homogeneous cones ([**24**]). We shall see another realization of the Vinberg cone $\mathcal{Q}$ in (3.6) and the last paragraph of Section 3.3.

Let $I = \{i_1, i_2, \ldots, i_k\}$ be a subset of $\{1, \ldots, r\}$ with $1 \leq i_1 < i_2 < \cdots < i_k \leq r$, and define

$$R^I := \{x \in \boldsymbol{R}^r\, ;\, x_i = 0 \text{ if } i \notin I\}\,. \tag{2.6}$$

We denote by $q^I_{\mathcal{Z}}$ the restriction of $q_{\mathcal{Z}}$ to the space $R^I \subset \boldsymbol{R}^r$. Clearly $q^I_{\mathcal{Z}} : R^I \to \mathcal{Z}^*$ is $\mathcal{Q}$-positive. The associated linear map $\phi_{q^I_{\mathcal{Z}}} : \mathcal{Z} \to \mathrm{Sym}(k, \boldsymbol{R})$ gives a submatrix of elements $y \in \mathcal{Z}$, that is, $\phi_{q^I_{\mathcal{Z}}}(y) = (y_{i_\alpha i_\beta})$, which we denote by y_I.

2.2. Riesz measures and Wishart laws associated to quadratic maps. In this section, for a given quadratic map q we define related Riesz measures and Wishart laws and we compute their Laplace transforms.

Definition 2.3. For a regular open convex cone $\Omega \subset \boldsymbol{R}^n$ and an Ω-positive quadratic map $q : \boldsymbol{R}^m \to \boldsymbol{R}^n$, let *the Riesz measure* μ_q *associated to* q be the image of the Lebesgue measure dx on $\boldsymbol{R}^m$ by q. Namely, the measure μ_q on $\boldsymbol{R}^n$ is defined in such a way that

$$\int_{\boldsymbol{R}^n} f(y)\, \mu_q(dy) = \int_{\boldsymbol{R}^m} f(q(x))\, dx \tag{2.7}$$

for a measurable function f on $\boldsymbol{R}^n$.

The terminology "the Riesz measure associated to q" is introduced by analogy to [**5**, Proposition VII.2.4]. The Ω-positivity of q implies that the support of μ_q is contained in the closure $\overline{\Omega}$ of the cone Ω.

Lemma 2.4. *Let $\phi : (\boldsymbol{R}^n)^* \to \mathrm{Sym}(m, \boldsymbol{R})$ be the linear map associated to q. Then, for $\eta \in \Omega^*$, the Laplace transform $L_{\mu_q}(\eta) := \int_{\boldsymbol{R}^n} e^{-\langle \eta, y\rangle} \mu_q(dy)$ of μ_q equals $\pi^{m/2}(\det \phi(\eta))^{-1/2}$.*

Proof. By definition, we have $L_{\mu_q}(\eta) = \int_{\boldsymbol{R}^m} e^{-{}^t x \phi(\eta) x}\, dx$. Since $\phi(\eta)$ is positive definite, the assertion follows from a formula of the Gaussian integral. □

Definition 2.5. The members of the exponential family $\{\gamma_{q,\theta}\}_{\theta \in \Omega^*}$ generated by μ_q are called *the Wishart laws on* Ω *associated to* q. Namely,

$$\gamma_{q,\theta}(dy) := \frac{e^{-\langle y, \theta\rangle}}{L_{\mu_q}(\theta)} \mu_q(dy) \qquad (y \in \boldsymbol{R}^n). \tag{2.8}$$

Remark 2.6. By (2.7), (2.8) and Lemma 2.4, we have for a measurable function f on $\boldsymbol{R}^n$

$$\int_{\boldsymbol{R}^n} f(y) \gamma_{q,\theta}(dy) = \pi^{-m/2} (\det \phi(\theta))^{1/2} \int_{\boldsymbol{R}^m} f(q(x)) e^{-{}^t x \phi(\theta) x}\, dx.$$

Putting $\Sigma := \phi(\theta)^{-1}$ and replacing the variable x by $x/\sqrt{2}$, we rewrite the right-hand side as

$$(2\pi)^{-m/2} (\det \Sigma)^{-1/2} \int_{\boldsymbol{R}^m} f(q(x)/2) e^{-{}^t x \Sigma^{-1} x/2}\, dx.$$

Therefore, if X is an $\boldsymbol{R}^m$-valued random variable with the normal law $N(0, \phi(\theta)^{-1})$, then $\gamma_{q,\theta}$ is nothing else but the law of $Y := q(X)/2$. In particular, the classical Wishart law as defined in [**21**, Definition 3.1.3] coincides with our $\gamma_{q,\theta}$ in Example 2.

PROPOSITION 2.7. *Let Y be an $\boldsymbol{R}^n$-valued random variable with the Wishart law $\gamma_{q,\theta}$. Then the Laplace transform $L_{\gamma_{q,\theta}}(\eta) = E(e^{-\langle Y,\eta\rangle})$ of $\gamma_{q,\theta}$ is given by*

$$L_{\gamma_{q,\theta}}(\eta) = \det(I_m + \phi(\theta)^{-1}\phi(\eta))^{-1/2}$$

for $\eta \in -\theta + \Omega^*$.

PROOF. By definition, we have $L_{\gamma_{q,\theta}}(\eta) = L_{\mu_q}(\theta)^{-1} L_{\mu_q}(\eta + \theta)$. Thus the formula follows from Lemma 2.4 and the observation that $(\det \phi(\theta))^{-1} \det \phi(\eta + \theta) = \det\{\phi(\theta)^{-1}(\phi(\theta) + \phi(\eta))\} = \det(I_m + \phi(\theta)^{-1}\phi(\eta))$. □

2.3. Moments of Wishart laws: quadratic case.

First we shall consider the mean and the covariance of the Wishart law $\gamma_{q,\theta}$. It is well known ([**19**]) that the mean $E(\langle Y, \eta\rangle)$ is given by the directional derivative $-D_\eta \log L_\mu(\theta)$, while the covariance $E((\langle Y,\eta\rangle - M)(\langle Y,\eta'\rangle - M'))$ equals $D_\eta D_{\eta'} \log L_\mu(\theta)$. Using Lemma 2.4 we obtain

THEOREM 2.8. *Let Y be an $\boldsymbol{R}^n$-valued random variable with the Wishart law $\gamma_{q,\theta}$.*

(i) *For $\eta \in (\boldsymbol{R}^n)^*$, one has*

$$E(\langle Y, \eta\rangle) = \operatorname{tr} \phi(\theta)^{-1}\phi(\eta)/2.$$

(ii) *For $\eta, \eta' \in (\boldsymbol{R}^n)^*$, one has*

$$E((\langle Y,\eta\rangle - M)(\langle Y,\eta'\rangle - M')) = \operatorname{tr} \phi(\theta)^{-1}\phi(\eta)\phi(\theta)^{-1}\phi(\eta')/2,$$

where $M := E(\langle Y, \eta\rangle)$ and $M' := E(\langle Y, \eta'\rangle)$.

Computation of higher moments of the Wishart law $\gamma_{q,\theta}$ boils down to writing explicitly higher derivatives of the Laplace transform $L_{\mu_q}(\theta) = \pi^{m/2}(\det \phi(\theta))^{-1/2}$. It can be done similarly as in [**9**, Lemma 5]. For an element π of the symmetric group $\mathfrak{S}_N$, we write $C(\pi)$ for the set of cycles of π.

THEOREM 2.9. *Let Y be an $\boldsymbol{R}^n$-valued random variable with the Wishart law $\gamma_{q,\theta}$. For $\eta_1, \eta_2, \ldots, \eta_N \in (\boldsymbol{R}^n)^*$, one has*

$$E(\langle Y,\eta_1\rangle\langle Y,\eta_2\rangle \ldots \langle Y,\eta_N\rangle) = \sum_{\pi\in\mathfrak{S}_N} \left(\frac{1}{2}\right)^{\sharp C(\pi)} \prod_{c\in C(\pi)} \operatorname{tr}\Bigl(\prod_{j\in c} \phi(\theta)^{-1}\phi(\eta_j)\Bigr).$$

2.4. Wishart laws associated to virtual quadratic maps.

We shall consider virtual quadratic maps, that is 'formal linear combinations' of quadratic maps, and the associated Wishart laws. First we introduce the notion of direct sum of quadratic maps. Let $q_i : \boldsymbol{R}^{m_i} \to \boldsymbol{R}^n$ $(i = 1, \dots, s)$ be Ω-positive quadratic maps. Then the direct sum $q = q_1 \oplus q_2 \oplus \cdots \oplus q_s$ is an $\boldsymbol{R}^n$-valued quadratic map on $\boldsymbol{R}^{m_1} \oplus \boldsymbol{R}^{m_2} \oplus \cdots \oplus \boldsymbol{R}^{m_s}$ given by

$$q(x) := q_1(x_1) + q_2(x_2) + \cdots + q_s(x_s) \quad \left(x = \sum_{i=1}^{s} x_i,\ x_i \in \boldsymbol{R}^{m_i} \right).$$

It is easy to see that q is also Ω-positive. If $q_1 = q_2 = \cdots = q_s$, then the direct sum q is denoted by $q_1^{\oplus s}$.

The linear map $\phi : (\boldsymbol{R}^n)^* \to \mathrm{Sym}(m, \boldsymbol{R})$ $(m := \sum_{i=1}^{s} m_i)$ associated to the direct sum $q = \sum^{\oplus} q_i$ is given by

$$\phi(\eta) = \begin{pmatrix} \phi_1(\eta) & & & \\ & \phi_2(\eta) & & \\ & & \ddots & \\ & & & \phi_s(\eta) \end{pmatrix} \qquad (\eta \in (\boldsymbol{R}^n)^*). \tag{2.9}$$

Conversely, if a symmetric matrix $\phi(\eta)$ is expressed by $\phi_1(\eta), \dots, \phi_s(\eta)$ as above for all $\eta \in (\boldsymbol{R}^n)^*$, then the corresponding quadratic map q is the direct sum of $q_1, \dots, q_s$. In Example 1, the quadratic map $q : \boldsymbol{R}^4 \to \boldsymbol{R}^3$ is the direct sum of 4 quadratic maps $q_i : \boldsymbol{R} \ni x \mapsto x^2 v_i \in \boldsymbol{R}^3$ $(i = 1, \dots, 4)$, where

$$v_1 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad v_2 := \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad v_3 := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad q_4 := \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

In Example 2, we see that $q_{r,s} : \mathrm{Mat}(r, s; \boldsymbol{R}) \to \mathrm{Sym}(r, \boldsymbol{R})$ is naturally identified with $q_{r,1}^{\oplus s}$. In Example 3 with $\mathcal{Z}$ given by (2.4), we have $q_{\mathcal{Z}} = q_{\mathcal{Z}}^{\{1\}} \oplus q_{\mathcal{Z}}^{\{2,3\}}$, while we do not have such a decomposition for the case (2.5).

Let $q_i : \boldsymbol{R}^{m_i} \to \boldsymbol{R}^n$ $(i = 1, 2)$ be Ω-positive quadratic maps, and q the direct sum $q_1 \oplus q_2$. Then it is easy to see that the measure μ_q equals the convolution $\mu_{q_1} * \mu_{q_2}$. Thus, for $\theta \in \Omega^*$ we have $L_{\mu_q}(\theta) = L_{\mu_{q_1}}(\theta) L_{\mu_{q_2}}(\theta)$ and $\gamma_{q,\theta} = \gamma_{q_1,\theta} * \gamma_{q_2,\theta}$. In general, if we set $q = q_1^{\oplus s_1} \oplus q_2^{\oplus s_2} \oplus \cdots \oplus q_t^{\oplus s_t}$ for Ω-positive quadratic maps $q_i : \boldsymbol{R}^{m_i} \to \boldsymbol{R}^n$ $(i = 1, 2, \dots, t)$ and positive integers $s_1, s_2, \dots, s_t$, then we have

$$L_{\mu_q}(\theta) = \prod_{i=1}^{t} L_{\mu_{q_i}}(\theta)^{s_i} \quad (\theta \in \Omega^*). \tag{2.10}$$

Now we remark that, even though s_i's are not positive integers, there may exist a positive measure μ_q on $\overline{\Omega}$ for which the relation (2.10) holds.

DEFINITION 2.10. For real numbers $s_1, \dots, s_t$, we call a formal sum $q = q_1^{\oplus s_1} \oplus q_2^{\oplus s_2} \oplus \cdots \oplus q_t^{\oplus s_t}$ a *virtual Ω-positive quadratic map.* If a positive measure μ_q satisfying

(2.10) exists, then μ_q is called *the Riesz measure associated to* q. In this case, *the Wishart laws* $\gamma_{q,\theta}$ $(\theta \in \Omega^*)$ are defined again as members of the exponential family generated by μ_q.

Observe that by the injectivity of the Laplace transform, the Riesz measure μ_q associated to a virtual q is unique if it exists.

PROPOSITION 2.11. *Let* $q_i : \boldsymbol{R}^{m_i} \to \boldsymbol{R}^n$ $(i = 1, \dots, t)$ *be* Ω*-positive quadratic maps. Assume that there exists a measure* μ_q *associated to the virtual quadratic map* $q = q_1^{\oplus s_1} \oplus \cdots \oplus q_t^{\oplus s_t}$ *for certain* $s_1, \dots, s_t \in \boldsymbol{R}$. *Let* Y *be an* $\boldsymbol{R}^n$*-valued random variable with the Wishart law* $\gamma_{q,\theta}$. *Then the Laplace transform* $L_{\gamma_{q,\theta}}(\eta) = E(e^{-\langle Y,\eta\rangle})$ *of the law* $\gamma_{q,\theta}$ *is given by*

$$L_{\gamma_{q,\theta}}(\eta) = \prod_{i=1}^{t} L_{\gamma_{q_i,\theta}}(\eta)^{s_i} = \prod_{i=1}^{t} \det(I_{m_i} + \phi_i(\theta)^{-1}\phi_i(\eta))^{-s_i/2}$$

for $\eta \in -\theta + \Omega^*$.

PROOF. We have $L_{\gamma_{q,\theta}}(\eta) = L_{\mu_q}(\theta)^{-1} L_{\mu_q}(\eta+\theta)$ by definition, and the right-hand side equals $\prod_{i=1}^{t} \bigl(L_{\mu_{q_i}}(\theta)^{-1} L_{\mu_{q_i}}(\eta+\theta)\bigr)^{s_i}$ by (2.10). Since $L_{\gamma_{q_i,\theta}}(\eta) = L_{\mu_{q_i}}(\theta)^{-1} L_{\mu_{q_i}}(\eta+\theta)$, we obtain the first equality. The second equality follows from Proposition 2.7. $\square$

2.5. Moments of Wishart laws: general case.

Since we see immediately from (2.10) that

$$\log L_{\mu_q}(\theta) = \sum_{i=1}^{t} s_i \log L_{\mu_{q_i}}(\theta),$$

the virtual version of Theorem 2.8 is given as follows:

PROPOSITION 2.12. *Under the same assumption of Proposition 2.11, one has*

(i) $E(\langle Y,\eta\rangle) = \sum_{i=1}^{t} s_i \operatorname{tr} \phi_i(\theta)^{-1}\phi_i(\eta)/2$ *for* $\eta \in (\boldsymbol{R}^n)^*$,

(ii) $E((\langle Y,\eta\rangle - M)(\langle Y,\eta'\rangle - M')) = \sum_{i=1}^{t} s_i \operatorname{tr} \phi_i(\theta)^{-1}\phi_i(\eta)\phi_i(\theta)^{-1}\phi_i(\eta')/2$ *for* $\eta, \eta' \in (\boldsymbol{R}^n)^*$, *where* $M := E(\langle Y,\eta\rangle)$ *and* $M' := E(\langle Y,\eta'\rangle)$.

As for higher moments, we generalize the formula in Theorem 2.9 as follows:

THEOREM 2.13. *Under the same assumption of Proposition* 2.11*, one has*

$$E(\langle Y,\eta_1\rangle\langle Y,\eta_2\rangle \cdots \langle Y,\eta_N\rangle)$$
$$= \sum_{\pi \in \mathfrak{S}_N} \left(\frac{1}{2}\right)^{\sharp C(\pi)} \prod_{c \in C(\pi)} \left\{ \sum_{i=1}^{t} s_i \operatorname{tr}\left(\prod_{j \in c} \phi_i(\theta)^{-1}\phi_i(\eta_j) \right) \right\}. \tag{2.11}$$

for $\eta_1, \eta_2, \dots, \eta_N \in (\boldsymbol{R}^n)^*$.

Theorem 2.13 easily follows by (2.9) from Theorem 2.9 when $s_1, \ldots, s_t$ are positive integers, that is, q is a true quadratic map. To prove (2.11) for general case, one verifies that the quantity $E(\langle Y, \eta_1\rangle\langle Y, \eta_2\rangle \cdots \langle Y, \eta_N\rangle)$ is a polynomial of $s_1, \ldots, s_t$. For this purpose, we make some calculations involving the semi-invariants or the cummulants (cf. [**19**]). We omit the details.

Another possibility is to prove first the following Proposition and next apply polarization.

Proposition 2.14. *One has*

$$E(\langle Y, \eta\rangle^N) = \sum_{\ell=1}^{N} \frac{1}{\ell!} \sum_{k_1+k_2+\cdots+k_\ell=N} \frac{(-1)^N N!}{k_1 k_2 \cdots k_\ell} \prod_{j=1}^{\ell} \left(\sum_{i=1}^{t} \frac{s_i}{2} \operatorname{tr}\bigl(\phi_i(\theta)^{-1}\phi_i(\eta)\bigr)^{k_j} \right). \quad (2.12)$$

2.6. Group equivariance of the Wishart laws.

Let $G(\Omega)$ be the linear automorphism group $\{g \in GL(n, \boldsymbol{R})\,;\, g\Omega = \Omega\}$ of Ω. For an Ω-positive quadratic map $q : \boldsymbol{R}^m \to \boldsymbol{R}^n$ and $g \in G(\Omega)$, the quadratic map $g \circ q : \boldsymbol{R}^m \to \boldsymbol{R}^n$ is again Ω-positive. It is easy to see that the Riesz measure $\mu_{g\circ q}$ is the image of μ_q by g, that is,

$$\mu_{g\circ q}(A) = \mu_q(g^{-1}A) \quad (2.13)$$

for a measurable set $A \subset \boldsymbol{R}^n$. Let us discuss the Wishart laws $\gamma_{g\circ q,\theta}$ for $\theta \in \Omega^*$. For $\eta \in (\boldsymbol{R}^n)^*$, we denote by $g^*\eta$ the linear form $\eta \circ g \in (\boldsymbol{R}^n)^*$. If $\eta \in \Omega^*$, then $g^*\eta \in \Omega^*$ because $\langle y, g^*\eta\rangle = \langle gy, \eta\rangle > 0$ for $y \in \overline{\Omega} \setminus \{0\}$. We observe

$$L_{\mu_q}(g^*\theta) = \int_{\boldsymbol{R}^m} e^{-\langle q(x), g^*\theta\rangle} dx = \int_{\boldsymbol{R}^m} e^{-\langle g\circ q(x), \theta\rangle} dx = L_{\mu_{g\circ q}}(\theta). \quad (2.14)$$

Therefore, denoting by 1_A the characteristic function of a measurable set $A \subset \boldsymbol{R}^n$, we have

$$\begin{aligned}\gamma_{g\circ q,\theta}(A) &= \frac{1}{L_{\mu_{g\circ q}}(\theta)} \int_{\boldsymbol{R}^m} 1_A(g \circ q(x)) e^{-\langle g\circ q(x), \theta\rangle} dx \\ &= \frac{1}{L_{\mu_q}(g^*\theta)} \int_{\boldsymbol{R}^m} 1_{g^{-1}A}(q(x)) e^{-\langle q(x), g^*\theta\rangle} dx = \gamma_{q,g^*\theta}(g^{-1}A). \quad (2.15)\end{aligned}$$

We restate (2.15) as follows.

Lemma 2.15. *Let g be an element of $G(\Omega)$. If a random variable Y obeys the Wishart law $\gamma_{q,\theta}$, the law of gY is $\gamma_{g\circ q,\,(g^{-1})^*\theta}$.*

Let $q_i : \boldsymbol{R}^{m_i} \to \boldsymbol{R}^n$ $(i = 1, \ldots, t)$ be Ω-positive quadratic maps, and q the virtual quadratic map $q_1^{\oplus s_1} \oplus \cdots \oplus q_t^{\oplus s_t}$ with $s_1, \ldots, s_t \in \boldsymbol{R}$. Then we define $g \circ q$ to be the virtual quadratic map $(g \circ q_1)^{\oplus s_1} \oplus \cdots \oplus (g \circ q_t)^{\oplus s_t}$.

Proposition 2.16. *If the Riesz measure μ_q exists, then the Riesz measure $\mu_{g\circ q}$ exists and equals the image of μ_q by g. Moreover $\gamma_{g\circ q,\,(g^{-1})^*\theta}$ is the image of $\gamma_{q,\theta}$ by g.*

Proof. Let μ' be the image of μ_q by g. For $\theta \in \Omega^*$, we have

$$\begin{aligned} L_{\mu'}(\theta) &= \int_{\boldsymbol{R}^n} e^{-\langle y,\theta\rangle}\mu'(dy) = \int_{\boldsymbol{R}^n} e^{-\langle gy,\theta\rangle}\mu_q(dy) \\ &= \int_{\boldsymbol{R}^n} e^{-\langle y,g^*\theta\rangle}\mu_q(dy) = L_{\mu_q}(g^*\theta). \end{aligned} \tag{2.16}$$

By (2.10) and (2.14), the last term equals $\prod_{i=1}^t L_{\mu_{q_i}}(g^*\theta)^{s_i} = \prod_{i=1}^t L_{\mu_{g\circ q_i}}(\theta)^{s_i}$. Thus we get $L_{\mu'}(\theta) = \prod_{i=1}^t L_{\mu_{g\circ q_i}}(\theta)^{s_i}$ which means $\mu' = \mu_{g\circ q}$ by (2.10).

Let γ' be the image of $\gamma_{q,\theta}$ by g. Similarly to (2.16), we have $L_{\gamma'}(\eta) = L_{\gamma_{q,\theta}}(g^*\eta)$ for $\eta \in -(g^{-1})^*\theta + \Omega^*$, while $L_{\gamma_{g\circ q_i,(g^{-1})^*\theta}}(\eta) = L_{\gamma_{q_i,\theta}}(g^*\eta)$ by Lemma 2.15. On the other hand, we see from Proposition 2.11 that

$$L_{\gamma_{q,\theta}}(g^*\eta) = \prod_{i=1}^t L_{\gamma_{q_i,\theta}}(g^*\eta)^{s_i} = \prod_{i=1}^t L_{\gamma_{g\circ q_i,(g^{-1})^*\theta}}(\eta)^{s_i} = L_{\gamma_{g\circ q,(g^{-1})^*\theta}}(\eta).$$

Thus we get $L_{\gamma'}(\eta) = L_{\gamma_{g\circ q,(g^{-1})^*\theta}}(\eta)$, so that $\gamma' = \gamma_{g\circ q,(g^{-1})^*\theta}$ by the injectivity of the Laplace transform. □

Let $\mathrm{Aut}(\Omega, q)$ be the set of pairs $(g_1, g_2) \in G(\Omega) \times GL(m, \boldsymbol{R})$ for which $g_1 \circ q = q \circ g_2$. Then $\mathrm{Aut}(\Omega, q)$ forms a Lie subgroup of $GL(n, \boldsymbol{R}) \times GL(m, \boldsymbol{R})$, and we have a group homomorphism

$$pr_1 : \mathrm{Aut}(\Omega, q) \ni (g_1, g_2) \mapsto g_1 \in G(\Omega).$$

The condition $(g_1, g_2) \in \mathrm{Aut}(\Omega, q)$ is also equivalent to

$$\phi_q(g_1^*\eta) = {}^{\mathsf{t}}g_2\phi_q(\eta)g_2 \quad (\eta \in (\boldsymbol{R}^n)^*). \tag{2.17}$$

Then we obtain

$$\det \phi_q(g_1^*\eta) = C \det \phi_q(\eta) \quad (\eta \in (\boldsymbol{R}^n)^*) \tag{2.18}$$

with $C = (\det g_2)^2$, which means that $\det \phi_q(\eta)$ is a relatively invariant polynomial on $(\boldsymbol{R}^n)^*$ under the contragredient action of $pr_1(\mathrm{Aut}(\Omega, q))$. The following proposition describes a transformation rule of the family of the Wishart laws $\{\gamma_{q,\theta}\}_{\theta\in\Omega^*}$ under the group $pr_1(\mathrm{Aut}(\Omega, q))$.

Proposition 2.17. *For a measurable set $A \subset \boldsymbol{R}^n$ and $(g_1, g_2) \in \mathrm{Aut}(\Omega, q)$, one has*

(i) $\mu_q(g_1^{-1}A) = \mu_{g_1\circ q}(A) = |\det g_2|^{-1}\mu_q(A)$,
(ii) $\gamma_{q,g_1^*\theta}(A) = \gamma_{q,\theta}(g_1 A)$.

Proof. (i) Because of (2.13), we only have to show the second equality. By definition, we have

$$\mu_{g_1\circ q}(A) = \int_{\boldsymbol{R}^m} 1_A(g_1 \circ q(x))\,dx = \int_{\boldsymbol{R}^m} 1_A(q \circ g_2(x))\,dx.$$

Putting $x' = g_2 x$, the last term equals

$$|\det g_2|^{-1} \int_{\boldsymbol{R}^m} 1_A(q(x'))\,dx' = |\det g_2|^{-1}\mu_q(A),$$

whence (i) follows.

(ii) By (2.15), we get for $y \in \boldsymbol{R}^n$

$$\gamma_{q,g_1^*\theta}(dy) = \gamma_{g_1\circ q,\theta}(g_1 dy) = \frac{e^{-\langle g_1 y,\theta\rangle}}{L_{\mu_{g_1\circ q}}(\theta)}\mu_{g_1\circ q}(g_1 dy),$$

Since $\mu_{g_1\circ q} = |\det g_2|^{-1}\mu_q$ by (i), the last term equals

$$\frac{e^{-\langle g_1 y,\theta\rangle}}{|\det g_2|^{-1}L_{\mu_q}(\theta)}|\det g_2|^{-1}\mu_q(g_1 dy) = \gamma_{q,\theta}(g_1 dy).$$

Hence (ii) is verified. □

3. Homogeneous Case.

3.1. Homogeneous quadratic map.

Definition 3.1. An Ω-positive map $q : \boldsymbol{R}^m \to \boldsymbol{R}^n$ is said to be *homogeneous* if, for any y, $y' \in \Omega$, there exists $(g_1, g_2) \in \mathrm{Aut}(\Omega, q)$ for which $g_1 y = y'$. In other words, q is homogeneous if $pr_1(\mathrm{Aut}(\Omega, q))$ acts on Ω transitively.

In this case, Ω is clearly a homogeneous cone, that is, a linear group on $\boldsymbol{R}^n$ acts on the cone Ω transitively. Then the dual cone $\Omega^* \subset (\boldsymbol{R}^n)^*$ is also a homogeneous cone on which the group $pr_1(\mathrm{Aut}(\Omega, q))$ acts transitively by the contragredient action ([**25**]). We see from (2.1) and (2.17) that the quadratic map q is homogeneous if and only if the associated linear map $\phi_q : (\boldsymbol{R}^n)^* \to \mathrm{Sym}(m, \boldsymbol{R})$ is a representation of the dual cone Ω^* in the sense of Rothaus [**23**] (see also [**16**]).

A typical example of a homogeneous cone is $\mathrm{Sym}_r^+ \subset \mathrm{Sym}(r, \boldsymbol{R})$. For $A \in GL(r, \boldsymbol{R})$, we denote by $\rho(A)$ the linear map on $\mathrm{Sym}(r, \boldsymbol{R})$ defined by $\rho(A)y := Ay\,{}^{\mathrm{t}}A$ ($y \in \mathrm{Sym}(r, \boldsymbol{R})$). Then the group $\rho(GL(r, \boldsymbol{R}))$ acts on Sym_r^+ transitively. Moreover, the linear automorphism group $G(\mathrm{Sym}_r^+)$ equals $\rho(GL(r, \boldsymbol{R}))$. We see that the quadratic map $q_{r,s} : \mathrm{Mat}(r, s; \boldsymbol{R}) \to \mathrm{Sym}(r, \boldsymbol{R})$ in Example 2 is homogeneous. Indeed, we have a surjective homomorphism $GL(r,\boldsymbol{R}) \times O(s) \ni (A,B) \mapsto (\rho(A), \tau_{r,s}(A,B)) \in \mathrm{Aut}(\mathrm{Sym}_r^+, q_{r,s})$, where $\tau_{r,s}(A,B)$ is a linear map on $\mathrm{Mat}(r,s;\boldsymbol{R})$ given by $\tau_{r,s}(A,B)x := AxB^{-1}$ ($x \in \mathrm{Mat}(r,s;\boldsymbol{R})$). Thus we have $pr_1(\mathrm{Aut}(\mathrm{Sym}_r^+, q_{r,s})) = \rho(GL(r,\boldsymbol{R}))$, which acts on Sym_r^+ transitively.

For a subset $I \subset \{1,\dots,r\}$, we denote by q^I the restriction of $q_{r,1} : \boldsymbol{R}^r \to \mathrm{Sym}(r,\boldsymbol{R})$ to the space $R^I \subset \boldsymbol{R}^r$ defined in (2.6). The map q^I coincides with $q^I_{\mathcal{Z}}$ in Example 3 with $\mathcal{Z} = \mathrm{Sym}(r,\boldsymbol{R})$. Let us observe that q^I is homogeneous in general. Let P^I be the linear group consisting of $A \in GL(r,\boldsymbol{R})$ for which $AR^I = R^I$. For example, if $r=3$, we have

$$P^{\{1\}} = \left\{ A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix} ; A \in GL(3,\boldsymbol{R}) \right\}, \tag{3.1}$$

$$P^{\{2,3\}} = \left\{ A = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} ; A \in GL(3,\boldsymbol{R}) \right\}. \tag{3.2}$$

Since we have a homomorphism $P^I \ni A \mapsto (\rho(A), A) \in \mathrm{Aut}(\mathrm{Sym}_r^+, q^I)$, it is enough to show that $\rho(P^I)$ acts on Sym_r^+ transitively. Put $k := \sharp I$ and take a permutation matrix $w_0 \in \mathfrak{S}_r \subset GL(r,\boldsymbol{R})$ sending $R^{\{r-k+1,\dots,r\}}$ onto R^I. Then we have $P^I = w_0 P^{\{r-k+1,\dots,r\}} w_0^{-1}$, and

$$P^{\{r-k+1,\dots,r\}} = \left\{ \begin{pmatrix} A_1 & 0 \\ A_2 & A_3 \end{pmatrix} ; \begin{array}{l} A_1 \in GL(k,\boldsymbol{R}), \quad A_2 \in \mathrm{Mat}(r-k+1,k;\boldsymbol{R}) \\ A_3 \in GL(r-k+1,\boldsymbol{R}) \end{array} \right\}.$$

Since $P^{\{r-k+1,\dots,r\}}$ contains the group of lower triangular matrices, $\rho(P^{\{r-k+1,\dots,r\}})$ acts on Sym_r^+ transitively. Therefore $\rho(P^I) = \rho(w_0)\rho(P^{\{r-k+1,\dots,r\}})\rho(w_0)^{-1}$ also acts on Sym_r^+ transitively, so that q^I is homogeneous.

Coming back to the examples (3.1) and (3.2), we note that $q = q^{\{1\}} \oplus q^{\{2,3\}}$ is not homogeneous as Sym_3^+-positive quadratic map, while both $q^{\{1\}}$ and $q^{\{2,3\}}$ are. Indeed, the image of q generates the space $\mathcal{Z} \subset \mathrm{Sym}(3,\boldsymbol{R})$ in (2.4). Thus, if $(g_1, g_2) \in \mathrm{Aut}(\mathrm{Sym}_3^+, q)$, then g_1 must preserve both $\mathcal{Z}$ and Sym_3^+. Let us take $y \in \mathrm{Sym}_3^+ \setminus \mathcal{Z}$. Then g_1 does not send $I_3 \in \mathrm{Sym}_3^+$ to y because $I_3 \in \mathcal{Z}$. Thus the action of $pr_1(\mathrm{Aut}(\mathrm{Sym}_3^+, q))$ on Sym_3^+ is not transitive.

On the other hand, if we regard q as a map from $R^{\{1\}} \oplus R^{\{2,3\}}$ to $\mathcal{Z}$, then q is a homogeneous $\mathcal{P}$-positive quadratic map, where $\mathcal{P} := \mathcal{Z} \cap \mathrm{Sym}_3^+$. In fact, since $(\rho(A), A) \in \mathrm{Aut}(\mathcal{P}, q^{\{1\}}) \cap \mathrm{Aut}(\mathcal{P}, q^{\{2,3\}})$ for $A \in P^{\{1\}} \cap P^{\{2,3\}}$, we have $\rho(A) \in pr_1(\mathrm{Aut}(\mathcal{P}, q))$. Therefore $pr_1(\mathrm{Aut}(\mathcal{P}, q))$ contains a group $\rho(P^{\{1\}} \cap P^{\{2,3\}})$ which acts on $\mathcal{P}$ transitively.

In Example 1, the quadratic map $q : \boldsymbol{R}^4 \to \boldsymbol{R}^3$ is not homogeneous because $\Omega \subset \boldsymbol{R}^4$ in (2.2) is not a homogeneous cone ([**14**]).

3.2. Matrix realization of homogeneous cones.

In this section, we shall discuss a homogeneous cone realized as $\mathcal{P}_{\mathcal{V}} = \mathcal{Z}_{\mathcal{V}} \cap \mathrm{Sym}_N^+$ with $\mathcal{Z}_{\mathcal{V}} \subset \mathrm{Sym}(N,\boldsymbol{R})$ constructed from an appropriate system $\mathcal{V} = \{\mathcal{V}_{lk}\}$ of vector spaces in a specific way explained below, following [**15**, section 3.1]. The investigation of such cones is fundamental because all homogeneous cones are linearly equivalent to some $\mathcal{P}_{\mathcal{V}}$ due to [**15**, Theorem D].

Let us take a partition $N = n_1 + n_2 + \cdots + n_r$ of a positive integer N, and consider a system of vector spaces $\mathcal{V}_{lk} \subset \mathrm{Mat}(n_l, n_k; \boldsymbol{R})$ $(1 \le k < l \le r)$ satisfying the following three conditions:

(V1) $A \in \mathcal{V}_{lk},\ B \in \mathcal{V}_{kj} \Rightarrow AB \in \mathcal{V}_{lj} \quad (1 \le j < k < l \le r)$,
(V2) $A \in \mathcal{V}_{lj},\ B \in \mathcal{V}_{kj} \Rightarrow A\,{}^{\mathrm{t}}B \in \mathcal{V}_{lk} \quad (1 \le j < k < l \le r)$,
(V3) $A \in \mathcal{V}_{lk} \Rightarrow A\,{}^{\mathrm{t}}A \in \boldsymbol{R}I_{n_l} \quad (1 \le k < l \le r)$.

Let $\mathcal{Z}_{\mathcal{V}}$ be the subspace of $\mathrm{Sym}(N, \boldsymbol{R})$ defined by

$$\mathcal{Z}_{\mathcal{V}} := \left\{ y = \begin{pmatrix} Y_{11} & {}^{\mathrm{t}}Y_{21} & \cdots & {}^{\mathrm{t}}Y_{r1} \\ Y_{21} & Y_{22} & & {}^{\mathrm{t}}Y_{r2} \\ \vdots & & \ddots & \\ Y_{r1} & Y_{r2} & \cdots & Y_{rr} \end{pmatrix} ; \begin{array}{l} Y_{kk} = y_{kk} I_{n_k},\ y_{kk} \in \boldsymbol{R}\ (k = 1, \ldots, r) \\ Y_{lk} \in \mathcal{V}_{lk}\ (1 \le k < l \le r) \end{array} \right\}.$$

We set $\mathcal{P}_{\mathcal{V}} := \mathcal{Z}_{\mathcal{V}} \cap \mathrm{Sym}_N^+$. Then $\mathcal{P}_{\mathcal{V}}$ is a regular open convex cone in the vector space $\mathcal{Z}_{\mathcal{V}}$. Let H_N be the group of real lower triangular matrices with positive diagonals, and $H_{\mathcal{V}}$ a Lie subgroup of H_N defined by

$$H_{\mathcal{V}} := \left\{ T = \begin{pmatrix} T_{11} & & & \\ T_{21} & T_{22} & & \\ \vdots & & \ddots & \\ T_{r1} & T_{r2} & \cdots & T_{rr} \end{pmatrix} ; \begin{array}{l} T_{kk} = t_{kk} I_{n_k},\ t_{kk} > 0\ (k = 1, \ldots, r) \\ T_{lk} \in \mathcal{V}_{lk}\ (1 \le k < l \le r) \end{array} \right\}.$$

If $T \in H_{\mathcal{V}}$ and $y \in \mathcal{Z}_{\mathcal{V}}$, then $\rho(T)y = Ty\,{}^{\mathrm{t}}T$ belongs to $\mathcal{Z}_{\mathcal{V}}$ thanks to (V1)–(V3). Moreover $\rho(H_{\mathcal{V}})$ acts on the cone $\mathcal{P}_{\mathcal{V}} \subset \mathcal{Z}_{\mathcal{V}}$ simply transitively (cf. [**15**, Proposition 3.2]).

Keeping (V3) in mind, we define an inner product on the vector space $\mathcal{V}_{lk}$ $(1 \le k < l \le r)$ by the equality

$$A\,{}^{\mathrm{t}}A = (A|A) I_{n_l} = \|A\|^2 I_{n_l} \quad (A \in \mathcal{V}_{lk}). \tag{3.3}$$

For $y, y' \in \mathcal{Z}_{\mathcal{V}}$, we set

$$\langle y, y' \rangle := \sum_{k=1}^{r} y_{kk} y'_{kk} + 2 \sum_{1 \le k < l \le r} (Y_{lk} | Y'_{lk}), \tag{3.4}$$

where y_{kk} and Y_{lk} (respectively y'_{kk} and Y'_{lk}) denote the components of y (respectively y'). Note that the inner product is not equal to $\mathrm{tr}\, yy'$ unless $n_1 = \cdots = n_r = 1$. By this coupling, we identify the dual space $\mathcal{Z}_{\mathcal{V}}^*$ with $\mathcal{Z}_{\mathcal{V}}$. Let us observe that I_N belongs to the dual cone $\mathcal{P}_{\mathcal{V}}^*$ of $\mathcal{P}_{\mathcal{V}}$, that is,

$$0 < \langle y, I_N \rangle = y_{11} + \cdots + y_{rr} \quad (y \in \overline{\mathcal{P}}_{\mathcal{V}} \setminus \{0\}).$$

Indeed, since each y_{kk} is a diagonal entry of the non-negative matrix $y \in \overline{\mathcal{P}}_{\mathcal{V}} \setminus \{0\}$, we have $y_{kk} \geq 0$. Suppose $\sum_{k=1}^{r} y_{kk} = 0$. Then $y_{11} = \cdots = y_{rr} = 0$ and $\operatorname{tr} y = \sum_{k=1}^{r} n_k y_{kk} = 0$. This together with the non-negativity of y implies $y = 0$, which is a contradiction.

For $T \in H_{\mathcal{V}}$, define $\rho^*(T) \in GL(\mathcal{Z}_{\mathcal{V}})$ by $\langle y, \rho^*(T)\eta \rangle = \langle \rho(T)y, \eta \rangle$ $(y, \eta \in \mathcal{Z}_{\mathcal{V}})$. By [**25**], we have $\mathcal{P}_{\mathcal{V}}^* = \{\rho^*(T)I_N \,;\, T \in H_{\mathcal{V}}\}$. Moreover, the map $H_{\mathcal{V}} \ni T \mapsto \rho^*(T)I_N \in \mathcal{P}_{\mathcal{V}}^*$ is a diffeomorphism. For $\underline{\sigma} = (\sigma_1, \ldots, \sigma_r) \in \boldsymbol{C}^r$, we define the one-dimensional representation $\chi_{\underline{\sigma}} : H_{\mathcal{V}} \to \boldsymbol{C}^{\times}$ by $\chi_{\underline{\sigma}}(T) := (t_{11})^{2\sigma_1} \cdots (t_{rr})^{2\sigma_r}$ $(T \in H_{\mathcal{V}})$. Note that any one-dimensional representation χ of $H_{\mathcal{V}}$ is of the form $\chi_{\underline{\sigma}}$, so that χ is determined by the values on the subgroup $A_{\mathcal{V}} \subset H_{\mathcal{V}}$ consisting of diagonal matrices.

Let us give some examples. When $n_1 = n_2 = \cdots = n_r = 1$ and $\mathcal{V}_{lk} = \boldsymbol{R}$ for all $1 \leq k < l \leq r$, the conditions (V1)–(V3) are clearly satisfied, and we have $\mathcal{Z}_{\mathcal{V}} = \mathrm{Sym}(r, \boldsymbol{R})$ and $\mathcal{P}_{\mathcal{V}} = \mathrm{Sym}_r^+$. For the case $r = 3$, $n_1 = n_2 = n_3 = 1$, $\mathcal{V}_{21} = \{0\}$ and $\mathcal{V}_{31} = \mathcal{V}_{32} = \boldsymbol{R}$, the space $\mathcal{Z}_{\mathcal{V}}$ equals $\mathcal{Z}$ in (2.5).

Let us set $r = 3$, $n_1 = 2$, $n_2 = n_3 = 1$,

$$\mathcal{V}_{21} = \{(v \ \ 0); v \in \boldsymbol{R}\}, \quad \mathcal{V}_{31} = \{(0 \ \ v); v \in \boldsymbol{R}\},$$

and $\mathcal{V}_{32} = \{0\}$. Then we have

$$\mathcal{Z}_{\mathcal{V}} = \left\{ \begin{pmatrix} y_{11} & 0 & y_{21} & 0 \\ 0 & y_{11} & 0 & y_{31} \\ y_{21} & 0 & y_{22} & 0 \\ 0 & y_{31} & 0 & y_{33} \end{pmatrix} ; \, y_{11}, \, y_{22}, \, y_{33}, \, y_{21}, \, y_{31} \in \boldsymbol{R} \right\} \tag{3.5}$$

and

$$\begin{aligned} \mathcal{P}_{\mathcal{V}} &= \{y \in \mathcal{Z}_{\mathcal{V}} \,;\, y \text{ is positive definite}\} \\ &= \left\{ y \in \mathcal{Z}_{\mathcal{V}} \,;\, y_{11} > 0, \ y_{11}y_{22} - (y_{21})^2 > 0, \ y_{11}y_{33} - (y_{31})^2 > 0 \right\}, \end{aligned} \tag{3.6}$$

which is exactly the Vinberg cone [**24**].

Set $r = 2$, $n_1 = m \geq 1$, $n_2 = 1$ and $\mathcal{V}_{21} = \mathrm{Mat}(1, m; \boldsymbol{R})$. Then

$$\mathcal{Z}_{\mathcal{V}} = \left\{ \begin{pmatrix} y_{11} & & & v_1 \\ & \ddots & & \vdots \\ & & y_{11} & v_m \\ v_1 & \cdots & v_m & y_{22} \end{pmatrix} ; \, y_{11}, \, y_{22}, \, v_1, \ldots, v_m \in \boldsymbol{R} \right\},$$

$$\mathcal{P}_{\mathcal{V}} = \left\{ y \in \mathcal{Z}_{\mathcal{V}} \,;\, y_{11} > 0, \ y_{11}y_{22} - (v_1)^2 - \cdots - (v_m)^2 > 0 \right\}, \tag{3.7}$$

so that we obtain the Lorentz cone of dimension $m + 2$.

3.3. Basic quadratic maps.

Let $W_{\mathcal{V}}^i$ $(i = 1, \dots, r)$ be the subspace of $\mathrm{Mat}(N, n_i; \boldsymbol{R})$ consisting of matrices x of the form

$$x = \begin{pmatrix} 0_{n_1+\cdots+n_{i-1}, n_i} \\ X_{ii} \\ \vdots \\ X_{ri} \end{pmatrix} \quad \begin{pmatrix} X_{ii} = x_{ii} I_{n_i},\ x_{ii} \in \boldsymbol{R} \\ X_{li} \in \mathcal{V}_{li}\ (l = i+1, \dots, r) \end{pmatrix}.$$

For example, when $\mathcal{Z}_{\mathcal{V}}$ is the one in (3.5), we have

$$W_{\mathcal{V}}^1 = \left\{ \begin{pmatrix} x_{11} & 0 \\ 0 & x_{11} \\ x_{21} & 0 \\ 0 & x_{31} \end{pmatrix} ;\, x_{11},\, x_{21},\, x_{31} \in \boldsymbol{R} \right\},$$

$$W_{\mathcal{V}}^2 = \left\{ \begin{pmatrix} 0 \\ 0 \\ x_{22} \\ 0 \end{pmatrix} ;\, x_{22} \in \boldsymbol{R} \right\}, \quad W_{\mathcal{V}}^3 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_{33} \end{pmatrix} ;\, x_{33} \in \boldsymbol{R} \right\}, \tag{3.8}$$

while for the case that $\mathcal{Z}_{\mathcal{V}}$ is the space $\mathcal{Z}$ in (2.5), we have

$$W_{\mathcal{V}}^1 = \left\{ \begin{pmatrix} x_{11} \\ 0 \\ x_{31} \end{pmatrix} ;\, x_{11},\, x_{31} \in \boldsymbol{R} \right\}, \quad W_{\mathcal{V}}^2 = \left\{ \begin{pmatrix} 0 \\ x_{22} \\ x_{32} \end{pmatrix} ;\, x_{22},\, x_{32} \in \boldsymbol{R} \right\},$$

$$W_{\mathcal{V}}^3 = \left\{ \begin{pmatrix} 0 \\ 0 \\ x_{33} \end{pmatrix} ;\, x_{33} \in \boldsymbol{R} \right\}.$$

For $T \in H_{\mathcal{V}}$ and $x \in W_{\mathcal{V}}^i$, we see from (V1) that $Tx \in W_{\mathcal{V}}^i$, which defines a representation $\tau_i : H_{\mathcal{V}} \to GL(W_{\mathcal{V}}^i)$.

DEFINITION 3.2. We define *the basic quadratic maps* $q_{\mathcal{V}}^1, \dots, q_{\mathcal{V}}^r$ *for* $\mathcal{P}_{\mathcal{V}}$ *in the following way.* Since $W_{\mathcal{V}}^i \subset \mathrm{Mat}(N, n_i; \boldsymbol{R})$, we can consider the restriction $q_{\mathcal{V}}^i$ of the Sym_N^+-positive quadratic map $q_{N,n_i} : \mathrm{Mat}(N, n_i; \boldsymbol{R}) \to \mathrm{Sym}(N, \boldsymbol{R})$ in Example 2 to the space $W_{\mathcal{V}}^i$. Thanks to (V2) and (V3), we have $q_{\mathcal{V}}^i(x) = x^{\mathrm{t}} x \in \mathcal{Z}_{\mathcal{V}}$ for $x \in W_{\mathcal{V}}^i$. Then the quadratic map $q_{\mathcal{V}}^i : W_{\mathcal{V}}^i \ni x \mapsto x^{\mathrm{t}} x \in \mathcal{Z}_{\mathcal{V}}$ is $\mathcal{P}_{\mathcal{V}}$-positive.

On the other hand, we observe

$$q_{\mathcal{V}}^i(\tau_i(T)x) = (Tx)^{\mathrm{t}}(Tx) = \rho(T) q_{\mathcal{V}}^i(x) \quad (x \in W_{\mathcal{V}}^i,\ T \in H_{\mathcal{V}}), \tag{3.9}$$

which yields the group homomorphism $H_{\mathcal{V}} \ni T \mapsto (\rho(T), \tau_i(T)) \in \mathrm{Aut}(\mathcal{P}_{\mathcal{V}}, q_{\mathcal{V}}^i)$. It follows that the quadratic map $q_{\mathcal{V}}^i$ is homogeneous.

Recalling the inner product on $\mathcal{V}_{lk}$ given by (3.3), we define an inner product on the space $W^i_{\mathcal{V}}$ via the natural isomorphism

$$W^i_{\mathcal{V}} \simeq \boldsymbol{R} \oplus \sum_{l>i}^{\oplus} \mathcal{V}_{li}. \tag{3.10}$$

Taking an orthonormal basis of $W^i_{\mathcal{V}}$ with respect to the inner product, we identify $W^i_{\mathcal{V}}$ with $\boldsymbol{R}^{m(i)}$, where $m(i) := \dim W^i_{\mathcal{V}}$. Then we consider the linear map $\phi^i_{\mathcal{V}} : \mathcal{Z}_{\mathcal{V}} \equiv \mathcal{Z}^*_{\mathcal{V}} \to \mathrm{Sym}(m(i), \boldsymbol{R})$ associated to the quadratic map $q^i_{\mathcal{V}}$. Note that, if we write n_{li} for $\dim \mathcal{V}_{li}$, we have $m(i) = 1 + \sum_{l>i} n_{li}$.

Proposition 3.3. (i) *One has*

$$\det \tau_i(T) = \chi_{\underline{m}(i)/2}(T) \qquad (T \in H_{\mathcal{V}}),$$

where

$$\underline{m}(i) := (0, \ldots, 0, 1, n_{i+1,i}, \ldots, n_{ri}) \in \boldsymbol{Z}^r. \tag{3.11}$$

(ii) *For $\eta = \rho^*(T) I_N \in \mathcal{P}^*_{\mathcal{V}}$ with $T \in H_{\mathcal{V}}$, one has*

$$\det \phi^i_{\mathcal{V}}(\eta) = \chi_{\underline{m}(i)}(T).$$

In particular, $(t_{rr})^2 = \det \phi^r_{\mathcal{V}}(\eta)$.

(iii) *For $1 \le i < r$, there exist integers $c_{i+1,i}, \ldots, c_{ri}$ such that*

$$(t_{ii})^2 = \det \phi^i_{\mathcal{V}}(\eta) \cdot (\det \phi^{i+1}_{\mathcal{V}}(\eta))^{c_{i+1,i}} \cdots (\det \phi^r_{\mathcal{V}}(\eta))^{c_{ri}}$$
$$(\eta = \rho^*(T) I_N \in \mathcal{P}^*_{\mathcal{V}},\ T \in H_{\mathcal{V}}).$$

(iv) *One has $\mathcal{P}^*_{\mathcal{V}} = \{\eta \in \mathcal{Z}_{\mathcal{V}} \,;\, \det \phi^i_{\mathcal{V}}(\eta) > 0 \ (i = 1, \ldots, r)\}$.*

Proof. (i) Since $H_{\mathcal{V}} \ni T \mapsto \det \tau_i(T) \in \boldsymbol{C}^\times$ is a one-dimensional representation, it is sufficient to check the equality for diagonal matrices $T \in A_{\mathcal{V}}$. In this case, the isomorphism (3.10) gives the eigenspace decomposition of $\tau_i(T)$, where $\boldsymbol{R}$ and $\mathcal{V}_{li}$ correspond to the eigenvalues t_{ii} and t_{ll} respectively. Therefore we have $\det \tau_i(T) = t_{ii} \prod_{l>i} t_{li}^{n_{li}} = \chi_{\underline{m}(i)/2}(T)$.

(ii) Thanks to (3.3), we have $(x|x) = \langle q^i_{\mathcal{V}}(x), I_N \rangle$, which implies $\phi^i_{\mathcal{V}}(I_N) = I_{m(i)}$. Thus we get $\det \phi^i_{\mathcal{V}}(\eta) = \det \phi^i_{\mathcal{V}}(\rho^*(T) I_N) = (\det \tau_i(T))^2 = \chi_{\underline{m}(i)}(T)$ by (2.17) and (i).

(iii) We see from (ii) that $(t_{ii})^2 = \det \phi^i_{\mathcal{V}}(\eta) \cdot (t_{i+1,i+1})^{-2n_{i+1,i}} \cdots (t_{rr})^{-2n_{ri}}$, whence we can deduce (iii) recursively.

(iv) It is known ([**25**, Chapter 3, Section 3] and [**7**, Section 1]) that a homogeneous cone is described as the subset of the ambient vector space consisting of points at which all relatively invariant (appropriately normalized) functions are positive. Thus the assertion follows from (iii). ☐

Example 4. Let $\mathcal{P}_{\mathcal{V}}$ be the Vinberg cone, that is, $\mathcal{Z}_{\mathcal{V}}$ is as in (3.5). Then we have

$$\phi_{\mathcal{V}}^1(\eta) = \begin{pmatrix} \eta_{11} & \eta_{21} & \eta_{31} \\ \eta_{21} & \eta_{22} & 0 \\ \eta_{31} & 0 & \eta_{33} \end{pmatrix}, \quad \phi_{\mathcal{V}}^2(\eta) = \eta_{22}, \quad \phi_{\mathcal{V}}^3(\eta) = \eta_{33}$$

for $\eta \in \mathcal{Z}_{\mathcal{V}}$. If $\eta = \rho^*(T)I_4$ for $T \in H_{\mathcal{V}}$, we have

$$(t_{11})^2(t_{22})^2(t_{33})^2 = \det \phi_{\mathcal{V}}^1(\eta), \quad (t_{22})^2 = \eta_{22}, \quad (t_{33})^2 = \eta_{33},$$

so that

$$(t_{11})^2 = \frac{\det \phi_{\mathcal{V}}^1(\eta)}{\eta_{22}\eta_{33}} = \eta_{11} - \frac{(\eta_{21})^2}{\eta_{22}} - \frac{(\eta_{31})^2}{\eta_{33}}.$$

On the other hand, we have by Proposition 3.3 (iv)

$$\mathcal{P}_{\mathcal{V}}^* = \left\{ \eta \in \mathcal{Z}_{\mathcal{V}} \,;\, \eta_{11}\eta_{22}\eta_{33} - \eta_{33}(\eta_{21})^2 - \eta_{22}(\eta_{31})^2 > 0,\ \eta_{22} > 0,\ \eta_{33} > 0 \right\}.$$

Therefore, if $\mathcal{Z}$ is the space in (2.5), the linear isomorphism

$$\iota : \mathcal{Z}_{\mathcal{V}} \ni \begin{pmatrix} \eta_{11} & 0 & \eta_{21} & 0 \\ 0 & \eta_{11} & 0 & \eta_{31} \\ \eta_{21} & 0 & \eta_{22} & 0 \\ 0 & \eta_{31} & 0 & \eta_{33} \end{pmatrix} \mapsto \begin{pmatrix} \eta_{33} & 0 & \eta_{31} \\ 0 & \eta_{22} & \eta_{21} \\ \eta_{31} & \eta_{21} & \eta_{11} \end{pmatrix} \in \mathcal{Z}$$

gives a bijection from $\mathcal{P}_{\mathcal{V}}^*$ onto $\mathcal{P} = \mathcal{Z} \cap \mathrm{Sym}_3^+$. The adjoint map $\iota^* : \mathcal{Z}^* \to \mathcal{Z}_{\mathcal{V}}^* \equiv \mathcal{Z}_{\mathcal{V}}$ gives the matrix realization of the Vinberg cone $\mathcal{Q}$ as the homogeneous cone $\mathcal{P}_{\mathcal{V}}$.

3.4. Standard quadratic maps and H-orbits in $\overline{\mathcal{P}_{\mathcal{V}}}$.

In this section a class of very important quadratic maps called *standard* is distinguished and analysed. Further we will see their applications to easy recognizing whether Riesz and Wishart measures are singular or absolutely continuous, as well as to prove the Bartlett decomposition of Wishart laws.

Let us introduce the virtual quadratic map

$$q_{\mathcal{V}}^{\underline{s}} := (q_{\mathcal{V}}^1)^{\oplus s_1} \oplus \cdots \oplus (q_{\mathcal{V}}^r)^{\oplus s_r} \tag{3.12}$$

for $\underline{s} = (s_1, \ldots, s_r) \in \boldsymbol{R}^r$. Take $\underline{\varepsilon} \in \{0,1\}^r$ with $\underline{\varepsilon} \neq (0, \ldots, 0)$. We write $I(\underline{\varepsilon})$ for the set $\{1 \le i \le r \,;\, \varepsilon_i = 1\}$. Then we identify $q_{\mathcal{V}}^{\underline{\varepsilon}}$ with a direct sum $\sum_{i \in I(\underline{\varepsilon})}^{\oplus} q_{\mathcal{V}}^i$ on the space $W_{\mathcal{V}}^{\underline{\varepsilon}} := \sum_{i \in I(\underline{\varepsilon})}^{\oplus} W_{\mathcal{V}}^i$.

Definition 3.4. We call the maps $q_{\mathcal{V}}^{\underline{\varepsilon}}$ ($\underline{\varepsilon} \in \{0,1\}^r$, $\underline{\varepsilon} \neq (0, \ldots, 0)$) *standard quadratic maps.*

Recalling (3.10), we have the isomorphism $W_{\mathcal{V}}^{\underline{\varepsilon}} \simeq \sum_{i\in I(\underline{\varepsilon})}^{\oplus}(\boldsymbol{R} \oplus \sum_{l>i}^{\oplus} \mathcal{V}_{li})$, which enables us to describe $y = q_{\mathcal{V}}^{\underline{\varepsilon}}(x) \in \mathcal{Z}_{\mathcal{V}}$ $(x \in W_{\mathcal{V}}^{\underline{\varepsilon}})$ as the matrix composed of the blocks

$$Y_{lk} = \sum_{i\le k,\ i\in I(\underline{\varepsilon})} X_{li}{}^{\mathsf{t}}X_{ki} \quad (1 \le k \le l \le r), \tag{3.13}$$

where $X_{ii} := x_{ii}I_{n_i}$ for $i \in I(\underline{\varepsilon})$. For the case $l = k$, we have $Y_{kk} = y_{kk}I_{n_k}$ and

$$y_{kk} = \sum_{i\le k,\ i\in I(\underline{\varepsilon})} \|X_{ki}\|^2 \tag{3.14}$$

thanks to (3.3), where we put $\|X_{ii}\| := |x_{ii}|$. For each $x \in W_{\mathcal{V}}^{\underline{\varepsilon}}$, let $T_x \in \mathrm{Mat}(N, \boldsymbol{R})$ be a lower triangular matrix whose (k,i)-block component is X_{ki} for $k > i$ with $i \in I(\varepsilon)$, and other components are zero. Then we have

$$q_{\mathcal{V}}^{\underline{\varepsilon}}(x) = T_x{}^{\mathsf{t}}T_x. \tag{3.15}$$

For example, if $r = 3$ and $\underline{\varepsilon} = (1,0,1)$, then an element x of $W_{\mathcal{V}}^{\underline{\varepsilon}} = W_{\mathcal{V}}^1 \oplus W_{\mathcal{V}}^3$ is of the form

$$x = \begin{pmatrix} X_{11} \\ X_{21} \\ X_{31} \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \\ X_{33} \end{pmatrix} \qquad \begin{pmatrix} X_{11} = x_{11}I_{n_1},\ X_{33} = x_{33}I_{n_3} \\ x_{11},\, x_{33} \in \boldsymbol{R},\ X_{21} \in \mathcal{V}_{21},\, X_{31} \in \mathcal{V}_{31} \end{pmatrix},$$

and we have

$$T_x = \begin{pmatrix} X_{11} & & \\ X_{21} & 0 & \\ X_{31} & 0 & X_{33} \end{pmatrix} \in \mathrm{Mat}(N, \boldsymbol{R}).$$

For $\underline{\varepsilon} \in \{0,1\}^r$, let $E_{\underline{\varepsilon}}$ be the element of $\mathcal{Z}_{\mathcal{V}}$ given by

$$E_{\underline{\varepsilon}} := \begin{pmatrix} \varepsilon_1 I_{n_1} & & \\ & \ddots & \\ & & \varepsilon_r I_{n_r} \end{pmatrix},$$

and $\mathcal{O}_{\underline{\varepsilon}}$ the $H_{\mathcal{V}}$-orbit $\rho(H_{\mathcal{V}})E_{\underline{\varepsilon}} \subset \mathcal{Z}_{\mathcal{V}}$ through $E_{\underline{\varepsilon}}$. In particular, the orbit $\mathcal{O}_{(0,\dots,0)}$ is the origin $\{0\}$, while $\mathcal{O}_{(1,\dots,1)} = \rho(H_{\mathcal{V}})I_N$ equals the cone $\mathcal{P}_{\mathcal{V}}$. It is shown in [**12**, Theorem 3.5] that the $H_{\mathcal{V}}$-orbit decomposition of the closure $\overline{\mathcal{P}_{\mathcal{V}}}$ is given as

$$\overline{\mathcal{P}_{\mathcal{V}}} = \bigsqcup_{\underline{\varepsilon}\in\{0,1\}^r} \mathcal{O}_{\underline{\varepsilon}}.$$

Proposition 3.5. *If* $\underline{\varepsilon} \neq (0, \dots, 0)$, *the image of the quadratic map* $q_{\mathcal{V}}^{\underline{\varepsilon}}$ *equals the closure* $\overline{\mathcal{O}_{\underline{\varepsilon}}}$ *of the orbit* $\mathcal{O}_{\underline{\varepsilon}}$.

Proof. We define

$$W_{\mathcal{V}}^{\underline{\varepsilon},+} := \left\{ x \in W_{\mathcal{V}}^{\underline{\varepsilon}} \, ; \, x_{ii} > 0 \ (i \in I(\underline{\varepsilon})) \right\}. \tag{3.16}$$

For $y = \rho(T)E_{\underline{\varepsilon}} = TE_{\underline{\varepsilon}}\,{}^{\mathrm{t}}T \in \mathcal{O}_{\underline{\varepsilon}}$ with $T \in H_{\mathcal{V}}$, we take a unique $x \in W_{\mathcal{V}}^{\underline{\varepsilon},+}$ for which T_x equals $TE_{\underline{\varepsilon}}$. Then we have $y = q_{\mathcal{V}}^{\underline{\varepsilon}}(x)$ by (3.15). Conversely, for any $x \in W_{\mathcal{V}}^{\underline{\varepsilon},+}$, we put $T := T_x + I - E_{\underline{\varepsilon}} \in H_{\mathcal{V}}$ so that $q_{\mathcal{V}}^{\underline{\varepsilon}}(x) = TE_{\underline{\varepsilon}}\,{}^{\mathrm{t}}T \in \mathcal{O}_{\underline{\varepsilon}}$. Therefore we obtain $\mathcal{O}_{\underline{\varepsilon}} = q_{\mathcal{V}}^{\underline{\varepsilon}}(W_{\mathcal{V}}^{\underline{\varepsilon},+})$. On the other hand, putting $W_{\mathcal{V}}^{\underline{\varepsilon},*} := \left\{ x \in W_{\mathcal{V}}^{\underline{\varepsilon}} \, ; \, x_{ii} \neq 0 \ (i \in I(\underline{\varepsilon})) \right\}$, we see easily that $q_{\mathcal{V}}^{\underline{\varepsilon}}(W_{\mathcal{V}}^{\underline{\varepsilon},*}) = q_{\mathcal{V}}^{\underline{\varepsilon}}(W_{\mathcal{V}}^{\underline{\varepsilon},+})$. Since $W_{\mathcal{V}}^{\underline{\varepsilon},*}$ is an open dense subset of $W_{\mathcal{V}}^{\underline{\varepsilon}}$, the orbit $\mathcal{O}_{\underline{\varepsilon}}$ is dense in the image of the quadratic map $q_{\mathcal{V}}^{\underline{\varepsilon}}$, which is necessarily closed. Indeed, introducing the projective imbedding ι_V of a vector space V by $\iota_V : V \ni y \mapsto [1, y] \in \boldsymbol{P}V := (\boldsymbol{R} \times V \setminus \{(0,0)\})/\boldsymbol{R}^{\times}$, we can extend $q_{\mathcal{V}}^{\underline{\varepsilon}} : W_{\mathcal{V}}^{\underline{\varepsilon}} \to \mathcal{Z}_{\mathcal{V}}$ to the map $\tilde{q}_{\mathcal{V}}^{\underline{\varepsilon}} : \boldsymbol{P}W_{\mathcal{V}}^{\underline{\varepsilon}} \ni [t, x] \mapsto [t^2, q_{\mathcal{V}}^{\underline{\varepsilon}}(x)] \in \boldsymbol{P}\mathcal{Z}_{\mathcal{V}}$ because $q_{\mathcal{V}}^{\underline{\varepsilon}}(x) \neq 0$ for $x \neq 0$. The image $\tilde{q}_{\mathcal{V}}^{\underline{\varepsilon}}(\boldsymbol{P}W_{\mathcal{V}}^{\underline{\varepsilon}})$ is compact, so that $q_{\mathcal{V}}^{\underline{\varepsilon}}(W_{\mathcal{V}}^{\underline{\varepsilon}}) = \iota_{\mathcal{Z}_{\mathcal{V}}}^{-1}(\tilde{q}_{\mathcal{V}}^{\underline{\varepsilon}}(\boldsymbol{P}W_{\mathcal{V}}^{\underline{\varepsilon}}))$ is closed. □

Remark 3.6. In the proof of Proposition 3.5, we see that the quadratic map $q_{\mathcal{V}}^{\underline{\varepsilon}}$ gives a surjective map from $W_{\mathcal{V}}^{\underline{\varepsilon},+}$ onto $\mathcal{O}_{\underline{\varepsilon}}$. The map is also one-to-one thanks to [**12**, Lemma 3.3 (ii)], while the map $q_{\mathcal{V}}^{\underline{\varepsilon}} : W_{\mathcal{V}}^{\underline{\varepsilon},*} \to \mathcal{O}_{\underline{\varepsilon}}$ is $2^{\sharp I(\underline{\varepsilon})}$-to-one. Actually, a large part of the content of this section is presented in language of normal j-algebra in [**12**, Sections 3 and 4].

We define the representation $\tau_{\underline{\varepsilon}} : H_{\mathcal{V}} \to GL(W_{\mathcal{V}}^{\underline{\varepsilon}})$ as the direct sum of the representations $(\tau_i, W_{\mathcal{V}}^i)$ of $H_{\mathcal{V}}$ for $i \in I(\underline{\varepsilon})$. Then we have by (3.9)

$$q_{\mathcal{V}}^{\underline{\varepsilon}}(\tau_{\underline{\varepsilon}}(T)x) = \rho(T) q_{\mathcal{V}}^{\underline{\varepsilon}}(x) \quad (x \in W_{\mathcal{V}}^{\underline{\varepsilon}}, \ T \in H_{\mathcal{V}}), \tag{3.17}$$

which implies that $q_{\mathcal{V}}^{\underline{\varepsilon}}$ is homogeneous.

The open set $W_{\mathcal{V}}^{\underline{\varepsilon},+} \subset W_{\mathcal{V}}^{\underline{\varepsilon}}$ is preserved by the action of $\tau_{\underline{\varepsilon}}(H_{\mathcal{V}})$. We put

$$R_+(\underline{\varepsilon}) := \{ \underline{u} = (u_1, \dots, u_r) \in \boldsymbol{R}^r \, ; \, u_i = 0 \ (\text{if } \varepsilon_i = 0), \quad u_i > 0 \ (\text{if } \varepsilon_i = 1) \}.$$

For $\underline{u} \in R_+(\underline{\varepsilon})$, let $\mathcal{M}_{\underline{u}}^{\underline{\varepsilon}}$ be the measure on $W_{\mathcal{V}}^{\underline{\varepsilon},+}$ given by

$$\begin{aligned} \mathcal{M}_{\underline{u}}^{\underline{\varepsilon}}(dx) &:= \prod_{i \in I(\underline{\varepsilon})} \left\{ \frac{2(x_{ii})^{2u_i - 1}\, dx_{ii}}{\Gamma(u_i)} \cdot \prod_{l > i} \frac{dX_{li}}{\pi^{n_{li}/2}} \right\} \\ &= \frac{\prod_{i \in I(\underline{\varepsilon})} (x_{ii})^{2u_i - 1}}{\Gamma_{\underline{\varepsilon}}(\underline{u})}\, dx \quad (x \in W_{\mathcal{V}}^{\underline{\varepsilon},+}), \end{aligned} \tag{3.18}$$

where $\Gamma_{\underline{\varepsilon}}(\underline{u}) = \pi^{\dim W_{\mathcal{V}}^{\underline{\varepsilon}}/2} \prod_{i \in I(\underline{\varepsilon})} (\Gamma(u_i)/2\sqrt{\pi})$. When $\underline{u} = \underline{\varepsilon}/2$, the measure $\mathcal{M}_{\underline{\varepsilon}/2}^{\underline{\varepsilon}}$ equals a constant multiple of the Lebesgue measure, that is, $\mathcal{M}_{\underline{\varepsilon}/2}^{\underline{\varepsilon}}(dx) = 2^{\sharp I(\underline{\varepsilon})} \pi^{-\dim W_{\mathcal{V}}^{\underline{\varepsilon}}/2} dx$.

We define $\underline{p}(\underline{\varepsilon}) := (p_1(\underline{\varepsilon}), p_2(\underline{\varepsilon}), \dots, p_r(\underline{\varepsilon}))$ by

$$p_k(\underline{\varepsilon}) := \sum_{i<k} \varepsilon_i n_{ki}. \tag{3.19}$$

LEMMA 3.7. (i) *For a measurable set $A \subset W_{\mathcal{V}}^{\underline{\varepsilon},+}$, one has*

$$\mathcal{M}_{\underline{u}}^{\underline{\varepsilon}}(\tau_{\underline{\varepsilon}}(T)A) = \chi_{\underline{u}+\underline{p}(\underline{\varepsilon})/2}(T)\mathcal{M}_{\underline{u}}^{\underline{\varepsilon}}(A) \quad (T \in H_{\mathcal{V}}).$$

(ii) *One has*

$$\int_{W_{\mathcal{V}}^{\underline{\varepsilon},+}} e^{-\|x\|^2}\, \mathcal{M}_{\underline{u}}^{\underline{\varepsilon}}(dx) = 1. \tag{3.20}$$

PROOF. (i) If $x' = \tau_{\underline{\varepsilon}}(T)x \in W_{\mathcal{V}}^{\underline{\varepsilon},+}$ with $x \in W_{\mathcal{V}}^{\underline{\varepsilon},+}$, we have $x'_{ii} = t_{ii}x_{ii}$ for $i \in I(\underline{\varepsilon})$. Thus

$$\prod_{i\in I(\underline{\varepsilon})} (x'_{ii})^{2u_i-1} = \chi_{\underline{u}-\underline{\varepsilon}/2}(T) \prod_{i\in I(\underline{\varepsilon})} (x_{ii})^{2u_i-1}. \tag{3.21}$$

On the other hand, we observe that

$$dx' = |\det \tau_{\underline{\varepsilon}}(T)|dx = \Big(\prod_{i\in I(\underline{\varepsilon})} \det \tau_i(T)\Big)dx,$$

and the last term equals $\big(\prod_{i\in I(\underline{\varepsilon})} \chi_{\underline{m}(i)/2}(T)\big)dx$ by Proposition 3.3 (i). Since

$$\sum_{i\in I(\varepsilon)} \underline{m}(i) = \sum_{i=1}^{r} \varepsilon_i \underline{m}(i) = \underline{\varepsilon} + \underline{p}(\underline{\varepsilon}),$$

we have $dx' = \chi_{\underline{\varepsilon}/2+\underline{p}(\underline{\varepsilon})/2}(T)dx$, which together with (3.21) implies (i).

(ii) By definition, we have

$$\|x\|^2 = \sum_{i\in I(\underline{\varepsilon})} \Big\{(x_{ii})^2 + \sum_{l>i} \|X_{li}\|^2\Big\}.$$

Thus the left-hand side of (3.20) equals

$$\prod_{i\in I(\underline{\varepsilon})} \left\{\frac{2}{\Gamma(u_i)} \int_0^\infty e^{-(x_{ii})^2}(x_{ii})^{2u_i-1}dx_{ii} \prod_{l>i} \int_{\mathcal{V}_{li}} e^{-\|X_{li}\|^2} \frac{dX_{li}}{\pi^{n_{li}/2}}\right\}.$$

Therefore we obtain (3.20) from $\int_{\mathcal{V}_{li}} e^{-\|X_{li}\|^2} dX_{li} = \pi^{n_{li}/2}$ and $\int_0^\infty e^{-(x_{ii})^2}(x_{ii})^{2u_i-1} \cdot dx_{ii} = \Gamma(u_i)/2$. $\square$

REMARK 3.8. Lemma 3.7 tells us that $e^{-\|x\|^2}\,\mathcal{M}_{\underline{u}}^{\underline{\varepsilon}}(dx)$ is a probability measure on $W_{\mathcal{V}}^{\underline{\varepsilon},+}$. Actually, we see from the proof that if $X^{\underline{u}}$ is an $W_{\mathcal{V}}^{\underline{\varepsilon},+}$-valued random variable with the law $e^{-\|x\|^2}\,\mathcal{M}_{\underline{u}}^{\underline{\varepsilon}}(dx)$, then its components are independent and satisfy $\sqrt{2}X_{li}^{\underline{u}} \sim N(0, I_{n_{li}})$, and $(\sqrt{2}X_{ii}^{\underline{u}})^2 \sim \chi^2(2u_i)$, where $\chi^2(u)$ denotes the chi-square law with the density $2^{-u}\Gamma(u/2)^{-1}e^{-t/2}t^{u-1}$ $(t > 0)$. We shall see later that any Wishart law associated to a homogeneous quadratic map is the image of this measure by an appropriate quadratic map.

For instance, let us consider the case where $\mathcal{Z}_{\mathcal{V}}$ is the one in (3.5) and $\underline{\varepsilon} = (1, 0, 1)$. Then, keeping (3.8) in mind, we see that the measure $\mathcal{M}_{\underline{u}}^{\underline{\varepsilon}}$ on $W_{\mathcal{V}}^{\underline{\varepsilon},+} \subset W_{\mathcal{V}}^{\underline{\varepsilon}} = W_{\mathcal{V}}^1 \oplus W_{\mathcal{V}}^3$ is given by

$$\mathcal{M}_{\underline{u}}^{\underline{\varepsilon}}(dx) = \frac{4}{\pi\Gamma(u_1)\Gamma(u_3)}(x_{11})^{2u_1-1}(x_{33})^{2u_3-1}\, dx_{11}dx_{21}dx_{31}dx_{33}$$
$$(x_{11} > 0,\ x_{33} > 0,\ x_{21},\ x_{31} \in \boldsymbol{R})$$

for $\underline{u} = (u_1, 0, u_3)$ with $u_1 > 0$, $u_3 > 0$.

3.5. Gindikin-Riesz distributions.

For $\underline{\sigma} = (\sigma_1, \ldots, \sigma_r) \in \boldsymbol{C}^r$, we denote $(\sigma_r, \ldots, \sigma_1)$ by $\underline{\sigma}^*$. Let $\Delta_{\underline{\sigma}}^*$ be the function on the cone $\mathcal{P}_{\mathcal{V}}^*$ given by

$$\Delta_{\underline{\sigma}}^*(\rho^*(T)I_N) = \chi_{\underline{\sigma}^*}(T) \quad (T \in H_{\mathcal{V}}). \tag{3.22}$$

By Proposition 3.3 (ii) and (iii), $\Delta_{\underline{\sigma}}^*(\eta)$ can be expressed as a product of powers of the polynomials $\det \phi_{\mathcal{V}}^i(\eta)$. Putting

$$E_{\underline{\eta}} := \begin{pmatrix} \eta_1 I_{n_1} & & \\ & \ddots & \\ & & \eta_r I_{n_r} \end{pmatrix} \in \mathcal{Z}_{\mathcal{V}}$$

for $\underline{\eta} = (\eta_1, \ldots, \eta_r) \in \boldsymbol{R}_{>0}^r$, we have

$$\Delta_{\underline{\sigma}}^*(E_{\underline{\eta}}) = (\eta_1)^{\sigma_r}(\eta_2)^{\sigma_{r-1}} \cdots (\eta_r)^{\sigma_1}. \tag{3.23}$$

DEFINITION 3.9. For each $\underline{\sigma} \in \boldsymbol{C}^r$, a tempered distribution $\mathcal{R}_{\underline{\sigma}} \in \mathcal{S}'(\mathcal{Z}_{\mathcal{V}})$ whose Laplace transform $L_{\mathcal{R}_{\underline{\sigma}}}(\theta) = \mathcal{R}_{\underline{\sigma}}(e^{-\langle y,\theta\rangle})$ is given by

$$L_{\mathcal{R}_{\underline{\sigma}}}(\theta) = \Delta_{-\underline{\sigma}^*}^*(\theta) \quad (\theta \in \mathcal{P}_{\mathcal{V}}^*) \tag{3.24}$$

is called a *Gindikin-Riesz distribution* on the homogeneous cone $\mathcal{P}_{\mathcal{V}}$.

Gindikin ([**7**], [**8**]) constructed $\mathcal{R}_{\underline{\sigma}}$ as a composition of an absolutely continuous complex measure on $\mathcal{P}_{\mathcal{V}}$ with a differential operator. The support of $\mathcal{R}_{\underline{\sigma}}$ is contained in $\overline{\mathcal{P}_{\mathcal{V}}}$, and $\mathcal{R}_{\underline{\sigma}}$ is relatively invariant under the action of $\rho(H_{\mathcal{V}})$, that is,

$$\mathcal{R}_{\underline{\sigma}}(f \circ \rho(T)) = \chi_{-\underline{\sigma}}(T)\mathcal{R}_{\underline{\sigma}}(f) \tag{3.25}$$

for $T \in H_{\mathcal{V}}$ and $f \in \mathcal{S}(\mathcal{Z}_{\mathcal{V}})$.

PROPOSITION 3.10. *For non-zero $\underline{\varepsilon} \in \{0,1\}^r$ and $\underline{u} \in R_+(\underline{\varepsilon})$, put $\underline{\sigma} := \underline{u} + \underline{p}(\underline{\varepsilon})/2$. Then $\mathcal{R}_{\underline{\sigma}}$ is the image of $\mathcal{M}_{\underline{u}}^{\underline{\varepsilon}}$ by the standard quadratic map $q_{\mathcal{V}}^{\underline{\varepsilon}}$.*

PROOF. By (3.24), it is sufficient to show that

$$\int_{W_{\mathcal{V}}^{\underline{\varepsilon},+}} e^{-\langle q_{\mathcal{V}}^{\underline{\varepsilon}}(x),\theta\rangle}\,\mathcal{M}_{\underline{u}}^{\underline{\varepsilon}}(dx) = \Delta^*_{-\underline{\sigma}^*}(\theta)$$

for $\theta \in \mathcal{P}_{\mathcal{V}}^*$. Take $T \in H_{\mathcal{V}}$ for which $\theta = \rho^*(T)I_N$. Then the left-hand side is

$$\int_{W_{\mathcal{V}}^{\underline{\varepsilon},+}} e^{-\langle q_{\mathcal{V}}^{\underline{\varepsilon}}(x),\rho^*(T)I_N\rangle}\,\mathcal{M}_{\underline{u}}^{\underline{\varepsilon}}(dx) = \int_{W_{\mathcal{V}}^{\underline{\varepsilon},+}} e^{-\langle q_{\mathcal{V}}^{\underline{\varepsilon}}(\tau_{\underline{\varepsilon}}(T)x),I_N\rangle}\,\mathcal{M}_{\underline{u}}^{\underline{\varepsilon}}(dx)$$

by (3.17), and it is equal to

$$\chi_{-(\underline{u}+\underline{p}(\underline{\varepsilon})/2)}(T)\int_{W_{\mathcal{V}}^{\underline{\varepsilon},+}} e^{-\langle q_{\mathcal{V}}^{\underline{\varepsilon}}(x),I_N\rangle}\,\mathcal{M}_{\underline{u}}^{\underline{\varepsilon}}(dx)$$

by Lemma 3.7 (i). Since $\langle q_{\mathcal{V}}^{\underline{\varepsilon}}(x), I_N\rangle = \|x\|^2$ by (3.14), we see from Lemma 3.7 (ii) that

$$\int_{W_{\mathcal{V}}^{\underline{\varepsilon},+}} e^{-\langle q_{\mathcal{V}}^{\underline{\varepsilon}}(x),\theta\rangle}\,\mathcal{M}_{\underline{u}}^{\underline{\varepsilon}}(dx) = \chi_{-(\underline{u}+\underline{p}(\underline{\varepsilon})/2)}(T) = \Delta^*_{-\underline{\sigma}^*}(\theta). \qquad \square$$

We set

$$\begin{aligned}\Xi(\underline{\varepsilon}) &:= \left\{\underline{\sigma} = \underline{u} + \underline{p}(\underline{\varepsilon})/2\,;\, \underline{u} \in R_+(\underline{\varepsilon})\right\} \\ &= \left\{\underline{\sigma} \in \boldsymbol{R}^r\,;\, \sigma_i = p_i(\underline{\varepsilon})/2 \text{ (if } \varepsilon_i = 0), \quad \sigma_i > p_i(\underline{\varepsilon})/2 \text{ (if } \varepsilon_i = 1)\right\}.\end{aligned} \tag{3.26}$$

If $\underline{\varepsilon} \neq (0,\dots,0)$ and $\underline{\sigma} \in \Xi(\underline{\varepsilon})$, then $\mathcal{R}_{\underline{\sigma}}$ is a positive measure on the orbit $\mathcal{O}_{\underline{\varepsilon}}$ by Proposition 3.10. For the case $\varepsilon = (0,\dots,0)$, we have $\Xi(0,\dots,0) = \{(0,\dots,0)\}$ and $\mathcal{R}_{(0,\dots,0)}$ is the Dirac measure at the origin $\{0\}$. It is proven in [**12**] that they exhaust all the cases that $\mathcal{R}_{\underline{\sigma}}$ is a positive measure.

THEOREM 3.11 ([**12**, Theorem 6.2]). *The Gindikin-Riesz distribution $\mathcal{R}_{\underline{\sigma}}$ is a positive measure if and only if $\underline{\sigma} \in \Xi := \bigsqcup_{\underline{\varepsilon}\in\{0,1\}^r} \Xi(\underline{\varepsilon})$. Moreover, if $\underline{\sigma} \in \Xi(\underline{\varepsilon})$, then $\mathcal{R}_{\underline{\sigma}}$ is a measure on $\mathcal{O}_{\underline{\varepsilon}}$.*

The parameter set Ξ is also described as

$$\Xi = \left\{ \sum_{i=1}^{r} \varepsilon_i (0, \ldots, 0, u_i, n_{i+1,i}/2, \ldots, n_{ri}/2) \,;\, \varepsilon_i \in \{0,1\},\, u_i > 0 \ (i = 1, \ldots, r) \right\}.$$

3.6. Riesz measures and Gindikin-Riesz distributions.
Let us investigate a relation of the Riesz measures μ_q associated to homogeneous $\mathcal{P}_{\mathcal{V}}$-positive quadratic maps q and the Gindikin-Riesz distributions on $\mathcal{P}_{\mathcal{V}}$.

Proposition 3.12. *For $i = 1, \ldots, r$, one has $\mu_{q^i_{\mathcal{V}}} = \pi^{m(i)/2} \mathcal{R}_{\underline{m}(i)/2}$.*

Proof. From Lemma 2.4 and Proposition 3.3 (ii), we have

$$L_{\mu_{q^i_{\mathcal{V}}}}(\theta) = \pi^{m(i)/2} \det \phi^i_{\mathcal{V}}(\theta)^{-1/2} = \pi^{m(i)/2} \Delta^*_{-\underline{m}(i)^*/2}(\theta) \quad (\theta \in \mathcal{P}^*_{\mathcal{V}}), \tag{3.27}$$

which implies the statement. □

Assume that there exists the Riesz measure $\mu_{q^{\underline{s}}_{\mathcal{V}}}$ associated to a virtual quadratic map $q^{\underline{s}}_{\mathcal{V}} = (q^1_{\mathcal{V}})^{\oplus s_1} \oplus \cdots \oplus (q^r_{\mathcal{V}})^{\oplus s_r}$. By (2.10) and (3.27), we have for $\theta \in \mathcal{P}^*_{\mathcal{V}}$

$$L_{\mu_{q^{\underline{s}}_{\mathcal{V}}}}(\theta) = \prod_{i=1}^{r} \left(\pi^{m(i)/2} \Delta^*_{-\underline{m}(i)^*/2}(\theta) \right)^{s_i}.$$

We put

$$\underline{\sigma} := \sum_{i=1}^{r} s_i \underline{m}(i)/2 = \frac{1}{2} \sum_{i=1}^{r} s_i (0, \ldots, 0, 1, n_{i+1,i}, \ldots, n_{ri}). \tag{3.28}$$

Then the equality above can be rewritten as

$$L_{\mu_{q^{\underline{s}}_{\mathcal{V}}}}(\theta) = \pi^{|\underline{\sigma}|} \Delta^*_{-\underline{\sigma}^*}(\theta),$$

where $|\underline{\sigma}| := \sigma_1 + \cdots + \sigma_r$. Thus $\mu_{q^{\underline{s}}_{\mathcal{V}}}$ equals $\pi^{|\underline{\sigma}|} \mathcal{R}_{\underline{\sigma}}$, so that $\underline{\sigma}$ belongs to $\Xi(\underline{\varepsilon})$ for some $\underline{\varepsilon} \in \{0,1\}^r$ owing to Theorem 3.11. The converse argument is also valid. Therefore we obtain

Theorem 3.13. *For a virtual quadratic map $q^{\underline{s}}_{\mathcal{V}} = (q^1_{\mathcal{V}})^{\oplus s_1} \oplus \cdots \oplus (q^r_{\mathcal{V}})^{\oplus s_r}$, there exists the associated Riesz measure $\mu_{q^{\underline{s}}_{\mathcal{V}}}$ if and only if $\underline{\sigma} := \sum_{i=1}^{r} s_i \underline{m}(i)/2$ belongs to Ξ. In this case $\mu_{q^{\underline{s}}_{\mathcal{V}}} = \pi^{|\underline{\sigma}|} \mathcal{R}_{\underline{\sigma}}$, and there exist $\underline{\varepsilon} \in \{0,1\}^r$ and $\underline{u} \in R_+(\underline{\varepsilon})$ for which $\underline{\sigma} = \underline{u} + \underline{p}(\underline{\varepsilon})/2$. If $\underline{\varepsilon} \neq (0, \ldots, 0)$, $\mu_{q^{\underline{s}}_{\mathcal{V}}}$ is the image of the measure $\pi^{|\underline{\sigma}|} \mathcal{M}^{\underline{\varepsilon}}_{\underline{u}}$ on $W^{\underline{\varepsilon}}_{\mathcal{V}}$ by the standard quadratic map $q^{\underline{\varepsilon}}_{\mathcal{V}}$.*

Let $q : \boldsymbol{R}^m \to \mathcal{Z}_{\mathcal{V}}$ be any homogeneous $\mathcal{P}_{\mathcal{V}}$-positive quadratic map. As is noted in

Section 3.1, the group $pr_1(\mathrm{Aut}(\mathcal{P}_\mathcal{V}, q))$ acts on the cone $\mathcal{P}_\mathcal{V}$ transitively. Assume first that $pr_1(\mathrm{Aut}(\mathcal{P}_\mathcal{V}, q))$ contains $\rho(H_\mathcal{V})$. Then the polynomial $\det \phi_q(\eta)$ is relatively invariant under the action of $\rho(H_\mathcal{V})$ by (2.18). Namely, for each $T \in H_\mathcal{V}$, there exists $c_T > 0$ such that $\det \phi_q(\rho^*(T)\eta) = c_T \det \phi_q(\eta)$ $(\eta \in \mathcal{Z}_\mathcal{V})$. It is easy to see that the correspondence $H_\mathcal{V} \ni T \mapsto c_T \in \boldsymbol{R}_{>0}$ is a one-dimensional representation, so that we have $c_T = \chi_{\underline{m}}(T)$ for some $\underline{m} \in \boldsymbol{R}^r$. Thus $\det \phi_q(\rho^*(T)I_n) = \chi_{\underline{m}}(T) \det \phi_q(I_N)$ for $T \in H_\mathcal{V}$, which means that $\det \phi_q(\eta) = C\Delta^*_{\underline{m}^*}(\eta)$ for $\eta \in \mathcal{P}^*_\mathcal{V}$, where $C := \det \phi_q(I_N)$. By (3.23), we have

$$\det \phi_q(E_{\underline{\eta}}) = C\,(\eta_1)^{m_1} \cdots (\eta_r)^{m_r},$$

which gives a practical way to determine m_i. Indeed, we see from this formula that m_i are non-negative integers. Comparing the degrees of both sides, we obtain $m = m_1 + \cdots + m_r$. Similarly to Proposition 3.12, we have

$$\mu_q = C^{-1/2}\pi^{m/2}\mathcal{R}_{\underline{m}/2}. \tag{3.29}$$

Let us consider the virtual quadratic map $q^{\oplus s}$. The associated Riesz measure exists if and only if $s\underline{m}/2 \in \Xi$, and in this case

$$\mu_{q^{\oplus s}} = C^{-s/2}\pi^{sm/2}\mathcal{R}_{s\underline{m}/2}.$$

As for the general case, we have the following result.

Proposition 3.14. *Let $q : \boldsymbol{R}^m \to \mathcal{Z}_\mathcal{V}$ be a homogeneous $\mathcal{P}_\mathcal{V}$-positive quadratic map. Then there exist $g_0 \in G(\mathcal{P}_\mathcal{V})$, $\underline{m} \in \boldsymbol{Z}^r$, and $C > 0$ for which*

$$\det \phi_q((g_0^{-1})^*\eta) = C\Delta^*_{\underline{m}^*}(\eta) \quad (\eta \in \mathcal{P}_\mathcal{V}).$$

The Riesz measure $\mu_{q^{\oplus s}}$ associated to the virtual quadratic map $q^{\oplus s}$ exists if and only if $s\underline{m}/2 \in \Xi$. In this case, $\mu_{q^{\oplus s}}$ equals the image of $C^{-s/2}\pi^{sm/2}\mathcal{R}_{s\underline{m}/2}$ by g_0.

Proof. We note that $pr_1(\mathrm{Aut}(\mathcal{P}_\mathcal{V}, q))$ acts on the cone $\mathcal{P}_\mathcal{V}$ transitively, and that the identity component of $pr_1(\mathrm{Aut}(\mathcal{P}_\mathcal{V}, q))$ equals the identity component of an algebraic group (cf. [**16**, Theorem 2]). It follows that an Iwasawa subgroup (maximal connected split solvable subgroup) $\mathcal{H}$ of $pr_1(\mathrm{Aut}(\mathcal{P}_\mathcal{V}, q))$ acts on $\mathcal{P}_\mathcal{V}$ simply transitively ([**25**, Chapter 1]). Since $\mathcal{H}$ is also an Iwasawa subgroup of $G(\mathcal{P}_\mathcal{V})$, it is conjugate to another Iwasawa subgroup $\rho(H_\mathcal{V}) \subset G(\mathcal{P}_\mathcal{V})$. Namely, there exists $g_0 \in G(\mathcal{P}_\mathcal{V})$ for which $g_0^{-1}\mathcal{H}g_0 = \rho(H_\mathcal{V})$. Let q' be the $\mathcal{P}_\mathcal{V}$-positive quadratic map $g_0^{-1} \circ q : \boldsymbol{R}^m \to \mathcal{Z}_\mathcal{V}$. We have $\phi_{q'}(\eta) = \phi_q((g_0^{-1})^*\eta)$ for $\eta \in \mathcal{Z}_\mathcal{V}$ because ${}^t x\phi_{q'}(\eta)x = \langle q'(x), \eta\rangle = \langle q(x), (g_0^{-1})^*\eta\rangle = {}^t x\phi_q((g_0^{-1})^*\eta)x$ for $x \in \boldsymbol{R}^m$. It is easy to see that

$$\mathrm{Aut}(\mathcal{P}_\mathcal{V}, q') = \left\{(g_0^{-1}g_1g_0, g_2) \in GL(\mathcal{Z}_\mathcal{V}) \times GL(\boldsymbol{R}^m)\,;\, (g_1, g_2) \in \mathrm{Aut}(\mathcal{P}_\mathcal{V}, q)\right\}.$$

Then $pr_1(\mathrm{Aut}(\mathcal{P}_\mathcal{V}, q')) = g_0^{-1}pr_1(\mathrm{Aut}(\mathcal{P}_\mathcal{V}, q))g_0 \supset g_0^{-1}\mathcal{H}g_0 = \rho(H_\mathcal{V})$. Thus we can apply the argument preceding Proposition 3.14 for q', so that we have

$$\det \phi_q((g_0^{-1})^* \eta) = \det \phi_{q'}(\eta) = C\Delta^*_{\underline{m}^*}(\eta)$$

with some $C > 0$ and $\underline{m} \in \boldsymbol{Z}^r$. Moreover $\mu_{(q')^{\oplus s}}$ equals $C^{-s/2}\pi^{sm/2}\mathcal{R}_{s\underline{m}/2}$ if $s\underline{m}/2 \in \Xi$. Since $q^{\oplus s} = g_0 \circ (q')^{\oplus s}$, we get the last statement from Proposition 2.16. $\square$

Proposition 3.14 states that the Riesz measure μ_q associated to a homogeneous q is equal to some Gindikin-Riesz distribution up to a linear transform on $G(\mathcal{P}_\mathcal{V})$. On the other hand, Theorem 3.13 tells us that if a Gindikin-Riesz distribution is a positive measure, then it equals a Riesz measure associated to the virtual sum $q^{\underline{s}}_\mathcal{V} = (q^1_\mathcal{V})^{\oplus s_1} \oplus \cdots \oplus (q^r_\mathcal{V})^{\oplus s_r}$ of basic quadratic maps up to a constant multiple. For example, let us recall the homogeneous Sym^+_r-positive quadratic map $q^I : R^I \to \mathrm{Sym}(r, \boldsymbol{R})$ with $I \subset \{1, \ldots, r\}$ and the permutation matrix $w_0 \in \mathfrak{S}_r \subset GL(r, \boldsymbol{R})$ in Section 3.1. Putting $g_0 := \rho(w_0)$, we have $q^I = g_0 \circ q^{\{r-k+1,\ldots,r\}}$ ($k := \sharp I$), while $q^{\{r-k+1,\ldots,r\}}$ is exactly the basic quadratic map $q^{r-k+1}_\mathcal{V}$ for $\mathcal{Z}_\mathcal{V} = \mathrm{Sym}(r, \boldsymbol{R})$. Therefore, the Riesz measure $\mu_{(q^I)^{\oplus s}}$ exists if and only if $(0, \ldots, 0, \underbrace{s/2, \ldots, s/2}_{k}) \in \Xi$, that is, $s \in \{0, 1, \ldots, k-1\} \cup (k-1, +\infty)$. In this case, $\mu_{(q^I)^{\oplus s}}$ equals the image of $\pi^{s/2}\mathcal{R}_{(0,\ldots,0,s/2,\ldots,s/2)} = \mu_{(q^{r-k+1}_\mathcal{V})^{\oplus s}}$ by g_0.

Let $q_1 : \boldsymbol{R}^{m_1} \to \mathcal{Z}_\mathcal{V}$ and $q_2 : \boldsymbol{R}^{m_2} \to \mathcal{Z}_\mathcal{V}$ be two homogeneous $\mathcal{P}_\mathcal{V}$-positive quadratic maps. As we have seen in Section 3.1, the direct sum $q_1 \oplus q_2$ is not necessarily homogeneous. Let us assume that the group $pr_1(\mathrm{Aut}(\mathcal{P}_\mathcal{V}, q_1)) \cap pr_1(\mathrm{Aut}(\mathcal{P}_\mathcal{V}, q_2))$ acts on $\mathcal{P}_\mathcal{V}$ transitively. In this case, we see easily that $q_1 \oplus q_2$ is homogeneous. As in Proposition 3.14, we can take $g_0 \in G(\mathcal{P}_\mathcal{V})$ for which $g_0\rho(H_\mathcal{V})g_0^{-1} \subset pr_1(\mathrm{Aut}(\mathcal{P}_\mathcal{V}, q_1)) \cap pr_1(\mathrm{Aut}(\mathcal{P}_\mathcal{V}, q_2))$. Then we have $\det \phi_{q_1}((g_0^{-1})^*\eta) = C_1\Delta^*_{\underline{m}'}(\eta)$ and $\det \phi_{q_2}((g_0^{-1})^*\eta) = C_2\Delta^*_{\underline{m}''}(\eta)$ for $\eta \in (\boldsymbol{R}^n)^*$ with some $C_1, C_2 > 0$ and $\underline{m}', \underline{m}'' \in \boldsymbol{Z}^r$. Now we consider a virtual quadratic map $q = q_1^{\oplus s_1} \oplus q_2^{\oplus s_2}$. We see that the associated Riesz measure μ_q exists if and only if $s_1\underline{m}' + s_2\underline{m}''/2 \in \Xi$, and in this case, μ_q is the image of $C_1^{-s_1/2}C_2^{-s_2/2}\pi^{(s_1m_1+s_2m_2)/2}\mathcal{R}_{(s_1\underline{m}'+s_2\underline{m}'')/2}$ by g_0. Obviously, the same argument is valid for general quadratic maps $q = q_1^{\oplus s_1} \oplus q_2^{\oplus s_2} \oplus \cdots \oplus q_t^{\oplus s_t}$.

3.7. Bartlett decomposition of the Wishart laws.

Let $q : \boldsymbol{R}^m \to \mathcal{Z}_\mathcal{V}$ be a homogeneous $\mathcal{P}_\mathcal{V}$-positive quadratic map. Then the Wishart law $\gamma_{q,\theta}$ ($\theta \in \mathcal{P}^*_\mathcal{V}$) is the image of the normal law $N(0, \phi(\theta)^{-1})$ on the vector space $\boldsymbol{R}^m$ by the quadratic map $q/2$, see Remark 2.6.

However, this description of the Wishart law does not permit us to determine its support in general. In this section, we shall give another construction of the Wishart random matrices, which is a generalization of the Bartlett decomposition ([**2**], [**21**, Theorem 3.2.14]) and has the advantage of controlling the support of the underlying Wishart law. Moreover, the result is valid for virtual quadratic maps.

First we consider the virtual quadratic map $q^{\underline{s}}_\mathcal{V} = (q^1_\mathcal{V})^{\oplus s_1} \oplus \cdots \oplus (q^r_\mathcal{V})^{\oplus s_r}$ whose associated Riesz measure $\mu_{q^{\underline{s}}_\mathcal{V}}$ exists. Then $\underline{\sigma} = \sum_{i=1}^r s_i\underline{m}(i)/2$ belongs to Ξ and we have $\mu_{q^{\underline{s}}_\mathcal{V}} = \pi^{|\underline{\sigma}|}\mathcal{R}_{\underline{\sigma}}$ by Theorem 3.13. Moreover, we have $L_{\mu_{q^{\underline{s}}_\mathcal{V}}}(\theta) = \pi^{|\underline{\sigma}|}\Delta^*_{-\underline{\sigma}^*}(\theta)$. Therefore we obtain from (2.8) that

$$\gamma_{q^{\underline{s}}_{\mathcal{V}},\theta}(dy) = e^{-\langle y,\theta\rangle}\Delta^*_{\underline{\sigma}^*}(\theta)\mathcal{R}_{\underline{\sigma}}(dy) \quad (y \in \boldsymbol{R}^n). \tag{3.30}$$

We remark that distributions of this type are considered in [**11**] for the case when $\mathcal{P}_{\mathcal{V}}$ is a symmetric cone. Assume that $\gamma_{q^{\underline{s}}_{\mathcal{V}},\theta}$ is not the Dirac measure. Then $\underline{\sigma} \neq (0,\dots,0)$, so that we can take a non-zero $\underline{\varepsilon} \in \{0,1\}^r$ and $\underline{u} \in R_+(\underline{\varepsilon})$ for which $\underline{\sigma} = \underline{u} + \underline{p}(\underline{\varepsilon})/2$. Recall the standard quadratic map $q^{\underline{\varepsilon}}_{\mathcal{V}} : W^{\underline{\varepsilon}}_{\mathcal{V}} \to \mathcal{Z}_{\mathcal{V}}$ and the subset $W^{\underline{\varepsilon},+}_{\mathcal{V}} \subset W^{\underline{\varepsilon}}_{\mathcal{V}}$ introduced in Section 3.4. As noted in (3.15), each element $x \in W^{\underline{\varepsilon},+}_{\mathcal{V}}$ is identified with a lower triangular matrix T_x for which $q^{\underline{\varepsilon}}_{\mathcal{V}}(x) = T_x\,{}^{\mathrm{t}}T_x$. Thus, the $W^{\underline{\varepsilon},+}_{\mathcal{V}}$-valued random variable $X^{\underline{u}}$ in the following theorem can be regarded as a triangular random matrix, similarly to the Bartlett decomposition of the classical Wishart laws.

THEOREM 3.15. *Let $\underline{\sigma} = \underline{u} + \underline{p}(\underline{\varepsilon})/2$ and $X^{\underline{u}}$ be an $W^{\underline{\varepsilon},+}_{\mathcal{V}}$-valued random variable whose components are independent and satisfy $(X^{\underline{u}}_{ii})^2 \sim \chi^2(2u_i)$ and $X^{\underline{u}}_{li} \sim N(0, I_{n_{li}})$ for $i \in I(\underline{\varepsilon})$ and $l > i$.*

- (i) *The Wishart law $\gamma_{q^{\underline{s}}_{\mathcal{V}},I_N}$ is the law of $Y = q^{\underline{\varepsilon}}_{\mathcal{V}}(X^{\underline{u}})/2$ and is supported by $\mathcal{O}_{\underline{\varepsilon}}$.*
- (ii) *For $\theta = \rho^*(T)I_N \in \mathcal{P}^*_{\mathcal{V}}$ with $T \in H_{\mathcal{V}}$, the Wishart law $\gamma_{q^{\underline{s}}_{\mathcal{V}},\theta}$ is the law of $Y' = \rho(T)^{-1} \circ q^{\underline{\varepsilon}}_{\mathcal{V}}(X^{\underline{u}})/2$ and is supported by $\mathcal{O}_{\underline{\varepsilon}}$.*

PROOF. For a measurable function f on $\mathcal{Z}_{\mathcal{V}}$, we see from (3.30) and Proposition 3.10 that

$$\int_{\mathcal{Z}_{\mathcal{V}}} f(y)\gamma_{q^{\underline{s}}_{\mathcal{V}},I_N}(dy) = \int_{\mathcal{Z}_{\mathcal{V}}} f(y)e^{-\langle y,I_N\rangle}\mathcal{R}_{\underline{\sigma}}(dy) = \int_{W^{\underline{\varepsilon},+}_{\mathcal{V}}} f(q^{\underline{\varepsilon}}_{\mathcal{V}}(x))e^{-\langle q^{\underline{\varepsilon}}_{\mathcal{V}}(x),I_N\rangle}\,\mathcal{M}^{\underline{\varepsilon}}_{\underline{u}}(dx).$$

Since $\langle q^{\underline{\varepsilon}}_{\mathcal{V}}(x), I_N\rangle = \|x\|^2$, by the change of variable of x by $x/\sqrt{2}$, we rewrite the last term as

$$\int_{W^{\underline{\varepsilon},+}_{\mathcal{V}}} f(q^{\underline{\varepsilon}}_{\mathcal{V}}(x)/2)e^{-\|x\|^2/2}\,\mathcal{M}^{\underline{\varepsilon}}_{\underline{u}}(dx/\sqrt{2}\,).$$

Keeping Remark 3.8 in mind, we see that the law of the random variable $X^{\underline{u}}$ is $e^{-\|x\|^2/2}\,\mathcal{M}^{\underline{\varepsilon}}_{\underline{u}}(dx/\sqrt{2}\,)$. Hence (i) holds. To show (ii), it suffices to check that $\gamma_{q^{\underline{s}}_{\mathcal{V}},\theta}$ is the image of $\gamma_{q^{\underline{s}}_{\mathcal{V}},I_N}$ by $\rho(T)^{-1}$. Since $\gamma_{q^{\underline{s}}_{\mathcal{V}},I_N}(dy) = e^{-\langle y,I_N\rangle}\mathcal{R}_{\underline{\sigma}}(dy)$, we have

$$\gamma_{q^{\underline{s}}_{\mathcal{V}},I_N}(\rho(T)\,dy) = e^{-\langle \rho(T)y,I_N\rangle}\mathcal{R}_{\underline{\sigma}}(\rho(T)\,dy) = e^{-\langle y,\theta\rangle}\chi_{\underline{\sigma}}(T)\mathcal{R}_{\underline{\sigma}}(dy)$$

by (3.25). Therefore (3.30) together with (3.22) leads us to the assertion (ii). $\square$

Now we consider the Wishart laws $\gamma_{q^{\oplus s},\theta}$, where q is a general homogeneous $\mathcal{P}_{\mathcal{V}}$-positive quadratic map. First we show a refinement of the first part of Proposition 3.14.

LEMMA 3.16. *Let $q : \boldsymbol{R}^m \to \mathcal{Z}_{\mathcal{V}}$ be a homogeneous $\mathcal{P}_{\mathcal{V}}$-positive quadratic map, and θ an element of $\mathcal{P}^*_{\mathcal{V}}$. Then there exist $g_0 \in G(\mathcal{P}_{\mathcal{V}})$, $\underline{m} \in \boldsymbol{Z}^r$ and $C > 0$ for which*

$$\det \phi_q((g_0^{-1})^*\eta) = C\Delta^*_{\underline{m}^*}(\eta) \quad (\eta \in \mathcal{P}^*_{\mathcal{V}}) \tag{3.31}$$

and $g_0^*\theta = I_N$.

Proof. It is shown in the proof of Proposition 3.14 that there exists $a_0 \in G(\mathcal{P}_{\mathcal{V}})$ for which $a_0\rho(H_{\mathcal{V}})a_0^{-1} \subset pr_1(\mathrm{Aut}(\mathcal{P}_{\mathcal{V}}, q))$. Since $a_0^*\theta \in \mathcal{P}^*_{\mathcal{V}}$, we take $T_0 \in H_{\mathcal{V}}$ for which $\rho^*(T_0)I_N = a_0^*\theta$. Put $g_0 := a_0\rho(T_0)^{-1} \in G(\mathcal{P}_{\mathcal{V}})$. Then $g_0\rho(H_{\mathcal{V}})g_0^{-1} \subset pr_1(\mathrm{Aut}(\mathcal{P}_{\mathcal{V}}, q))$ and $g_0^*\theta = \rho^*(T_0)^{-1}a_0^*\theta = I_N$. Similarly to Proposition 3.14, we see that g_0 together with an appropriate $\underline{m}$ and $C > 0$ satisfies the required properties. □

Assume that there exists the Riesz measure $\mu_{q^{\oplus s}}$ associated to a virtual quadratic map $q^{\oplus s}$, and that $\mu_{q^{\oplus s}}$ is not the Dirac measure. Then we can take non-zero $\underline{\varepsilon} \in \{0,1\}^r$ and $\underline{u} \in R_+(\underline{\varepsilon})$ such that $s\underline{m}/2 = \underline{u} + \underline{p}(\underline{\varepsilon})/2$ as in Theorem 3.13. Using these data together with g_0 in Lemma 3.16, we obtain the Bartlett decomposition of the Wishart law $\gamma_{q^{\oplus s},\theta}$.

Theorem 3.17. *Let* $s\underline{m}/2 = \underline{u} + \underline{p}(\underline{\varepsilon})/2$ *and* $X^{\underline{u}}$ *be the* $W^{\underline{\varepsilon},+}_{\mathcal{V}}$*-valued random variable in Theorem* 3.15. *Then the Wishart law* $\gamma_{q^{\oplus s},\theta}$ *is the law of* $Y = g_0 \circ q^{\underline{\varepsilon}}_{\mathcal{V}}(X^{\underline{u}})/2$.

Proof. Put $q' := g_0^{-1} \circ q$. As is seen in the proof of Proposition 3.14, the Riesz measure $\mu_{(q')^{\oplus s}}$ equals $C^{-s/2}\pi^{sm/2}\mathcal{R}_{s\underline{m}/2}$. Thus, similarly to the proof of Theorem 3.15 (i), we see that $\gamma_{(q')^{\oplus s},I_N}(dy) = e^{-\langle y, I_N\rangle}\mathcal{R}_{s\underline{m}/2}(dy)$, and that $\gamma_{(q')^{\oplus s},I_N}$ is the law of $Y = q^{\underline{\varepsilon}}_{\mathcal{V}}(X^{\underline{u}})/2$. Since $q^{\oplus s} = g_0 \circ (q')^{\oplus s}$, Theorem 3.17 follows from Proposition 2.16. □

We have seen that Riesz measures and Wishart laws associated to a homogeneous quadratic map are obtained (up to linear transforms as in Proposition 3.14 and Theorem 3.17) as the ones associated to a virtual quadratic map $q^{\underline{s}}_{\mathcal{V}} = (q^1_{\mathcal{V}})^{\oplus s_1} \oplus \cdots \oplus (q^r_{\mathcal{V}})^{\oplus s_r}$, that is, a virtual sum of basic quadratic maps. However, it does not mean that every homogeneous quadratic map is equal to a direct sum of basic quadratic maps. The structure of homogeneous quadratic maps is more rich than the maps generated by basic quadratic maps. Let us study the following example.

Example 5. Let $\mathrm{Herm}(2, \boldsymbol{C})$ be the vector space of Hermitian matrices of size 2, and $\Omega \subset \mathrm{Herm}(2, \boldsymbol{C})$ the subset of positive definite matrices. Then we see that

$$\Omega = \left\{ \begin{pmatrix} y_1 & y_3 + iy_4 \\ y_3 - iy_4 & y_2 \end{pmatrix} ;\, y_1 > 0,\ y_1y_2 - (y_3)^2 - (y_4)^2 > 0 \right\},$$

so that Ω is the 4-dimensional Lorentz cone. Recalling (3.7), we have the linear isomorphism

$$\iota : \mathrm{Herm}(2, \boldsymbol{C}) \ni \begin{pmatrix} y_1 & y_3 - iy_4 \\ y_3 + iy_4 & y_2 \end{pmatrix} \mapsto \begin{pmatrix} y_1 & 0 & y_3 \\ 0 & y_1 & y_4 \\ y_3 & y_4 & y_2 \end{pmatrix} \in \mathcal{Z}_{\mathcal{V}}$$

which gives a matrix realization of Ω. Let us consider the quadratic map $\tilde{q} : \boldsymbol{C}^2 \ni z \mapsto$

$z\,{}^{\mathrm{t}}\bar{z} \in \mathrm{Herm}(2, \boldsymbol{C})$, which is clearly Ω-positive. We have a group homomorphism

$$\mathrm{GL}(2, \boldsymbol{C}) \ni A \mapsto (\tilde{\rho}(A), A) \in \mathrm{Aut}(\Omega, \tilde{q}),$$

where $\tilde{\rho}(A) \in GL(\mathrm{Herm}(2, \boldsymbol{C}))$ is defined by $\tilde{\rho}(A)(Z) := AZ\,{}^{\mathrm{t}}\bar{A}$ ($Z \in \mathrm{Herm}(2, \boldsymbol{C})$). Since $\tilde{\rho}(GL(2, \boldsymbol{C}))$ acts on Ω transitively, the quadratic map $\tilde{q}$ is homogeneous. Keeping the natural isomorphism $\boldsymbol{C}^2 \simeq \boldsymbol{R}^4$ in mind, we define the quadratic map $q : \boldsymbol{R}^4 \to \mathcal{Z}_{\mathcal{V}}$ by

$$\begin{aligned} q(x) &:= \iota \circ \tilde{q} \begin{pmatrix} x_1 + ix_2 \\ x_3 + ix_4 \end{pmatrix} \\ &= \iota \begin{pmatrix} (x_1)^2 + (x_2)^2 & (x_1x_3 + x_2x_4) - i(x_1x_4 - x_2x_3) \\ (x_1x_3 + x_2x_4) + i(x_1x_4 - x_2x_3) & (x_3)^2 + (x_4)^2 \end{pmatrix} \\ &= \begin{pmatrix} (x_1)^2 + (x_2)^2 & 0 & x_1x_3 + x_2x_4 \\ 0 & (x_1)^2 + (x_2)^2 & x_1x_4 - x_2x_3 \\ x_1x_3 + x_2x_4 & x_1x_4 - x_2x_3 & (x_3)^2 + (x_4)^2 \end{pmatrix}. \end{aligned}$$

Then we have

$$\phi_q(\eta) = \begin{pmatrix} \eta_1 & 0 & \eta_3 & \eta_4 \\ 0 & \eta_1 & -\eta_4 & \eta_3 \\ \eta_3 & -\eta_4 & \eta_2 & 0 \\ \eta_4 & \eta_3 & 0 & \eta_2 \end{pmatrix}$$

for $\eta \in \mathcal{Z}_{\mathcal{V}}$. It is easily checked that the map $\phi_q \circ \iota : \mathrm{Herm}(2, \boldsymbol{C}) \to \mathrm{Sym}(4, \boldsymbol{R})$ is a Jordan algebra representation. For $\eta \in \Omega^*$ we have

$$L_{\mu_q}(-\eta) = \pi^2 (\det \phi_q(\eta))^{-1/2} = \pi^2 (\eta_1\eta_2 - (\eta_3)^2 - (\eta_4)^2)^{-1} \tag{3.32}$$

by Lemma 2.4. On the other hand, the basic quadratic maps $q^i_{\mathcal{V}} : W^i_{\mathcal{V}} \to \mathcal{Z}_{\mathcal{V}}$ $(i = 1, 2)$ are given by

$$q^1_{\mathcal{V}} \begin{pmatrix} x_1 & 0 \\ 0 & x_1 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} (x_1)^2 & 0 & x_1x_3 \\ 0 & (x_1)^2 & x_1x_4 \\ x_1x_3 & x_1x_4 & (x_3)^2 + (x_4)^2 \end{pmatrix}, \quad q^2_{\mathcal{V}} \begin{pmatrix} 0 \\ 0 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (x_2)^2 \end{pmatrix},$$

so that we have for $\eta \in \mathcal{Z}^*_{\mathcal{V}}$

$$\phi^1_{\mathcal{V}}(\eta) = \begin{pmatrix} \eta_1 & \eta_3 & \eta_4 \\ \eta_3 & \eta_2 & 0 \\ \eta_4 & 0 & \eta_2 \end{pmatrix}, \quad \phi^2_{\mathcal{V}}(\eta) = \eta_2.$$

Thus we obtain

$$L_{\mu_{q^1_{\mathcal{V}}}}(-\eta) = \pi^{3/2}(\eta_2)^{-1/2}(\eta_1\eta_2 - (\eta_3)^2 - (\eta_4)^2)^{-1/2}, \quad L_{\mu_{q^2_{\mathcal{V}}}}(-\eta) = \pi^{1/2}(\eta_2)^{-1/2} \tag{3.33}$$

for $\eta \in \mathcal{P}^*_{\mathcal{V}}$. Comparing (3.32) and (3.33), we see that $\mu_q = \mu_{(q^1_{\mathcal{V}})^{\oplus 2} \oplus (q^2_{\mathcal{V}})^{\oplus(-2)}}$, whereas the quadratic map q is by no means equal to the virtual quadratic map $(q^1_{\mathcal{V}})^{\oplus 2} \oplus (q^2_{\mathcal{V}})^{\oplus(-2)}$. We see also from (3.32) and (3.33) that $\mu_{q \oplus (q^2_{\mathcal{V}})^{\oplus 2}} = \mu_{(q^1_{\mathcal{V}})^{\oplus 2}}$, whereas the two (true) quadratic maps $q \oplus (q^2_{\mathcal{V}})^{\oplus 2}$ and $(q^1_{\mathcal{V}})^{\oplus 2}$ do not coincide even up to linear transforms $g_0 \in G(\Omega)$, as the domains of these maps are different.

Therefore two different quadratic maps may correspond to the same Riesz measure.

3.8. Density function for the non-singular case.

Since the orbit $\mathcal{O}_{\underline{\varepsilon}} = \rho(H)E_{\underline{\varepsilon}}$ is contained in the boundary $\partial\mathcal{P}_{\mathcal{V}}$ of the homogeneous cone $\mathcal{P}_{\mathcal{V}}$ unless $\varepsilon = (1, \dots, 1)$, the Gindikin-Riesz distribution $\mathcal{R}_{\underline{\sigma}}$ is a singular measure for $\underline{\sigma} \in \Xi(\underline{\varepsilon})$ with $\underline{\varepsilon} \neq (1, \dots, 1)$ thanks to Proposition 3.10. On the other hand, if $\underline{\sigma} \in \Xi(1, \dots, 1)$, that is,

$$\sigma_i > p_i/2 \quad (i = 1, \dots, r),$$

where $p_i := p_i(1, \dots, 1) = \sum_{l>i} n_{li}$, then the Gindikin-Riesz distribution is an absolutely continuous measure with respect to the Lebesgue measure, and the density function is given explicitly in [**7**] as follows.

Noting that the group $H_{\mathcal{V}}$ acts on $\mathcal{P}_{\mathcal{V}}$ simply transitively, we define the function $\Delta_{\underline{\sigma}} : \mathcal{P}_{\mathcal{V}} \to \boldsymbol{C}^\times$ for $\underline{\sigma} = (\sigma_1, \dots, \sigma_r) \in \boldsymbol{C}^r$ by $\Delta_{\underline{\sigma}}(\rho(T)I_N) := \chi_{\underline{\sigma}}(T)$ $(T \in H_{\mathcal{V}})$. For $y = \rho(T)I_N = T\,{}^{\mathrm{t}}T \in \mathcal{P}_{\mathcal{V}}$, we can express $\Delta_{\underline{\sigma}}(y)$ as a product of powers of principal minors of y (cf. [**5**, p. 122]). Define $\underline{d} = (d_1, \dots, d_r) \in \boldsymbol{Z}^r/2$ by $d_k := 1 + (\sum_{l>k} n_{lk} + \sum_{i<k} n_{ki})/2$. Then $\Delta_{-\underline{d}}(y)dy$ gives a $G(\mathcal{P}_{\mathcal{V}})$-invariant measure on $\mathcal{P}_{\mathcal{V}}$ ([**7**, Proposition 2.2]). Take $\underline{\sigma} \in \Xi(1, \dots, 1)$. We see from [**7**, Theorem 2.1] that the integral

$$\Gamma_{\mathcal{P}_{\mathcal{V}}}(\underline{\sigma}) := \int_{\mathcal{P}_{\mathcal{V}}} e^{-\langle y, I_N\rangle} \Delta_{\underline{\sigma}-\underline{d}}(y)\, dy$$

converges and equals $\pi^{(\dim \mathcal{Z}_{\mathcal{V}} - r)/2} \prod_{i=1}^r \Gamma(\sigma_i - p_i/2)$. By [**7**, Proposition 2.3], we see that

$$\mathcal{R}_{\underline{\sigma}}(dy) = \frac{\Delta_{\underline{\sigma}-\underline{d}}(y)}{\Gamma_{\mathcal{P}_{\mathcal{V}}}(\underline{\sigma})}\, dy \quad (y \in \mathcal{P}_{\mathcal{V}}). \tag{3.34}$$

Owing to (3.30) and (3.34), we conclude the following proposition.

PROPOSITION 3.18. *Let $q^{\underline{s}}_{\mathcal{V}} = (q^1_{\mathcal{V}})^{\oplus s_1} \oplus \cdots \oplus (q^r_{\mathcal{V}})^{\oplus s_r}$ be the virtual quadratic map such that $\underline{\sigma} = \sum_{i=1}^r s_i \underline{m}(i)/2$ belongs to $\Xi(1, \dots, 1)$, that is, $\sigma_i > p_i/2$ $(i = 1, \dots, r)$. Then one has for $\theta \in \mathcal{P}^*_{\mathcal{V}}$*

$$\gamma_{q^{\underline{s}}_{\mathcal{V}}, \theta}(dy) = \frac{e^{-\langle y, \theta\rangle} \Delta^*_{\underline{\sigma}^*}(\theta) \Delta_{\underline{\sigma}-\underline{d}}(y)}{\Gamma_{\mathcal{P}_{\mathcal{V}}}(\underline{\sigma})}\, dy \quad (y \in \mathcal{P}_{\mathcal{V}}). \tag{3.35}$$

Note that the formula (3.35) served as a definition of a Wishart law in [**1**].

Example 6. Let $\mathcal{Z}_{\mathcal{V}}$ be the space defined in (3.5). If $y = \rho(T)I_N = T^{\mathrm{t}}T \in \mathcal{P}_{\mathcal{V}}$ with $T \in H_{\mathcal{V}}$, then we see easily that

$$y_{11} = (t_{11})^2, \quad y_{11}y_{22} - (y_{21})^2 = (t_{11})^2(t_{22})^2, \quad y_{11}y_{33} - (y_{31})^2 = (t_{11})^2(t_{33})^2,$$

so that $\Delta_{\underline{\sigma}}(y) = (t_{11})^{2\sigma_1}(t_{22})^{2\sigma_2}(t_{33})^{2\sigma_3}$ equals

$$(y_{11})^{\sigma_1-\sigma_2-\sigma_3}(y_{11}y_{22} - (y_{21})^2)^{\sigma_2}(y_{11}y_{33} - (y_{31})^2)^{\sigma_3}.$$

On the other hand, we have $(p_1,\, p_2,\, p_3) = (0, 1, 1)$ and $(d_1,\, d_2,\, d_3) = (2,\, 3/2,\, 3/2)$. Thus we have by (3.34)

$$\mathcal{R}_{(\sigma_1,\sigma_2,\sigma_3)}(dy) = \frac{(y_{11})^{\sigma_1-\sigma_2-\sigma_3+1}(y_{11}y_{22} - (y_{21})^2)^{\sigma_2-3/2}(y_{11}y_{33} - (y_{31})^2)^{\sigma_3-3/2}}{\pi\Gamma(\sigma_1)\Gamma(\sigma_2 - 1/2)\Gamma(\sigma_3 - 1/2)}\, dy$$

if $\sigma_1 > 0$, $\sigma_2 > 1/2$ and $\sigma_3 > 1/2$. Let us consider the Wishart laws associated to the virtual quadratic map $(q^1_{\mathcal{V}})^{\oplus s}$, where $q^1_{\mathcal{V}} : W^1_{\mathcal{V}} \to \mathcal{Z}_{\mathcal{V}}$ is the basic quadratic map. Since $\underline{m}(1) = (1,\, 1,\, 1)$, we observe that $s\underline{m}(1)/2 \in \Xi$ if and only if $s \in \{0, 1\} \cup (1, +\infty)$. If $s = 0$, the associated Wishart law is the Dirac measure. If $s = 1$, the associated Wishart law $\gamma_{q^1_{\mathcal{V}},\theta}$ $(\theta \in \mathcal{P}^*_{\mathcal{V}})$ is described as the image of the normal law $N(0, \phi^1_{\mathcal{V}}(\theta)^{-1})$ on $W^1_{\mathcal{V}} \equiv \boldsymbol{R}^3$ by the quadratic map $q^1_{\mathcal{V}}/2$, where $\phi^1_{\mathcal{V}}(\theta)$ is given in Example 4 after Proposition 3.3.

If $s > 1$, then $s\underline{m}(1)/2 = (s/2,\, s/2,\, s/2)$ belongs to $\Xi(1, 1, 1)$. Since

$$\Delta^*_{s\underline{m}(1)^*/2}(\theta) = \det \phi^1_{\mathcal{V}}(\theta)^{s/2} = \big(\theta_{11}\theta_{22}\theta_{33} - \theta_{33}(\theta_{21})^2 - \theta_{22}(\theta_{31})^2\big)^{s/2} \quad (\theta \in \mathcal{P}^*_{\mathcal{V}}),$$

we have

$$\begin{aligned}\gamma_{(q^1_{\mathcal{V}})^{\oplus s},\theta}(dy) = {} & \frac{e^{-\langle y,\theta\rangle}(\theta_{11}\theta_{22}\theta_{33} - \theta_{33}(\theta_{21})^2 - \theta_{22}(\theta_{31})^2)^{s/2}}{\pi\Gamma(s/2)\Gamma((s-1)/2)\Gamma((s-1)/2)} \\ & \times (y_{11})^{1-s/2}(y_{11}y_{22} - (y_{21})^2)^{(s-3)/2}(y_{11}y_{33} - (y_{31})^2)^{(s-3)/2}\, dy \quad (y \in \mathcal{P}_{\mathcal{V}})\end{aligned}$$

by Proposition 3.18.

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