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Erratum and addendum to "Commutators of C^∞-diffeomorphisms preserving a submanifold"

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Abstract. Let $D^\infty(M,N)$ be the group of C^∞-diffeomorphisms of a compact manifold M preserving a submanifold N. We give a condition for $D^\infty(M,N)$ to be uniformly perfect.

1. Introduction and statement of results.

This paper gives a correction of Theorem 1.4 in our previous paper [**1**] in expanded form and also subsequent supplements.

Let M be a connected C^∞-manifold without boundary and let $D_c^\infty(M)$ denote the group of all C^∞-diffeomorphisms of M which are isotopic to the identity through C^∞-diffeomorphisms with compact support. It is known that M. Herman [**5**] and W. Thurston [**9**] proved $D_c^\infty(M)$ is perfect, which means that every element of $D_c^\infty(M)$ can be represented by a product of commutators.

Let (M,N) be a manifold pair and $D_c^\infty(M,N)$ be the group of C^∞-diffeomorphisms of M preserving N which are isotopic to the identity through compactly supported C^∞-diffeomorphisms preserving N. In the previous paper [**1**], we proved that the group $D_c^\infty(M,N)$ is perfect if the dimension of N is positive. In this paper we consider the conditions for $D_c^\infty(M,N)$ to be uniformly perfect. A group G is said to be uniformly perfect if each element of G can be represented as a product of a bounded number of commutators of elements in G.

Let $\pi : D^\infty(M,N) \to D^\infty(N)$ be the map given by the restriction. First we shall prove the following.

THEOREM 1.1. *Let M be an m-dimensional compact manifold without boundary and N an n-dimensional C^∞-submanifold such that both groups $D^\infty(N)$*

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and $D_c^\infty(M-N)$ *are uniformly perfect. If the connected components of* $\ker\pi$ *are finite, then* $D^\infty(M,N)$ *is a uniformly perfect group for* $n \geq 1$.

Secondary we shall prove that the converse of Theorem 1.1 is valid when N is the disjoint union of circles in M.

Theorem 1.2. *Let* M *be an* m*-dimensional compact manifold without boundary and* N *be the disjoint union of circles in* M. *If the connected components of* $\ker\pi$ *are infinite, then* $D^\infty(M,N)$ *is not a uniformly perfect group.*

2. The proof of Theorem 1.1.

In this section we prove Theorem 1.1 and give some examples for this result. First we recall the uniform perfectness of $D_c^\infty(\boldsymbol{R}^m,\boldsymbol{R}^n)$ $(n \geq 1)$.

Theorem 2.1 ([**1**, Theorem 4.2]). $D_c^\infty(\boldsymbol{R}^m,\boldsymbol{R}^n)$ *is uniformly perfect for* $n \geq 1$. *In fact, any* $f \in D_c^\infty(\boldsymbol{R}^m,\boldsymbol{R}^n)$ *can be represented by two commutators of elements in* $D_c^\infty(\boldsymbol{R}^m,\boldsymbol{R}^n)$.

By the result of R. Palais [**7**], π is epimorphic and in fact it is a locally trivial fibration. Applying Theorem 2.1, we can prove Theorem 1.1 as follows.

Proof of Theorem 1.1. Take any element $f \in D^\infty(M,N)$ and put $\bar{f} = \pi(f)$. Since $D^\infty(N)$ is uniformly perfect by the assumption, there exists a bounded number k such that $\bar{f}$ can be written as

$$\bar{f} = \prod_{j=1}^{k}[\bar{g}_j,\bar{h}_j] \quad \text{for } \bar{g}_j,\bar{h}_j \in D^\infty(N).$$

Then there exist diffeomorphisms g_j and h_j of M preserving N such that $\pi(g_j) = \bar{g}_j$, $\pi(h_j) = \bar{h}_j$. Let $\hat{f} = (\prod_{j=1}^{k}[g_j,h_j])^{-1} \circ f$. Thus we have $\hat{f} \in \ker\pi$.

First we consider the case that $\hat{f}$ is isotopic to the identity in $\ker\pi$. Let $\hat{f}_t$ $(0 \leq t \leq 1)$ be an isotopy in $\ker\pi$ satisfying $\hat{f}_0 = id$ and $\hat{f}_1 = \hat{f}$. Take a tubular neighborhood W of N which is identified with the normal bundle of N. Let $q : W \to N$ be the bundle projection.

Let ℓ be the category number of N and $\mathcal{U} = \{U_i\}_{i=1}^{\ell+1}$ be an open covering of N such that each connected component of U_i is diffeomorphic to an open ball B^n in N. Then we may assume that U_i is diffeomorphic to B^n and $q^{-1}(U_i)$ is diffeomorphic to the product of open balls $B^n \times B^{m-n}$. Let $\{\varphi_i\}_{i=1}^{\ell+1}$ be a partition of unity subordinate to the covering $\mathcal{U}$.

Let $\hat{\varphi}_i$ be a C^∞-function on W defined by $\hat{\varphi}_i(p) = \varphi_i(q(p))$. Let Φ_i $(1 \leq i \leq$

$\ell+1$) be real valued functions on W given by $\Phi_i(p) = 1-(\hat{\varphi}_1(p)+\cdots+\hat{\varphi}_i(p))$, and let h_i be a smooth map from W to M defined by $h_i(p) = \hat{f}_{\Phi_i(p)}(p)$.

Let $\{x_1^i,\ldots,x_n^i,y_1^i,\ldots,y_{m-n}^i\}$ be a coordinate on $q^{-1}(U_i)$ such that $\{x_1^i,\ldots,x_n^i\}$ is a coordinate on U_i. Since $\hat{f}_t$ is an isotopy in $\ker\pi$, if $p\in N$ we have the following.

$$\begin{aligned} &x_j^i(\hat{f}_t(p)) = x_j^i(h_i(p)) = x_j^i(p) && (1\le i\le \ell+1,\ 1\le j\le n),\\ &y_k^i(\hat{f}_t(p)) = y_k^i(h_i(p)) = 0 && (1\le i\le \ell+1,\ n+1\le k\le m-n) \quad \text{and}\\ &\frac{\partial\hat{\varphi}_i}{\partial y_k^i}(p) = 0 && (1\le i\le \ell+1,\ n+1\le k\le m-n). \end{aligned}$$

Then the Jacobian matrix of h_i is non-singular on N. Thus h_i is diffeomorphic on a neighborhood of N. Since $\hat{f}_t$ is an isotopy in $\ker\pi$, from Chapter 8 Theorem 1.3 in [**6**], we can find $\hat{h}_i\in\ker\pi$ such that $\hat{f}\circ\hat{h}_i^{-1}$ is supported in $q^{-1}(\bigcup_{j=1}^i U_j)$, which coincides with h_i on a neighborhood of N.

Now we define $\tilde{f}_i\in\ker\pi$ ($i=1,2,\ldots,\ell+1$) supported in $q^{-1}(U_i)$ such that for $p\in M$,

$$\begin{aligned} &\tilde{f}_1(p) = \big(\hat{f}\circ\hat{h}_1^{-1}\big)(p) \quad \text{and}\\ &\tilde{f}_i(p) = \big((\tilde{f}_1\circ\cdots\circ\tilde{f}_{i-1})^{-1}\circ\hat{f}\circ\hat{h}_i^{-1}\big)(p) \quad (i=2,\ldots,\ell+1). \end{aligned}$$

Then we have $\hat{f} = \tilde{f}_1\circ\cdots\circ\tilde{f}_{\ell+1}$ on a neighborhood of N. From Theorem 2.1, each $\tilde{f}_i$ ($i=1,\ldots,\ell+1$) can be represented by two commutators of elements in $D_c^\infty(q^{-1}(U_i),U_i)$. Put $\tilde{f}_{\ell+2} = (\tilde{f}_1\circ\cdots\circ\tilde{f}_{\ell+1})^{-1}\circ\hat{f}$. Then $\tilde{f}_{\ell+2}$ is in $D_c^\infty(M-N)$. By the assumption of Theorem 1.1, there exist a bounded number s such that $\tilde{f}_{\ell+2}$ can be represented by s commutators of elements in $D_c^\infty(M-N)$. Hence f can be represented by $k+2(\ell+1)+s$ commutators of elements in $D^\infty(M,N)$.

Next we consider the case that $\hat{f}$ is not connected to the identity in $\ker\pi$. Let a be the number of the connected components of $\ker\pi$. Take elements, say $g_1,\ldots,g_a$, from each connected component of $\ker\pi$ and fix them. Then from Theorem 1.1 of [**1**], each g_i can be written by t_i commutators of elements in $D^\infty(M,N)$. Put $t=\max\{t_1,\ldots,t_a\}$. For any element $g\in\ker\pi$, there exists g_i ($i=1,\ldots,a$) satisfying that g and g_i are in the same connected component of $\ker\pi$. Since $g\circ(g_i)^{-1}$ is in the identity component of $\ker\pi$, g can be written by $2(\ell+1)+s+t$ commutators. Hence for any element $f\in D^\infty(M,N)$, above $\hat{f}$ can be written by $2(\ell+1)+s+t$ commutators. Therefore any $f\in D^\infty(M,N)$ can be written by $k+2(\ell+1)+s+t$ commutators of elements in $D^\infty(M,N)$. Since k,ℓ,s and t are bounded numbers, this completes the proof of Theorem 1.1. □

Remark 2.2. T. Tsuboi ([**10**], [**11**]) studied the uniform perfectness of $\mathrm{Diff}_c^r(M)$. He has proved that it is uniformly perfect if $1 \leq r \leq \infty$ ($r \neq \dim M+1$) and M is one of the following cases

(1) an odd dimensional compact manifold without boundary,
(2) an even dimensional compact manifold without boundary of dimension ≥ 6,
(3) the interior of a compact manifold W which has a handle decomposition only with handles of indices not greater than $(\dim W - 1)/2$.

Remark 2.3.

(1) The proof of Theorem 1.1 is valid when M is a compact C^∞-manifold with boundary and $N = \partial M$.
(2) The condition that the connected components of $\ker \pi$ are finite is necessary for the proof of Theorem 1.1 in general. The statement of Theorem 1.4 in [**1**] should be added this condition.

Example 2.4. (1) It is known that for $\mathrm{int}\, D^m$ and S^{m-1} ($m \geq 2$), $D_c^\infty(\mathrm{int}\, D^m)$ and $D^\infty(S^{m-1})$ are uniformly perfect groups (cf. Tsuboi [**10**]). Furthermore the number of the connected component of $D^\infty(D^m, rel\ S^{m-1})$ is finite for $m \neq 4$ (see Smale [**8**], Hatcher [**4**], and Cerf [**2**]), where $D^\infty(D^m, rel\ S^{m-1})$ denotes the subgroup of $D^\infty(D^m, S^{m-1})$ consisting of C^∞-diffeomorphisms of D^m which are the identity on S^{m-1}. Therefore, from Theorem 1.1 $D^\infty(D^m, S^{m-1})$ is a uniformly perfect group for $m \neq 4$.

(2) Since $D^\infty(S^2 \times [0,1], rel\ \partial(S^2 \times [0,1]))$ has two connected components (Hatcher [**4**]), $D^\infty(S^2 \times [0,1], \partial(S^2 \times [0,1]))$ is a uniformly perfect group.

(3) Let N be the Hopf link in S^3. Then the map $\pi : D^\infty(S^3, N) \to D^\infty(N)$ induces the surjective homomorphism $\pi_* : \pi_1(D^\infty(S^3, N), id) \to \pi_1(D^\infty(N), id)(\cong \boldsymbol{Z} \times \boldsymbol{Z})$ since any element in $\pi_1(D^\infty(N), id)$ is come from an element of $\pi_1(D^\infty(M, N), id)$ generated by a flow along a corresponding Seifert fibered space with N as fiber via π_*. Thus $D^\infty(S^3, rel\ N)$ is connected. Furthermore $D_c^\infty(S^3 - N)$ is uniformly perfect since $S^3 - N$ is diffeomorphic to $T^2 \times \boldsymbol{R}$. Therefore, from Theorem 1.1 $D^\infty(S^3, N)$ is a uniformly perfect group. This fact has been pointed out by Y. Mitsumatsu.

3. The proof of Theorem 1.2.

In this section we prove Theorem 1.2 and give some examples.

Put $G = D^\infty(M, N)$. We construct a quasimorphism of G to $\boldsymbol{Z}(\subset \boldsymbol{R})$. Then we can see that G is not uniformly perfect (cf. J. M. Gambaudo, É. Ghys [**3**]). Let $\hat{G}$ denote the universal covering group of G. Then $\hat{G}$ is expressed as the quotient group of PG by the normal subgroup of null-homotopic loops. Here PG is the path space of G starting from the identity with the product of paths given by the pointwise multiplication. By the assumption, N is the disjoint union of circles $S_1, \ldots, S_k$ in M. Take a base point e_i of S_i for each i.

Let $F = \{F_t | 0 \leq t \leq 1\}$ be an element of $\hat{G}$. Put $f_t = \pi(F_t)$ $(0 \leq t \leq 1)$. Then f_t is written as

$$f_t = (f_t^1, \ldots, f_t^k) \in D^\infty(S_1) \times \cdots \times D^\infty(S_k).$$

Let $p^i(t) = f_t^i(e_i) \in S_i$ $(i = 1, \ldots, k)$. Note that $p^i = \{p^i(t) | 0 \leq t \leq 1\}$ is a path in S_i starting from e_i.

Since S_i is a circle, we can identify the universal covering space $\hat{S}_i$ with the real line $\boldsymbol{R}$. Note that $p^i(0) = e_i$. Let $\hat{p}^i = \{\hat{p}^i(t)\}$ be the lifting of p^i with $\hat{p}^i(0) = 0$. Put

$$\psi^i(F) = \hat{p}^i(1).$$

Let n_i be the integer satisfying $n_i \leq \hat{p}^i(1) < n_i + 1$ $(1 \leq i \leq k)$. Define a map $\varphi^i : \hat{G} \to \boldsymbol{Z}$ to be $\varphi^i(F) = n_i$.

Now we prove that φ^i is a quasimorphism. Let $H = \{H_t | 0 \leq t \leq 1\}$ be another element of $\hat{G}$. Put $h_t = \pi(H_t)$ which is written as $h_t = (h_t^1, \ldots, h_t^k)$. Let $q^i(t) = h_t^i(e_i)$ and let $\hat{q}^i = \{\hat{q}^i(t)\}$ be the lifting of q^i with $\hat{q}^i(0) = 0$.

Let $F \sharp H$ be a path in G given by

$$(F \sharp H)_t = \begin{cases} F_{2t} & \left(0 \leq t \leq \dfrac{1}{2}\right) \\ H_{2t-1} \cdot F_1 & \left(\dfrac{1}{2} \leq t \leq 1\right). \end{cases}$$

If α is a path from x to y and β is a path from y to z, let $\alpha \vee \beta$ denote the path composition. Then

$$H \cdot F \simeq (H_0 \vee H) \cdot (F \vee F_1) = F \vee (H \cdot F_1) = F \sharp H. \tag{3.1}$$

Here $\alpha \simeq \beta$ means that the paths α and β in G with the same initial point and the terminal point are homotopic relative to the set of the initial point and the terminal point.

Since π is a group homomorphism, it follows from (3.1) that

$$h^i \cdot f^i \simeq f^i \sharp h^i \quad (i = 1, \ldots, k). \tag{3.2}$$

Here we need the following lemma.

Lemma 3.1. *Let $E : \boldsymbol{R} \to S^1$ denote the exponential map. Let $h = \{h_t |\ 0 \leq t \leq 1\}$ be a path in $D^\infty(S^1)$ such that $h_0 = 1$. For a real number a, put $\bar{a} = E(a)$. Set $q(t) = h_t(1)$ and $s(t) = h_t(\bar{a})$ $(0 \leq t \leq 1)$. Let $\hat{q}$ and $\hat{s}$ be the liftings of q and s such that $\hat{q}(0) = 0$ and $\hat{s}(0) = a$, respectively. Then $|\hat{s}(t) - \hat{q}(t) - a| < 1$.*

Proof. Assume that $\hat{s}(t_1) - \hat{q}(t_1) \geq a + 1$ for some t_1. Since $\hat{s}(0) - \hat{q}(0) = a$, there exists a real number $t_0 \in (0, t_1]$ with $\hat{s}(t_0) - \hat{q}(t_0) \in \boldsymbol{Z}$. Then

$$E(\hat{s}(t_0) - \hat{q}(t_0)) = s(t_0) \cdot q(t_0)^{-1} = 1.$$

Thus we have $h_{t_0}(\bar{a}) = h_{t_0}(1)$. Therefore $\bar{a} = 1$ and $s(t) = q(t)$. Since $\hat{s}(0) - \hat{q}(0) = a$, we have that $\hat{s}(t_1) - \hat{q}(t_1) = a$. This is a contradiction. We can argue similarly when $a - 1 < \hat{s}(t_1) - \hat{q}(t_1)$. Then Lemma 3.1 follows. □

Proof of Theorem 1.2 continued. Set $s^i(t) = h^i_t(p^i(1))$ and let $\hat{s}^i$ be the lifting of s^i such that $\hat{s}^i(0) = \hat{p}^i(1)$. By Lemma 3.1 we have

$$\left|\hat{s}^i(t) - \hat{q}^i(t) - \hat{p}^i(1)\right| < 1. \tag{3.3}$$

It follows from the definition that

$$(f^i \sharp h^i)_t(e_i) = \begin{cases} p^i(2t) & \left(0 \leq t \leq \dfrac{1}{2}\right) \\ s^i(2t-1) & \left(\dfrac{1}{2} \leq t \leq 1\right). \end{cases}$$

Combining (3.2) we have

$$\psi^i(H \cdot F) = \psi^i(F \sharp H) = \psi^i(F) + \psi^i(H \cdot F_1) = \hat{p}^i(1) + \hat{s}^i(1).$$

By (3.3) we obtain

$$|\psi^i(H \cdot F) - \psi^i(H) - \psi^i(F)| < 1.$$

Thus we have that

$$|\varphi^i(H \cdot F) - \varphi^i(F) - \varphi^i(H)| < 3.$$

Therefore φ^i is a quasimorphism.

Let ΩG denote the loop group of G. Then ΩG is a normal subgroup of PG and $PG/\Omega G \cong \hat{G}/\pi_0(\Omega G) \cong G$. Put $\varphi = (\varphi^1, \cdots, \varphi^k)$. Note that the map φ restricted to $\pi_0(\Omega G)$ coincides with the homomorphism

$$\pi_0(\Omega G) \cong \pi_1(G) \xrightarrow{\pi_*} \pi_1(D^\infty(N)) \cong \boldsymbol{Z}^k.$$

Consider the following homotopy exact sequence:

$$\pi_1(G) \xrightarrow{\pi_*} \pi_1(D^\infty(N)) \to \pi_0(\ker\ \pi) \to 1.$$

Then the cokernel of the homomorphism π_* is isomorphic to $\boldsymbol{Z}^k/\varphi(\Omega G)$. Since the connected components of $\ker \pi$ are infinite, $\boldsymbol{Z}^k/\varphi(\Omega G)$ has infinite cyclic group $\boldsymbol{Z}^\ell$ of rank ℓ as a direct summand for a positive number ℓ. Since π is a locally trivial fibration, π induces the epimorphism of the universal coverings. Then the induced map $\varphi_* : G \to \boldsymbol{Z}^\ell$ is a non-trivial map and each component of φ_* is a quasimorphism. Hence we have a quasimorphism of G to $\boldsymbol{Z}$. This completes the proof of Theorem 1.2. □

Example 3.2. (1) We see that the group $D^\infty(S^1 \times [0,1], \partial(S^1 \times [0,1]))$ satisfies the conditions of Theorem 1.2. Therefore $D^\infty(S^1 \times [0,1], \partial(S^1 \times [0,1]))$ is not a uniformly perfect group. This fact has been pointed out by F. L. Roux privately.

(2) Let $M^n = S^1 \times L^{n-1}$ be the product manifold of the circle S^1 and an orientable closed manifold L^{n-1} $(n \geq 3)$. Take a base point $(x, y) \in S^1 \times L$. Let $N = S_1 \cup S_2$ be the union of two circles in M^n as follows. $S_1 = S^1 \times \{y\}$, and S_2 is a circle in $\{x\} \times L$ not passing through (x, y). Let h be a diffeomorphism of M^n obtained by rotating one time along S_1 supporting in a small neighborhood of S_1. Then we can prove that h generates elements of infinite order in $\pi_0(D^\infty(M^n\, rel\, N))$. Therefore $D^\infty(M, N)$ is not uniformly perfect from Theorem 1.2.

References

[1] K. Abe and K. Fukui, Commutators of C^∞-diffeomorphisms preserving a submanifold, J. Math. Soc. Japan, **61** (2009), 427–436.

[2] J. Cerf, Topologie de certains espaces de plongements, Bull. Soc. Math. France, **89** (1961), 227–380.

[3] J.-M. Gambaudo and É. Ghys, Commutators and diffeomorphisms of surfaces, Ergodic Theory Dynam. Systems, **24** (2004), 1591–1617.

[4] A. E. Hatcher, A proof of the Smale conjecture, Diff$(S^3) \simeq$ O(4), Ann. of Math. (2), **117** (1983), 553–607.

[5] M.-R. Herman, Simplicité du groupe des difféomorphismes de classe C^∞, isotopes à l'identité, du tore de dimension n, C. R. Acad. Sci. Paris Sér. A-B, **273** (1971), 232–234.
[6] M. W. Hirsch, Differential Topology, Grad. Text in Math., **33**, Springer-Verlag, New York, 1976.
[7] R. S. Palais, Local triviality of the restriction map for embeddings, Comment. Math. Helv., **34** (1960), 305–312.
[8] S. Smale, Diffeomorphisms of the 2-sphere, Proc. Amer. Math. Soc., **10** (1959), 621–626.
[9] W. Thurston, Foliations and groups of diffeomorphisms, Bull. Amer. Math. Soc., **80** (1974), 304–307.
[10] T. Tsuboi, On the uniform perfectness of diffeomorphism groups, In: Groups of Diffeomorphisms, (ed. Y. Mitsumatsu *et al.*), Adv. Stud. Pure Math., **52**, Math. Soc. Japan, 2008, pp. 505–524.
[11] T. Tsuboi, On the uniform perfectness of the groups of diffeomorphisms of even-dimensional manifolds, Comment. Math. Helv., **87** (2012), 141–185.

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