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On general boundary conditions for one-dimensional diffusions with symmetry

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Abstract. We give a simple proof of the symmetry of a minimal diffusion X^0 on a one-dimensional open interval I with respect to the attached canonical measure m along with the identification of the Dirichlet form of X^0 on $L^2(I;m)$ in terms of the triplet (s,m,k) attached to X^0. The L^2-generators of X^0 and its reflecting extension X^r are then readily described. We next use the associated reproducing kernels in connecting the L^2-setting to the traditional C_b-setting and thereby deduce characterizations of the domains of C_b-generators of X^0 and X^r by means of boundary conditions. We finally identify the C_b-generators for all other possible symmetric diffusion extensions of X^0 and construct by that means all diffusion extensions of X^0 in [**IM2**].

1. Introduction.

It is well known that the minimal diffusion X^0 on a one-dimensional open interval $I=(r_1,r_2)$ can be described in terms of a triplet (s,m,k) of a canonical scale s, a canonical measure m and a killing measure k, and furthermore all possible diffusion extensions of X^0 to the closed interval $[r_1,r_2]$ can be characterized by means of general boundary conditions imposed at both boundaries r_1 and r_2 ([**IM2**]). The description and characterization were formulated in the framework of the space C_b of all bounded (finely) continuous functions.

But it was shown only quite recently in [**F2**] that the minimal diffusion process X^0 with no killing inside ($k=0$) is symmetric with respect to the attached canonical measure m and further the Dirichlet form of X^0 on $L^2(I;m)$ is identified in terms of m and the attached canonical scale s. Itô's construction of the symmetric resolvent density of X^0 in [**I1**] was utilized in the proof. Afterward the stated results have been extended in [**CF**, Section 5.3] to a general minimal diffusion admitting killings inside by using a method of resurrection and killing.

In Section 2.2 of the present paper, we give a more direct and simpler proof of the above mentioned results in [**CF**] by showing the m-symmetry of part processes of X^0 on each relatively compact subintervals of I and calculating corresponding symmetric forms just via integrations by parts. Based on them, we aim at characterizing all possible symmetric diffusion extensions of X^0 in terms of boundary conditions both in L^2-setting

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and C_b-one.

As an immediate consequence of the identification of the Dirichlet form of X^0 on $L^2(I;m)$ with the form $(\mathcal{E}^0,\mathcal{F}^0)$ introduced in Section 2.1, we can identify the L^2-generator of X^0 in Section 2.3. Section 2.4 will deal with the reflecting extension X^r of X^0. Denote by I^* the interval obtained from I by adding its regular boundaries and extend m to I^* by setting $m(I^*\setminus I)=0$. X^r is the m-symmetric diffusion on I^* extending X^0 whose Dirichlet form on $L^2(I^*;m)=L^2(I;m)$ is $(\mathcal{E}^r,\mathcal{F}^r)$ introduced in Section 2.1, which is actually the active reflected Dirichlet space of $(\mathcal{E}^0,\mathcal{F}^0)$ in the sense of [**CF**, Chapter 6]. We shall give the identification of the L^2-generator of X^r in two ways: simply using an integration by parts and alternatively using its general characterization in terms of the zero flux condition formulated in [**CF**, Section 7.3].

We then make use of the associated reproducing kernels in connecting the L^2-setting to the traditional C_b-setting and thereby deduce in Section 3 and Section 4 characterzations of the domains of C_b-generators of X^0 and X^r, respectively, by means of boundary conditions. As compared to their L^2-counterparts, they involve an extra boundary condition that the function in the domain of the generator should vanish at each exit boundary. This extra condition is implicit but hidden in an integrability condition for the descriptions of L^2-generators.

In Section 5, we deal with symmetric diffusion extensions of X^0 other than X^r. X^r is known to be a unique m-symmetric diffusion extension of X^0 from I to I^* that admits no sojourn nor killing at points of $I^*\setminus I$ ([**CF**, Section 7.7]). When both r_1 and r_2 are regular, there are three one-point extensions of X^0 admitting no sojourn nor killing on the boundary; the diffusion reflected at r_1 (resp. r_2) and absorbed at r_2 (resp. r_1) and the extension of X^0 to the one-point-compactification of I. On the other hand, to admit sojourn or killing at boundary points just amounts to extending m or k to the boundary points by allowing it to have positive masses there.

It turns out that those extensions mentioned above exhaust all possible symmetric diffusion extensions of X^0 obtained by adding regular boundary points or their identification to I. All of them are irreducible. We shall exhibit their generators of both kinds in terms of boundary conditions along with their constructions using Dirichlet forms.

Notice that the class of extensions of X^0 we deal with from Section 3 to Section 5 is different from the one in [**IM2**, Section 4.1, Section 4.7] where all possible diffusion extensions from I to $[r_1,r_2]$ were investigated. The latter includes the cases of non-regular boundaries while our class collects only extensions of X^0 to regular boundaries or their identification. However the condition at a regular boundary we shall derive in Theorem 5.5 coincides with the original one in [**IM2**]. We shall verify in Section 6 that actually all the diffusion extensions of X^0 considered in [**IM2**] are, except for trivial ones, symmetrizable and furthermore of Feller transition functions on appropriate state spaces containing entrance boundaries.

Extensions of X^0 involving jumps from boundaries into I have been intensively studied ([**Fe1**], [**Fe2**], [**IM1**], [**I2**]). They are not symmetrizable because they are non-reversible in time. Extensions of X^0 with symmetric jumps among boundaries were considered in [**Fe2**], [**F1**].

2. Minimal diffusion and its reflecting extension.

Let $I = (r_1, r_2) \subset \mathbb{R}$ be a one-dimensional open interval. A strictly increasing continuous function s on I is called a *canonical scale.* A positive Radon measure m on I with full topological support is called a *canonical measure.* We shall work with a triplet (s, m, k) where s is a canonical scale, m is a canonical measure and k is a positive Radon measure on I.

2.1. Two regular Dirichlet forms for a triplet (s, m, k).

Define the space $(\mathcal{F}^{(s)}, \mathcal{E}^{(s)})$ by

$$\mathcal{F}^{(s)} = \{u : u \text{ is absolutely continuous in } s \text{ and } \mathcal{E}^{(s)}(u,u) < \infty\}, \tag{2.1}$$

$$\mathcal{E}^{(s)}(u,v) = \int_I D_s u(x) D_s v(x)\ ds(x). \tag{2.2}$$

From the elementary identity $u(b) - u(a) = \int_a^b D_s u(x) ds(x)$, $a, b \in I$, we get

$$(u(b) - u(a))^2 \le |s(b) - s(a)|\ \mathcal{E}^{(s)}(u,u), \quad a, b \in I, \quad u \in \mathcal{F}^{(s)}. \tag{2.3}$$

(2.3) implies that, if $\{u_n\} \subset \mathcal{F}^{(s)}$ is $\mathcal{E}^{(s)}$-Cauchy and convergent at one point $a \in I$, then it is convergent to a function of $\mathcal{F}^{(s)}$ uniformly on each compact subinterval of I.

We call the boundary r_i *approachable* if $|s(r_i)| < \infty$, $i = 1, 2$. If r_i is approachable, then any $u \in \mathcal{F}^{(s)}$ admits a finite limit $u(r_i)$ by (2.3). Let us introduce the space

$$\mathcal{F}_0^{(s)} = \{u \in \mathcal{F}^{(s)} : u(r_i) = 0 \text{ whenever } r_i \text{ is approachable}\}. \tag{2.4}$$

We further write $(u,v)_k = \int_I uvdk,\ (u,v) = \int_I uvdm$, and let

$$\begin{cases} \mathcal{F}^{(s),k} = \mathcal{F}^{(s)} \cap L^2(I;k), & \mathcal{F}_0^{(s),k} = \mathcal{F}_0^{(s)} \cap L^2(I;k), \\ \mathcal{E}^{(s),k}(u,v) = \mathcal{E}^{(s)}(u,v) + (u,v)_k, & u, v \in \mathcal{F}^{(s),k}, \\ \mathcal{E}_\alpha^{(s),k}(u,v) = \mathcal{E}^{(s),k}(u,v) + \alpha(u,v), & \alpha > 0,\ u, v \in \mathcal{F}^{(s),k} \cap L^2(I;m). \end{cases} \tag{2.5}$$

We will be concerned with two forms $(\mathcal{E}^r, \mathcal{F}^r)$ and $(\mathcal{E}^0, \mathcal{F}^0)$ defined respectively by

$$\mathcal{F}^r = \mathcal{F}^{(s),k} \cap L^2(I;m), \quad \mathcal{E}^r(u,v) = \mathcal{E}^{(s),k}(u,v),\ u, v \in \mathcal{F}^r, \tag{2.6}$$

$$\mathcal{F}^0 = \mathcal{F}_0^{(s),k} \cap L^2(I;m), \quad \mathcal{E}^0(u,v) = \mathcal{E}^{(s),k}(u,v),\ u, v \in \mathcal{F}^0. \tag{2.7}$$

The assertions of the next lemma are proven in [**CF**, Section 2.2.3] in the case that $k = 0$ and the same proof is valid in the present general case.

LEMMA 2.1. (i) *If $\{u_n\} \subset \mathcal{F}^{(s),k}$ is $\mathcal{E}^{(s),k}$-Cauchy and convergent to a function u m-a.e. as $n \to \infty$, then $u \in \mathcal{F}^{(s),k}$ and* $\lim_{n\to\infty} \mathcal{E}^{(s),k}(u_n - u, u_n - u) = 0$.

(ii) *The form* $(\mathcal{E}^r, \mathcal{F}^r)$ *defined by* (2.6) *is a Dirichlet form on* $L^2(I;m)$.

(iii) *Consider the contractive real functions* $\varphi_\ell(t) = t - (-1/\ell) \vee t \wedge (1/\ell)$, $t \in \mathbb{R}$, $\ell \in \mathbb{N}$. *For any* $u \in \mathcal{F}^{(s),k}$, *the Cesàro mean sequence* $\{u_n\}$ *of a certain subsequence of* $\{\varphi_\ell(u)\}$ *is* $\mathcal{E}^{(s),k}$*-convergent to* u.

For a given triplet (s,m,k), we write

$$j = m + k. \tag{2.8}$$

r_i is called *regular* if r_i is approachable and the measure j is finite in a neighborhood of r_i. If r_i is approachable, then any $u \in \mathcal{F}^r = \mathcal{F}^{(s),k} \cap L^2(I;m)$ has a finite limit $u(r_i)$ which must vanish whenever r_i is non-regular. Therefore the space defined by (2.7) can be rewritten as

$$\mathcal{F}^0 = \{u \in \mathcal{F}^r : u(r_i) = 0, \text{ whenever } r_i \text{ is regular}\}. \tag{2.9}$$

Let I^* be the interval obtained from I by adding its boundary r_i to I whenever r_i is regular. m is extended to I^* by setting $m(I^* \setminus I) = 0$ so that $L^2(I^*;m) = L^2(I;m)$. For $u \in \mathcal{F}^r$, the inequality (2.3) is valid for any $a, b \in I^*$ and further, for any closed interval $K = [\alpha, \beta] \subset I^*$,

$$\sup_{x \in K} u(x)^2 \le C_K\, \mathcal{E}_1^r(u,u), \quad u \in \mathcal{F}^r \tag{2.10}$$

for some constant $C_K > 0$. In fact, we get from (2.3)

$$\sup_{\alpha \le y \le \beta} u(y)^2 \le 2(s(\beta) - s(\alpha))\mathcal{E}^{(s)}(u,u) + 2u(x)^2, \quad \alpha \le x \le \beta.$$

Integrating the both hand sides by dj on $[\alpha, \beta]$, we obtain (2.10). It follows from (2.10) that $\mathcal{F}^r$ is a subspace of $C(I^*)$ the space of all continuous functions on I^*.

$C_c(I)$ (resp. $C_c(I^*)$) will denote the space of continuous functions on I (resp. I^*) with compact support. $\sqrt{\mathcal{E}^{(s),k}(u,u)}$ will be designated by $\|u\|_{\mathcal{E}^{(s),k}}$ occasionally. We refer to [**CF**, Chapter 6] for the definition of an active reflected Dirichlet space and a Silverstein extension of $(\mathcal{E}^0, \mathcal{F}^0)$.

Theorem 2.2. (i) $(\mathcal{E}^r, \mathcal{F}^r)$ *is a regular, local, and irreducible Dirichlet form on* $L^2(I^*;m)$. *Each one point of* I^* *has a positive capacity relative to* $(\mathcal{E}^r, \mathcal{F}^r)$.
Let $\mathcal{F}^r_e$ *be the extended Dirichlet space of* $(\mathcal{E}^r, \mathcal{F}^r)$. *Then*

$$\mathcal{F}^r_e = \{u \in \mathcal{F}^{(s),k} : u(r_i) = 0 \textit{ whenever } r_i \textit{ is approachable but non-regular}\}. \tag{2.11}$$

(ii) $(\mathcal{E}^0, \mathcal{F}^0)$ *is a regular, local, and irreducible Dirichlet form on* $L^2(I;m)$. *Each one point of* I *has a positive capacity relative to* $(\mathcal{E}^0, \mathcal{F}^0)$.
Let $(\mathcal{F}^0_e, \mathcal{E}^0)$ *be its extended Dirichlet space. Then*

$$\mathcal{F}_e^0 = \mathcal{F}_0^{(s),k}, \qquad \mathcal{E}^0 = \mathcal{E}^{(s),k}. \tag{2.12}$$

(iii) $(\mathcal{E}^0, \mathcal{F}^0)$ *is the part of* $(\mathcal{E}^r, \mathcal{F}^r)$ *on* I.
(iv) $(\mathcal{E}^r, \mathcal{F}^r)$ *is the active reflected Dirichlet space of* $(\mathcal{E}^0, \mathcal{F}^0)$.
(v) *The Silverstein extension of* $(\mathcal{E}^0, \mathcal{F}^0)$ *is unique if and only if no regular boundary is present.*

Proof. It suffices to prove (i). Indeed, under (i), (iii) is just a restatement of (2.9). (ii) then follows either from a general theorem [**CF**, Theorems 3.3.9, 3.4.9] or from a repetition of the proof of (i) for $(\mathcal{E}^0, \mathcal{F}^0)$ in place of $(\mathcal{E}^r, \mathcal{F}^r)$. (iv) is shown in [**CF**, Section 6.5, (**3**°)] in the case that $k = 0$. (v) follows from (iv) and (2.9).

We proceed to a proof of (i). $(\mathcal{E}^r, \mathcal{F}^r)$ is a Dirichlet form on $L^2(I; m)$ by Lemma 2.1 (ii). Obviously it is local. Suppose a Borel set $A \subset I^*$ is invariant with respect to the L^2-semigroup associated with $(\mathcal{E}^r, \mathcal{F}^r)$. For any compact interval $K \subset I^*$, there exists a function $v \in \mathcal{F}^r$ with $v = 1$ on K. Since $u = v \cdot 1_A$ is an element of $\mathcal{F}^r \subset C(I^*)$ taking values only 0 or 1 on K, $\{x \in K : u(x) = 1\}$ is a closed and open subset of K, and consequently either $A \cap K$ or $A^c \cap K$ is m-negligible. By letting $K \uparrow I^*$, we get the irreducibility of $(\mathcal{E}^r, \mathcal{F}^r)$. Each one point of I^* has a positive capacity relative to $(\mathcal{E}^r, \mathcal{F}^r)$ owing to (2.10).

Denote by $\widehat{\mathcal{F}}$ the space appearing on the right hand side of (2.11). Then $\mathcal{F}^r \subset \widehat{\mathcal{F}}$ on account of the observation made right after (2.8) and accordingly, $\mathcal{F}_e^r \subset \widehat{\mathcal{F}}$ owing to Lemma 2.1 (i) and the inequality (2.3) holding for $a, b \in I^*$.

To prove the converse inclusion, take any $u \in \widehat{\mathcal{F}}$. We may assume without loss of generality that u is bounded, that is, $|u| \le M$ for some constant M.

We consider a sequence of functions $\psi_n \in C_c^1(\mathbb{R}_+)$ such that

$$\begin{cases} \psi_n(x) = 1 \text{ for } 0 \le x < n; & \psi_n(x) = 0 \text{ for } x > 2n+1; \\ |\psi_n'(x)| \le \frac{1}{n},\ n \le x \le 2n+1; & 0 \le \psi_n(x) \le 1,\ x \in \mathbb{R}_+. \end{cases}$$

Put $w_n(x) = u_n(x) \cdot \psi_n(|s(x)|)$ for $x \in I$, where u_n, $n \ge 1$, are the functions constructed in Lemma 2.1 (iii) for u. Then, $w_n \in \mathcal{F}^{(s),k} \cap C_c(I^*)$ because u_n vanishes on a neighborhood of r_i whenever r_i is approachable but non-regular, while so does $\psi_n(|s(x)|)$ whenever r_i is non-approachable. Further, since $u(x) - w_n(x) = u(x)(1 - \psi_n(|s(x)|)) + (u(x) - u_n(x))\psi_n(|s(x)|)$ and $|u(x) - u_n(x)| \le |u(x)|$,

$$\begin{aligned} \|u - w_n\|^2_{\mathcal{E}^{(s),k}} \le 4 &\int_I (D_s u(x))^2 (1 - \psi_n(|s(x)|)^2 ds(x) \\ &+ 8 \int_I u(x)^2 (\psi_n'(|s(x)|)^2 ds(x) + 4 \int_I (D_s u(x) - D_s u_n(x))^2 ds(x) \\ &+ 2 \int_I u(x)^2 (1 - \psi_n(|s(x)|))^2 dk(x) + 2 \int_I (u(x) - u_n(x))^2 dk(x) \end{aligned}$$

$$\leq 4\int_{|s(x)|\geq n}(D_s u(x))^2 ds(x) + 8M^2\int_{n\leq |s(x)|<2n+1}(\psi_n'(|s(x)|)^2 ds(x)$$
$$+ 2\int_{|s(x)|\geq n} u(x)^2 dk(x) + 4\|u-u_n\|^2_{\mathcal{E}^{(s)},k}$$
$$\leq 4\int_{|s(x)|\geq n}(D_s u(x))^2 ds(x) + 16M^2\frac{n+1}{n^2} + \int_{|s(x)|\geq n} u(x)^2 dk(x)$$
$$+ 4\|u-u_n\|^2_{\mathcal{E}^{(s)},k} \quad \longrightarrow 0, \qquad n\to\infty.$$

This shows that $\{w_n\}\subset \mathcal{F}^r\cap C_c(I^*)$ is $\mathcal{E}^{(s),k}$-Cauchy. Since w_n converges to u pointwise, we get $u\in\mathcal{F}^r_e$.

For any bounded $u\in\mathcal{F}^r$, the same functions $\{w_n, n\geq 1\}$ as above are in $\mathcal{F}^r\cap C_c(I^*)$ and $\mathcal{E}^r_1$-convergent to u as $n\to\infty$. Obviously $\mathcal{F}^r\cap C_c(I^*)$ is uniformly dense in $C_c(I^*)$. Thus $(\mathcal{E}^r,\mathcal{F}^r)$ is regular. □

In what follows, we adopt Feller's classification of the boundary for a given triplet (s,m,k): for $r_1<c<r_2$

$$\lambda_1=\int_{r_1}^c s(dx)\int_x^c j(dy),\quad \mu_1=\int_{r_1}^c j(dx)\int_x^c s(dy),\ r_1<c<r_2.$$

The left boundary r_1 of I is called

regular	if	$\lambda_1<\infty$,	$\mu_1<\infty$,
exit	if	$\lambda_1<\infty$,	$\mu_1=\infty$,
entrance	if	$\lambda_1=\infty$,	$\mu_1<\infty$,
natural	if	$\lambda_1=\infty$,	$\mu_1=\infty$.

An analogous classification of r_2 is in force. Notice that these names of the boundaries are slightly different from [**IM2**] but analogous to [**Fe1**], [**I1**].

It is easy to see that r_i is regular in Feller's sense if and only if it is regular in the previous sense, namely, it is approachable and j is finite in a neighborhood of r_i. Moreover, if r_i is exit, then it is approachable but non-regular, and in particular, $u(r_i)=0$ for any $u\in\mathcal{F}^r_e$ in view of (2.11).

2.2. m-symmetry and the Dirichlet form of a minimal diffusion X^0.

A Markov process $X^0=(X^0_t,\zeta^0,\boldsymbol{P}^0_x)$ on I is called a *minimal diffusion* if

(d.1) X^0 is a Hunt process on I,
(d.2) X^0 is a diffusion process: X^0_t is continuous in $t\in(0,\zeta^0)$ almost surely,
(d.3) X^0 is irreducible: $\boldsymbol{P}^0_x(\sigma_y<\infty)>0$ for any $x,y\in I$,
where $\sigma_y=\inf\{t>0: X^0_t=y\}$, $\inf\emptyset=\infty$.

Under (d.1) and (d.2), the condition (d.3) is equivalent to the requirement for each point $a\in I$ to be *regular* in the sense that, for $\alpha>0$, $\boldsymbol{E}_a[e^{-\alpha\sigma_{a+}}]=\boldsymbol{E}_a[e^{-\alpha\sigma_{a-}}]=1$

where $E_a[e^{-\alpha\sigma_{a\pm}}] = \lim_{b\to\pm a} E_a[e^{-\alpha\sigma_b}]$, ([**IM2**, Section 3.9]).

Denote by $\{R^0_\alpha; \alpha > 0\}$ the resolvent of a minimal diffusion X^0 and by $C_b(I)$ (resp. $\mathcal{B}_b(I)$) the space of all continuous (resp. Borel measurable) bounded functions on I. Then $R^0_\alpha(\mathcal{B}_b(I)) \subset C_b(I)$ due to the above regularity of each point of I ([**IM2**, Section 3.6]) and R^0_α is a one-to-one map from $C_b(I)$ into itself. Thus the generator $\mathcal{G}^0$ of X^0 is well defined by

$$\begin{cases} \mathcal{D}(\mathcal{G}^0) = R^0_\alpha(C_b(I)), \\ (\mathcal{G}^0 u)(x) = \alpha u(x) - f(x) \quad \text{for } u = R^0_\alpha f,\ f \in C_b(I),\ x \in I, \end{cases} \tag{2.13}$$

$\mathcal{G}^0$ so defined is independent of $\alpha > 0$ by the resolvent equation. Let us call $\mathcal{G}^0$ the C_b-*generator* of X^0. For X^0, the fine continuity is equivalent to the ordinary continuity so that $C_b(I)$ is the space of all bounded finely continuous functions on I. With this interpretation, the above definition of the C_b-generator is well formulated for a general Borel right process.

By Section 4.3 and Section 4.4 of [**IM2**], there exist, for a given minimal diffusion X^0, a canonical scale s, a canonical measure m and a positive Radon measure k called a *killing measure* on I such that

$$(\mathcal{G}^0 u)(x) = \frac{dD_s u - u dk}{dm}(x) \qquad x \in I, \quad \text{for any } u \in \mathcal{D}(\mathcal{G}^0), \tag{2.14}$$

in the sense that the Radon Nikodym derivative appearing on the right hand side has a version belonging to $C_b(I)$ which coincides with the left hand side. In particular, we have for $u = R^0_\alpha f$, $f \in C_b(I)$,

$$\alpha u(x) - \frac{dD_s u - u dk}{dm}(x) = f(x) \qquad x \in I. \tag{2.15}$$

The triplet (s, m, k) is unique up to a multiplicative constant in the sense that, for another such triplet $(\widetilde{s}, \widetilde{m}, \widetilde{k})$, there exists a constant $c > 0$ such that $d\widetilde{s} = c ds$, $d\widetilde{m} = c^{-1} dm$ and $d\widetilde{k} = c^{-1} dk$.

We call a triplet (s, m, k) satisfying (2.14) to be *attached to* the minimal diffusion X^0.

Although the generator $\mathcal{G}^0$ admits an explicit expression (2.14) on $\mathcal{D}(\mathcal{G}^0)$, it is not easy even to figure out how its domain $\mathcal{D}(\mathcal{G}^0) \subset C_b(I)$ looks like explicitly. We first make a detour by determining the L^2-generator of X^0.

Let $J = (j_1, j_2)$ with $r_1 < j_1 < j_2 < r_2$ and $\{R^J_\alpha,\ \alpha > 0\}$ be the resolvent kernel of the part process X^J of X^0 on J. The next key lemma is a counterpart of [**I1**, Theorem 5.9.2] and the proof is taken from [**CF**, Lemma 5.3.2] and [**F2**, Lemma 2.1].

LEMMA 2.3. *Let $u = R^J_\alpha f$ for $f \in C_b(I)$. Then $u \in C_c(I)$, and*

$$-dD_s u + u dk + \alpha u dm = f dm \quad \text{on} \quad J, \tag{2.16}$$

for a triplet (s,m,k) attached to X^0. Moreover

$$u(j_1+)=u(j_2-)=0. \tag{2.17}$$

PROOF. Let $\varphi_i(x)=\boldsymbol{E}_x^0[e^{-\alpha\tau_J};X_{\tau_J-}=j_i]$, where τ_J denotes the first leaving time from J. It suffices to show that

$$-dD_s\varphi_i+\varphi_i dk+\alpha\varphi_i dm=0 \quad \text{on} \quad J, \tag{2.18}$$

and

$$\varphi_1(j_1+)=1, \quad \varphi_1(j_2-)=0, \quad \varphi_2(j_1+)=0, \quad \varphi_2(j_2-)=1. \tag{2.19}$$

Indeed, (2.16) and (2.17) then follow from the identity

$$u=R_\alpha^0 f-R_\alpha^0 f(j_1)\varphi_1-R_\alpha^0 f(j_2)\varphi_2, \quad \text{on} \quad J.$$

To prove (2.18), observe that, for any $g\in C_b(I)$ vanishing on J,

$$R_\alpha^0 g(x)=R_\alpha^0 g(j_1)\varphi_1(x)+R_\alpha^0 g(j_2)\varphi_2(x), \quad x\in J.$$

If $g_1\in C_b(I)$, $g_1=0$ on (r_1,j_2) and $g_1>0$ on (j_2,r_2), then $R_\alpha^0 g_1$ is strictly positive on I by (d.2), (d.3) and moreover strictly increasing on J. A similar choice of $g_2\in C_b(I)$ gives $R_\alpha^0 g_2$ strictly decreasing, and $R_\alpha^0 g_1(j_1)R_\alpha^0 g_2(j_2)-R_\alpha^0 g_2(j_1)R_\alpha^0 g_1(j_2)<0$, so that $\varphi_i(x)$ is a linear combination of $R_\alpha^0 g_1(x)$ and $R_\alpha^0 g_2(x)$ for $x\in J$. From (2.15) for g_i in place of f, we then get (2.18).

(2.19) can be shown as follows. Put $\psi(x)=E_x^0[e^{-\alpha\sigma_{j_2}}]$, $x\in I$, and take $f\in C_b(I)$ vanishing on (r_1,j_2) and strictly positive on (j_2,r_2). Then $R_\alpha^0 f\in C_b(I)$ and $R_\alpha^0 f(x)=\psi(x)R_\alpha^0 f(j_2)$, $x<j_2$, so that $R_\alpha^0 f(j_2)=\psi(j_2-)R_\alpha^0 f(j_2)$. Since $R_\alpha^0 f(j_2)>0$ as above, we have $\psi(j_2-)=1$, which in turn implies that

$$P_{j_2-}^0(\sigma_{j_2}<\epsilon)=1, \quad \text{for any } \epsilon>0,$$

because $\psi(x)\le P_x^0(\sigma_{j_2}<\epsilon)+e^{-\alpha\epsilon}(1-P_x^0(\sigma_{j_2}<\epsilon))$, $x\in I$.

Now, for any $\epsilon>0$ and $x\in J$,

$$\begin{aligned}\varphi_2(x)&\ge E_x^0[e^{-\alpha\sigma_{j_2}};\sigma_{j_2}<\epsilon,\ \sigma_{j_1}\ge\epsilon]\\&\ge E_x^0[e^{-\alpha\sigma_{j_2}};\sigma_{j_2}<\epsilon]-E_x^0[e^{-\alpha\sigma_{j_2}};\sigma_{j_1}<\epsilon]\\&\ge e^{-\alpha\epsilon}P_x^0(\sigma_{j_2}<\epsilon)-E_{x_0}^0[e^{-\alpha\sigma_{j_2}};\sigma_{j_1}<\epsilon], \quad \text{for } j_1<x_0<x<j_2.\end{aligned}$$

By letting $x\uparrow j_2$ and then $\epsilon\downarrow 0$, we obtain the last identity of (2.19). We also have

$$\varphi_1(x)\le 1-P_x^0(\sigma_{j_1}>\sigma_{j_2})\le 1-P_x^0(\sigma_{j_2}<\epsilon)+P_{x_0}^0(\sigma_{j_1}<\epsilon),$$

which leads us to the second identity of (2.19) similarly. The first and third ones can be proved analogously. □

THEOREM 2.4. (i) *Let X^0 be a minimal diffusion on I and (s, m, k) be a triplet attached to X^0. X^0 is then m-symmetric. The Dirichlet form of X^0 on $L^2(I; m)$ coincides with $(\mathcal{E}^0, \mathcal{F}^0)$ defined by (2.7) in terms of the attached triplet (s, m, k).*

(ii) *Conversely, for an arbitrarily given triplet (s, m, k), the space $(\mathcal{E}^0, \mathcal{F}^0)$ defined by (2.7) is a regular Dirichlet form on $L^2(I; m)$, and the associated Hunt process X^0 on I is a minimal diffusion possessing (s, m, k) as a triplet attached to it.*

PROOF. (i) For $J = (j_1, j_2)$ with $r_1 < j_1 < j_2 < r_2$, let $u = R^J_\alpha f$, $v = R^J_\alpha g$ for $f, g \in C_c(I)$. We then get from (2.16)

$$-\int_J v dD_s u + \int_J uv dk + \alpha \int_J uv dm = \int_J v f dm.$$

By (2.17), $v(j_1+)D_s u(j_1+) - v(j_2-)D_s u(j_2-) = 0$ so that an integration by parts gives

$$\int_J (D_s u)(D_s v) ds + \int_J uv dk + \alpha \int_J uv dm = \int_J v f dm. \tag{2.20}$$

Thus $\int_J f R^J_\alpha g dm = \int_J R^J_\alpha f g dm$, which implies the m symmetry $\int_I f R^0_\alpha g dm = \int_I R^0_\alpha f g dm$ of the resolvent of X^0 by letting $J \uparrow I$ for non-negative f, g.

Now let $(\mathcal{E}, \mathcal{F})$ be the Dirichlet form of X^0 on $L^2(I; m)$. In view of [**FOT**, Therem 4.4.5], $R^J_\alpha f \in \mathcal{F}$ for $f \in C_c(I)$ and $\mathcal{E}_\alpha(R^J_\alpha f, R^J_\alpha f) = \int_I R^J_\alpha f f dm$ for a relatively compact open interval $J \subset I$, and the union of $R^J_\alpha(C_b(I))$ over all such J is $\mathcal{E}_1$-dense in $\mathcal{F}$. On the other hand, (2.20) reads for $f \in C_c(I)$

$$R^J_\alpha f \in \mathcal{F}^0, \quad \mathcal{E}^{(s),k}_\alpha(R^J_\alpha f, R^J_\alpha f) = \int_I R^J_\alpha f f dm,$$

and therefore we have $\mathcal{F} \subset \mathcal{F}^0$ and $\mathcal{E} = \mathcal{E}^0\big|_{\mathcal{F}\times\mathcal{F}}$.

On the other hand, we obtain in the same way as (2.20)

$$\mathcal{E}^{(s),k}_\alpha(R^J_\alpha f, v) = \int_I v f dm, \quad f \in C_c(I),$$

for any $v \in \mathcal{F}^0 \cap C_c(I)$ and any relatively compact open interval J containing the support of v. This means that $\mathcal{F}^0 \cap C_c(I) \subset \mathcal{F}$ and hence $\mathcal{F}^0 \subset \mathcal{F}$ because of the regularity of the form $(\mathcal{E}^0, \mathcal{F}^0)$.

(ii) By virtue of Theorem 2.2, $(\mathcal{E}^0, \mathcal{F}^0)$ is a regular, local and irreducible Dirichlet form on $L^2(I; m)$ for which each one-point of I has a positive capacity. Therefore the associated m-symmetric Hunt process X^0 on I is a minimal diffusion on account of [**CF**, Theorem 3.5.6, Theorem 4.3.4].

Let $(\widetilde{s}, \widetilde{m}, \widetilde{k})$ be a triplet attached to X^0. By (i), X^0 is $\widetilde{m}$-symmetric so that we may assume $\widetilde{m} = m$ on account of the uniqueness of a symmetrizing measure for a strongly irreducible Markov process due to Ying-Zhao [**YZ**]. By (i) again, the Dirichlet form $(\widetilde{\mathcal{E}}^0, \widetilde{\mathcal{F}}^0)$ of X^0 on $L^2(I;m)$ is defined by (2.7) in terms of $(\widetilde{s}, \widetilde{k})$ in place of (s,k). Since $(\widetilde{\mathcal{E}}^0, \widetilde{\mathcal{F}}^0) = (\mathcal{E}^0, \mathcal{F}^0)$, we can conclude that $\widetilde{s} = s$, $\widetilde{k} = k$. □

2.3. L^2-generators of X^0 and its reflecting extension X^r.

Let X^0 be a minimal diffusion on I and (s,m,k) the triplet attached to it. Denote by $\mathcal{A}^0$ the infinitesimal generator of the strongly continuous contraction semigroup of X^0 on $L^2(I;m)$:

$$u \in \mathcal{D}(\mathcal{A}^0) \quad \text{and} \quad \mathcal{A}^0 u = f \in L^2(I;m)$$

if and only if

$$u \in \mathcal{F}^0, \quad \mathcal{E}^0(u,v) = -(f,v), \text{ for any } v \in \mathcal{F}^0 \cap C_c(I),$$

on account of the regularity of $(\mathcal{E}^0, \mathcal{F}^0)$. $\mathcal{A}^0$ is simply called the L^2-*generator* of X^0. We immediately deduce from Theorem 2.4, (2.9) and the above characterization of $\mathcal{A}^0$ the following. We write

$$u(r_i) = \lim_{x \to r_i,\, x \in I} u(x).$$

Corollary 2.5. *The L^2-generator $\mathcal{A}^0$ of the minimal diffusion X^0 on I can be described in terms of the attached triplet (s,m,k) as follows: $u \in \mathcal{D}(\mathcal{A}^0)$ if and only if*

$$\begin{cases} u \in \mathcal{F}^{(s),k} \cap L^2(I;m), \quad \dfrac{dD_s u - u dk}{dm} \in L^2(I;m), \quad \textit{and} \\ u(r_i) = 0 \quad \textit{whenever } r_i \textit{ is regular.} \end{cases} \tag{2.21}$$

In this case,

$$\mathcal{A}^0 u = \frac{dD_s u - u dk}{dm}, \quad u \in \mathcal{D}(\mathcal{A}^0). \tag{2.22}$$

Next let $X^r = (X_t^r, \boldsymbol{P}_x^r)$ be the m-symmetric diffusion process on I^* associated with the regular, local and irreducible Dirichlet form $(\mathcal{E}^r, \mathcal{F}^r)$ on $L^2(I^*;m)$ defined by (2.6). By Theorem 2.2 (iii), X^0 is the part of X^r on I. In other words, X^r is an m-symmetric extension of X^0 to I^*. Based on the property (iv) of Theorem 2.2, X^r is called the *reflecting extension* of X^0.

Since each point of I^* has a positive capacity with respect to $(\mathcal{E}^r, \mathcal{F}^r)$, X^r enjoys a strong irreducibility property

$$\boldsymbol{P}_x^r(\sigma_y < \infty) > 0, \quad \text{for any } x, y \in I^*, \tag{2.23}$$

which in turn implies that $\boldsymbol{E}_a^r[e^{-\alpha\sigma_{a\pm}}] = 1$ for any $a \in I$, $\boldsymbol{E}_{r_1}^r[e^{-\alpha\sigma_{r_1+}}] = 1$ whenever $r_1 \in I^*$, and $\boldsymbol{E}_{r_2}^r[e^{-\alpha\sigma_{r_2-}}] = 1$ whenever $r_2 \in I^*$, for $\alpha > 0$.

Consequently, if we define the space $C_b(I^*)$ by

$$C_b(I^*) = \Big\{u \in C_b(I) : u(r_i) = \lim_{x \to r_i,\, x \in I} u(x) \text{ whenever } r_i \in I^*\Big\}, \tag{2.24}$$

then

$$R_\alpha^r(\mathcal{B}_b(I)) \subset C_b(I^*), \tag{2.25}$$

where $\{R_\alpha^r; \alpha > 0\}$ denotes the resolvent kernel of X^r.

Denote by $\mathcal{A}^r$ the infinitesimal generator of the strongly continuous contraction semigroup of X^r on $L^2(I^*; m) = L^2(I; m)$:

$$u \in \mathcal{D}(\mathcal{A}^r) \quad \text{and} \quad \mathcal{A}^r u = f \in L^2(I; m)$$

if and only if

$$u \in \mathcal{F}^r, \quad \mathcal{E}^r(u, v) = -(f, v), \text{ for any } v \in \mathcal{F}^r \cap C_c(I^*), \tag{2.26}$$

on account of the regularity of $(\mathcal{E}^r, \mathcal{F}^r)$. $\mathcal{A}^r$ will be called the L^2-generator of X^r.

We write

$$D_s u(r_i) = \lim_{x \to r_i, x \in I} D_s u(x).$$

THEOREM 2.6. *The L^2-generator $\mathcal{A}^r$ of the reflecting extenion X^r of X^0 to I^* can be described in terms of the triplet (s, m, k) as follows*:
$u \in \mathcal{D}(\mathcal{A}^r)$ if and only if

$$u \in \mathcal{F}^{(s),k} \cap L^2(I; m), \quad \frac{dD_s u - udk}{dm} \in L^2(I; m) \tag{2.27}$$

and

$$D_s u(r_i) = 0 \quad \textit{whenever } r_i \textit{ is regular.} \tag{2.28}$$

In this case,

$$\mathcal{A}^r u = \frac{dD_s u - udk}{dm}, \quad u \in \mathcal{D}(\mathcal{A}^r). \tag{2.29}$$

PROOF. This is immediate from the characterization (2.26). Indeed, (2.27) and (2.29) follow from (2.26) for $v \in \mathcal{F}^r \cap C_c(I)$. If r_1 is regular, we then obtain (2.28) with

$r_i = r_1$ from (2.26) for $v \in \mathcal{F}^r \cap C_c([r_1, r_2))$ using an integration by parts. The case that r_2 is regular can be treated analogously. The converse implication is also clear.

Here we give an alternative proof of this theorem by using general notations and a criterion presented in [**CF**, Chapter 7]. For a regular boundary r_i, let

$$u_\alpha^{r_i}(x) = \boldsymbol{E}_x^r[e^{-\alpha\tau_I}; X_{\tau_I} = r_i], \quad x \in I.$$

Denote by $\mathcal{D}(\mathcal{L})$ the collection of functions satisfying condition (2.27) and define the linear operator $\mathcal{L}$ as the generalized differential operator appearing there. For $u \in \mathcal{D}(\mathcal{L})$, define its *flux* $\mathcal{N}(u)(r_i)$ at a regular boundary r_i by

$$\mathcal{N}(u)(r_i) = \mathcal{E}^{(s),k}(u, u_\alpha^{r_i}) + \int_I \mathcal{L}u \cdot u_\alpha^{r_i} dm.$$

By virtue of a general criterion [**CF**, (7.3.21)], we know that $u \in \mathcal{D}(\mathcal{A}^r)$ if and only if

$$u \in \mathcal{D}(\mathcal{L}), \quad \mathcal{N}(u)(r_i) = 0, \quad \text{whenever } r_i \text{ is regular.} \tag{2.30}$$

On the other hand, we can verify in exactly the same way as [**CF**, Section 7.6 (**2°**)] that

$$\begin{cases} \mathcal{N}(u)(r_1) = -D_s u(r_1) & \text{when } r_1 \text{ is regular} \\ \mathcal{N}(u)(r_2) = D_s u(r_2) & \text{when } r_2 \text{ is regular.} \end{cases} \tag{2.31}$$

(2.30) and (2.31) lead us to the conclusions of Theorem 2.6. □

3. C_b-generator of X^0.

For a given triplet (s, m, k) on I, consider a homogeneous equation

$$\alpha u(x) - \frac{dD_s u - u dk}{dm}(x) = 0, \quad x \in I,\ \alpha > 0. \tag{3.1}$$

There exists a positive strictly increasing (resp. decreasing) solution u_1 (resp. u_2) of (3.1). When r_i is regular, there are many solutions u_i; among them are the extremal ones $\underline{u}_i$ with $\underline{u}_i(r_i) = 0$, $D_s\underline{u}_i(r_i) \neq 0$ and $\overline{u}_i$ with $D_s\overline{u}(r_i) = 0$, $\overline{u}_i(r_i) > 0$, both being unique up to positive multiplicative constants. Otherwise u_i is unique up to a positive multiplicative constant (see [**I1**, Section 5.13]).

The following table on the behaviors of u_i for the right boundary r_2 is taken from [**IM2**, p 130]. [**I1**, Section 5.13] has the same table in the case that $k = 0$. We remark that the results and analytic arguments leading to them in [**I1**, Section 5.12, Section 5.13, Section 5.14] remain valid in general by replacing m there with $j = m + k$.

	regular	exit	entrance	natural
$u_1(r_2)$	$\in (0,\infty)$	$\in (0,\infty)$	$=\infty$	$=\infty$
$D_s u_1(r_2)$	$\in (0,\infty)$	$=\infty$	$\in (0,\infty)$	$=\infty$
$u_2(r_2)$	$<\infty$	$=0$	$\in (0,\infty)$	$=0$
$-D_s u_2(r_2)$	$<\infty$	$\in (0,\infty)$	$=0$	$=0$

Let X^0 be a minimal diffusion on I with an attached triplet (s,m,k). By Theorem 2.4, X^0 is m-symmetric and its Dirichlet form on $L^2(I;m)$ is $(\mathcal{E}^0,\mathcal{F}^0)$ given by (2.7). Due to the inequality (2.10), the Hilbert space $(\mathcal{F}^0,\mathcal{E}^0_\alpha)$ admits a *reproducing kernel* $g^0_\alpha(x,y)$, $x,y\in I$: for each $y\in I$,

$$g^0_\alpha(\cdot,y)\in\mathcal{F}^0,\quad \mathcal{E}^0_\alpha(g^0_\alpha(\cdot,y),v)=v(y),\quad \text{for any } v\in\mathcal{F}^0. \tag{3.2}$$

If r_i is either regular or exit, then r_i is approachable and hence it follows from the first property of (3.2) that

$$g^0_\alpha(r_i,y)=0,\quad \text{whenever } r_i \text{ is either regular or exit.} \tag{3.3}$$

LEMMA 3.1. (i) *$g^0_\alpha(x,y)$ admits an expression*

$$g^0_\alpha(x,y)=\begin{cases} W(u_1,u_2)^{-1}\, u_1(x)u_2(y) & \text{if } x\le y,\quad x,y\in I,\\ W(u_1,u_2)^{-1}\, u_2(x)u_1(y) & \text{if } x\ge y\quad x,y\in I,\end{cases} \tag{3.4}$$

where $W(u_1,u_2)(x)=D_su_1(x)u_2(x)-D_su_2(x)u_1(x)$ is the Wronskian of u_1,u_2 which is positive and independent of $x\in I$. Here u_i should be chosen to be

$$u_i=\underline{u}_i,\quad \text{whenever } r_i \text{ is regular,} \tag{3.5}$$

(ii) *$g^0_\alpha(x,y)$ is a density function of the resolvent kernel R^0_α of X^0 with respect to m:*

$$R^0_\alpha f(x)=\int_I g^0_\alpha(x,y)f(y)m(dy),\quad x\in I,\ f\in C_b(I). \tag{3.6}$$

PROOF. (i) Let $p^y_\alpha(x)=\boldsymbol{E}^0_x[e^{-\alpha\sigma_y}]$, $x\in I$. The function p^y_α is characterized as

$$p^y_\alpha\in\mathcal{F}^0,\quad p^y_\alpha(y)=1,\quad \mathcal{E}^0_\alpha(p^y_\alpha,v)=0,\ \forall v\in\mathcal{F}^0,\ v(y)=0, \tag{3.7}$$

which compared with (3.2) yields

$$p^y_\alpha(x)=g^0_\alpha(x,y)/g^0_\alpha(y,y),\quad x\in I, \tag{3.8}$$

by noting that

$$g_\alpha^0(y,y) = \mathcal{E}_\alpha^0(g_\alpha^0(\cdot,y), g_\alpha^0(\cdot,y)) > 0. \tag{3.9}$$

(3.7) further implies

$$p_\alpha^y(x) = \begin{cases} u_1(x)/u_1(y) & \text{if } x \le y, \quad x,y \in I, \\ u_2(x)/u_2(y) & \text{if } x \ge y \quad x,y \in I, \end{cases} \tag{3.10}$$

(3.8), (3.10) and the symmetry of $g_\alpha^0(x,y)$ lead us to, for $x < y$,

$$\frac{u_1(x)}{u_1(y)} g_\alpha^0(y,y) = g_\alpha^0(x,y) = g_\alpha^0(y,x) = \frac{u_2(y)}{u_2(x)} g_\alpha^0(x,x),$$

which means that $C =: g_\alpha^0(x,x)/u_1(x)u_2(x)$ is independent of $x \in I$ and $g_\alpha^0(x,y)$ admits an expression

$$g_\alpha^0(x,y) = \begin{cases} Cu_1(x)u_2(y) & \text{if } x \le y, \quad x,y \in I, \\ Cu_2(x)u_1(y) & \text{if } x \ge y \quad x,y \in I. \end{cases} \tag{3.11}$$

In particular, (3.5) follows from (3.3).

In order to determine the above constant C, we substitute (3.11) into the equation (3.9). The left hand side equals $Cu_1(y)u_2(y)$. We compute the right hand side by choosing $y \in I$ to be a continuous point for m and k. Since u_1, u_2 are the solutions of (3.1), we perform integrations by parts on each of subintervals (r_1, y), (y, r_2) to see that the right hand side of (3.9) is equal to

$$\begin{aligned} &C^2 u_2(y)^2 (u_1(y) D_s u_1(y) - u_1(r_1+) D_s u_1(r_1+)) \\ &\quad + C^2 u_1(y)^2 (u_2(r_2-) D_s u_2(r_2-) - u_2(y) D_s u_2(y)). \end{aligned}$$

But two terms involving r_1 and r_2 in the above expression vanish on account of the preceding table and (3.5). Thus we arrive at the desired identity $CW(u_1, u_2) = 1$.

(ii) We write $G_\alpha^0 f(x) = \int_I g_\alpha^0(x,y) f(y) m(dy)$, $x \in I$, $f \in C_b(I)$. It suffices to prove $G_\alpha^0 f = R_\alpha^0 f$ for $f \in C_c(I)$. Let $u = G_\alpha^0 f$ for $f \in C_b(I)$. Denoting the Wronskian $W(u_1, u_2)$ by W, we have from (i)

$$Wu(x) = u_2(x) \int_{r_1}^{x+0} f(y) u_1(y) dm(y) + u_1(x) \int_{x+0}^{r_2} f(y) u_2(y) dm(y). \tag{3.12}$$

By differentiating products of functions of bounded variation, we simply get

$$WD_s u(x) = D_s u_2(x) \int_{r_1}^{x+0} f(y) u_1(y) dm(y) + D_s u_1(x) \int_{x+0}^{r_2} f(y) u_2(y) dm(y), \tag{3.13}$$

which means that the derivative can be taken under the integral sign:

$$D_s u(x) = \int_I \frac{dg_\alpha^0(x,y)}{ds(x)} f(y)m(dy), \quad x \in I.$$

By taking $f \in C_c(I)$, we can now integrate the both hand sides of the equation

$$\mathcal{E}_\alpha^0(g_\alpha^0(\cdot,y), g_\alpha^0(\cdot,y')) = g_\alpha^0(y,y')$$

in y and y' with respect to $f(y)m(dy)$ and $f(y')m(dy')$, respectively, to obtain $\mathcal{E}_\alpha^{(s),k}(u,u) = \int_I f(y)u(y)m(dy) < \infty$, namely, $u \in \mathcal{F}^0$. Similarly we have $\mathcal{E}_\alpha^0(u,v) = (f,v)$ for any $v \in \mathcal{F}^0$ proving $G_\alpha^0 f = R_\alpha^0 f$ for $f \in C_c(I)$. □

Notice that (3.3) and (3.4) imply that

$$u_i(r_i) = 0, \quad \text{whenever } r_i \text{ is exit,} \tag{3.14}$$

which is however contained in the preceding table already.

By (3.6), we see for $f \in C_b(I)$ and $x \in I$ that $WR_\alpha^0 f(x)$ equals the right hand side of (3.12) for $W = W(u_1,u_2)$. By taking the bound $\alpha R_\alpha^0 1(x) \le 1$ into account, we let $x \uparrow r_2$ to obtain

$$R_\alpha^0 f(r_2) = W^{-1}\, u_2(r_2) \int_I f u_1 dm. \tag{3.15}$$

An analogous identity holds for r_1 and we conclude from (3.5) and (3.14) that, for $f \in C_b(I)$,

$$R_\alpha^0 f(r_i) = 0, \quad \text{if } r_i \text{ is either regular or exit.} \tag{3.16}$$

We can now give a complete characterization of the C_b-generator $\mathcal{G}^0$ of the minimal diffusion X^0 on I.

THEOREM 3.2. *$u \in \mathcal{D}(\mathcal{G}^0)$ if and only if*

$$\begin{cases} u \in C_b(I), & \dfrac{dD_s u - u dk}{dm} \in C_b(I), \quad \textit{and} \\ u(r_i) = 0 & \textit{if } r_i \textit{ is either regular or exit.} \end{cases} \tag{3.17}$$

In this case,

$$\mathcal{G}^0 u = \frac{dD_s u - u dk}{dm}, \quad u \in \mathcal{D}(\mathcal{G}^0). \tag{3.18}$$

PROOF. To show the "only if" part, take any function $u \in \mathcal{D}(\mathcal{G}^0)$ so that $u = R_\alpha^0 f$

for some $f \in C_b(I)$. By virtue of Lemma 3.1, u satisfies (3.12) and (3.13). Since $dD_s u_i = u_i(\alpha dm + dk)$, $i = 1, 2$, we take the differentials of the both hand sides of (3.13) to get

$$\begin{aligned} W dD_s u(x) &= W u(x)(\alpha dm(x) + dk(x)) \\ &\qquad + D_s u_2(x) f(x) u_1(x) dm(x) - D_s u_1(x) f(x) u_2(x) dm(x) \\ &= W u(x)(\alpha dm(x) + dk(x)) - W f(x) dm(x), \end{aligned}$$

yielding the first property of (3.17) together with (3.18). The second property of (3.17) is a consequence of (3.16).

To prove the "if" part, take any function u satisfying condition (3.17). We then let $f = \alpha u - ((dD_s u - u dk)/dm)$, $v = R_\alpha^0 f$ and $w = u - v$. Since $v \in \mathcal{D}(\mathcal{G}^0)$ and hence $\alpha v - ((dD_s v - v dk)/dm) = f$ by (2.15), we see that w is a bounded solution of (3.1). Since $v(r_i)$ vanishes whenever r_i is regular or exit by the "only if" part, so does w.

We write $w = C_1 u_1 + C_2 u_2$ for some constants C_1, C_2. If both r_1, r_2 are either regular or exit, we have $w(r_1) = w(r_2) = 0$, which implies $C_1 = C_2 = 0$ because $u_1(r_1)u_2(r_2) - u_1(r_2)u_2(r_1) < 0$. If r_1 is either regular or exit but r_2 is either entrance or natural, then $u_1(r_2) = \infty$ from the above table so that $C_1 = 0$ and $0 = w(r_1) = C_2 u_2(r_1)$, yielding $C_2 = 0$ because $u_2(r_1) > 0$. If both boundaries are either entrance or natural, then we have $C_1 = C_2 = 0$ trivially. Thus $u = v \in \mathcal{D}(\mathcal{G}^0)$. $\square$

The appearance of the seminal paper by W. Feller on general boundary conditions for one-dimensional diffusions goes back to [**Fe1**], that treated the case where $(dD_s u - u dk)/dm$ is reduced to a simple differential operator $u'' + b(x)u'$. In [**Fe1**], the condition (3.17) was explicitly stated to characterize the range of the resolvent defined by (3.4), (3.5) and (3.6). However Theorem 3.2 in this generality appears here for the first time.

4. C_b-generator of X^r.

Let X^r be the reflecting extension of X^0 to I^*. m is extended to I^* by setting $m(I^* \setminus I) = 0$. X^r is m-symmetric diffusion on I^* whose Dirichlet form on $L^2(I^*; m) = L^2(I; m)$ is $(\mathcal{E}^r, \mathcal{F}^r)$ given by (2.6). Due to the inequality (2.10), the Hilbert space $(\mathcal{F}^r, \mathcal{E}_\alpha^r)$ admits a *reproducing kernel* $g_\alpha^r(x, y)$, $x, y \in I^*$: for each $y \in I^*$,

$$g_\alpha^r(\cdot, y) \in \mathcal{F}^r, \quad \mathcal{E}_\alpha^r(g_\alpha^r(\cdot, y), v) = v(y), \quad \text{for any } v \in \mathcal{F}^r. \tag{4.1}$$

We first note that

$$D_s g_\alpha^r(r_i, y) = 0, \quad \text{for each } y \in I, \quad \text{whenever } r_i \text{ is regular.} \tag{4.2}$$

We show this only when r_1 is regular. Denote $g_\alpha^r(x, y)$ by $u(x)$. Since we get from (4.1), $\mathcal{E}^r(u, v) = 0$ for any $v \in \mathcal{F}^r$ that vanishes on $[y, r_2)$, u is α-harmonic on (r_1, y) in the sense it satisfies (3.1) for any $x \in (r_1, y)$. Therefore, for any such v,

$$0 = \mathcal{E}_\alpha^{(s),k}(u,v) = \int_I D_s u D_s v ds + \int_I v d D_s u = v(r_1) D_s u(r_1).$$

Moreover $g_\alpha^r(\cdot, y)$ enjoys the property

$$g_\alpha^r(r_i, y) = 0, \quad \text{for each } y \in I, \quad \text{whenever } r_i \text{ is exit}, \tag{4.3}$$

because $g_\alpha^r(\cdot, y) \in \mathcal{F}^r$ by (4.1) and any function in the space $\mathcal{F}^r$ vanishes at an exit boundary as was stated at the end of Section 2.1.

We can prove the following lemma analogously to Lemma 3.1.

LEMMA 4.1. (i) $g_\alpha^r(x,y)$ *admits an expression*

$$g_\alpha^r(x,y) = \begin{cases} W(u_1,u_2)^{-1} u_1(x) u_2(y) & \text{if } x \le y, \quad x, y \in I^*, \\ W(u_1,u_2)^{-1} u_2(x) u_1(y) & \text{if } x \ge y \quad x, y \in I^*. \end{cases} \tag{4.4}$$

Here u_i should be chosen to be

$$u_i = \overline{u}_i, \quad \text{whenever } r_i \text{ is regular.} \tag{4.5}$$

(ii) $g_\alpha^r(x,y)$ *is a density function of the resolvent kernel R_α^r of X^r with respect to m*:

$$R_\alpha^r f(x) = \int_I g_\alpha^r(x,y) f(y) m(dy), \quad x \in I^*, \ f \in C_b(I). \tag{4.6}$$

(4.5) is a consequence of (4.2). It also follows from (4.3) that

$$u_i(r_i) = 0 \quad \text{whenever } r_i \text{ is exit}, \tag{4.7}$$

but this is already contained in the table of Section 3.

By (4.6), we see for $f \in C_b(I)$ and $x \in I$ that $W R_\alpha^r f(x)$ and $W D_s(R_\alpha^r f)(x)$ are equal to the right hand sides of (3.12) and (3.13), respectively, with u_i chosen in a way of (4.5). By noting the table of Section 3 and the bound $\alpha R_\alpha^r 1 \le 1$, we let $x \uparrow r_2$ to get

$$W\ R_\alpha^r f(r_2) = u_2(r_2) \int_I f u_1 dm, \ W\ D_s(R_\alpha^r f)(r_2) = D_s u_2(r_2) \int_I f u_1 dm. \tag{4.8}$$

Analogous identities hold for r_1 and we conclude from (4.5) and (4.7) that

$$\begin{cases} D_s(R_\alpha^r f)(r_i) = 0, & \text{whenever } r_i \text{ is regular}, \\ R_\alpha^r f(r_i) = 0, & \text{whenever } r_i \text{ is exit}. \end{cases} \tag{4.9}$$

We are in a position to identify C_b-generator $\mathcal{G}^r$ of the reflecting extension X^r of X^0 to I^*. Define the space $C_b(I^*)$ by (2.24). In view of (2.25), the C_b-generator $\mathcal{G}^r$ of

X^r is well defined by

$$\begin{cases} \mathcal{D}(\mathcal{G}^r) = R^r_\alpha(C_b(I^*)), \\ (\mathcal{G}^r u)(x) = \alpha u(x) - f(x), \quad \text{for } u = R^r_\alpha f,\ f \in C_b(I^*),\ x \in I^*, \end{cases}$$

analogously to (2.13).

Theorem 4.2. *$u \in \mathcal{D}(\mathcal{G}^r)$ if and only if*

$$u \in C_b(I^*), \quad \frac{dD_s u - u dk}{dm} \in C_b(I^*), \tag{4.10}$$

and

$$D_s u(r_i) = 0 \quad \textit{whenever } r_i \textit{ is regular}, \tag{4.11}$$

$$u(r_i) = 0 \quad \textit{whenever } r_i \textit{ is exit}. \tag{4.12}$$

In this case,

$$\mathcal{G}^r u(x) = \frac{dD_s u - u dk}{dm}(x), \quad x \in I^*,\ u \in \mathcal{D}(\mathcal{G}^r). \tag{4.13}$$

Proof. As for the "only if" part, (4.10) and (4.13) can be shown as in the proof of Theorem 3.2 by making use of Lemma 4.1. (4.11) and (4.12) follow from (4.9).

To prove the "if" part, take any function u satisfying conditions (4.11) and (4.12). We then let $f = \alpha u - ((dD_s u - u dk)/dm)$, $v = R^r_\alpha f$ and $w = u - v$. Since $v \in \mathcal{D}(\mathcal{G}^r)$ and $\alpha v - ((dD_s v - v dk)/dm) = f$ by (4.13), we see that w is a bounded solution of (3.1). Since v satisfies (4.11) and (4.12) by the "only if" part, so does w.

We write $w = C_1 u_1 + C_2 u_2$ for some constants C_1, C_2. Then $D_s w = C_1 D_s u_1 + C_2 D_s u_2$. If both r_1, r_2 are regular, we have $D_s w(r_1) = D_s w(r_2) = 0$, which implies $C_1 = C_2 = 0$ because $D_s u_1 > 0$, $D_s u_2 < 0$ and both $D_s u_1$, $D_s u_2$ are strictly increasing, and so

$$D_s u_1(r_1) D_s u_2(r_2) - D_s u_1(r_2) D_s u_2(r_1) > 0.$$

If r_1 is regular and r_2 is exit, then $0 = w(r_2) = C_1 u_1(r_2)$ by the table of Section 3 so that $C_1 = 0$ because $u_1(r_2) > 0$. Further $0 = D_s w(r_1) = C_2 D_s u_2(r_1)$ yielding $C_2 = 0$ because u_2 is a positive decreasing solution of (3.1) and so $D_s u_2(r_1) < 0$. If r_1 is regular and r_2 is either entrance or natural, then $u_1(r_2) = \infty$ by the same table and $C_1 = 0$ so that $C_2 = 0$ as in the previous case. If both r_1 and r_2 are exit, we have $0 = C_1 u_1(r_2) = C_2 u_2(r_1)$ yielding $C_1 = C_2 = 0$. We also get $w = 0$ in other cases trivially. Thus $u = v \in \mathcal{D}(\mathcal{G}^r)$.

□

5. Generators of other symmetric diffusion extensions of X^0.

Let X^0 be a minimal diffusion on I with attached triplet (s, m, k). We shall describe two kinds of generators for all possible symmetric diffusion extensions of X^0 other than X^r. The next two subsections will deal with one-point extensions of X^0 when both boundaries r_1 and r_2 are regular. The associated Dirichlet forms on $L^2(I; m)$ are Silverstein extensions of $(\mathcal{E}^0, \mathcal{F}^0)$ in the sense of [**CF**, Section 6.6].

5.1. Diffusion reflected at r_1 and absorbed at r_2.

Assume that both r_1 and r_2 are regular so that $I^* = [r_1, r_2]$. The diffusion $X^{r,0}$ on $[r_1, r_2)$ reflected at r_1 and absorbed at r_2 is by definition the part process of the reflecting extension X^r on $[r_1, r_2)$, namely, the process obtained from X^r by killing upon hitting the point r_2. $X^{r,0}$ is an m-symmetric extension of X^0 and, in view of (2.6), the Dirichlet form $(\mathcal{E}^{r,0}, \mathcal{F}^{r,0})$ of $X^{r,0}$ on $L^2([r_1, r_2); m) = L^2(I; m)$ is given by

$$\begin{cases} \mathcal{F}^{r,0} = \{u \in \mathcal{F}^{(s),k} \cap L^2(I; m) : u(r_2) = 0\}, \\ \mathcal{E}^{r,0}(u, v) = \mathcal{E}^{(s),k}(u, v), \ \ u, v \in \mathcal{F}^{r,0}. \end{cases} \tag{5.1}$$

Denote by $\mathcal{A}^{r,0}$, $\mathcal{G}^{r,0}$ the L^2-generator and C_b-generator of $X^{r,0}$, respectively, defined analogously to $\mathcal{A}^r$ in Section 2.3 and to $\mathcal{G}^r$ in Section 4.

Theorem 5.1. *(i) $u \in \mathcal{D}(\mathcal{A}^{r,0})$ if and only if*

$$u \in \mathcal{F}^{(s),k} \cap L^2(I; m), \quad \frac{dD_s u - u dk}{dm} \in L^2(I; m) \tag{5.2}$$

and

$$D_s u(r_1) = 0 \quad \textit{and} \quad u(r_2) = 0. \tag{5.3}$$

In this case,

$$\mathcal{A}^{r,0} u = \frac{dD_s u - u dk}{dm}, \quad u \in \mathcal{D}(\mathcal{A}^{r,0}). \tag{5.4}$$

(ii) $u \in \mathcal{D}(\mathcal{G}^{r,0})$ if and only if

$$u \in C_b([r_1, r_2)), \quad \frac{dD_s u - u dk}{dm} \in C_b([r_1, r_2)), \tag{5.5}$$

and

$$D_s u(r_1) = 0 \quad \textit{and} \quad u(r_2) = 0. \tag{5.6}$$

In this case,

$$\mathcal{G}^{r,0}u(x) = \frac{dD_s u - u dk}{dm}(x), \quad x \in [r_1, r_2),\ u \in \mathcal{D}(\mathcal{G}^{r,0}). \tag{5.7}$$

Proof. (i) This follows from (5.1) and the definition of $\mathcal{A}^{r,0}$. The first condition of (5.3) is obtained by integration by parts of the left hand side of the equation $\mathcal{E}^{r,0}(u,v) = -(f,v)$, $v \in \mathcal{F}^{r,0}$.

(ii) Since $m(I) < \infty$, we have $\mathcal{D}(\mathcal{G}^{r,0}) \subset \mathcal{D}(\mathcal{A}^{r,0})$ and so the "only if" part of (ii) follows from that part of (i). To show the "if" part, we assume as in the proof of Theorem 4.2 that $w = C_1u_1 + C_2u_2$ satisfies (5.6). Then $C_1D_su_1(r_1) + C_2D_su_2(r_1) = 0$, $C_1u_1(r_2) + C_2u_2(r_2) = 0$, which yields $C_1 = C_2 = 0$ because the coefficient matrix of this equation has a positive determinant. □

The interchange of r_1 and r_2 in the above theorem yields the description of the both kinds of generators of the m-symmetric diffusion extension of X^0 absorbed at r_1 and reflected r_2.

5.2. One-point extension to the one-point-compactification.

We assume that both boundaries r_1 and r_2 of I are regular and consider the active reflected Dirichlet space $(\mathcal{E}^r, \mathcal{F}^r)$ defined by (2.6). Let $C([r_1, r_2])$ be the space of continuous function on $[r_1, r_2]$ with the topology of the uniform convergence. Because of the inequality (2.10), we have the continuous embedding

$$\mathcal{F}^r \subset C([r_1, r_2]). \tag{5.8}$$

Denote by $\dot{I}$ the one-point-compactification of I and extend m to $\dot{I}$ by setting $m(\dot{I} \setminus I) = 0$. Define the subspace of $(\mathcal{E}^r, \mathcal{F}^r)$ by

$$\dot{\mathcal{F}} = \{u \in \mathcal{F}^r : u(r_1) = u(r_2)\}, \quad \dot{\mathcal{E}} = \mathcal{E}^r\big|_{\dot{\mathcal{F}} \times \dot{\mathcal{F}}}. \tag{5.9}$$

Since $\dot{\mathcal{F}}$ is an subalgebra of $C(\dot{I})$ containing the constant function 1 and the space $\mathcal{F}^0$ that seperate the points of $\dot{I}$, $\dot{\mathcal{F}}$ is uniformly dense in $C(\dot{I})$. Hence $(\dot{\mathcal{E}}, \dot{\mathcal{F}})$ is a regular, local irreducible Dirichlet form on $L^2(\dot{I}; m) = L^2(I; m)$.

Let $\dot{X}$ be the m-symmetric diffusion on $\dot{I}$ associated with $(\dot{\mathcal{E}}, \dot{\mathcal{F}})$. Since the part of the latter on I equals $(\mathcal{E}^0, \mathcal{F}^0)$, the part of $\dot{X}$ on I equals X^0, namely, $\dot{X}$ is an m-symmetric extension of the minimal diffusion X^0 to $\dot{I}$. We denote by $\dot{\mathcal{A}}$ and $\dot{\mathcal{G}}$ the L^2-generator and C_b-generator of $\dot{X}$, respectively. As $m(I) < \infty$, we have the inclusion

$$\mathcal{D}(\dot{\mathcal{G}}) = \dot{R}_\alpha(C(\dot{I})) \subset \dot{R}_\alpha(L^2(I : m)) = \mathcal{D}(\dot{\mathcal{A}}), \tag{5.10}$$

where $\dot{R}_\alpha$ denotes the resolvent of $\dot{X}$.

Theorem 5.2. (i) *$u \in \mathcal{D}(\dot{\mathcal{A}})$ if and only if*

$$u \in \mathcal{F}^{(s),k} \cap L^2(I; m), \quad \frac{dD_s u - u dk}{dm} \in L^2(I; m) \tag{5.11}$$

and

$$u(r_1) = u(r_2) \quad \textit{and} \quad D_s u(r_1) = D_s u(r_2). \tag{5.12}$$

In this case,

$$\dot{\mathcal{A}} u = \frac{dD_s u - u dk}{dm}, \quad u \in \mathcal{D}(\dot{\mathcal{A}}). \tag{5.13}$$

(ii) $u \in \mathcal{D}(\dot{\mathcal{G}})$ *if and only if*

$$u \in C(\dot{I}), \quad \frac{dD_s u - u dk}{dm} \in C(\dot{I}), \tag{5.14}$$

and

$$u(r_1) = u(r_2) \quad \textit{and} \quad D_s u(r_1) = D_s u(r_2). \tag{5.15}$$

In this case,

$$\dot{\mathcal{G}} u(x) = \frac{dD_s u - u dk}{dm}(x), \quad x \in \dot{I}, \ u \in \mathcal{D}(\dot{\mathcal{G}}). \tag{5.16}$$

Proof. (i) This follows from (5.9) and the definition of $\dot{\mathcal{A}}$. The second condition of (5.12) is obtained by integration by parts of the left hand side of the equation $\dot{\mathcal{E}}(u, v) = -(f, v)$, $v \in \dot{\mathcal{F}}$.

(ii) Because of (5.10), the "only if" part of (ii) follows from that part of (i). To show the "if" part, we assume as in the proof of Theorem 5.2 that $w = C_1 u_1 + C_2 u_2$ satisfies (5.15). Then

$$\begin{cases} C_1(u_1(r_1) - u_1(r_2)) + C_2(u_2(r_1) - u_2(r_2)) = 0, \\ C_1(D_s u_1(r_1) - D_s u_1(r_2)) + C_2(D_s u_2(r_1) - D_s u_2(r_2)) = 0, \end{cases}$$

which yields $C_1 = C_2 = 0$ because the coefficient matrix of this equation has a negative determinant. □

When X^0 admits no killing inside so that $k = 0$, the diffusion $\dot{X}$ on $\dot{I}$ can be constructed by piecing together the excursions of X^0 starting at $K = \{r_1, r_2\}$ and ending at K which evolves as a Poisson point process according to Theorems 7.5.6 and 7.5.9 of [**CF**]. Generally, we first construct a process on $\dot{I}$ corresponding to $k = 0$ in the above way, then its canonical subprocess with respect to its positive continuous additive functional with Revuz measure k is the desired process $\dot{X}$ on $\dot{I}$.

5.3. Diffusions with sojourn and killing on boundaries.

We assume that the left boundary r_1 of I is regular but the right boundary r_2 is non-regular. We exhibit two kinds of generators of a reflecting extension of X^0 allowing sojourn and killing at r_1. Allowing sojourn and killing at r_2 when it is regular and at the boundary for the one-point extensions of Section 5.1 and Section 5.2 can be dealt with in quite analogous manners and will be omitted.

Let m^* and k^* be extensions of m and k from I to $I^* = [r_1, r_2)$, respectively allowing point masses at r_1 so that

$$m^*(r_1)\ (= m^*(\{r_1\})) \geq 0, \quad k^*(r_1)\ (= k^*(\{r_1\})) \geq 0. \tag{5.17}$$

Define the Dirichlet form $(\mathcal{E}^*, \mathcal{F}^*)$ on $L^2(I^*; m^*)$ by

$$\begin{cases} \mathcal{F}^* = \mathcal{F}^{(s)} \cap L^2(I^*; k^*) \cap L^2(I^*; m^*), \\ \mathcal{E}^*(u, v) = \mathcal{E}^{(s),k}(u, v) + u(r_1)v(r_1)k^*(r_1), \quad u, v \in \mathcal{F}^*. \end{cases} \tag{5.18}$$

$(\mathcal{E}^*, \mathcal{F}^*)$ is then a regular, local irreducible Dirichlet form on $L^2(I^*; m^*)$ and it admits an associated m^*-symmetric diffusion process $X^* = (X_t^*, \boldsymbol{P}_x^*)$ on I^*.

We denote by $\mathcal{A}^*$ the L^2-generator of X^*, which can be readily identified.

Theorem 5.3. *$u \in \mathcal{D}(\mathcal{A}^*)$ if and only if*

$$u \in \mathcal{F}^{(s)} \cap L^2(I^*; k^*) \cap L^2(I^*; m^*), \quad \frac{dD_s u - u dk}{dm} \in L^2(I; m) \tag{5.19}$$

and

$$D_s u(r_1) - u(r_1)k^*(r_1) = A^* u(r_1) m^*(r_1). \tag{5.20}$$

In this case, for $u \in \mathcal{D}(\mathcal{A}^*)$,

$$\begin{cases} \mathcal{A}^* u(x) = \dfrac{dD_s u - u dk}{dm}(x), & x \in I, \\ \mathcal{A}^* u(r_1) = (D_s u(r_1) - u(r_1)k^*(r_1))/m^*(r_1), & \text{if } m^*(r_1) > 0. \end{cases} \tag{5.21}$$

Proof. $u \in \mathcal{D}(\mathcal{A}^*)$ and $\mathcal{A}^* u = f \in L^2(I^*; m^*)$ if and only if $u \in \mathcal{F}^*$ and

$$\mathcal{E}^{(s)}(u, v) + \int_I uv dk + u(r_1)v(r_1)k^*(r_1) = -\int_I fv dm + f(r_1)v(r_1)m^*(r_1)$$

for any $v \in \mathcal{F}^* \cap C_c([r_1, r_2))$. By taking $v \in \mathcal{F}^0$, we get (5.19) and the first identity of (5.21). Then, by taking $v \in \mathcal{F}^*$ with $v(r_1) \neq 0$, we arrive at (5.20) and the second identity of (5.21). The converse implication is also clear. □

Denote by $\{R_\alpha^*, \alpha > 0\}$ the resolvent operator (and resolvent kernel as well) of X^*, and put

$$C_b(I^*) = \{u \in C_b(I) : u \text{ is right continuous at } r_1\}.$$

Just as in the case of the reflecting extension X^r of X^0, the C_b-generator $\mathcal{G}^*$ of X^* is well defined by $\mathcal{D}(\mathcal{G}^*) = R^*_\alpha(C_b(I^*))$, and for $u = R^*_\alpha f$, $f \in C_b(I^*)$,

$$(\mathcal{G}^* u)(x) = \alpha u(x) - f(x), \quad \text{for } x \in I^* = [r_1, r_2). \tag{5.22}$$

Analogously to Section 4, we have the following. For $\alpha > 0$, the Hilbert space $(\mathcal{F}^*, \mathcal{E}^*_\alpha)$ admits a reproducing kernel $g^*_\alpha(x,y),\ x,y \in [r_1, r_2)$: for each $y \in [r_1, r_2)$,

$$g^*_\alpha(\cdot, y) \in \mathcal{F}^*, \quad \mathcal{E}^*_\alpha(g^*_\alpha(\cdot, y), v) = v(y), \quad \text{for any } v \in \mathcal{F}^*. \tag{5.23}$$

Moreover, for each $y \in I$, $g^*_\alpha(\cdot, y)$ enjoys the properties

$$-D_s g^*_\alpha(r_1, y) + g^*_\alpha(r_1, y)(k^*(r_1) + \alpha m^*(r_1)) = 0, \tag{5.24}$$

$$g^*_\alpha(r_2, y) = 0, \quad \text{for each } y \in I, \text{ if } r_2 \text{ is exit.} \tag{5.25}$$

Lemma 5.4. (i) *$g^*_\alpha(x,y)$ admits an expression*

$$g^*_\alpha(x,y) = \begin{cases} W(u_1,u_2)^{-1} u_1(x) u_2(y) & \text{if } x \le y, \quad x, y \in I^*, \\ W(u_1,u_2)^{-1} u_2(x) u_1(y) & \text{if } x \ge y \quad x, y \in I^*. \end{cases} \tag{5.26}$$

Here u_1 should be chosen to satisfy

$$-D_s u_1(r_1) + u_1(r_1)(k^*(r_1) + \alpha m^*(r_1)) = 0. \tag{5.27}$$

(ii) *For $f \in C_b(I^*)$ and $x \in I^*$, $R^*_\alpha f(x)$ admits an expression*

$$R^*_\alpha f(x) = R^*_\alpha(1_I f)(x) + R^*_\alpha(1_{\{r_1\}} f)(x)$$

with

$$R^*_\alpha(1_I f)(x) = \int_I g^*_\alpha(x,y) f(y) m(dy). \tag{5.28}$$

Furthermore the function

$$w(x) = R^*_\alpha(1_{\{r_1\}} f)(x) \ (= R^*_\alpha(x, \{r_1\}) f(r_1)),\ x \in I^*,$$

satisfies (3.1) *on I and*

$$-D_s w(r_1) + w(r_1)(k^*(r_1) + \alpha m^*(r_1)) = f(r_1) m^*(r_1). \tag{5.29}$$

$R^*_\alpha(x,\{r_1\})$ vanishes identically if $m^*(r_1)=0$, while it is a strictly positive decreasing solution of (3.1) *if* $m^*(r_1)>0$.

Proof. (i) We first express $g^*_\alpha(x,y)$ as (3.11) of Lemma 3.1 for some constant C. (5.27) then follows from (5.24). We next substitute the expression (3.11) into $g^*_\alpha(y,y)=\mathcal{E}^*_\alpha(g^*_\alpha(\cdot,y),g^*_\alpha(\cdot,y))$ to obtain $C=W(u_1,u_2)^{-1}$ by taking (5.27) into account.

(ii) (5.28) follows from (i) as in the proof of Lemma 3.1 (ii). The function w defined above satisfies the equation

$$\mathcal{E}^*_\alpha(w,v)=v(r_1)f(r_1)m(r_1),\quad \text{for any } v\in\mathcal{F}^*.$$

From this, we draw the conclusion that w satisfies (3.1) on I, (5.29) at r_1 as well as the last statement of (ii). $\square$

Here we make a remark that a positive strictly increasing solution u_1 of (3.1) satisfying (5.27) can be taken as follows:

$$u_1=\begin{cases}\overline{u}_1 & \text{if } k^*(r_1)+m^*(r_1)=0,\\[2mm] \overline{u}_1+\dfrac{\overline{u}_1(r_1)}{D_s\underline{u}_1(r_1)}(k^*(r_1)+\alpha m^*(r_1))\,\underline{u}_1 & \text{if } k^*(r_1)+m^*(r_1)>0.\end{cases}\tag{5.30}$$

We also remark that a function w satisfying (3.1) and (5.29) can be taken using a positive decreasing solution u_2 of (3.1) with $u_2(r_1)=1$ as

$$w=\frac{f(r_1)m^*(r_1)}{-D_su_2(r_1)+k^*(r_1)+\alpha m^*(r_1)}\,u_2,\quad \text{when } m^*(r_1)>0.\tag{5.31}$$

It follows from (5.25) and (5.26) that

$$u_2(r_2)=0\quad \text{if } r_2 \text{ is exit},\tag{5.32}$$

which is already contained in the table of Section 3 however. It also follows from the above lemma just as in Section 4 that for $f\in C(I^*)$

$$WR^*_\alpha(1_If)(r_1)=u_1(r_1)\int_I fu_2dm,\quad WD_s(R^*_\alpha(1_If))(r_1)=D_su_1(r_1)\int_I fu_2dm.\tag{5.33}$$

We also have

$$WR^*_\alpha(1_If)(r_2)=u_2(r_2)\int_I fu_1dm.\tag{5.34}$$

Combining (5.33) with (5.27), (5.29) and (5.22), we arrive at, for $f\in C_b(I^*)$,

$$D_s(R^*_\alpha f)(r_1)-R^*_\alpha f(r_1)k^*(r_1)=\mathcal{G}^*(R^*_\alpha f)(r_1)m^*(r_1).\tag{5.35}$$

On the other hand, we get from (5.32) and (5.34) that $R^*_\alpha(1_I f)(r_2) = 0$ if r_2 is exit. When $m^*(r_1) > 0$, $R^*_\alpha(x, \{r_1\})$ is a positive strictly decreasing solution of (3.1) so that $R^*_\alpha(r_2, \{r_1\}) = 0$ by the table of Section 3 provided that r_2 is exit. Hence, for any $f \in C_b(I^*)$,

$$R^*_\alpha f(r_2) = 0, \quad \text{if } r_2 \text{ is exit.} \tag{5.36}$$

THEOREM 5.5. *$u \in \mathcal{D}(\mathcal{G}^*)$ if and only if*

$$u \in C_b(I^*), \quad \frac{dD_s u - udk}{dm} \in C_b(I^*), \tag{5.37}$$

and

$$\begin{cases} D_s u(r_1) - u(r_1)k^*(r_1) = \mathcal{G}^* u(r_1) m^*(r_1), \\ u(r_2) = 0, \quad \textit{if } r_2 \textit{ is exit}, \end{cases} \tag{5.38}$$

where $\mathcal{G}^* u(r_1)$ denotes the value of the function $(dD_s u - udk)/dm$ $(\in C_b([r_1, r_2)))$ at r_1.

In this case, for $u \in \mathcal{D}(\mathcal{G}^*)$,

$$\begin{cases} \mathcal{G}^* u(x) = \dfrac{dD_s u - udk}{dm}(x), & x \in I, \\ \mathcal{G}^* u(r_1) = (D_s u(r_1) - u(r_1)k^*(r_1))/m^*(r_1), & \textit{if } m^*(r_1) > 0. \end{cases} \tag{5.39}$$

PROOF. As for the "only if" part, (5.37) and the first identity of (5.39) can be shown as in the proof of Theorem 3.2 by making use of Lemma 5.4. (5.38) and the second identity of (5.39) follow from (5.35) and (5.36).

To prove the "if" part, take any function u satisfying conditions (5.37) and (5.38). We then let $f = \alpha u - ((dD_s u - udk)/dm)$, $v = R^*_\alpha f$ and $w = u - v$. Since $v \in \mathcal{D}(\mathcal{G}^*)$ and $\alpha v - ((dD_s v - vdk)/dm) = f$ by (5.39), we see that w is a bounded solution of (3.1). Since v satisfies (5.38) by the "only if" part, so does w.

We write $w = C_1 u_1 + C_2 u_2$ for some constants C_1, C_2. If r_2 is exit, then, by the second condition of (5.38) and the table of Section 3, we have $C_1 = 0$. From the first condition of (5.38),

$$C_2(D_s u_2(r_1) - u_2(r_1)k^*(r_1) - \mathcal{G}^* u_2(r_1) m^*(r_1)) = 0.$$

Since $\mathcal{G}^* u_2(r_1) = \lim_{x \downarrow r_1} (dD_s u_2 - u_2 dk)/dm(x) = \alpha u_2(r_1) > 0$ and $D_s u_2(r_1) < 0$, the quantity inside the brace is negative and hence $C_2 = 0$. If r_2 is either entrance or natural, then $u_1(r_2) = \infty$ by the table of Section 3 and consequently $C_1 = 0$. We also get $C_2 = 0$ as above. □

6. On diffusion extensions by Itô-McKean.

So far we have considered a minimal diffusion X^0 on $I = (r_1, r_2)$ with attached triplet (s, m, k) and its symmetric diffusion extensions X only to regular boundaries or their identification using Dirichlet forms. The collection of all such symmetric diffusions X will be denoted by $\mathrm{Ext}_{\mathrm{DF}}(X^0)$. By convention, $\mathrm{Ext}_{\mathrm{DF}}(X^0)$ includes X^0 but excludes $\dot{X}$ of Section 5.2 a one-pont extension of X^0 to $\dot{I}$. Our general boundary condition at a regular boundary obtained in Theorem 5.5 recovers the corresponding one in [**IM2**, Section 4.4, Section 4.7], where possible diffusion extensions X of X^0 to $[r_1, r_2]$ were investigated. The class of all such X considered in [**IM2**] will be denoted by $\mathrm{Ext}_{\mathrm{IM}}(X^0)$.

$\mathrm{Ext}_{\mathrm{IM}}(X^0)$ contains an extension X with a trivial boundary condition $\mathcal{G}u(r_i) + ku(r_i) = 0$, $0 \le k < \infty$, at a non-entrance boundary r_i, which means that X starting at r_i remains there until its lifetime. We modify such X by killing it at time σ_{r_i} whenever it is finite and discarding r_i from the state space. The resulting modified family is designated as $\mathrm{Ext}'_{\mathrm{IM}}(X^0)$. Notice that, when I has an entrance boundary, it persists to belong to the state space of any $X \in \mathrm{Ext}'_{\mathrm{IM}}(X^0)$ without being removed.

On the other hand, when I has entrance boundaries, they can be added to the state space of any $X \in \mathrm{Ext}_{\mathrm{DF}}(X^0)$ to produce a symmetric extension $\widetilde{X}$ of X but possessing the same Dirichlet form as X. For simplicity, we explain this procedure only for $X = X^0 \in \mathrm{Ext}_{\mathrm{DF}}(X^0)$. When r_1 is entrance, there exists a diffusion $\widetilde{X}^0 = (\widetilde{X}^0_t, \widetilde{\boldsymbol{P}}^0_x)$ on the extended state space $[r_1, r_2)$ such that

$$\widetilde{X}^0\big|_I = X^0 \quad \text{and} \quad \widetilde{\boldsymbol{P}}^0_{r_1}(\widetilde{X}^0_t \in I \text{ for any } t \in (0, \widetilde{\zeta}^0)) = 1. \tag{6.1}$$

In particular, the part process of $\widetilde{X}^0$ on I equals X^0 and the one-point set $\{r_1\}$ is polar for $\widetilde{X}^0$ so that $\widetilde{X}^0$ can be viewed as an m-symmetric diffusion extension of X^0 from I to $[r_1, r_2)$ but possessing the same associated Dirichlet form $(\mathcal{E}^0, \mathcal{F}^0)$ on $L^2(I; m)$ as X^0. In fact, since r_1 is entrance, it is easy to get from the table in the beginning of Section 3 the properties of X^0 that

$$\lim_{\epsilon \downarrow 0} \boldsymbol{E}^0_{r_1+}[e^{-\sigma_{r_1+\epsilon}}] = 1, \quad \lim_{\epsilon \downarrow 0} \boldsymbol{P}^0_{r_1+\epsilon}(\sigma_{r_1+} < \infty) = 0.$$

Using these properties, the above mentioned extension $\widetilde{X}^0$ of X^0 can be constructed as in [**IM2**, Problem 3.6.3] by defining $(\widetilde{X}^0_t, \widetilde{\boldsymbol{P}}^0_{r_1})$ to be a kind of limit of $(X^0_t, \boldsymbol{P}^0_{r_1+1/n})$ as $n \to \infty$ using the direct product $\prod_{n=1}^{\infty} \boldsymbol{P}^0_{r_1+1/n}$.

The C_b-generator of $\widetilde{X}^0$ can be readily identified as follows. Due to a 0-1 law, property (6.1) implies $\widetilde{\boldsymbol{E}}^0_{r_1}[e^{-\sigma_{r_1+}}] = 1$ (cf. [**IM2**, 3.3, 3a)]). Therefore the resolvent $\{\widetilde{R}^0_\alpha; \alpha > 0\}$ of $\widetilde{X}^0$ satisfies $\widetilde{R}^0_\alpha(\mathcal{B}_b(I)) \subset C_b([r_1, r_2))$. We introduce the C_b-generator $\widetilde{\mathcal{G}}^0$ of $\widetilde{X}^0$ by

$$\begin{cases} \mathcal{D}(\widetilde{\mathcal{G}}^0) = \widetilde{R}^0_\alpha(C_b([r_1, r_2)), \\ (\widetilde{\mathcal{G}}^0 u)(x) = \alpha u(x) - f(x), \quad \text{for } u = \widetilde{R}^0_\alpha f, \ f \in C_b([r_1, r_2)), \ x \in [r_1, r_2). \end{cases}$$

Then we see just as in the proof of Theorem 3.2 that $u \in \mathcal{D}(\widetilde{\mathcal{G}}^0)$ if and only if u satisfies the condition (3.17) with $C_b([r_1, r_2))$ in place of $C_b(I)$.

If both r_1 and r_2 are entrance, we can replace the above $\widetilde{X}^0$ by its further m-symmetric extension to $[r_1, r_2]$ so that the resulting diffusion $\widetilde{X}^0$ has the same Dirichlet form $(\mathcal{E}^0, \mathcal{F}^0)$ as X^0 and its C_b-generator is characterized as Theorem 3.2 but with $C_b([r_1, r_2])$ in place of $C_b(I)$.

We denote by $\widetilde{\mathrm{Ext}}_{\mathrm{DF}}(X^0)$ the collection of all $X \in \mathrm{Ext}_{\mathrm{DF}}(X^0)$ but being modified to be $\widetilde{X}$ as above by adding entrance boundaries whenever they are present. We can then readily verify that

$$\mathrm{Ext}'_{\mathrm{IM}}(X^0) = \widetilde{\mathrm{Ext}}_{\mathrm{DF}}(X^0). \tag{6.2}$$

Thus every element X of $\mathrm{Ext}'_{\mathrm{IM}}(X^0)$ is symmetric with respect to m or its extension m^* to regular boundaries. Furthermore we can verify that the transition function P_t of $X \in \mathrm{Ext}'_{\mathrm{IM}}(X^0)$ determines a Feller semigroup on the space $C_\infty(\widehat{I})$. Here $\widehat{I}$ denotes the interval obtained from I by adding the boundaries r_i to it only in the following two cases:

(I) r_i is regular and X is not absorbed at r_i,
(II) r_i is entrance.

$C_\infty(\widehat{I})$ denotes the space of all continuous functions on $\widehat{I}$ vanishing at infinity of $\widehat{I}$.

Indeed, combining general expressions (3.6), (5.28), (5.31) of the resolvent R_α of X with [**I1**, Theorem 5.14.1] and the table of Section 3, we see that R_α makes invariant the space of bounded continuous functions on I vanishing at a natural boundary. Therefore, on account of the observations we have made on the C_b-generator of X, we can conclude that $R_\alpha(C_\infty(\widehat{I})) \subset C_\infty(\widehat{I})$. Moreover $\lim_{\alpha\to\infty} \alpha R_\alpha f(x) = f(x)$, $x \in \widehat{I}$, $f \in C_\infty(\widehat{I})$, by the path continuity of X. Hence $\{R_\alpha; \alpha > 0\}$ becomes a strongly continuous contraction resolvent on $C_\infty(\widehat{I})$ and consequently $\{P_t; t > 0\}$ is a strongly continuous contraction semigroup on $C_\infty(\widehat{I})$ with infinitesimal generator $\widehat{\mathcal{G}} = \alpha I - R_\alpha^{-1}$, $\mathcal{D}(\widehat{\mathcal{G}}) = R_\alpha(C_\infty(\widehat{I}))$.

PROPOSITION 6.1. *The transition function $\{P_t; t > 0\}$ of $X \in \mathrm{Ext}'_{\mathrm{IM}}(X^0)$ determines a strongly continuous contraction semigroup on $C_\infty(\widehat{I})$.*

Let $\widehat{\mathcal{G}}$ be its infinitesimal generator. $u \in \mathcal{D}(\widehat{\mathcal{G}})$ if and only if

$$u \in C_\infty(\widehat{I}), \quad \frac{dD_s u - u dk}{dm} \in C_\infty(\widehat{I}), \tag{6.3}$$

and

$$D_s u(r_i) - u(r_i)k^*(r_i) = \widehat{\mathcal{G}}u(r_i)m^*(r_i), \quad \text{if } r_i \text{ is regular and } r_i \in \widehat{I}, \tag{6.4}$$

where $\widehat{\mathcal{G}}u(r_i)$ denotes the value of the function $(dD_s u - u dk)/dm$ ($\in C_\infty(\widehat{I})$) at r_i, and $m^*(r_i)$, $k^*(r_i)$ are non-negative parameters.

In this case, it holds for $u \in \mathcal{D}(\widehat{\mathcal{G}})$ that

$$
\begin{cases}
\widehat{\mathcal{G}}u(x) = \dfrac{dD_s u - u dk}{dm}(x), & x \in I, \\
\widehat{\mathcal{G}}u(r_i) = (D_s u(r_i) - u(r_i)k^*(r_i))/m^*(r_i), & \text{if } m^*(r_i) > 0.
\end{cases} \tag{6.5}
$$

Proof. The first assertion has been shown above. We have already characterized the C_b-generator of $X \in \mathrm{Ext}'_{\mathrm{IM}}(X^0)$ which leads us to the characterizaion (6.3), (6.4) of $\mathcal{D}(\widehat{\mathcal{G}})$ and the expression (6.5) of $\widehat{\mathcal{G}}$ because of the inclusion $C_\infty(\widehat{I}) \subset C_b(\widehat{I})$. They are quite analogous to those in Theorem 5.5. The boundary condition (6.4) involves only the regular boundaries belonging to $\widehat{I}$, while (6.3) contains implicitly the condition that $u(r_i) = 0$ if either r_i is exit or regular but not in $\widehat{I}$. □

Conversely, given a linear operator $\widehat{\mathcal{G}}$ on $C_\infty(\widehat{I})$ satisfying (6.3), (6.4) and (6.5), we can solve the equation $(\alpha - \widehat{\mathcal{G}})u = f$ in the space $C_\infty(\widehat{I})$ using the functions $g^*_\alpha(x, y)$ and $w(x)$ defined by (5.26) and (5.31), respectively. But it is not easy to verify that $\mathcal{D}(\widehat{\mathcal{G}})$ is dense in $C_\infty(\widehat{I})$ unless the associated Dirichlet form is utilized.

The Dirichlet form method gives us a direct and quickest way to construct the diffusion $X \in \mathrm{Ext}'_{\mathrm{IM}}(X^0)$. The constructed diffusion X has a Feller transition function by the above proposition.

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