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Fixed point subalgebras of lattice vertex operator algebras by an automorphism of order three

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Abstract. We study the fixed point subalgebra of a certain class of lattice vertex operator algebras by an automorphism of order 3, which is a lift of a fixed-point-free isometry of the underlying lattice. We classify the irreducible modules for the subalgebra. Moreover, the rationality and the C_2-cofiniteness of the subalgebra are established. Our result contains the case of the vertex operator algebra associated with the Leech lattice.

1. Introduction.

Let V be a vertex operator algebra. For an automorphism g of V of finite order, the space $V^g = \{v \in V \mid gv = v\}$ of fixed points is a subalgebra of V called an orbifold of the vertex operator algebra V. It is conjectured in [**7**] that every irreducible V^g-module is contained in some irreducible untwisted or twisted V-module. It is also conjectured that if V is rational and C_2-cofinite, then so is V^g. These conjectures have important meanings in the theory of vertex operator algebras. However, it is difficult to investigate an orbifold in general, even if the original vertex operator algebra V is well understood.

In the case where V is the lattice vertex operator algebra V_Γ associated with a positive definite even lattice Γ and the automorphism g is a canonical lift θ of the -1 isometry $\alpha \mapsto -\alpha$ of the lattice Γ, the orbifold $V_\Gamma^\theta = V_\Gamma^+$ has been studied extensively. In fact, the representation theory of V_Γ^+, that is, the classification of irreducible modules [**3**], [**19**] and the determination of fusion rules [**1**], [**4**], together with the C_2-cofiniteness [**2**], [**38**] of V_Γ^+ are established.

In this paper we study an orbifold of a certain class of lattice vertex operator algebras by an automorphism of order 3. We start with a lattice $L \cong \sqrt{2}$ (A_2-lattice) and a fixed-point-free isometry τ of L of order 3. There are 12 cosets of

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L in its dual lattice $L^\perp$. Using an even $\mathbb{Z}_2 \times \mathbb{Z}_2$-code C of length ℓ and a self-orthogonal $\mathbb{Z}_3$-code D of the same length, we construct a positive definite even lattice $L_{C\times D} \subset (L^\perp)^{\oplus \ell}$ of rank 2ℓ from the 12 cosets of L in $L^\perp$. We also consider an action of τ on $\mathbb{Z}_2 \times \mathbb{Z}_2$. The isometry τ induces a fixed-point-free isometry $(\tau, \ldots, \tau)$ of $L_{C\times D}$ provided that C is invariant under the corresponding action of $(\tau, \ldots, \tau)$ on $(\mathbb{Z}_2 \times \mathbb{Z}_2)^\ell$. For simplicity of notation, we denote $(\tau, \ldots, \tau)$ by τ also.

Our main concern is to classify the irreducible modules for the orbifold $V_{L_{C\times D}}^\tau$ of the lattice vertex operator algebra $V_{L_{C\times D}}$ by an automorphism τ of order 3 which is a lift of the isometry τ of $L_{C\times D}$. The vertex operator algebra $V_{L_{C\times D}}$ is simple, rational, C_2-cofinite, and of CFT type. The dual lattice $(L_{C\times D})^\perp$ of $L_{C\times D}$ is equal to $L_{C^\perp\times D^\perp}$, where $C^\perp$ (resp. $D^\perp$) is the dual code of C (resp. D). Then $V_{L_{(\lambda+C)\times(\gamma+D)}}$, $\lambda + C \in C^\perp/C$, $\gamma + D \in D^\perp/D$ form a complete set of representatives of equivalence classes of irreducible $V_{L_{C\times D}}$-modules. Such a $V_{L_{(\lambda+C)\times(\gamma+D)}}$ is τ-stable if and only if $\lambda \in C$. One can also construct irreducible τ^i-twisted $V_{L_{C\times D}}$-modules $V_{L_{C\times D}}^{T,\eta}(\tau^i)$, $\eta \in D^\perp \pmod D$ for $i = 1, 2$ by the method of [**12**], [**29**].

The orbifold $V_{L_{C\times D}}^\tau$ is a simple vertex operator algebra. The following is a list of known irreducible $V_{L_{C\times D}}^\tau$-modules. Let $\zeta_3 = \exp(2\pi\sqrt{-1}/3)$.

(1) $V_{L_{C\times(\gamma+D)}}(\varepsilon) = \{u \in V_{L_{C\times(\gamma+D)}} \mid \tau u = \zeta_3^\varepsilon u\}$, $\gamma + D \in D^\perp/D$, $\varepsilon \in \mathbb{Z}_3$.
(2) $V_{L_{(\lambda+C)\times(\gamma+D)}}$, $0 \neq \lambda + C \in (C^\perp/C)_{\equiv_\tau}$, $\gamma + D \in D^\perp/D$, where $(C^\perp/C)_{\equiv_\tau}$ is the set of τ-orbits in $C^\perp/C$.
(3) $V_{L_{C\times D}}^{T,\eta}(\tau^i)[\varepsilon] = \{u \in V_{L_{C\times D}}^{T,\eta}(\tau^i) \mid \tau^i u = \zeta_3^\varepsilon u\}$, $\eta \in D^\perp \pmod D$, $\varepsilon \in \mathbb{Z}_3$, $i = 1, 2$.

These irreducible $V_{L_{C\times D}}^\tau$-modules are inequivalent each other [**20**], [**34**]. The above mentioned conjecture says that any irreducible $V_{L_{C\times D}}^\tau$-module is isomorphic to one of these.

In our argument we deal with not only simple current extension [**13**] but also certain nonsimple current extension. Simple current extension is rather easy, whereas nonsimple current extension is complicated and difficult to study. In order to avoid the difficulty, we restrict ourselves to the special case where C is a τ-invariant self-dual $\mathbb{Z}_2 \times \mathbb{Z}_2$-code with minimum weight at least 4 and D is a self-dual $\mathbb{Z}_3$-code. In this case the lattice $L_{C\times D}$ is unimodular and there is a unique irreducible $V_{L_{C\times D}}$-module, namely, $V_{L_{C\times D}}$ itself. Likewise, there is a unique irreducible τ^i-twisted $V_{L_{C\times D}}$-module $V_{L_{C\times D}}^{T,\mathbf{0}}(\tau^i)$, $i = 1, 2$, where $\mathbf{0}$ is the zero codeword. Under this hypothesis we have the following theorem (Theorem 7.10).

Theorem. *Suppose C is a τ-invariant self-dual $\mathbb{Z}_2\times\mathbb{Z}_2$-code with minimum weight at least* 4 *and D is a self-dual $\mathbb{Z}_3$-code. Then the vertex operator algebra*

$V_{L_{C\times D}}^{\tau}$ is simple, rational, C_2-cofinite, and of CFT type. Moreover, every irreducible $V_{L_{C\times D}}^{\tau}$-module is isomorphic to one of $V_{L_{C\times D}}(\varepsilon)$, $V_{L_{C\times D}}^{T,\mathbf{0}}(\tau^i)[\varepsilon]$, $\varepsilon \in \mathbb{Z}_3$, $i = 1, 2$.

One of the most important examples of orbifold is the fixed point subalgebra V_{Λ}^{θ} of the Leech lattice vertex operator algebra V_{Λ} by the automorphism θ of order 2. This orbifold was first studied by I. Frenkel, J. Lepowsky and A. Meurman, and in fact it was used for the construction of the moonshine vertex operator algebra $V^{\natural}$ [**22**]. We note that the Leech lattice Λ can be expressed as $L_{C\times D}$ for some C and D which satisfy the hypothesis of the theorem (Remark 7.11).

A remarkable property of $V^{\natural}$ is that its automorphism group $\mathrm{Aut}\, V^{\natural}$ is isomorphic to the Monster $\mathbb{M}$. The construction of $V^{\natural}$ in [**22**] is based on a $2B$-element of $\mathbb{M}$. In [**22**, Introduction], it is suggested that an analogous construction may be possible for some appropriate elements in $\mathbb{M}$ of order 3, 5, 7, and 13. The classification of irreducible modules, the determination of fusion rules, the rationality and the C_2-cofiniteness for the orbifold V_{Λ}^{g} by such an element g should play an important role in those expected construction. This is the motivation for the present work.

The organization of the paper is as follows. Section 2 is devoted to the preliminaries. In Section 2.1 we collect basic terminology for later use. In Section 2.2 we introduce the lattice $L_{C\times D}$ and study its properties. In Section 2.3 we introduce a central extension $\hat{L}_{C\times D}$ of $L_{C\times D}$ by a group $\langle \kappa_{36} \rangle$ of order 36 and discuss an action of a lift of the isometry τ of the lattice $L_{C\times D}$. In Section 2.4 we study the vertex operator algebra $V_{L_{C\times D}}$ and its irreducible modules. The automorphism τ of $\hat{L}_{C\times D}$ naturally induces an automorphism of $V_{L_{C\times D}}$ of order 3, which is again denoted by τ.

In Section 3 we discuss in detail the irreducible τ^i-twisted $V_{L_{C\times D}}$-modules $V_{L_{C\times D}}^{T,\eta}(\tau^i)$, $i = 1, 2$, which are obtained by the method of [**12**], [**29**]. We describe those irreducible τ^i-twisted $V_{L_{C\times D}}$-modules as modules for $(V_L^{\tau})^{\otimes \ell}$ (Theorem 3.13). The classification of irreducible modules for the orbifold V_L^{τ} was accomplished in [**36**]. Our argument here is based on the result.

In Section 4 we determine certain fusion rules for V_L^{τ} (Proposition 4.5), which will be necessary in Section 5. In fact, these fusion rules are crucial for our arguments.

The proof of the main theorem is divided into three steps. In Section 5 we begin with the classification of irreducible modules for the orbifold $V_{L^{\oplus \ell}}^{\tau}$ (Proposition 5.3). This is the case where both of C and D are the zero code. The rationality and the C_2-cofiniteness of $V_{L^{\oplus \ell}}^{\tau}$ are also obtained. Moreover, some of the fusion rules are computed (Proposition 5.7).

In Section 6 we classify the irreducible modules for $V_{L_{\mathbf{0}\times D}}^{\tau}$ (Theorem 6.2), which is the case where C is the zero code $\{\mathbf{0}\}$. In this case only simple current extension is involved and the argument is relatively straightforward. The rationality and the C_2-cofiniteness of $V_{L_{\mathbf{0}\times D}}^{\tau}$ (Theorem 6.2), together with some of the fusion rules are also obtained (Proposition 6.3).

Section 7 consists of two subsections. In Section 7.1 we use Zhu's theory to study the irreducible $V_{L\oplus\ell}^{\tau}$-modules contained in a $V_{L_{C(\mu)\times\mathbf{0}}}^{\tau}$-module, where $C(\mu)$ is the $\mathbb{Z}_2\times\mathbb{Z}_2$-code generated by μ and $\tau(\mu)$. The results obtained here will be necessary in Section 7.2. We do not discuss the classification of irreducible modules nor the rationality for the vertex operator algebra $V_{L_{C(\mu)\times\mathbf{0}}}^{\tau}$. Note that $V_{L_{C(\mu)\times\mathbf{0}}}^{\tau}$ is a nonsimple current extension of $V_{L\oplus\ell}^{\tau}$.

In Section 7.2 we study the orbifold $V_{L_{C\times D}}^{\tau}$ and prove the main theorem (Theorem 7.10) under the hypothesis that C is a τ-invariant self-dual $\mathbb{Z}_2\times\mathbb{Z}_2$-code with minimum weight at least 4 and D is a self-dual $\mathbb{Z}_3$-code. We need to assume that D is self-dual for the proof of Proposition 7.8. Our argument fails if the minimum weight of C is 2 (Remark 7.2). The case $L_{C\times D}\cong E_8$-lattice is such an example (Remark 7.12).

We should make a few remarks on the simplicity and the CFT type property. Most of the vertex operator algebras discussed in this paper are clearly simple and of CFT type. In such a case we omit the proof of these properties.

This paper is the detailed version of our paper [**37**].

2. Preliminaries.

Throughout this paper, $\zeta_n=\exp(2\pi\sqrt{-1}/n)$ is a primitive n-th root of unity for a positive integer n. For simplicity, $0,1$ and 2 are sometimes understood to be elements of $\mathbb{Z}_3$.

2.1. Basic terminology.

Let g be an automorphism of a vertex operator algebra $(V,Y,\mathbf{1},\omega)$ of finite order T. Set $V^r=\{v\in V\mid gv=\zeta_T^r v\}$, so that $V=\bigoplus_{r\in\mathbb{Z}/T\mathbb{Z}}V^r$.

For subsets A,B of V and a subset X of a weak g-twisted V-module M, set $A\cdot B=\mathrm{span}_{\mathbb{C}}\{u_nv\mid u\in A, v\in B, n\in\mathbb{Z}\}$ and $A\cdot X=\mathrm{span}_{\mathbb{C}}\{u_nw\mid u\in A, w\in X, n\in(1/T)\mathbb{Z}\}$. Then it follows that $(A\cdot B)\cdot X=A\cdot(B\cdot X)$ by [**32**, Lemma 3.12] and [**36**, Lemmas 2.5 and 2.6].

Let $\mathbb{N}$ be the set of nonnegative integers. A $(1/T)\mathbb{N}$-graded weak g-twisted V-module here is called an admissible g-twisted V-module in [**14**]. Without loss we can shift the grading of a $(1/T)\mathbb{N}$-graded weak g-twisted V-module M so that $M(0)\neq 0$ if $M\neq 0$. We call such an $M(0)$ the *top level* of M.

A vertex operator algebra V is said to be *rational* if every $\mathbb{N}$-graded weak V-module is a direct sum of irreducible $\mathbb{N}$-graded weak V-modules. If the dimension of the quotient space $V/\mathrm{span}_{\mathbb{C}}\{u_{-2}v\mid u,v\in V\}$ is finite, V is said to be C_2-

cofinite [**39**]. If $V = \bigoplus_{n=0}^{\infty} V_n$ and $V_0 = \mathbb{C}\mathbf{1}$, then V is said to be of *CFT type.* Here $V_n = \{u \in V \mid \omega_1 u = nu\}$ is the homogeneous subspace of weight n. If V is C_2-cofinite and of CFT type, then the classification of irreducible V-modules means the classification of irreducible weak V-modules [**2**, Proposition 5.6 and Corollary 5.7].

For $h \in \operatorname{Aut} V$ and a weak (resp. $(1/T)\mathbb{N}$-graded weak) g-twisted V-module (M, Y_M), we define a weak (resp. $(1/T)\mathbb{N}$-graded weak) $h^{-1}gh$-twisted V-module $(M \circ h, Y_{M\circ h})$ by $M \circ h = M$ as vector spaces and $Y_{M\circ h}(u, x) = Y(hu, x)$. If M is irreducible, so is $M \circ h$.

Let G be an automorphism group of V and V^G the vertex operator subalgebra of G-invariants of V. A set $\mathcal{S}$ of irreducible V-modules is said to be G-*stable* if for any $M \in \mathcal{S}$ and $h \in G$ there exists $W \in \mathcal{S}$ such that $M \circ h \cong W$. An irreducible V-module M is said to be G-stable if $M \circ g \cong M$ for all $g \in G$. It is shown in [**17**, Theorem 4.4] that if V is simple and G is of finite order, then V^G is simple.

We denote by $I_V\binom{M^3}{M^1\ M^2}$ the set of all intertwining operators of type $\binom{M^3}{M^1\ M^2}$ [**21**]. Let $\mathcal{M}$ be the set of all irreducible V-modules up to isomorphism and $\mathbb{Z}\mathcal{M}$ be a free $\mathbb{Z}$-module with basis $\mathcal{M}$. For $M^1, M^2 \in \mathcal{M}$,

$$M^1 \times M^2 = \sum_{M^3\in\mathcal{M}} \dim_{\mathbb{C}} I_V\binom{M^3}{M^1\ M^2} M^3 \in \mathbb{Z}\mathcal{M}$$

is the fusion rule. We write $\sum_{M\in\mathcal{M}} S_M M \geq \sum_{M\in\mathcal{M}} T_M M$ when $S_M \geq T_M$ for all $M \in \mathcal{M}$.

2.2. Lattice $L_{C\times D}$.

We follow the notation in [**10**], [**24**], [**25**], [**36**]. Let $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ be $\sqrt{2}$ times an ordinary root lattice of type A_2 and let $\{\beta_1, \beta_2\}$ be a $\mathbb{Z}$-basis of L such that $\langle\beta_1, \beta_1\rangle = \langle\beta_2, \beta_2\rangle = 4$ and $\langle\beta_1, \beta_2\rangle = -2$. Set $\beta_0 = -\beta_1 - \beta_2$. Let τ be an isometry of L induced by the permutation $\beta_1 \mapsto \beta_2 \mapsto \beta_0 \mapsto \beta_1$. Then τ is fixed-point-free and of order 3.

There are 12 cosets of L in its dual lattice $L^\perp = \{\alpha \in \mathbb{Q} \otimes_{\mathbb{Z}} L \mid \langle\alpha, L\rangle \subset \mathbb{Z}\}$. These 12 cosets are parametrized by $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_3$. Let $(L^\perp)^{\oplus\ell}$ be an orthogonal sum of ℓ copies of $L^\perp$. We shall construct a lattice $L_{C\times D}$ in $(L^\perp)^{\oplus\ell}$ from those 12 cosets of L by using a $\mathbb{Z}_2 \times \mathbb{Z}_2$-code C and a $\mathbb{Z}_3$-code D. We shall also introduce certain isometry groups of $(L^\perp)^{\oplus\ell}$.

First, τ can be extended to an isometry of $L^\perp$. Let H_ℓ be a direct product of ℓ copies of the group $\langle\tau\rangle$ generated by τ. Each element $g = (g_1, \dots, g_\ell)$ of H_ℓ transforms $\alpha = \alpha_1 + \cdots + \alpha_\ell \in (L^\perp)^{\oplus\ell}$ as $g(\alpha) = g_1(\alpha_1) + \cdots + g_\ell(\alpha_\ell)$, where $g_s \in \langle\tau\rangle$ and α_s is the s-th component of α. For convenience, we denote $(\tau, \dots, \tau) \in H_\ell$ simply by τ also. A symmetric group $\mathfrak{S}_\ell$ of degree ℓ acts on $(L^\perp)^{\oplus\ell}$

by permuting the components. Let G_ℓ be an isometry group of $(L^\perp)^{\oplus \ell}$ generated by H_ℓ and $\mathfrak{S}_\ell$, which is a semidirect product $H_\ell \rtimes \mathfrak{S}_\ell$ of H_ℓ by $\mathfrak{S}_\ell$.

Now, we discuss a $\mathbb{Z}_2 \times \mathbb{Z}_2$-code and a $\mathbb{Z}_3$-code. A $\mathbb{Z}_2 \times \mathbb{Z}_2$-code of length ℓ means an additive subgroup of $\mathcal{K}^\ell$, where $\mathcal{K} = \{0, a, b, c\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is Klein's four-group. We call it a $\mathcal{K}$-code also. Note that $b + c = a$ in $\mathcal{K}$. For $x, y \in \mathcal{K}$, define

$$x \cdot y = \begin{cases} 1 & \text{if } x \neq y,\ x \neq 0,\ y \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$x \cdot y \equiv m_1 n_2 + m_2 n_1 \qquad (\mathrm{mod}\ 2\mathbb{Z}) \tag{2.1}$$

if $x = m_1 c + m_2 b$, $y = n_1 c + n_2 b \in \mathcal{K}$ with $m_1, m_2, n_1, n_2 \in \mathbb{Z}$.

For $\lambda = (\lambda_1, \dots, \lambda_\ell)$, $\mu = (\mu_1, \dots, \mu_\ell) \in \mathcal{K}^\ell$, let $\langle \lambda, \mu \rangle_{\mathcal{K}} = \sum_{i=1}^{\ell} \lambda_i \cdot \mu_i \in \mathbb{Z}_2$. The orthogonal form $(\lambda, \mu) \mapsto \langle \lambda, \mu \rangle_{\mathcal{K}}$ on $\mathcal{K}^\ell$ was used in [**24**], [**27**]. For a $\mathcal{K}$-code C of length ℓ, we define its dual code by

$$C^\perp = \{\lambda \in \mathcal{K}^\ell \mid \langle \lambda, \mu \rangle_{\mathcal{K}} = 0 \text{ for all } \mu \in C\}.$$

A $\mathcal{K}$-code C is said to be *self-orthogonal* if $C \subset C^\perp$ and self-dual if $C = C^\perp$. For $\lambda = (\lambda_1, \dots, \lambda_\ell) \in \mathcal{K}^\ell$, its *support* is defined to be $\mathrm{supp}_{\mathcal{K}}(\lambda) = \{i \mid \lambda_i \neq 0\}$. The cardinality of $\mathrm{supp}_{\mathcal{K}}(\lambda)$ is called the *weight* of λ. We denote the weight of λ by $\mathrm{wt}_{\mathcal{K}}(\lambda)$. In the case $\ell = 1$, we have $\mathrm{wt}_{\mathcal{K}}(x) = 0$ or 1 according to $x = 0$ or $x \in \{a, b, c\}$. A $\mathcal{K}$-code C is said to be *even* if $\mathrm{wt}_{\mathcal{K}}(\lambda)$ is even for every $\lambda \in C$.

We consider an action of τ on $\mathcal{K}$ such that $\tau(0) = 0$, $\tau(a) = b$, $\tau(b) = c$, and $\tau(c) = a$. Moreover, we consider a componentwise action of H_ℓ on $\mathcal{K}^\ell$, so that τ acts on $\mathcal{K}^\ell$ by $\tau(\lambda_1, \dots, \lambda_\ell) = (\tau(\lambda_1), \dots, \tau(\lambda_\ell))$. Then G_ℓ acts on $\mathcal{K}^\ell$ naturally. We denote by $(\mathcal{K}^\ell)_{\equiv_\tau}$ the set of all τ-orbits in $\mathcal{K}^\ell$. For simplicity of notation, we sometimes denote a τ-orbit in $\mathcal{K}^\ell$ by its representative $\lambda \in \mathcal{K}^\ell$.

The first assertion of the next lemma is [**27**, Lemma 2.8]. The second assertion follows from the fact that $\langle \lambda, \tau(\lambda) \rangle_{\mathcal{K}} \equiv \mathrm{wt}_{\mathcal{K}}(\lambda)\ (\mathrm{mod}\ 2\mathbb{Z})$ for $\lambda \in \mathcal{K}^\ell$.

Lemma 2.1. *Let C be a $\mathcal{K}$-code of length ℓ.*

(1) *If C is even, then C is self-orthogonal.*
(2) *If C is τ-invariant, then C is even if and only if C is self-orthogonal.*

A $\mathbb{Z}_3$-code of length ℓ is a subspace of the vector space $\mathbb{Z}_3^\ell$. For $\gamma = (\gamma_1, \dots, \gamma_\ell)$, $\delta = (\delta_1, \dots, \delta_\ell) \in \mathbb{Z}_3^\ell$, we consider the ordinary inner product $\langle \gamma, \delta \rangle_{\mathbb{Z}_3} = \sum_{i=1}^{\ell} \gamma_i \delta_i \in \mathbb{Z}_3$. The dual code $D^\perp$ of a $\mathbb{Z}_3$-code D is defined to be

$$D^{\perp} = \{\gamma \in \mathbb{Z}_3^{\ell} \mid \langle \gamma, \delta \rangle_{\mathbb{Z}_3} = 0 \text{ for all } \delta \in D\}.$$

Then D is said to be *self-orthogonal* if $D \subset D^{\perp}$ and *self-dual* if $D = D^{\perp}$.

We define the support and the weight of $\gamma = (\gamma_1, \dots, \gamma_\ell) \in \mathbb{Z}_3^{\ell}$ in the same way as before. Thus $\mathrm{supp}_{\mathbb{Z}_3}(\gamma) = \{i \mid \gamma_i \neq 0\}$ and $\mathrm{wt}_{\mathbb{Z}_3}(\gamma)$ is the cardinality of $\mathrm{supp}_{\mathbb{Z}_3}(\gamma)$. Note that $\mathrm{wt}_{\mathbb{Z}_3}(\gamma) \equiv \langle \gamma, \gamma \rangle \pmod{3\mathbb{Z}}$. Then the following lemma holds.

LEMMA 2.2. *Let D be a self-orthogonal $\mathbb{Z}_3$-code of length ℓ. Then* $\mathrm{wt}_{\mathbb{Z}_3}(\delta - \gamma) \equiv \mathrm{wt}_{\mathbb{Z}_3}(\delta) \pmod{3\mathbb{Z}}$ *for any $\gamma \in D$ and $\delta \in D^{\perp}$.*

We consider the trivial action of τ on $\mathbb{Z}_3$, that is, $\tau(j) = j$ for $j \in \mathbb{Z}_3$. Then H_ℓ acts trivially on $\mathbb{Z}_3^{\ell}$ and G_ℓ acts on $\mathbb{Z}_3^{\ell}$ naturally.

Take a $\mathbb{Z}$-basis $\tilde{\beta}_1 = \beta_1/2$, $\tilde{\beta}_2 = (\beta_1 - \beta_2)/6$ of $L^{\perp}$. Note that $\{2\tilde{\beta}_1, 6\tilde{\beta}_2\}$ is a $\mathbb{Z}$-basis of L. For $\alpha = m_1\tilde{\beta}_1 + m_2\tilde{\beta}_2$, $\beta = n_1\tilde{\beta}_1 + n_2\tilde{\beta}_2 \in L^{\perp}$, we have

$$\langle \alpha, \beta \rangle = m_1 n_1 + \frac{m_1 n_2 + m_2 n_1}{2} + \frac{m_2 n_2}{3}. \tag{2.2}$$

We also have $\tau(\tilde{\beta}_1) = \tilde{\beta}_1 - 3\tilde{\beta}_2$ and $\tau(\tilde{\beta}_2) = \tilde{\beta}_1 - 2\tilde{\beta}_2$. We use the same notation as in [**10**], [**24**], [**25**], [**36**] to denote the 12 cosets $L^{(x,i)}$, $x \in \mathcal{K}$, $i \in \mathbb{Z}_3$ of L in its dual lattice $L^{\perp}$. For each $x \in \mathcal{K}$ we assign $\beta(x) \in L^{\perp}$ by $\beta(0) = 0$, $\beta(a) = \beta_2/2$, $\beta(b) = \beta_0/2$, and $\beta(c) = \beta_1/2$. Then

$$L^{(x,i)} = \beta(x) + i\frac{-\beta_1 + \beta_2}{3} + L. \tag{2.3}$$

Since $\tilde{\beta}_1 = \beta(c) \in L^{(c,0)}$ and $\tilde{\beta}_2 = \beta(b) + (-\beta_1 + \beta_2)/3 + \beta_1 \in L^{(b,1)}$, we can describe $L^{(x,i)}$ by using the basis $\{\tilde{\beta}_1, \tilde{\beta}_2\}$ of $L^{\perp}$.

LEMMA 2.3. *For $x \in \mathcal{K}$ and $i \in \mathbb{Z}_3$,*

$$L^{(x,i)} = \big\{m_1\tilde{\beta}_1 + m_2\tilde{\beta}_2 \in L^{\perp} \mid x = m_1 c + m_2 b \text{ in } \mathcal{K} \text{ and } i = m_2 + 3\mathbb{Z}\big\}.$$

We also have the following lemma.

LEMMA 2.4. *Let $\alpha \in L^{(x,i)}$ and $\beta \in L^{(y,j)}$ with $x, y \in \mathcal{K}$, $i, j \in \mathbb{Z}_3$.*

(1) $\langle \alpha, \beta \rangle \equiv x \cdot y/2 + ij/3 \pmod{\mathbb{Z}}$.
(2) $\langle \alpha, \alpha \rangle \equiv \mathrm{wt}_{\mathcal{K}}(x) - 2i^2/3 \pmod{2\mathbb{Z}}$.

For $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathcal{K}^\ell$ and $\gamma = (\gamma_1, \ldots, \gamma_\ell) \in \mathbb{Z}_3^\ell$, let

$$L_{(\lambda,\gamma)} = L^{(\lambda_1,\gamma_1)} \oplus \cdots \oplus L^{(\lambda_\ell,\gamma_\ell)} \subset (L^\perp)^{\oplus \ell}.$$

Moreover, for $\mu \in \mathcal{K}^\ell, \delta \in \mathbb{Z}_3^\ell$, $P \subset \mathcal{K}^\ell$, and $Q \subset \mathbb{Z}_3^\ell$, set

$$L_{\mu\times Q} = \bigcup_{\gamma\in Q} L_{(\mu,\gamma)}, \quad L_{P\times\delta} = \bigcup_{\lambda\in P} L_{(\lambda,\delta)}, \quad L_{P\times Q} = \bigcup_{\lambda\in P,\gamma\in Q} L_{(\lambda,\gamma)}.$$

For a $\mathcal{K}$-code C of length ℓ and a $\mathbb{Z}_3$-code D of the same length, $L_{C\times D}$ is an additive subgroup of $(L^\perp)^{\oplus\ell}$. However, $L_{C\times D}$ is not an integral lattice in general. In the case where $C = \mathcal{K}^\ell$ and $D = \mathbb{Z}_3^\ell$, $L_{C\times D}$ coincides with $(L^\perp)^{\oplus\ell}$. If $C = \{\mathbf{0}\}$ and $D = \{\mathbf{0}\}$, then $L_{\{\mathbf{0}\}\times\{\mathbf{0}\}} = L_{(\mathbf{0},\mathbf{0})} = L^{\oplus\ell}$, where $\mathbf{0} = (0, \ldots, 0)$. In the case of $\ell = 1$, we note that $L_{\mathcal{K}\times\mathbf{0}} = \mathbb{Z}\tilde{\beta}_1 + \mathbb{Z}(3\tilde{\beta}_2)$, $L_{\mathbf{0}\times\mathbb{Z}_3} = \mathbb{Z}(2\tilde{\beta}_1) + \mathbb{Z}(2\tilde{\beta}_2)$, and $L = L_{\mathbf{0}\times\mathbf{0}} = \mathbb{Z}(2\tilde{\beta}_1) + \mathbb{Z}(6\tilde{\beta}_2)$.

Let $(L_{C\times D})^\perp = \{\alpha \in (\mathbb{Q} \otimes_\mathbb{Z} L)^{\oplus\ell} \mid \langle\alpha, L_{C\times D}\rangle \subset \mathbb{Z}\}$. The following lemma is a consequence of Lemma 2.4 (1).

Lemma 2.5. $(L_{C\times D})^\perp = L_{C^\perp\times D^\perp}$.

Thus $L_{C\times D}$ is an integral lattice if and only if both of C and D are self-orthogonal. The first assertion of the next lemma follows from Lemma 2.4 (2). The second assertion is a special case of the above lemma (see also [**24**, Theorems 5.6, 5.7]).

Lemma 2.6. (1) *If C is even and D is self-orthogonal, then $L_{C\times D}$ is an even lattice.*
(2) *If C and D are self-dual, then $L_{C\times D}$ is a unimodular lattice.*

2.3. Central extensions $\hat{L}_{C\times D}$, $\hat{L}_{C\times D,\tau^i}$, $i = 1, 2$.

Suppose C is a τ-invariant even $\mathcal{K}$-code of length ℓ and D is a self-orthogonal $\mathbb{Z}_3$-code of the same length. Then $L_{C\times D}$ is a positive definite even lattice by Lemma 2.6. The isometry τ of $L^\perp$ permutes the cosets $L^{(x,i)}$, $x \in \mathcal{K}$, $i \in \mathbb{Z}_3$ of L in $L^\perp$. In fact, $\tau(L^{(x,i)}) = L^{(\tau(x),i)}$ by our definition of the action of τ on $L^\perp$, $\mathcal{K}$ and $\mathbb{Z}_3$ introduced in Section 2.2. In particular, τ induces an isometry of $L_{C\times D}$, for we are assuming that C is τ-invariant. Note that τ is fixed-point-free on $L_{C\times D}$. We also have $g(L_{C\times D}) = L_{g(C)\times g(D)}$ for $g \in G_\ell$.

For any positive integer n, let $\langle\kappa_n\rangle$ be a cyclic group of order n with generator κ_n. We assume that ${\kappa_n}^{n/m} = \kappa_m$ if m is a divisor of n. We shall construct three central extensions $\widehat{L}_{C\times D}$ and $\hat{L}_{C\times D,\tau^i}$, $i = 1, 2$ of $L_{C\times D}$ by $\langle\kappa_{36}\rangle$ which will be used in later sections. We realize each of these central extensions as a subgroup of a central extension of $(L^{\perp})^{\oplus \ell}$ by $\langle \kappa_{36} \rangle$.

Define $\mathbb{Z}$-bilinear forms $\varepsilon_1, \varepsilon_2, \varepsilon_2', c_1, c_2, c_2' \colon L^{\perp} \times L^{\perp} \to \mathbb{Z}/36\mathbb{Z}$ as follows. For $\alpha = m_1\tilde{\beta}_1 + m_2\tilde{\beta}_2$, $\beta = n_1\tilde{\beta}_1 + n_2\tilde{\beta}_2 \in L^{\perp}$, set

$$\varepsilon_1(\alpha, \beta) = 27m_1n_1 + 27m_2n_1 + 9m_2n_2 + 36\mathbb{Z}, \tag{2.4}$$

$$\varepsilon_2(\alpha, \beta) = 6m_1n_1 + 6m_2n_1 + 14m_2n_2 + 36\mathbb{Z}, \tag{2.5}$$

$$\varepsilon_2'(\alpha, \beta) = 6m_1n_1 + 15m_1n_2 + 27m_2n_1 + 14m_2n_2 + 36\mathbb{Z}, \tag{2.6}$$

and

$$c_1(\alpha, \beta) = \varepsilon_1(\alpha, \beta) - \varepsilon_1(\beta, \alpha) = 9m_1n_2 + 27m_2n_1 + 36\mathbb{Z}, \tag{2.7}$$

$$c_2(\alpha, \beta) = \varepsilon_2(\alpha, \beta) - \varepsilon_2(\beta, \alpha) = 30m_1n_2 + 6m_2n_1 + 36\mathbb{Z}, \tag{2.8}$$

$$c_2'(\alpha, \beta) = \varepsilon_2'(\alpha, \beta) - \varepsilon_2'(\beta, \alpha) = 24m_1n_2 + 12m_2n_1 + 36\mathbb{Z}. \tag{2.9}$$

We also set

$$\begin{aligned} c_0(\alpha, \beta) &= 18\langle \alpha, \beta \rangle + 36\mathbb{Z} \\ &= 18m_1n_1 + 9m_1n_2 + 9m_2n_1 + 6m_2n_2 + 36\mathbb{Z}. \end{aligned} \tag{2.10}$$

All of these $\mathbb{Z}$-bilinear forms are τ-invariant. Since ε_1 is $\mathbb{Z}$-bilinear, it is a 2-cocycle. Let $\widehat{L^{\perp}} = \langle \kappa_{36} \rangle \times L^{\perp}$. We simply write $\kappa_{36}^p e^{\alpha}$ for $(\kappa_{36}^p, \alpha) \in \widehat{L^{\perp}}$. In particular, $\kappa_{36}^p = (\kappa_{36}^p, 0)$ and $e^{\alpha} = (1, \alpha)$. Define a multiplication on the set $\widehat{L^{\perp}}$ by

$$\left(\kappa_{36}^p e^{\alpha}\right) \cdot \left(\kappa_{36}^q e^{\beta}\right) = \kappa_{36}^{p+q+\varepsilon_1(\alpha,\beta)} e^{\alpha+\beta}. \tag{2.11}$$

Take ε_2 (resp. ε_2') in place of ε_1. Then we obtain a multiplicative group $\widehat{L^{\perp}}_{\tau}$ (resp. $\widehat{L^{\perp}}_{\tau^2}$). We use the same notation $\kappa_{36}^p e^{\alpha}$ to denote its element. As to its multiplication, we write $\times_{\tau}$ (resp. $\times_{\tau^2}$) so that

$$\left(\kappa_{36}^p e^{\alpha}\right) \times_{\tau} \left(\kappa_{36}^q e^{\beta}\right) = \kappa_{36}^{p+q+\varepsilon_2(\alpha,\beta)} e^{\alpha+\beta}, \tag{2.12}$$

$$\left(\kappa_{36}^p e^{\alpha}\right) \times_{\tau^2} \left(\kappa_{36}^q e^{\beta}\right) = \kappa_{36}^{p+q+\varepsilon_2'(\alpha,\beta)} e^{\alpha+\beta}. \tag{2.13}$$

For $a, b \in \widehat{L^{\perp}}$ or $\widehat{L^{\perp}}_{\tau^i}$, $i = 1, 2$, we simply write ab for the product in the group when there is no ambiguity. Define $^{-} \colon \widehat{L^{\perp}} \to L^{\perp}$ (resp. $\widehat{L^{\perp}}_{\tau^i} \to L^{\perp}$) by $\overline{\kappa_{36}^p e^{\alpha}} = \alpha$. Then $\widehat{L^{\perp}}$ (resp. $\widehat{L^{\perp}}_{\tau}$, $\widehat{L^{\perp}}_{\tau^2}$) is a central extension of $L^{\perp}$ by $\langle \kappa_{36} \rangle$ with associated commutator map c_1 (resp. c_2, c_2') ([**22**, Sections 5.1, 5.2], [**30**, Section 6.4]).

Note that

$$e^{\alpha}e^{\beta} = e^{\alpha+\beta} \text{ in } \widehat{L^{\perp}} \tag{2.14}$$

for $\alpha, \beta \in L_{\mathbf{0}\times\mathbb{Z}_3} = \mathbb{Z}(2\tilde{\beta}_1) + \mathbb{Z}(2\tilde{\beta}_2)$ by (2.11).

Define an automorphism of the group $\widehat{L^{\perp}}$ (resp. $\widehat{L^{\perp}}_{\tau^i}$, $i = 1, 2$) of order 3 by

$$\begin{aligned} \kappa_{36} &\mapsto \kappa_{36}, \\ e^{\alpha} &\mapsto e^{\tau(\alpha)} \end{aligned} \tag{2.15}$$

for $\alpha \in L^{\perp}$. Since ε_1 (resp. ε_2, ε_2') is τ-invariant, the map is in fact an automorphism of the group $\widehat{L^{\perp}}$ (resp. $\widehat{L^{\perp}_{\tau}}$, $\widehat{L^{\perp}}_{\tau^2}$) of order 3. By abuse of notation, we denote it by τ also.

REMARK 2.7. In [**11**, Remark 2.2], three bilinear forms ε_0, c_0 and c_0^{ν} were considered. Apply [**11**, (2.9), (2.10), (2.13)] to $L^{\perp}$ in place of L with $\nu = \tau$ or τ^2, $p = 3$ and $q = 36$. Then the bilinear form c^0 of [**11**, (2.9)] is identical with our c^0. Moreover, ε_0 and c_0^{ν} become

$$\begin{aligned} \varepsilon_0(\alpha, \beta) &= 30\langle \alpha, \tau(\beta)\rangle + 36\mathbb{Z} \\ &= 21m_1n_1 + 21m_2n_1 + 31m_2n_2 + 36\mathbb{Z}, \end{aligned} \tag{2.16}$$

$$\begin{aligned} \varepsilon_0'(\alpha, \beta) &= 30\langle \alpha, \tau^2(\beta)\rangle + 36\mathbb{Z} \\ &= 21m_1n_1 + 21m_1n_2 + 31m_2n_2 + 36\mathbb{Z}, \end{aligned} \tag{2.17}$$

$$\begin{aligned} c_0^{\tau}(\alpha, \beta) &= 12\langle \tau(\alpha) + 2\tau^2(\alpha), \beta\rangle + 36\mathbb{Z} \\ &= 18m_1n_1 + 30m_1n_2 + 24m_2n_1 + 30m_2n_2 + 36\mathbb{Z}, \end{aligned} \tag{2.18}$$

$$\begin{aligned} c_0^{\tau^2}(\alpha, \beta) &= 12\langle \tau^2(\alpha) + 2\tau^4(\alpha), \beta\rangle + 36\mathbb{Z} \\ &= 18m_1n_1 + 24m_1n_2 + 30m_2n_1 + 30m_2n_2 + 36\mathbb{Z} \end{aligned} \tag{2.19}$$

for $\alpha = m_1\tilde{\beta}_1 + m_2\tilde{\beta}_2$, $\beta = n_1\tilde{\beta}_1 + n_2\tilde{\beta}_2 \in L^{\perp}$. Here we write ε_0' for ε_0 of [**11**, (2.13)] in the case $\nu = \tau^2$. Note that q of [**11**, Remark 2.2] should be a multiple of 12 by (2.10). We take $q = 36$ so that every coefficient of m_in_j in (2.16) and (2.17) is an integer. These bilinear forms are related to our ones as follows.

$$\begin{aligned}
\varepsilon_0(\alpha,\beta) &= \varepsilon_1(\alpha,\beta) - \varepsilon_2(\alpha,\beta),\\
\varepsilon_0'(\alpha,\beta) &= \varepsilon_1(\alpha,\beta) - \varepsilon_2'(\alpha,\beta),\\
c_0(\alpha,\beta) &\equiv c_1(\alpha,\beta) - 36\langle\alpha,\tau(\beta)\rangle \pmod{36\mathbb{Z}},\\
c_0^{\tau}(\alpha,\beta) &\equiv c_2(\alpha,\beta) + 36\langle\alpha,\tau(\beta)\rangle \pmod{36\mathbb{Z}},\\
c_0^{\tau^2}(\alpha,\beta) &\equiv c_2'(\alpha,\beta) + 36\langle\alpha,\tau(\beta)\rangle \pmod{36\mathbb{Z}}.
\end{aligned}$$

We extend the $\mathbb{Z}$-bilinear forms $\varepsilon_1, \varepsilon_2, \varepsilon_2', c_1, c_2, c_2', c_0, \varepsilon_0, \varepsilon_0', c_0^{\tau}, c_0^{\tau^2}$ on $L^{\perp}$ to $(L^{\perp})^{\oplus\ell}$ naturally. For example,

$$\varepsilon_1(\alpha,\beta) = \sum_{s=1}^{\ell} \varepsilon_1(\alpha^{(s)},\beta^{(s)})$$

for $\alpha = \sum_{s=1}^{\ell}\alpha^{(s)}$, $\beta = \sum_{s=1}^{\ell}\beta^{(s)} \in (L^{\perp})^{\oplus\ell}$, where $\alpha^{(s)}$ and $\beta^{(s)}$ are in the s-th entry of $(L^{\perp})^{\oplus\ell}$. These $\mathbb{Z}$-bilinear forms are all τ-invariant.

Remark 2.8. If $\langle\alpha,\tau(\beta)\rangle \in \mathbb{Z}$, then Remark 2.7 implies that $c_1(\alpha,\beta) = c_0(\alpha,\beta)$, $c_2(\alpha,\beta) = c_0^{\tau}(\alpha,\beta)$, $c_2'(\alpha,\beta) = c_0^{\tau^2}(\alpha,\beta)$ and

$$\begin{aligned}
\varepsilon_0(\alpha,\beta) - \varepsilon_0(\beta,\alpha) &= c_0(\alpha,\beta) - c_0^{\tau}(\alpha,\beta),\\
\varepsilon_0'(\alpha,\beta) - \varepsilon_0'(\beta,\alpha) &= c_0(\alpha,\beta) - c_0^{\tau^2}(\alpha,\beta).
\end{aligned} \tag{2.20}$$

Let $(\widehat{L^{\perp}})^{\ell}$ be a direct product of ℓ copies of $\widehat{L^{\perp}}$ and let T be a subgroup in the center of $(\widehat{L^{\perp}})^{\ell}$ generated by $\kappa_{36}^{(r)}(\kappa_{36}^{(s)})^{-1}$, $1 \le r,s \le \ell$, where $\kappa_{36}^{(s)}$ denotes $\kappa_{36} \in L^{\perp}$ in the s-th entry of $(\widehat{L^{\perp}})^{\ell}$. We consider $(\widehat{L^{\perp}})^{\ell}/T$. For simplicity of notation, we write $e^{\alpha_1+\cdots+\alpha_\ell}$ for $(e^{\alpha_1},\ldots,e^{\alpha_\ell})T$ and κ_{36}^{p} for $(\kappa_{36}^{(1)})^{p}T$ in $(\widehat{L^{\perp}})^{\ell}/T$. Then any element of $(\widehat{L^{\perp}})^{\ell}/T$ can be expressed uniquely in the form $\kappa_{36}^{p}e^{\alpha}$ with $p \in \mathbb{Z}/36\mathbb{Z}$ and $\alpha \in (L^{\perp})^{\oplus\ell}$.

By (2.11) we have

$$e^{\alpha}e^{\beta} = \kappa_{36}^{\varepsilon_1(\alpha,\beta)}e^{\alpha+\beta} \tag{2.21}$$

in $(\widehat{L^{\perp}})^{\ell}/T$. For $\kappa_{36}^{p}e^{\alpha} \in (\widehat{L^{\perp}})^{\ell}/T$, let $\overline{\kappa_{36}^{p}e^{\alpha}} = \alpha \in (L^{\perp})^{\oplus\ell}$. Then

$$1 \to \langle\kappa_{36}\rangle \to (\widehat{L^{\perp}})^{\ell}/T \xrightarrow{-} (L^{\perp})^{\oplus\ell} \to 1$$

is a central extension of $(L^{\perp})^{\oplus \ell}$ by $\langle \kappa_{36} \rangle$ with associated commutator map c_1. We denote $(\widehat{L^{\perp}})^{\ell}/T$ by $\widehat{(L^{\perp})^{\oplus \ell}}$ also.

By (2.15), G_ℓ acts on the group $\widehat{(L^{\perp})^{\oplus \ell}}$ naturally. In particular, $\tau = (\tau, \ldots, \tau)$ acts on $\widehat{(L^{\perp})^{\oplus \ell}}$ as an automorphism of order 3. We have $\overline{g(a)} = g(\bar{a})$ for $g \in G_\ell$ and $a \in \widehat{(L^{\perp})^{\oplus \ell}}$.

By (2.10) and Remark 2.7, we have

$$\kappa_{36}^{c_1(\alpha,\beta)} = \kappa_{36}^{c_0(\alpha,\beta)} = \kappa_2^{\langle \alpha,\beta \rangle}$$

if $\langle \alpha, \tau(\beta) \rangle$ is an integer. This is the case for $\alpha, \beta \in L_{C \times D}$, since $L_{C \times D}$ is a τ-invariant integral lattice.

For any subset Q of $(L^{\perp})^{\oplus \ell}$, we set $\hat{Q} = \{a \in \widehat{(L^{\perp})^{\oplus \ell}} \mid \bar{a} \in Q\}$. In particular, $\hat{L}_{C \times D} = \{a \in \widehat{(L^{\perp})^{\oplus \ell}} \mid \bar{a} \in L_{C \times D}\}$. Then

$$1 \to \langle \kappa_{36} \rangle \to \hat{L}_{C \times D} \xrightarrow{\ \bar{}\ } L_{C \times D} \to 1 \tag{2.22}$$

is a central extension of $L_{C \times D}$ by $\langle \kappa_{36} \rangle$ with associated commutator map c_1.

Replace $\widehat{L^{\perp}}$ with $\widehat{L^{\perp}}_{\tau}$ (resp. $\widehat{L^{\perp}}_{\tau^2}$) and ε_1 with ε_2 (resp. ε_2') in the above argument. Then we obtain a central extension $\widehat{(L^{\perp})^{\oplus \ell}}_{\tau}$ (resp. $\widehat{(L^{\perp})^{\oplus \ell}}_{\tau^2}$) of $(L^{\perp})^{\oplus \ell}$ by $\langle \kappa_{36} \rangle$ with associated commutator map c_2 (resp. c_2'). We have $e^{\alpha} e^{\beta} = \kappa_{36}^{\varepsilon_2(\alpha,\beta)} e^{\alpha+\beta}$ in $\widehat{(L^{\perp})^{\oplus \ell}}_{\tau}$ by (2.12) (resp. $e^{\alpha} e^{\beta} = \kappa_{36}^{\varepsilon_2'(\alpha,\beta)} e^{\alpha+\beta}$ in $\widehat{(L^{\perp})^{\oplus \ell}}_{\tau^2}$ by (2.13)) for $\alpha, \beta \in (L^{\perp})^{\oplus \ell}$. We also consider $\hat{Q}_{\tau^i} = \{a \in \widehat{(L^{\perp})^{\oplus \ell}}_{\tau^i} \mid \bar{a} \in Q\}$, $i = 1, 2$ similarly for a subset Q of $(L^{\perp})^{\oplus \ell}$.

Note that τ induces an automorphism of $\hat{L}_{C \times D}$ of order 3. Let $\theta \in \operatorname{Aut} \hat{L}_{C \times D}$ be a distinguished lift of the isometry -1 of $L_{C \times D}$ defined by [**22**, (10.3.12)]

$$\theta \colon \hat{L}_{C \times D} \to \hat{L}_{C \times D}; \quad a \mapsto a^{-1} \kappa_2^{\langle \bar{a}, \bar{a} \rangle / 2}. \tag{2.23}$$

Then $\theta^2 = 1, \overline{\theta(a)} = -\bar{a}$ for $a \in \hat{L}_{C \times D}$, and $\theta(\kappa_{36}) = \kappa_{36}$. Moreover, $\theta\tau = \tau\theta$ since $\langle \cdot\, , \cdot \rangle$ is τ-invariant. Thus we have obtained the following lemma.

LEMMA 2.9. *$\hat{L}_{C \times D}$ is a central extension of $L_{C \times D}$ by $\langle \kappa_{36} \rangle$ with commutation relation*

$$ab = \kappa_2^{\langle \bar{a}, \bar{b} \rangle} ba, \quad a, b \in \hat{L}_{C \times D}. \tag{2.24}$$

Moreover, τ and θ are automorphisms of $\hat{L}_{C \times D}$ such that $\tau^3 = \theta^2 = 1$, $\tau(\kappa_{36}) = \theta(\kappa_{36}) = \kappa_{36}$, $\overline{\tau(a)} = \tau(\bar{a})$, $\overline{\theta(a)} = -\bar{a}$ for $a \in \hat{L}_{C \times D}$, and $\theta\tau = \tau\theta$.

The sublattice $L_{\mathbf{0}\times D}$ of $L_{C\times D}$ has nice properties. For $\alpha,\beta\in L_{\mathbf{0}\times D}$, we have $e^\alpha e^\beta = e^{\alpha+\beta}$ by (2.14) and $\tau(e^\alpha) = e^{\tau(\alpha)}$ by (2.15). Furthermore, $\theta(e^\alpha) = e^{-\alpha}$ by (2.23), since $\langle\alpha,\alpha\rangle\in 4\mathbb{Z}$ for $\alpha\in L_{\mathbf{0}\times D}$.

Now, set $\mathbb{C}\{(L^\perp)^{\oplus\ell}\} = \mathbb{C}[\widehat{(L^\perp)^{\oplus\ell}}]/(\kappa_{36}-\zeta_{36})\mathbb{C}[\widehat{(L^\perp)^{\oplus\ell}}]$, which is a twisted group algebra of $(L^\perp)^{\oplus\ell}$. By abuse of notation, we denote the image of $e^\alpha\in\widehat{(L^\perp)^{\oplus\ell}}$ in $\mathbb{C}\{(L^\perp)^{\oplus\ell}\}$ by the same symbol e^α for $\alpha\in(L^\perp)^{\oplus\ell}$. The automorphisms τ and θ also induce automorphisms of $\mathbb{C}\{(L^\perp)^{\oplus\ell}\}$. We use the same symbols τ and θ to denote those automorphisms. For any subset P of $(L^\perp)^{\oplus\ell}$, we set $\mathbb{C}\{P\} = \mathrm{span}_{\mathbb{C}}\{e^\alpha \mid \alpha\in P\}\subset\mathbb{C}\{(L^\perp)^{\oplus l}\}$.

The following lemma is a direct consequence of Lemma 2.9.

Lemma 2.10. *$\mathbb{C}\{L_{C\times D}\}$ is a twisted group algebra of $L_{C\times D}$ such that*

$$e^\alpha e^\beta = (-1)^{\langle\alpha,\beta\rangle} e^\beta e^\alpha,\quad \alpha,\beta\in L_{C\times D}.$$

Moreover, τ and θ are automorphisms of $\mathbb{C}\{L_{C\times D}\}$ such that $\tau^3=\theta^2=1$ and $\theta\tau=\tau\theta$.

2.4. Vertex operator algebra $V_{L_{C\times D}}$.

We use the standard notation for the vertex operator algebra $(V_\Gamma, Y(\,\cdot\,,x))$ associated with a positive definite even lattice Γ and its module $V_{\Gamma^\perp}$ ([**22**, Chapter 8], [**30**, Section 6.4]). Let C be a τ-invariant even $\mathcal{K}$-code of length ℓ and D be a self-orthogonal $\mathbb{Z}_3$-code of the same length. Thus the lattice $L_{C\times D}$ is a τ-invariant positive definite even lattice by Lemma 2.6. We use the twisted group algebra $\mathbb{C}\{L_{C\times D}\}$ of Lemma 2.10 for the vertex operator algebra $V_{L_{C\times D}} = M(1)\otimes\mathbb{C}\{L_{C\times D}\}$. We identify $V_{L^{\oplus\ell}}$ with $V_L^{\otimes\ell}$ and $V_{(L^\perp)^{\oplus\ell}}$ with $V_{L^\perp}^{\otimes\ell}$.

Recall the action of the group G_ℓ on $(L^\perp)^{\oplus\ell}$, $\mathcal{K}^\ell$ and $\mathbb{Z}_3^\ell$ discussed in Section 2.2. For $g\in G_\ell$, define a linear isomorphism on $V_{(L^\perp)^{\oplus\ell}} = M(1)\otimes\mathbb{C}\{(L^\perp)^{\oplus\ell}\}$ by

$$\alpha^1(-n_1)\cdots\alpha^k(-n_k)e^\beta\mapsto(g\alpha^1)(-n_1)\cdots(g\alpha^k)(-n_k)g(e^\beta).$$

For simplicity of notation, we denote it by g also. Then

$$g(Y_{(L_{C\times D})^\perp}(u,x)v) = Y_{(L_{g(C)\times g(D)})^\perp}(gu,x)gv$$

for $u\in V_{L_{C\times D}}$ and $v\in V_{(L_{C\times D})^\perp}$. Hence $g\colon V_{L_{C\times D}}\mapsto V_{L_{g(C)\times g(D)}}$ is an isomorphism of vertex operator algebras. In particular, τ is an automorphism of $V_{L_{C\times D}}$. Our purpose is the classification of irreducible modules for the fixed point subalgebra $V_{L_{C\times D}}^\tau = \{u\in V_{L_{C\times D}}\mid\tau u=u\}$ of $V_{L_{C\times D}}$ by the automorphism τ.

We also note that

$$g\colon V_{L_{(\lambda+C)\times(\gamma+D)}} \mapsto V_{L_{(g(\lambda)+g(C))\times(g(\gamma)+g(D))}}$$

for $\lambda \in C^{\perp}$ and $\gamma \in D^{\perp}$ is a map from $V_{L_{C\times D}}$-modules to $V_{L_{g(C)\times g(D)}}$-modules. In the case where C and D are g-invariant, we have

$$V_{L_{(\lambda+C)\times(\gamma+D)}} \circ g \cong g^{-1}\big(V_{L_{(\lambda+C)\times(\gamma+D)}}\big) = V_{L_{(g^{-1}(\lambda)+C)\times(g^{-1}(\gamma)+D)}}. \tag{2.25}$$

By [**8**, Theorem 3.1] and Lemma 2.5, we have the following proposition.

Proposition 2.11. *$\{V_{L_{(\lambda+C)\times(\gamma+D)}} \mid \lambda + C \in C^{\perp}/C, \gamma + D \in D^{\perp}/D\}$ is a set of all irreducible $V_{L_{C\times D}}$-modules up to isomorphism.*

The following lemma is a straightforward consequence of (2.25).

Lemma 2.12. *We have $V_{L_{(\lambda+C)\times(\gamma+D)}} \circ \tau \cong V_{L_{(\tau^{-1}(\lambda)+C)\times(\gamma+D)}}$. In particular, $V_{L_{(\lambda+C)\times(\gamma+D)}}$ is τ-stable if and only if $\lambda \in C$.*

For $\varepsilon = 0,1,2$, let $V_{L_{C\times(\gamma+D)}}(\varepsilon) = \{u \in V_{L_{C\times(\gamma+D)}} \mid \tau u = \zeta_3^{\varepsilon} u\}$. These are irreducible $V_{L_{C\times D}}^{\tau}$-modules.

The following proposition is clear.

Lemma 2.13. *As $(V_L)^{\otimes \ell}$-modules, we have*

$$V_{L_{(\lambda+C)\times(\gamma+D)}} = \bigoplus_{\mu\in\lambda+C,\delta\in\gamma+D} V_{L_{(\mu,\delta)}}.$$

The fusion rules for $V_{L_{C\times D}}$ are known by [**11**, Corollary 12.10].

Lemma 2.14. *For $\lambda^1, \lambda^2 \in C^{\perp}$ and $\gamma^1, \gamma^2 \in D^{\perp}$, we have*

$$V_{L_{(\lambda^1+C)\times(\gamma^1+D)}} \times V_{L_{(\lambda^2+C)\times(\gamma^2+D)}} = V_{L_{(\lambda^1+\lambda^2+C)\times(\gamma^1+\gamma^2+D)}}.$$

3. Irreducible τ^i-twisted $V_{L_{C\times D}}$-modules, $i = 1, 2$.

As before, we assume that C is a τ-invariant even $\mathcal{K}$-code of length ℓ and D is a self-orthogonal $\mathbb{Z}_3$-code of the same length. We shall describe a decomposition of every irreducible τ^i-twisted $V_{L_{C\times D}}$-module constructed by the method of [**12**], [**29**] into a direct sum of irreducible $(V_L^{\tau})^{\otimes \ell}$-modules, $i = 1, 2$. The argument in the τ^2-twisted case is parallel to that in the τ-twisted case. Thus we deal with mainly the τ-twisted ones.

By our construction $\widehat{L^{\oplus \ell}}$ (resp. $\widehat{L^{\oplus \ell}}_{\tau}$) is a subgroup of $\hat{L}_{C\times D}$ (resp. $\hat{L}_{C\times D,\tau}$).

In [**10**], [**36**], we have considered irreducible τ-twisted V_L-modules $V_L^{T_{\chi_j}}(\tau)$, $j = 0,1,2$. In order to apply the results obtained in these previous papers, we need to examine the relation between $\hat{L}$ (resp. $\hat{L}_\tau$) of [**10**] and $\widehat{L^{\oplus\ell}}$ (resp. $\widehat{L^{\oplus\ell}}_\tau$).

In [**10**, (2.1)], $\hat{L}$ was a central extension of L by $\langle\kappa_6\rangle$ with trivial associated commutator map $L \times L \to \mathbb{Z}/6\mathbb{Z}$ and a section $L \to \hat{L}$; $\alpha \mapsto e^\alpha$ was chosen so that $e^\alpha e^\beta = e^{\alpha+\beta}$ and $\tau(e^\alpha) = e^{\tau(\alpha)}$. In our case we have $e^\alpha e^\beta = e^{\alpha+\beta}$ and $\tau(e^\alpha) = e^{\tau(\alpha)}$ in $\widehat{L^\perp}$ for $\alpha, \beta \in L$ by (2.14) and (2.15). Thus for each $1 \le s \le \ell$, the map

$$\begin{aligned} &\kappa_6 \mapsto \kappa_{36}^6 = \kappa_6, \\ &e^\alpha \mapsto (1, \dots, e^\alpha, \dots, 1)T \end{aligned}$$

is an injective group homomorphism of $\hat{L}$ to $\widehat{L^{\oplus\ell}}$, where $(1, \dots, e^\alpha, \dots, 1)$ is the element of $(\widehat{L^\perp})^\ell$ whose s-th component is e^α and the other components are 1. This injective homomorphism is compatible with the action of τ.

The embedding of $\hat{L}$ into $\widehat{L^{\oplus\ell}}$ gives rise to an embedding $v \mapsto 1\otimes\cdots\otimes v\otimes\cdots\otimes 1$ of the vertex operator algebra V_L into $V_L^{\otimes\ell} \cong V_{L^{\oplus\ell}}$ which maps V_L isomorphically to the s-th component of $V_L^{\otimes\ell}$ for each $1 \le s \le \ell$. This embedding is again compatible with the action of τ.

We denote the bilinear form ε_0 on L of [**10**, (4.4)] by ε' for a while. Thus $\varepsilon'(\alpha,\beta) = 5\langle\tau^2\alpha,\beta\rangle + 6\mathbb{Z}$. In [**10**], the multiplications $a \times b$ in $\hat{L}$ and $a \times_\tau b$ in $\hat{L}_\tau$ are related as $a \times b = \kappa_6^{\varepsilon'(\bar{a},\bar{b})} a \times_\tau b$. Since $\kappa_{36}^{\varepsilon_0(\alpha,\beta)} = \kappa_6^{\varepsilon'(\alpha,\beta)}$ for $\alpha, \beta \in L$ by (2.16), it follows from (2.20) that the map $\kappa_6 \mapsto \kappa_{36}^6 = \kappa_6$, $e^\alpha \mapsto e^\alpha$ for $\alpha \in L$ is an injective group homomorphism of $\hat{L}_\tau$ to the s-th component of $\widehat{L^{\oplus\ell}}_\tau$ for each $1 \le s \le \ell$.

Now, $V_L^{\otimes\ell} \cong V_{L^{\oplus\ell}} \subset V_{L_{C\times D}}$. Since $\tau = (\tau, \dots, \tau)$ and since the irreducible τ-twisted V_L-modules $V_L^{T_{\chi_j}}(\tau)$, $j = 0,1,2$ of [**10**], [**36**] were constructed by the same method as in [**12**], [**29**], the above argument shows that the action of V_L on $V_L^{T_{\chi_j}}(\tau)$ is realized in the action of the s-th component of $V_L^{\otimes\ell}$ on the irreducible τ-twisted $V_{L_{C\times D}}$-modules $V_{L_{C\times D}}^{T,\eta}(\tau)$ constructed by (3.24) below.

We can verify the following properties of the $\mathbb{Z}$-bilinear form c_2. In fact, it is sufficient to show the assertions for the case $\ell = 1$. Note that Lemma 2.3 implies

$$L_{\mathcal{K}^\ell\times\mathbf{0}} = \left\{ \sum_{s=1}^{l} \left(m_1^{(s)}\tilde{\beta}_1^{(s)} + 3m_2^{(s)}\tilde{\beta}_2^{(s)}\right) \middle| m_1^{(s)}, m_2^{(s)} \in \mathbb{Z} \right\}.$$

Lemma 3.1. (1) *For $\alpha \in (L^\perp)^{\oplus\ell}$, we have $c_2(\alpha,\beta) = 0$ for all $\beta \in L^{\oplus\ell}$ if* *and only if* $\alpha \in L_{\mathcal{K}^\ell \times \mathbf{0}}$.

(2) *For* $\alpha = \sum_{s=1}^{\ell} \left(m_1^{(s)} \tilde{\beta}_1^{(s)} + 3m_2^{(s)} \tilde{\beta}_2^{(s)}\right) \in L_{\mathcal{K}^\ell \times \mathbf{0}}$ *and* $\beta \in L_{(\mathbf{0},\gamma)}$ *with* $\gamma \in \mathbb{Z}_3^\ell$, *we have*

$$c_2(\alpha, \beta) = 12\langle (m_1^{(s)})_{s=1}^{\ell}, \gamma \rangle_{\mathbb{Z}_3} + 36\mathbb{Z}.$$

(3) *For* $\alpha \in L_{(\lambda,\mathbf{0})}$, $\beta \in L_{(\mu,\mathbf{0})}$ *with* $\lambda, \mu \in \mathcal{K}^\ell$, *we have*

$$c_2(\alpha, \beta) = 18\langle \lambda, \mu \rangle_{\mathcal{K}} + 36\mathbb{Z}.$$

We now follow [**29**]. The commutator map $C(\alpha, \beta)$ of [**29**] is ${\kappa_{36}}^{c_2(\alpha,\beta)}$ in our notation. Let $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L_{C \times D}$, so that $\mathfrak{h} = (\mathbb{C} \otimes_{\mathbb{Z}} L)^{\oplus \ell}$. We extend τ to an isometry of $\mathfrak{h}$ linearly. Then τ is fixed-point-free on $\mathfrak{h}$ and N of [**29**] is identical with $L_{C \times D}$ in our case.

Let $R = \{\alpha \in L_{C \times D} \mid c_2(\alpha, \beta) = 0 \text{ for all } \beta \in L_{C \times D}\}$ be the radical of the alternating $\mathbb{Z}$-bilinear form c_2 on $L_{C \times D}$, which is identical with the R of [**29**, Section 6]. Since C is self-orthogonal, Lemma 3.1 implies the following assertion.

Lemma 3.2. *The radical R of the alternating $\mathbb{Z}$-bilinear form c_2 on $L_{C \times D}$ consists of the elements*

$$\sum_{s=1}^{\ell} \left(m_1^{(s)} \tilde{\beta}_1^{(s)} + 3m_2^{(s)} \tilde{\beta}_2^{(s)}\right) \in L_{C \times \mathbf{0}}$$

with $m_1^{(s)}, m_2^{(s)} \in \mathbb{Z}$ *such that* $(m_1^{(s)} + 3\mathbb{Z})_{s=1}^{\ell} \in D^\perp$.

By Lemma 3.1 we also have the following lemma. Thus we can choose $L_{C \times \mathbf{0}}$ as the group A of [**29**, Proposition 6.2].

Lemma 3.3. *$L_{C \times \mathbf{0}}$ is a subgroup of $L_{C \times D}$ which is maximal subject to the condition that the alternating $\mathbb{Z}$-bilinear form c_2 is trivial on it.*

We shall consider $(1-\tau)L_{C \times D} = \cup_{(\lambda,\gamma) \in C \times D} (1-\tau)L_{(\lambda,\gamma)}$, which corresponds to the subgroup denoted by M in [**29**, Section 6]. For $m_1, m_2 \in \mathbb{Z}$, we have

$$(1-\tau)(m_1 \tilde{\beta}_1 + m_2 \tilde{\beta}_2) = -m_2 \tilde{\beta}_1 + 3(m_1 + m_2)\tilde{\beta}_2$$

and hence $(1-\tau)L^\perp = L_{\mathcal{K} \times \mathbf{0}}$ and $(1-\tau)L = \mathbb{Z}(6\tilde{\beta}_1) + \mathbb{Z}(6\tilde{\beta}_2)$. More precisely,

$$
\begin{aligned}
(1-\tau)\Big(\frac{-\beta_1+\beta_2}{3}\Big) &= 2\tilde{\beta}_1 - 6\tilde{\beta}_2,\\
(1-\tau)(\beta(a)) &= \beta(c) + 2\tilde{\beta}_1 - 6\tilde{\beta}_2,\\
(1-\tau)(\beta(b)) &= \beta(a) + 2\tilde{\beta}_1 - 6\tilde{\beta}_1 + 6\tilde{\beta}_2,\\
(1-\tau)(\beta(c)) &= \beta(b) + 2\tilde{\beta}_1.
\end{aligned}
$$

Then we see from (2.3) that

$$
(1-\tau)L^{(x,i)} = \mathrm{wt}_{\mathcal{K}}(x)(\beta(\tau^2(x)) + 2\tilde{\beta}_1) + 2i\tilde{\beta}_1 + \mathbb{Z}(6\tilde{\beta}_1) + \mathbb{Z}(6\tilde{\beta}_2) \tag{3.1}
$$

for $x \in \mathcal{K}$ and $i \in \mathbb{Z}_3$, where $\mathrm{wt}_{\mathcal{K}}(x) = 1$ if $x \in \{a, b, c\}$ and 0 otherwise. Thus,

$$
\begin{aligned}
(1-\tau)L_{(\lambda,\gamma)} = \sum_{s=1}^{\ell} \big(&\mathrm{wt}_{\mathcal{K}}(\lambda_s)\big(\beta(\tau^2(\lambda_s)) + 2\tilde{\beta}_1^{(s)}\big)\\
&+ 2\gamma_s\tilde{\beta}_1^{(s)} + \mathbb{Z}\big(6\tilde{\beta}_1^{(s)}\big) + \mathbb{Z}\big(6\tilde{\beta}_2^{(s)}\big)\big)
\end{aligned} \tag{3.2}
$$

for $\lambda = (\lambda_s)_{s=1}^{\ell} \in \mathcal{K}^{\ell}$ and $\gamma = (\gamma_s)_{s=1}^{\ell} \in \mathbb{Z}_3^{\ell}$. We also note that

$$
L^{(\tau^2(x),0)} = (1-\tau)L^{(x,0)} \cup (1-\tau)L^{(x,1)} \cup (1-\tau)L^{(x,2)} \tag{3.3}
$$

is a disjoint union for $x \in \mathcal{K}$ by (2.3) and (3.1). Thus,

$$
L_{C\times\mathbf{0}} = \bigcup_{\substack{\lambda\in C\\ \gamma\in\mathbb{Z}_3^{\ell}}} (1-\tau)L_{(\lambda,\gamma)}; \quad \text{disjoint}. \tag{3.4}
$$

Define a $\mathbb{Z}$-linear map $\varphi\colon L^{\perp} \to \mathbb{Z}_3$ by

$$
\varphi(m_1\tilde{\beta}_1 + m_2\tilde{\beta}_2) = m_1 + 3\mathbb{Z} \tag{3.5}
$$

for $m_1, m_2 \in \mathbb{Z}$. We can verify that $\varphi(\beta(\tau^2(x)) + 2\tilde{\beta}_1) = 0$ if $x \in \{a, b, c\}$. Hence (3.1) implies the following lemma.

LEMMA 3.4. *$\varphi((1-\tau)L^{(x,i)}) = \{2i\}$ for $x \in \mathcal{K}$, $i \in \mathbb{Z}_3$.*

We extend $\varphi\colon L^{\perp} \to \mathbb{Z}_3$ to a homomorphism of additive groups $\varphi\colon (L^{\perp})^{\oplus\ell} \to \mathbb{Z}_3^{\ell}$ componentwise, so that it maps the s-th component $L^{\perp}$ to $\mathbb{Z}_3$ by (3.5). Set

$M_0 = (1-\tau)L_{C\times\mathbf{0}}$ and $M = (1-\tau)L_{C\times D}$. By Lemma 3.4, we have $\varphi((1-\tau)L_{(\lambda,\gamma)}) = \{2\gamma\}$ for $\lambda \in \mathcal{K}^\ell$ and $\gamma \in \mathbb{Z}_3^\ell$. Thus the following lemma holds by (3.4) and Lemma 3.2.

Lemma 3.5. *The restriction $\varphi|_{L_{C\times\mathbf{0}}}\colon L_{C\times\mathbf{0}} \to \mathbb{Z}_3^\ell$ of φ to $L_{C\times\mathbf{0}}$ is a surjective homomorphism and its kernel is M_0. Moreover, $\varphi(M) = D$ and $\varphi(R) = D^\perp$. That is, φ gives the following surjections.*

$$\varphi\colon\quad \begin{array}{ccccccc} M_0 & \subset & M & \subset & R & \subset & L_{C\times\mathbf{0}} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \{\mathbf{0}\} & \subset & D & \subset & D^\perp & \subset & \mathbb{Z}_3^\ell. \end{array} \tag{3.6}$$

Since $6\tilde{\beta}_1^{(s)} = 3\beta_1^{(s)}$ and $6\tilde{\beta}_2^{(s)} = \beta_1^{(s)} - \beta_2^{(s)}$, M_0 contains $\beta_1^{(s)} - \beta_2^{(s)}$, $\beta_2^{(s)} - \beta_0^{(s)}$ and $3\beta_i^{(s)}$, $i = 0,1,2$ by (3.2). Let $\gamma = (\gamma_1,\dots,\gamma_\ell) \in \mathbb{Z}_3^\ell$. Then the inverse image of $\{2\gamma\}$ under $\varphi|_{L_{C\times\mathbf{0}}}$ is $\sum_{s=1}^{\ell} \gamma_s\beta_i^{(s)} + M_0$. By Lemma 3.5, φ induces an isomorphism $L_{C\times\mathbf{0}}/M_0 \cong \mathbb{Z}_3^\ell$. Taking the inverse image of D, $D^\perp$ and $\mathbb{Z}_3^\ell$, respectively, we have the following coset decompositions.

$$M = \bigcup_{\gamma\in D} \big(\gamma_1\beta_i^{(1)} + \cdots + \gamma_\ell\beta_i^{(\ell)} + M_0\big), \tag{3.7}$$

$$R = \bigcup_{\gamma\in D^\perp} \big(\gamma_1\beta_i^{(1)} + \cdots + \gamma_\ell\beta_i^{(\ell)} + M_0\big), \tag{3.8}$$

$$L_{C\times\mathbf{0}} = \bigcup_{\gamma\in \mathbb{Z}_3^\ell} \big(\gamma_1\beta_i^{(1)} + \cdots + \gamma_\ell\beta_i^{(\ell)} + M_0\big). \tag{3.9}$$

Recall that $\hat{Q}_\tau$ denotes the inverse image of Q under the homomorphism $\hat{L}_{C\times D,\tau} \xrightarrow{-} L_{C\times D}$ for a subset Q of $L_{C\times D}$. Lemma 3.3 implies that the inverse image $\hat{L}_{C\times\mathbf{0},\tau}$ of $L_{C\times\mathbf{0}}$ is isomorphic to $L_{C\times\mathbf{0}} \times \langle\kappa_{36}\rangle$, which is a maximal abelian subgroup of $\hat{L}_{C\times D,\tau}$. The inverse image $\hat{R}_\tau$ of R is the center of the group $\hat{L}_{C\times D,\tau}$.

A central subgroup K defined in [**12**, Remark 4.2] is crucial for the construction of a certain class of irreducible $\hat{L}_{C\times D,\tau}$-modules (see also [**22**, Section 7.4], [**29**, Section 6]). Let $K = \{a\tau(a)^{-1} \mid a \in \hat{L}_{C\times D,\tau}\}$. Then K is a subgroup of the center $\hat{R}_\tau$ of $\hat{L}_{C\times D,\tau}$ and $K \cap \langle\kappa_{36}\rangle = 1$ [**12**, Remark 4.2]. Indeed, $\overline{a\tau(a)^{-1}} = \overline{a} - \tau(\overline{a}) \in M$. If $a\tau(a)^{-1} \in \langle\kappa_{36}\rangle$, then $\overline{a} = \tau(\overline{a})$ and so $\overline{a} = 0$. Hence $a \in \langle\kappa_{36}\rangle$ and $a\tau(a)^{-1} = 1$. Thus $K \cap \langle\kappa_{36}\rangle = 1$. Since K lies in $\hat{R}_\tau$, $b\tau(b)^{-1}$ commutes with $\tau(a)^{-1}$ for $a, b \in \hat{L}_{C\times D,\tau}$ and

$$a\tau(a)^{-1}b\tau(b)^{-1} = ab\tau(b)^{-1}\tau(a)^{-1} = ab\tau(ab)^{-1}. \tag{3.10}$$

Thus K is a group. Now the inverse image $\hat{M}_\tau$ of M in $\hat{L}_{C\times D,\tau}$ is $K \times \langle \kappa_{36}\rangle \cong M \times \langle \kappa_{36}\rangle$. Clearly, K is τ-invariant. Moreover, K is θ-invariant since θ commutes with τ.

We shall construct an irreducible $\hat{L}_{C\times D,\tau}$-module T_ψ as in [**29**, Proposition 6.2]. Since $\hat{M}_\tau = K \times \langle \kappa_{36}\rangle$, there is a unique group homomorphism $\rho\colon \hat{M}_\tau \to \mathbb{C}^\times$ such that $\rho(\kappa_{36}) = \zeta_{36}$ and $\rho(a) = 1$ for $a \in K$. Note that $(1+\tau+\tau^2)\alpha = 0$ for $\alpha \in L_{C\times D}$. Thus ρ is the homomorphism denoted by τ in [**29**, Proposition 6.1]. Let $\chi\colon \hat{R}_\tau \to \mathbb{C}^\times$ be a homomorphism extending ρ and $\psi\colon \hat{L}_{C\times\mathbf{0},\tau} \to \mathbb{C}^\times$ be a homomorphism extending χ. Then $\psi(\kappa_{36}) = \zeta_{36}$ and ψ is 1 on K. Such an extension ψ exists, since in the central extension

$$1 \to \langle \kappa_{36}\rangle \to \hat{L}_{C\times D,\tau}/K \to L_{C\times D}/M \to 1$$

with associated commutator map $\bar{c}_2$ defined by $\bar{c}_2(\alpha+M,\beta+M) = c_2(\alpha,\beta)$, the subgroup $\hat{L}_{C\times\mathbf{0},\tau}/K$ splits by Lemma 3.3. That is, $\hat{L}_{C\times\mathbf{0},\tau}/K \cong (L_{C\times\mathbf{0}}/M) \times \langle \kappa_{36}\rangle$ and $\hat{R}_\tau/K \cong (R/M) \times \langle \kappa_{36}\rangle$. Let $\mathbb{C}_\psi$ be a one dimensional $\hat{L}_{C\times\mathbf{0},\tau}$-module with character ψ and $T_\psi = \mathbb{C}[\hat{L}_{C\times D,\tau}] \otimes_{\mathbb{C}[\hat{L}_{C\times\mathbf{0},\tau}]} \mathbb{C}_\psi$ be the $\hat{L}_{C\times D,\tau}$-module induced from $\mathbb{C}_\psi$.

We need to know ψ and T_ψ in detail. For this purpose, set $K_0 = \{a\tau(a)^{-1} \mid a \in \hat{L}_{C\times\mathbf{0},\tau}\}$. Then K_0 is a subgroup of K with $\hat{M}_{0,\tau} = K_0 \times \langle \kappa_{36}\rangle$, where $\hat{M}_{0,\tau}$ denotes the inverse image of M_0 in $\hat{L}_{C\times D,\tau}$. Moreover, K_0 is θ- and τ-invariant. We shall describe the group $\hat{L}_{C\times D,\tau}/K_0$ explicitly.

We can verify that $\varepsilon_2(\alpha,\tau(\alpha)) = \varepsilon_2(\alpha,\alpha)$ and $e^\alpha\tau(e^\alpha)^{-1} = e^{(1-\tau)\alpha}$ in $\widehat{L^\perp}_\tau$ for $\alpha \in L^\perp$ by (2.5), (2.12) and (2.15). Hence

$$e^\beta\tau(e^\beta)^{-1} = e^{(1-\tau)\beta} \quad \text{in } \widehat{(L^\perp)^{\oplus\ell}}_\tau \tag{3.11}$$

for $\beta \in (L^\perp)^{\oplus\ell}$. In the case of $\beta = -\beta_1^{(s)} + \beta_2^{(s)}$, we have

$$e^{-\beta_1^{(s)}+\beta_2^{(s)}}\tau\big(e^{-\beta_1^{(s)}+\beta_2^{(s)}}\big)^{-1} = e^{3\beta_2^{(s)}} \quad \text{in } \hat{L}_{C\times D,\tau}. \tag{3.12}$$

For $\gamma = (\gamma_1,\ldots,\gamma_\ell) \in D$, set

$$a(\gamma) = \sum_{s=1}^{\ell} j_s \frac{-\beta_1^{(s)}+\beta_2^{(s)}}{3} \in L_{\mathbf{0}\times D}, \tag{3.13}$$

where $j_s = 0,1,2$ such that $\gamma_s = j_s + 3\mathbb{Z}$. These $a(\gamma), \gamma \in D$ form a complete set of coset representatives of $L_{C\times\mathbf{0}}$ in $L_{C\times D}$, and so

$$\hat{L}_{C\times D,\tau} = \bigcup_{\gamma\in D} e^{a(\gamma)}\hat{L}_{C\times\mathbf{0},\tau}. \tag{3.14}$$

Then using (3.10) we see that

$$K = \bigcup_{\gamma\in D} e^{a(\gamma)}\tau(e^{a(\gamma)})^{-1}K_0. \tag{3.15}$$

Moreover, it follows from (3.11) that

$$e^{a(\gamma)}\tau(e^{a(\gamma)})^{-1} = e^{\sum_{s=1}^{\ell} j_s\beta_2^{(s)}} \quad \text{in } \hat{L}_{C\times D,\tau}. \tag{3.16}$$

Now, using (2.5) and (2.12) we can verify that

$$(e^{\beta_i})^m = \kappa_3^{m(m-1)}e^{m\beta_i} \quad \text{in } \widehat{L^\perp}_\tau \tag{3.17}$$

for $m \in \mathbb{Z}$, $i = 0, 1, 2$.

By (2.15), (3.12) and (3.17), we have the following lemma.

Lemma 3.6. *The following assertions hold in $\hat{L}_{C\times D,\tau}$ for $1 \le s \le \ell$.*

(1) $e^{\beta_1^{(s)}} \equiv e^{\beta_2^{(s)}} \equiv e^{\beta_0^{(s)}} \pmod{K_0}$.
(2) $(\kappa_3 e^{\beta_i^{(s)}})^3 = e^{3\beta_i^{(s)}} \in K_0$, $i = 0, 1, 2$.
(3) $(\kappa_3 e^{\beta_i^{(s)}})^{-1} = \kappa_3 e^{-\beta_i^{(s)}}$, $i = 0, 1, 2$.

By (3.17), $(\kappa_3 e^{\beta_i^{(s)}})^m = \kappa_3^{m^2}e^{m\beta_i^{(s)}}$ in $\hat{L}_{C\times D,\tau}$ for any integer m. Now, let $m_s \in \mathbb{Z}, 1 \le s \le \ell$. Then $e^{\sum_{s=1}^{\ell} m_s\beta_i^{(s)}} = e^{m_1\beta_i^{(1)}}\cdots e^{m_\ell\beta_i^{(\ell)}}$ in $\hat{L}_{C\times D,\tau}$, since $\varepsilon_2(\beta_i^{(s)}, \beta_i^{(t)}) = 0$ if $s \ne t$. Thus

$$\left(\kappa_3 e^{\beta_i^{(1)}}\right)^{m_1}\cdots\left(\kappa_3 e^{\beta_i^{(\ell)}}\right)^{m_\ell} = \kappa_3^{\sum_{s=1}^{\ell} m_s^2}e^{\sum_{s=1}^{\ell} m_s\beta_i^{(s)}} \quad \text{in } \hat{L}_{C\times D,\tau} \tag{3.18}$$

for any $(m_1, \dots, m_\ell) \in \mathbb{Z}^\ell$. The above lemma implies that $(\kappa_3\beta_i^{(s)})^m K_0$ and $e^{m\beta_i^{(s)}}K_0$ depend only on $m \pmod{3\mathbb{Z}}$. Hence (3.18) is reduced to

$$\left(\kappa_3 e^{\beta_i^{(1)}}\right)^{\gamma_1}\cdots\left(\kappa_3 e^{\beta_i^{(\ell)}}\right)^{\gamma_\ell}K_0 = \kappa_3^{\langle\gamma,\gamma\rangle_{\mathbb{Z}_3}}e^{\sum_{s=1}^{\ell}\gamma_s\beta_i^{(s)}}K_0 \tag{3.19}$$

modulo K_0 for $\gamma = (\gamma_1, \dots, \gamma_\ell) \in \mathbb{Z}_3^\ell$. If $\gamma \in D$, then $\langle\gamma,\gamma\rangle_{\mathbb{Z}_3} = 0$ since D is self-orthogonal. Therefore, (3.15) and (3.16) give that

$$K = \bigcup_{\gamma \in D} \bigl(\kappa_3 e^{\beta_i^{(1)}}\bigr)^{\gamma_1} \cdots \bigl(\kappa_3 e^{\beta_i^{(\ell)}}\bigr)^{\gamma_\ell} K_0. \tag{3.20}$$

Motivated by the above result, we set

$$K_1 = \bigcup_{\gamma \in D^\perp} \bigl(\kappa_3 e^{\beta_i^{(1)}}\bigr)^{\gamma_1} \cdots \bigl(\kappa_3 e^{\beta_i^{(\ell)}}\bigr)^{\gamma_\ell} K_0, \tag{3.21}$$

$$K_2 = \bigcup_{\gamma \in \mathbb{Z}_3^\ell} \bigl(\kappa_3 e^{\beta_i^{(1)}}\bigr)^{\gamma_1} \cdots \bigl(\kappa_3 e^{\beta_i^{(\ell)}}\bigr)^{\gamma_\ell} K_0 \tag{3.22}$$

with $\gamma = (\gamma_1, \ldots, \gamma_\ell)$. Then the following lemma holds.

LEMMA 3.7. (1) *K_2 is a subgroup of $\hat{L}_{C\times\mathbf{0},\tau}$ such that $K_2 \cap \langle \kappa_{36} \rangle = 1$ and $\hat{L}_{C\times\mathbf{0},\tau} = K_2 \times \langle \kappa_{36} \rangle$. Moreover,*

$$K_2/K_0 = \bigl\langle \kappa_3 e^{\beta_i^{(1)}} K_0/K_0 \bigr\rangle \times \cdots \times \bigl\langle \kappa_3 e^{\beta_i^{(\ell)}} K_0/K_0 \bigr\rangle,$$

which is isomorphic to $L_{C\times\mathbf{0}}/M_0 \cong \mathbb{Z}_3^\ell$.

(2) *K_1 is a subgroup of K_2 such that $\hat{R}_\tau = K_1 \times \langle \kappa_{36} \rangle$. Moreover, K_1/K_0 is isomorphic to $R/M_0 \cong D^\perp$.*

Let $\psi\colon \hat{L}_{C\times\mathbf{0},\tau} \to \mathbb{C}^\times$ be a homomorphism of abelian groups such that $\psi(\kappa_{36}) = \zeta_{36}$ and $\psi(a) = 1$ for $a \in K_0$. Then $\psi(\kappa_3 e^{\beta_i^{(s)}}) = \zeta_3^{\eta_s}$, $1 \le s \le \ell$ for some $\eta = (\eta_1, \ldots, \eta_\ell) \in \mathbb{Z}_3^\ell$ by Lemma 3.7. We denote such a homomorphism ψ by ψ_η. In fact, $\eta \mapsto \psi_\eta$ is an isomorphism of the additive group $\mathbb{Z}_3^\ell$ onto the multiplicative group of all homomorphisms $\psi\colon \hat{L}_{C\times\mathbf{0},\tau} \to \mathbb{C}^\times$ with $\psi(\kappa_{36}) = \zeta_{36}$ and $\psi(a) = 1$ for $a \in K_0$. The homomorphism ψ_η is determined by the three conditions (i) $\psi_\eta(\kappa_{36}) = \zeta_{36}$, (ii) ψ_η is 1 on K_0, and (iii) $\psi_\eta(\kappa_3 e^{\beta_i^{(s)}}) = \zeta_3^{\eta_s}$.

REMARK 3.8. The conditions (i), (ii), and (iii) for ψ_η are consistent with the conditions for χ_j in [**10**, Section 4].

As before, let $\mathbb{C}_{\psi_\eta}$ be a one dimensional $\hat{L}_{C\times\mathbf{0},\tau}$-module affording the character ψ_η and $T_{\psi_\eta} = \mathbb{C}[\hat{L}_{C\times D,\tau}] \otimes_{\mathbb{C}[\hat{L}_{C\times\mathbf{0},\tau}]} \mathbb{C}_{\psi_\eta}$ be the $\hat{L}_{C\times D,\tau}$-module induced from $\mathbb{C}_{\psi_\eta}$. It follows from (3.14) that $\{e^{a(\gamma)} \otimes 1_\eta \mid \gamma \in D\}$ is a basis of T_{ψ_η}, where 1_η denotes a fixed nonzero vector in $\mathbb{C}_{\psi_\eta}$. For $b \in \hat{L}_{C\times\mathbf{0},\tau}$, we have $be^{a(\gamma)} = \kappa_{36}^{c_2(\bar{b},a(\gamma))} e^{a(\gamma)} b$ and the action of b on $e^{a(\gamma)} \otimes 1_\eta$ is

$$b \cdot (e^{a(\gamma)} \otimes 1_\eta) = \zeta_{36}^{c_2(\bar{b},a(\gamma))} \psi_\eta(b)(e^{a(\gamma)} \otimes 1_\eta).$$

For $\delta \in D$, we have $e^{a(\delta)}e^{a(\gamma)} \in e^{a(\delta+\gamma)}\hat{L}_{C\times\mathbf{0},\tau}$ by (2.12) since $a(\delta)+a(\gamma) \equiv a(\delta+\gamma) \pmod{L^{\oplus\ell}}$. Then T_{ψ_η} is an irreducible $\hat{L}_{C\times D,\tau}$-module and the following lemma holds.

Lemma 3.9. (1) *κ_{36} and K_0 act on T_{ψ_η} as ζ_{36} and 1, respectively. Moreover, K (resp. K_1) acts on T_{ψ_η} as 1 if and only if $\eta \in D^\perp$ (resp. $\eta \in D$).*

(2) *For $\eta,\eta' \in \mathbb{Z}_3^\ell$, the $\hat{L}_{C\times D,\tau}$-modules T_{ψ_η} and $T_{\psi_{\eta'}}$ are equivalent if and only if $\eta \equiv \eta' \pmod{D}$, which is also equivalent to the condition that ψ_η and $\psi_{\eta'}$ agree on K_1.*

(3) *The action of $\kappa_3 e^{\pm\beta_i^{(s)}}$ on $e^{a(\gamma)}\otimes 1_\eta$ is such that*

$$\kappa_3 e^{\pm\beta_i^{(s)}}\cdot(e^{a(\gamma)}\otimes 1_\eta) = \zeta_3^{\pm(\eta_s-\gamma_s)}e^{a(\gamma)}\otimes 1_\eta.$$

That is, $\mathbb{C}e^{a(\gamma)}\otimes 1_\eta$ is a one dimensional $\hat{L}_{C\times\mathbf{0},\tau}$-module with character $\psi_{\eta-\gamma}$.

By the above lemma, $e^{\pm\beta_i^{(s)}}$ acts on $e^{a(\gamma)}\otimes 1_\eta \in T_{\psi_\eta}$ as

$$e^{\pm\beta_i^{(s)}}\cdot(e^{a(\gamma)}\otimes 1_\eta) = \zeta_3^{-1\pm(\eta_s-\gamma_s)}e^{a(\gamma)}\otimes 1_\eta. \tag{3.23}$$

Remark 3.10. T_{ψ_η}, $\eta \in D^\perp$ are exactly the modules T of [**29**, Proposition 6.2] in our case.

Recall that $\mathfrak{h} = \mathbb{C}\otimes_{\mathbb{Z}} L_{C\times D} = (\mathbb{C}\otimes_{\mathbb{Z}} L)^{\oplus\ell}$. As before, we use $\alpha^{(s)}$ to denote the element $\alpha \in \mathbb{C}\otimes_{\mathbb{Z}} L$ in the s-th entry of $(\mathbb{C}\otimes_{\mathbb{Z}} L)^{\oplus\ell}$. Let

$$h_1^{(s)} = \frac{1}{3}\big(\beta_1^{(s)}+\zeta_3^2\beta_2^{(s)}+\zeta_3\beta_0^{(s)}\big), \quad h_2^{(s)} = \frac{1}{3}\big(\beta_1^{(s)}+\zeta_3\beta_2^{(s)}+\zeta_3^2\beta_0^{(s)}\big).$$

Then $\tau h_j^{(s)} = \zeta_3^j h_j^{(s)}$, $\langle h_j^{(s)}, h_j^{(t)}\rangle = 0$, and $\langle h_1^{(s)}, h_2^{(t)}\rangle = 2\delta_{s,t}$. Set

$$\mathfrak{h}_{(n)} = \{\alpha \in \mathfrak{h} \mid \tau\alpha = \zeta_3^n\alpha\}$$

for $n \in \mathbb{Z}$. The index n of $\mathfrak{h}_{(n)}$ is considered to be modulo 3. Then $\mathfrak{h}_{(0)} = 0$ and $\mathfrak{h} = \mathfrak{h}_{(1)}\oplus\mathfrak{h}_{(2)}$ with $\mathfrak{h}_{(n)} = \mathbb{C}h_n^{(1)}\oplus\cdots\oplus\mathbb{C}h_n^{(\ell)}$, $n=1,2$. If $\alpha \in \mathfrak{h}$, we write $\alpha_{(n)}$ for the component of α in $\mathfrak{h}_{(n)}$. In this notation we have $(\beta_i^{(s)})_{(1)} = \zeta_3^{i-1}h_1^{(s)}$ and $(\beta_i^{(s)})_{(2)} = \zeta_3^{2(i-1)}h_2^{(s)}$, $i=0,1,2$.

The τ-twisted affine Lie algebra $\hat{\mathfrak{h}}[\tau]$ is defined to be

$$\hat{\mathfrak{h}}[\tau] = \left(\bigoplus_{n\in\mathbb{Z}} \mathfrak{h}_{(n)} \otimes t^{n/3}\right) \oplus \mathbb{C}c$$

with the bracket

$$[x \otimes t^m, y \otimes t^n] = m\langle x, y\rangle \delta_{m+n,0} c$$

for $x \in \mathfrak{h}_{(3m)}$, $y \in \mathfrak{h}_{(3n)}$, m, $n \in (1/3)\mathbb{Z}$ and $[c, \hat{\mathfrak{h}}[\tau]] = 0$. The isometry τ acts on $\hat{\mathfrak{h}}[\tau]$ by $\tau(x \otimes t^{n/3}) = \zeta_3^n x \otimes t^{n/3}$ and $\tau(c) = c$. Set

$$\hat{\mathfrak{h}}[\tau]^+ = \bigoplus_{n>0} \mathfrak{h}_{(n)} \otimes t^{n/3}, \quad \hat{\mathfrak{h}}[\tau]^- = \bigoplus_{n<0} \mathfrak{h}_{(n)} \otimes t^{n/3}, \quad \hat{\mathfrak{h}}[\tau]^0 = \mathbb{C}c$$

and consider the $\hat{\mathfrak{h}}[\tau]$-module

$$S[\tau] = U(\hat{\mathfrak{h}}[\tau]) \otimes_{U(\hat{\mathfrak{h}}[\tau]^+ \oplus \hat{\mathfrak{h}}[\tau]^0)} \mathbb{C}$$

induced from the $\hat{\mathfrak{h}}[\tau]^+ \oplus \hat{\mathfrak{h}}[\tau]^0$-module $\mathbb{C}$, where $\hat{\mathfrak{h}}[\tau]^+$ acts as 0 and $\hat{\mathfrak{h}}[\tau]^0$ acts as 1 on $\mathbb{C}$. The weight gradation on $S[\tau]$ is given by $\mathrm{wt}(x \otimes t^n) = -n$ and $\mathrm{wt}(1) = \ell/9$ for $n \in (1/3)\mathbb{Z}$ and $x \in \mathfrak{h}_{(3n)}$ [**12**, (4.6), (4.10)]. For $\alpha \in \mathfrak{h}$ and $n \in (1/3)\mathbb{Z}$, we write $\alpha(n)$ for the operator on $S[\tau]$ induced by the action of $\alpha_{(3n)} \otimes t^n$. The weight of the operator $h_i^{(s)}(i/3 + n)$ is $-i/3 - n$. The group H_ℓ acts as

$$(\tau^{j_1}, \dots, \tau^{j_\ell})\left(h_i^{(s)}\left(\frac{i}{3} + n\right)\right) = \zeta_3^{j_s i} h_i^{(s)}\left(\frac{i}{3} + n\right).$$

Set

$$V_{L_{C\times D}}^{T,\eta}(\tau) = S[\tau] \otimes T_{\psi_\eta} \tag{3.24}$$

for $\eta \in D^\perp$. By [**12**, Theorem 7.1] and [**29**, Proposition 6.2], we can define a τ-twisted vertex operator $Y^\tau(\,\cdot\,, x)\colon V_{L_{C\times D}} \to \mathrm{End}(V_{L_{C\times D}}^{T,\eta}(\tau))\{x\}$ so that $(V_{L_{C\times D}}^{T,\eta}(\tau), Y^\tau)$, $\eta \in D^\perp$ is an irreducible τ-twisted $V_{L_{C\times D}}$-module. The weight of any element in T_{ψ_η} is defined to be 0. Hence the weight of elements in $V_{L_{C\times D}}^{T,\eta}(\tau)$ is given by $\mathrm{wt}(u \otimes v) = \mathrm{wt}(u)$ for $u \in S[\tau]$ and $v \in T_{\psi_\eta}$.

We define an action of H_ℓ on $\mathbb{C}e^{a(\gamma)} \otimes 1_\eta$ by

$$(\tau^{j_1}, \dots, \tau^{j_\ell})(e^{a(\gamma)} \otimes 1_\eta) = \zeta_3^{2\langle (j_s)_{s=1}^\ell, \eta - \gamma\rangle_{\mathbb{Z}_3}} e^{a(\gamma)} \otimes 1_\eta$$

and extend to $T_{\psi_\eta} = \oplus_{\gamma\in D}\mathbb{C}e^{a(\gamma)} \otimes 1_\eta$ by linearity. Note that Lemma 2.2 implies $\tau(e^{a(\gamma)} \otimes 1_\eta) = \zeta_3^{2\,\mathrm{wt}_{\mathbb{Z}_3}(\eta)} e^{a(\gamma)} \otimes 1_\eta$ for $\gamma \in D$. Thus τ acts on T_{ψ_η} as a scalar $\zeta_3^{2\,\mathrm{wt}_{\mathbb{Z}_3}(\eta)}$, which depends only on the coset $\eta + D \in D^\perp/D$. The group H_ℓ acts on the vector space $V_{L_{C\times D}}^{T,\eta}(\tau)$ by

$$g(u \otimes v) = g(u) \otimes g(v) \tag{3.25}$$

for $g \in H_\ell$, $u \in S[\tau]$ and $v \in T_{\psi_\eta}$. Then, $\tau(Y^\tau(u,x)w) = Y^\tau(\tau u,x)\tau w$ for $u \in V_{L_{C\times D}}$ and $w \in V_{L_{C\times D}}^{T,\eta}(\tau)$ by [**11**, Section 4].

We have discussed only irreducible τ-twisted $V_{L_{C\times D}}$-modules so far. Now, we deal with irreducible τ^2-twisted ones. Actually, we can construct $|D^\perp/D|$ inequivalent irreducible τ^2-twisted $V_{L_{C\times D}}$-modules $(V_{L_{C\times D}}^{T,\eta}(\tau^2), Y^{\tau^2})$, $\eta \in D^\perp \pmod D$ similarly. Indeed, replace τ with τ^2 in the above argument and proceed in the same way. We can construct a class of irreducible $\hat{L}_{C\times D,\tau^2}$-modules T'_{ψ_η}, $\eta \in D^\perp$. Let $h_1'^{(s)} = h_2^{(s)}$ and $h_2'^{(s)} = h_1^{(s)}$, $1 \le s \le \ell$. Set $\mathfrak{h}'_{(n)} = \{\alpha \in \mathfrak{h} \mid \tau^2\alpha = \zeta_3^n\alpha\}$ for $n \in \mathbb{Z}$ (see [**10**, Section 4.3]). Take $h_1'^{(s)}$ and $h_2'^{(s)}$ instead of $h_1^{(s)}$ and $h_2^{(s)}$, respectively and consider $S[\tau^2]$. Then

$$V_{L_{C\times D}}^{T,\eta}(\tau^2) = S[\tau^2] \otimes T'_{\psi_\eta}.$$

We define an action of H_ℓ on $\mathbb{C}e^{a(\gamma)} \otimes 1_\eta$ by

$$(\tau^{j_1}, \ldots, \tau^{j_\ell})(e^{a(\gamma)} \otimes 1_\eta) = \zeta_3^{\langle (j_s)_{s=1}^\ell, \eta-\gamma\rangle_{\mathbb{Z}_3}} e^{a(\gamma)} \otimes 1_\eta$$

and extend to $T'_{\psi_\eta} = \oplus_{\gamma\in D}\mathbb{C}e^{a(\gamma)} \otimes 1_\eta$ by linearity. Thus $\tau^2(v) = \zeta_3^{2\,\mathrm{wt}_{\mathbb{Z}_3}(\eta)} v$ for $v \in T'_{\psi_\eta}$. Now, H_ℓ acts on the vector space $V_{L_{C\times D}}^{T,\eta}(\tau^2)$ by

$$g(u \otimes v) = g(u) \otimes g(v) \tag{3.26}$$

for $g \in H_\ell$, $u \in S[\tau^2]$ and $v \in T'_{\psi_\eta}$. We have $\tau(Y^{\tau^2}(u,x)w) = Y^{\tau^2}(\tau u,x)\tau w$ for $u \in V_{L_{C\times D}}$ and $w \in V_{L_{C\times D}}^{T,\eta}(\tau^2)$.

Since $V_{L_{C\times D}}$ is rational and C_2-cofinite, the number of irreducible τ^i-twisted $V_{L_{C\times D}}$-modules is bounded above by the number of τ-stable irreducible $V_{L_{C\times D}}$-modules by [**15**, Theorem 10.2] for each $i = 1, 2$. Now, $\{V_{L_{C\times(\eta+D)}} \mid \eta \in D^\perp \pmod D\}$ is the set of all τ-stable irreducible $V_{L_{C\times D}}$-modules up to isomorphism. Hence we have the following theorem.

THEOREM 3.11. *For $i = 1, 2$, there are exactly $|D^{\perp}/D|$ inequivalent irreducible τ^i-twisted $V_{L_{C\times D}}$-modules. They are represented by $(V_{L_{C\times D}}^{T,\eta}(\tau^i), Y^{\tau^i})$, $\eta \in D^{\perp} \pmod{D}$.*

The map $\widehat{L^{\oplus \ell}}_{\tau} \to \widehat{L}_{C\times \mathbf{0},\tau}/K_0$; $a \mapsto aK_0$ is surjective by (3.9) and (3.11). For $\alpha \in L_{\mathcal{K}\times\mathbf{0}}$, note that $\alpha \in L$ if $(1-\tau)\alpha \in L$. Then

$$\{a \times_{\tau} \tau(a)^{-1} \mid a \in \hat{L}_{\mathcal{K}\times \mathbf{0},\tau}\} \cap \hat{L}_{\tau} = \{a \times_{\tau} \tau(a)^{-1} \mid a \in \hat{L}_{\tau}\}$$

and the following lemma holds.

LEMMA 3.12. *The map $\widehat{L^{\oplus \ell}}_{\tau} \to \widehat{L}_{C\times\mathbf{0},\tau}$; $a \mapsto a$ induces an isomorphism $\widehat{L^{\oplus \ell}}_{\tau}/\{a \times_{\tau} \tau(a)^{-1} \mid a \in \widehat{L^{\oplus \ell}}_{\tau}\} \cong \widehat{L}_{C\times\mathbf{0},\tau}/K_0$.*

For $i = 1, 2$ and $\varepsilon \in \mathbb{Z}_3$, set

$$V_{L_{C\times D}}^{T,\eta}(\tau^i)[\varepsilon] = \big\{u \in V_{L_{C\times D}}^{T,\eta}(\tau^i) \mid \tau^i u = \zeta_3^{\varepsilon} u\big\}.$$

These are irreducible $V_{L_{C\times D}}^{\tau}$-modules.

In the case where $\ell = 1$ with $C = \{\mathbf{0}\}$ and $D = \{\mathbf{0}\}$, $V_{L_{C\times D}}^{T,\eta}(\tau^i)[\varepsilon]$ reduces to $V_L^{T,j}(\tau^i)[\varepsilon]$, $j = 0, 1, 2$. The relation between our $V_L^{T,j}(\tau^i)[\varepsilon]$ and $V_L^{T_{\chi_j}}(\tau)(\varepsilon)$, $V_L^{T_{\chi'_j}}(\tau^2)(\varepsilon)$ in [**10**], [**36**] is as follows (see [**36**, (1-1)] also).

$$\begin{aligned}
V_L^{T,0}(\tau)[\varepsilon] &= V_L^{T_{\chi_0}}(\tau)(\varepsilon),\\
V_L^{T,0}(\tau^2)[\varepsilon] &= V_L^{T_{\chi'_0}}(\tau^2)(\varepsilon),\\
V_L^{T,j}(\tau)[\varepsilon] &= V_L^{T_{\chi_j}}(\tau)(\varepsilon+1), \quad j = 1, 2,\\
V_L^{T,j}(\tau^2)[\varepsilon] &= V_L^{T_{\chi'_j}}(\tau^2)(\varepsilon+1), \quad j = 1, 2
\end{aligned}$$

for $\varepsilon \in \mathbb{Z}_3$. Recall that the action of τ on T_{χ_j} and $T_{\chi'_j}$ was defined to be 1 in [**10**], [**36**], while τ acts on T_{ψ_η} (resp. T'_{ψ_η}) as $\zeta_3^{2\,\mathrm{wt}_{\mathbb{Z}_3}(\eta)}$ (resp. $\zeta_3^{\mathrm{wt}_{\mathbb{Z}_3}(\eta)}$). The new notation is suitable for the description of the fusion rules in later sections.

Since $V_{L^{\oplus \ell}} = V_L^{\otimes \ell}$ is a vertex operator subalgebra of $V_{L_{C\times D}}$, $V_{L_{C\times D}}^{T,\eta}(\tau)$ is a τ-twisted $V_{L^{\oplus\ell}}$-module and for each $\gamma \in D$, $S[\tau] \otimes (e^{a(\gamma)} \otimes 1_\eta)$ is a τ-twisted $V_{L^{\oplus\ell}}$-submodule of $V_{L_{C\times D}}^{T,\eta}(\tau)$. By Lemma 3.9 (3) and Lemma 3.12, the $\hat{L}_{C\times\mathbf{0},\tau}$-module $\mathbb{C}e^{a(\gamma)} \otimes 1_\eta$ is isomorphic to $\mathbb{C}_{\psi_{\eta-\gamma}}$ as $\widehat{L^{\oplus\ell}}_{\tau}$-modules. This implies that

$S[\tau] \otimes (e^{a(\gamma)} \otimes 1_\eta) \cong V_{L^{\oplus \ell}}^{T,\eta-\gamma}(\tau)$ as τ-twisted $V_{L^{\oplus \ell}}$-modules. Thus,

$$V_{L_{C\times D}}^{T,\eta}(\tau) \cong \bigoplus_{\gamma \in D} V_{L^{\oplus \ell}}^{T,\eta-\gamma}(\tau) \tag{3.27}$$

as τ-twisted $V_{L^{\oplus \ell}}$-modules. For $\rho = (\rho_1, \ldots, \rho_\ell) \in \mathbb{Z}_3^\ell$, H_ℓ acts on $V_{L^{\oplus \ell}}^{T,\rho}(\tau)$. Note that H_ℓ is an automorphism group of $V_{L^{\oplus \ell}}$ and $g(Y^\tau(u,x)w) = Y^\tau(gu,x)gw$ for $g \in H_\ell$, $u \in V_{L^{\oplus \ell}}$ and $w \in V_{L^{\oplus \ell}}^{T,\rho}(\tau)$ by the definition of $(V_{L^{\oplus \ell}}^{T,\rho}(\tau), Y^\tau)$. Thus, $V_{L^{\oplus \ell}}^{T,\rho}(\tau) \circ g \cong g^{-1}(V_{L^{\oplus \ell}}^{T,\rho}(\tau)) = V_{L^{\oplus \ell}}^{T,\rho}(\tau)$ for $g \in H_\ell$. Note that $(V_{L^{\oplus \ell}})^{H_\ell} = (V_L^\tau)^{\otimes \ell}$. We have the following decomposition of $V_{L^{\oplus \ell}}^{T,\rho}(\tau)$ into a direct sum of irreducible $(V_L^\tau)^{\otimes \ell}$-modules.

$$V_{L^{\oplus \ell}}^{T,\rho}(\tau) \cong \bigoplus_{(\varepsilon_1,\ldots,\varepsilon_\ell) \in \mathbb{Z}_3^\ell} V_L^{T,\rho_1}(\tau)[\varepsilon_1] \otimes \cdots \otimes V_L^{T,\rho_\ell}(\tau)[\varepsilon_\ell]. \tag{3.28}$$

It follows from [**34**, Theorem 2] that $V_L^{T,\rho_1}(\tau)[\varepsilon_1] \otimes \cdots \otimes V_L^{T,\rho_\ell}(\tau)[\varepsilon_\ell]$, $(\varepsilon_1, \ldots, \varepsilon_\ell) \in \mathbb{Z}_3^\ell$ in (3.28) are all inequivalent irreducible $(V_L^\tau)^{\otimes \ell}$-modules.

The corresponding results for τ^2-twisted $V_{L_{C\times D}}$-modules can be verified by a similar argument as above. Thus we have obtained the following theorem.

THEOREM 3.13. *For $i = 1, 2$ and $\eta = (\eta_1, \ldots, \eta_\ell) \in D^\perp \pmod D$, the irreducible τ^i-twisted $V_{L_{C\times D}}$-module $(V_{L_{C\times D}}^{T,\eta}(\tau^i), Y^{\tau^i})$ is decomposed into a direct sum of irreducible $(V_L^\tau)^{\otimes \ell}$-modules as follows.*

$$V_{L_{C\times D}}^{T,\eta}(\tau^i) \cong \bigoplus_{(\gamma_1,\ldots,\gamma_\ell) \in D} \bigoplus_{(\varepsilon_1,\ldots,\varepsilon_\ell) \in \mathbb{Z}_3^\ell} V_L^{T,\eta_1 - i\gamma_1}(\tau^i)[\varepsilon_1] \otimes \cdots \otimes V_L^{T,\eta_\ell - i\gamma_\ell}(\tau^i)[\varepsilon_\ell].$$

Moreover, for the irreducible $V_{L_{C\times D}}^\tau$-module $V_{L_{C\times D}}^{T,\eta}(\tau^i)[r]$, $r = 0, 1, 2$ we have

$$\begin{aligned} &V_{L_{C\times D}}^{T,\eta}(\tau^i)[r] \\ &\quad \cong \bigoplus_{(\gamma_1,\ldots,\gamma_\ell) \in D} \bigoplus_{\varepsilon_1 + \cdots + \varepsilon_\ell \equiv r \pmod 3} V_L^{T,\eta_1 - i\gamma_1}(\tau^i)[\varepsilon_1] \otimes \cdots \otimes V_L^{T,\eta_\ell - i\gamma_\ell}(\tau^i)[\varepsilon_\ell]. \end{aligned}$$

4. Modules of V_L^τ.

In this section we recall the classification of irreducible V_L^τ-modules in [**36**] and compute some fusion rules for V_L^τ.

PROPOSITION 4.1 ([**36**]). *V_L^τ is a simple, rational, C_2-cofinite, and CFT type vertex operator algebra. There are exactly 30 inequivalent irreducible V_L^τ-modules. Their representatives are $V_{L^{(0,j)}}(\varepsilon)$, $V_{L^{(c,j)}}$ and $V_L^{T,k}(\tau^i)[\varepsilon]$ for $i = 1, 2$ and $j, k, \varepsilon = 0, 1, 2$.*

We need the structure of each irreducible V_L^τ-module to compute certain fusion rules. Let $M_k^i, W_k^i, M_t^j, W_t^j, M_k^0(\varepsilon)$ and $W_k^0(\varepsilon)$ be as in [**10**], [**36**]. Then $M_k^0, M_k^0(0)$ and M_t^0 are simple vertex operator algebras. Set $M^0 = M_k^0(0) \otimes M_t^0$ and $W^0 = W_k^0(0) \otimes W_t^0$. Then $V_L^\tau = M^0 \oplus W^0$ and

$$\begin{aligned} V_{L^{(0,j)}}(\varepsilon) &\cong M_k^0(\varepsilon) \otimes M_t^j \oplus W_k^0(\varepsilon) \otimes W_t^j, \\ V_{L^{(c,j)}} &\cong M_k^c \otimes M_t^j \oplus W_k^c \otimes W_t^j, \quad j, \varepsilon = 0, 1, 2 \end{aligned} \tag{4.1}$$

as M^0-modules [**36**, Section 4].

Moreover, let $M_T(\tau^i), W_T(\tau^i), M_T(\tau^i)(\varepsilon)$ and $W_T(\tau^i)(\varepsilon)$ be as in [**10**], [**36**]. Then, for $j, \varepsilon \in \mathbb{Z}_3$,

$$\begin{aligned} V_L^{T,j}(\tau)[\varepsilon] &\cong M_T(\tau)(\varepsilon) \otimes M_t^{-j} \oplus W_T(\tau)(\varepsilon) \otimes W_t^{-j}, \\ V_L^{T,j}(\tau^2)[\varepsilon] &\cong M_T(\tau^2)(\varepsilon) \otimes M_t^j \oplus W_T(\tau^2)(\varepsilon) \otimes W_t^j \end{aligned} \tag{4.2}$$

as M^0-modules [**36**, Section 4].

PROPOSITION 4.2 ([**10**]). *$M_k^0(0)$ is a simple, rational, C_2-cofinite, and CFT type vertex operator algebra. There are exactly 20 inequivalent irreducible $M_k^0(0)$-modules. Their representatives are $M_k^0(\varepsilon), W_k^0(\varepsilon), M_k^c, W_k^c, M_T(\tau^i)(\varepsilon)$, and $W_T(\tau^i)(\varepsilon)$ for $\varepsilon = 0, 1, 2$ and $i = 1, 2$.*

PROPOSITION 4.3 ([**33**]). *M_t^0 is a simple, rational, C_2-cofinite, and CFT type vertex operator algebra. There are exactly 6 inequivalent irreducible M_t^0-modules. Their representatives are M_t^j and W_t^j for $j = 0, 1, 2$. The fusion rules for M_t^0 are as follows.*

$$\begin{aligned} M_t^i \times M_t^j &= M_t^{i+j}, \\ M_t^i \times W_t^j &= W_t^{i+j}, \\ W_t^i \times W_t^j &= M_t^{i+j} + W_t^{i+j} \end{aligned} \tag{4.3}$$

for $i, j = 0, 1, 2$.

We compute some fusion rules for V_L^τ.

Lemma 4.4. *Let $\varepsilon, \varepsilon_1, \varepsilon_2, j, j_1, j_2, k \in \mathbb{Z}_3$ and $i = 1, 2$. Then*

$$V_{L^{(0,j_1)}}(\varepsilon_1) \times V_{L^{(0,j_2)}}(\varepsilon_2) \leq V_{L^{(0,j_1+j_2)}}(\varepsilon_1 + \varepsilon_2), \tag{4.4}$$

$$V_{L^{(0,j_1)}}(\varepsilon) \times V_{L^{(c,j_2)}} \leq V_{L^{(c,j_1+j_2)}}, \tag{4.5}$$

$$V_{L^{(c,j_1)}} \times V_{L^{(c,j_2)}} \leq \sum_{\rho=0}^{2} V_{L^{(0,j_1+j_2)}}(\rho) + 2V_{L^{(c,j_1+j_2)}}, \tag{4.6}$$

$$V_{L^{(0,j)}}(\varepsilon_1) \times V_L^{T,k}(\tau^i)[\varepsilon_2] \leq V_L^{T,k-ij}(\tau^i)[i\varepsilon_1 + \varepsilon_2], \tag{4.7}$$

$$V_{L^{(c,j)}} \times V_L^{T,k}(\tau^i)[\varepsilon] \leq \sum_{\rho=0}^{2} V_L^{T,k-ij}(\tau^i)[\rho]. \tag{4.8}$$

Proof. We have the following fusion rules of irreducible $M_k^0(0)$-modules.

$$\begin{aligned} M_k^0(\varepsilon_1) \times M_k^0(\varepsilon_2) &= M_k^0(\varepsilon_1 + \varepsilon_2), \\ M_k^0(\varepsilon) \times M_k^c &= M_k^c, \\ M_k^c \times M_k^c &= \sum_{\rho=0}^{2} M_k^0(\rho) + 2M_k^c, \\ M_k^0(\varepsilon_1) \times M_T(\tau^i)(\varepsilon_2) &\leq M_T(\tau^i)(i\varepsilon_1 + \varepsilon_2), \\ M_k^c \times M_T(\tau^i)(\varepsilon) &\leq \sum_{\rho=0}^{2} M_T(\tau^i)(\rho). \end{aligned} \tag{4.9}$$

The first three fusion rules can be found in [**35**, Theorem 4] and we can show the last two formulas by applying the same method used there. We shall sketch the proof. In [**35**], $M_k^0(0)$ and $M_k^0(\varepsilon)$ are denoted by $\mathcal{W}$ and $M_k^{0(\varepsilon)}$, respectively, and M_k^a is used instead of M_k^c. Let $A(M_k^0(0))$ be the Zhu algebra of $M_k^0(0)$ and let $A(M_k^0(\varepsilon_1))$, $A(M_k^c)$ be the $A(M_k^0(0))$-bimodules introduced in [**23**]. In [**10**], it is shown that $A(M_k^0(0))$ is generated by two elements $[\omega]$ and $[J]$. Their action on the top level of every irreducible $M_k^0(0)$-module are also computed there. Using these data and [**31**, Proposition 2.10], the same argument as in [**35**, Theorem 4] shows the last two formulas in (4.9).

By (4.3), (4.9), and [**18**, Proposition 2.10], we have fusion rules for M^0 as follows.

$$
\begin{aligned}
M_k^0(\varepsilon_1)\otimes M_t^{k_1}\times M_k^0(\varepsilon_2)\otimes M_t^{k_2} &= M_k^0(\varepsilon_1+\varepsilon_2)\otimes M_t^{k_1+k_2},\\
M_k^0(\varepsilon)\otimes M_t^{k_1}\times M_k^c\otimes M_t^{k_2} &= M_k^c\otimes M_t^{k_1+k_2},\\
M_k^c\otimes M_t^{k_1}\times M_k^c\otimes M_t^{k_2} &= \sum_{\rho=0}^{2} M_k^0(\rho)\otimes M_t^{k_1+k_2} + 2M_k^c\otimes M_t^{k_1+k_2},\\
M_k^0(\varepsilon_1)\otimes M_t^{k_1}\times M_T(\tau^i)(\varepsilon_2)\otimes M_t^{k_2} &\le M_T(\tau^i)(i\varepsilon_1+\varepsilon_2)\otimes M_t^{k_1+k_2},\\
M_k^c\otimes M_t^{k_1}\times M_T(\tau^i)(\varepsilon)\otimes M_t^{k_2} &\le \sum_{\rho=0}^{2} M_T(\tau^i)(\rho)\otimes M_t^{k_1+k_2}, \qquad (4.10)
\end{aligned}
$$

where $k_1,k_2\in\mathbb{Z}_3$. Let N be an irreducible V_L^τ-module. By Propositions 4.1–4.3, (4.1), (4.2), and [**36**, (3.25)], there exist irreducible M^0-modules M_N and W_N such that

$$
\begin{aligned}
N &= M_N\oplus W_N,\\
W^0\times M_N &= W_N,\\
W^0\times W_N &= M_N + W_N
\end{aligned}
$$

as M^0-modules. These M_N and W_N are uniquely determined by N.

For V_L^τ-modules N^1,N^2 and N^3,

$$
\dim_{\mathbb{C}} I_{V_L^\tau}\binom{N^3}{N^1\ N^2} \le \dim_{\mathbb{C}} I_{M^0}\binom{N^3}{M_{N^1}\ M_{N^2}} \qquad (4.11)
$$

by [**11**, Proposition 11.9] and

$$
I_{M^0}\binom{N^3}{M_{N^1}\ M_{N^2}} \cong I_{M^0}\binom{M_{N^3}}{M_{N^1}\ M_{N^2}} \oplus I_{M^0}\binom{W_{N^3}}{M_{N^1}\ M_{N^2}} \qquad (4.12)
$$

as vector spaces. The assertion follows from (4.1), (4.2), (4.10), (4.11), and (4.12). □

For $\mu\in\mathcal{K}^\ell$, $C(\mu)$ denotes the $\mathcal{K}$-code generated by μ and $\tau(\mu)$. Note that $C(\mu)$ is τ-invariant since $\mu+\tau(\mu)+\tau^2(\mu)=\mathbf{0}$, where $\mathbf{0}=(0,\dots,0)$. For $\gamma\in\mathbb{Z}_3^\ell$, $D(\gamma)$ denotes the $\mathbb{Z}_3$-code generated by γ. These symbols will be used in this section, Sections 5, and 7.

Proposition 4.5. *Let $\varepsilon,\varepsilon_1,\varepsilon_2,j,j_1,j_2,k\in\mathbb{Z}_3$ and $i=1,2$. Then*

$$V_{L^{(0,j_1)}}(\varepsilon_1) \times V_{L^{(0,j_2)}}(\varepsilon_2) = V_{L^{(0,j_1+j_2)}}(\varepsilon_1+\varepsilon_2), \tag{4.13}$$

$$V_{L^{(0,j_1)}}(\varepsilon) \times V_{L^{(c,j_2)}} = V_{L^{(c,j_1+j_2)}}, \tag{4.14}$$

$$V_{L^{(c,j_1)}} \times V_{L^{(c,j_2)}} = \sum_{\rho=0}^{2} V_{L^{(0,j_1+j_2)}}(\rho) + 2V_{L^{(c,j_1+j_2)}}, \tag{4.15}$$

$$V_{L^{(0,j)}}(\varepsilon_1) \times V_L^{T,k}(\tau^i)[\varepsilon_2] = V_L^{T,k-ij}(\tau^i)[i\varepsilon_1+\varepsilon_2], \tag{4.16}$$

$$V_{L^{(c,j)}} \times V_L^{T,k}(\tau^i)[\varepsilon] = \sum_{\rho=0}^{2} V_L^{T,k-ij}(\tau^i)[\rho]. \tag{4.17}$$

Proof. Restricting intertwining operators for V_L in Lemma 2.14 to V_L^τ-modules, we have

$$\begin{aligned} V_{L^{(0,j_1)}}(\varepsilon_1) \times V_{L^{(0,j_2)}}(\varepsilon_2) &\geq V_{L^{(0,j_1+j_2)}}(\varepsilon_1+\varepsilon_2), \\ V_{L^{(0,j_1)}}(\varepsilon) \times V_{L^{(c,j_2)}} &\geq V_{L^{(c,j_1+j_2)}}, \\ V_{L^{(c,j_1)}} \times V_{L^{(c,j_2)}} &\geq \sum_{\rho=0}^{2} V_{L^{(0,j_1+j_2)}}(\rho) + 2V_{L^{(c,j_1+j_2)}}, \end{aligned} \tag{4.18}$$

where $\dim_{\mathbb{C}} I_{V_L^\tau}\binom{V_{L^{(c,j_1+j_2)}}}{V_{L^{(c,j_1)}}\ V_{L^{(c,j_2)}}} \geq 2$ follows from the same arguments as in the proof of [**35**, Lemma 6 (2)]. By Lemma 4.4 and (4.18), we have (4.13)–(4.15).

We shall show (4.16) and (4.17) for $i=1$. Note that $L_{\mathbf{0}\times D(1^6)}$ and $L_{C(c^6)\times D(1^6)}$ are even lattices by Lemma 2.6, where $(c^6)=(c,c,c,c,c,c)\in\mathcal{K}^6$ and $(1^6)=(1,1,1,1,1,1)\in\mathbb{Z}_3^6$. We use the lattice vertex operator algebras $V_{L_{\mathbf{0}\times D(1^6)}}$ and $V_{L_{C(c^6)\times D(1^6)}}$ instead of $V_{L_{\mathbf{0}\times D(1)}}$ and $V_{L_{C(c)\times D(1)}}$ since the lattices $L_{\mathbf{0}\times D(1)}$ and $L_{C(c)\times D(1)}$ are not even. By Theorem 3.13,

$$V_{L_{\mathbf{0}\times D(1^6)}}^{T,\mathbf{0}}(\tau) \cong \bigoplus_{k=0}^{2} \bigoplus_{(\rho_1,\ldots,\rho_6)\in\mathbb{Z}_3^6} \bigotimes_{s=1}^{6} V_L^{T,-k}(\tau)[\rho_s]. \tag{4.19}$$

For $j,k,\varepsilon_1,\varepsilon_2\in\mathbb{Z}_3$,

$$V_{L^{(0,j)}}(\varepsilon_1)^{\otimes 6}\cdot V_L^{T,k}(\tau)[\varepsilon_2]^{\otimes 6} \subset V_L^{T,k-j}(\tau)[\varepsilon_1+\varepsilon_2]^{\otimes 6}$$

in $V_{L_{\mathbf{0}\times D(1^6)}}^{T,\mathbf{0}}(\tau)$ by (4.7) and (4.19). Since $V_{L_{\mathbf{0}\times D(1^6)}}^{T,\mathbf{0}}(\tau)$ is irreducible, we have $V_{L_{\mathbf{0}\times D(1^6)}}\cdot V_L^{T,k}(\tau)[\varepsilon_2]^{\otimes 6} = V_{L_{\mathbf{0}\times D(1^6)}}^{T,\mathbf{0}}(\tau)$ and

$$V_{L^{(0,j)}}(\varepsilon_1)^{\otimes 6} \cdot V_L^{T,k}(\tau)[\varepsilon_2]^{\otimes 6} = V_L^{T,k-j}(\tau)[\varepsilon_1+\varepsilon_2]^{\otimes 6} \tag{4.20}$$

in $V^{T,\mathbf{0}}_{L_{\mathbf{0}\times D(1^6)}}(\tau)$. Let $\mathrm{pr}\colon V^{T,\mathbf{0}}_{L_{\mathbf{0}\times D(1^6)}}(\tau) \to V_L^{T,k-j}(\tau)[\varepsilon_1+\varepsilon_2]^{\otimes 6}$ be a projection. For $u \in V_{L^{(0,j)}}(\varepsilon_1)^{\otimes 6}$, $v \in V_L^{T,k}(\tau)[\varepsilon_2]^{\otimes 6}$, set $f(u,x)v = \mathrm{pr}\, Y_{V^{T,\mathbf{0}}_{L_{\mathbf{0}\times D(1^6)}}(\tau)}(u,x)v$. Then $f(\cdot,x)$ is a nonzero intertwining operator of type $\binom{V_L^{T,k-j}(\tau)[\varepsilon_1+\varepsilon_2]^{\otimes 6}}{V_{L^{(0,j)}}(\varepsilon_1)^{\otimes 6}\ V_L^{T,k}(\tau)[\varepsilon_2]^{\otimes 6}}$ for $(V_L^{\tau})^{\otimes 6}$ by (4.20). It follows from [**11**, Proposition 11.9] and [**18**, Proposition 2.10] that (4.16) holds for $i=1$.

By Theorem 3.13,

$$V^{T,\mathbf{0}}_{L_{C(c^6)\times D(1^6)}}(\tau) \cong \bigoplus_{k=0}^{2} \bigoplus_{(\rho_1,\ldots,\rho_6)\in\mathbb{Z}_3^6} \bigotimes_{m=1}^{6} V_L^{T,-k}(\tau)[\rho_m]. \tag{4.21}$$

Since $V_{L_{C(c^6)\times D(1^6)}}$ is simple, it follows from Lemma 2.13 and (4.5) that

$$\left(\bigotimes_{m=1}^{6} V_{L^{(0,0)}}(\nu_m)\right) \cdot V_{L^{(c,j)}}^{\otimes 6} = V_{L^{(c,j)}}^{\otimes 6}$$

in $V_{L_{C(c^6)\times D(1^6)}}$ for $\nu_1,\ldots,\nu_6 \in \mathbb{Z}_3$. Therefore,

$$\begin{aligned} V_{L^{(c,j)}}^{\otimes 6} \cdot V_L^{T,k}(\tau)[\varepsilon]^{\otimes 6} &= \left(\left(\bigotimes_{m=1}^{6} V_{L^{(0,0)}}(\nu_m)\right) \cdot V_{L^{(c,j)}}^{\otimes 6}\right) \cdot V_L^{T,k}(\tau)[\varepsilon]^{\otimes 6} \\ &= \left(\bigotimes_{m=1}^{6} V_{L^{(0,0)}}(\nu_m)\right) \cdot \left(V_{L^{(c,j)}}^{\otimes 6} \cdot V_L^{T,k}(\tau)[\varepsilon]^{\otimes 6}\right) \end{aligned} \tag{4.22}$$

in $V^{T,\mathbf{0}}_{L_{C(c^6)\times D(1^6)}}(\tau)$. For $j,k,\varepsilon \in \mathbb{Z}_3$, (4.8) and (4.21) imply

$$V_{L^{(c,j)}}^{\otimes 6} \cdot V_L^{T,k}(\tau)[\varepsilon]^{\otimes 6} \subset \bigoplus_{(\rho_1,\ldots,\rho_6)\in\mathbb{Z}_3^6} \bigotimes_{m=1}^{6} V_L^{T,k-j}(\tau)[\rho_m] \tag{4.23}$$

and for $\nu_1,\ldots,\nu_6,\rho_1,\ldots,\rho_6 \in \mathbb{Z}_3$, (4.7) implies

$$\left(\bigotimes_{m=1}^{6} V_{L^{(0,0)}}(\nu_m)\right) \cdot \left(\bigotimes_{m=1}^{6} V_L^{T,k-j}(\tau)[\rho_m]\right) \subset \bigotimes_{m=1}^{6} V_L^{T,k-j}(\tau)[\nu_m+\rho_m] \tag{4.24}$$

in $V^{T,\mathbf{0}}_{L_{C(c^6)\times D(1^6)}}(\tau)$. Since $V^{T,\mathbf{0}}_{L_{C(c^6)\times D(1^6)}}(\tau)$ is irreducible, $V_{L^{(c,j)}}^{\otimes 6}\cdot V_L^{T,k}(\tau)[\varepsilon]^{\otimes 6}$ is a nonzero $(V_L^\tau)^{\otimes 6}$-module. Since $\bigotimes_{m=1}^6 V_L^{T,k-j}(\tau)[\rho_m]$, $(\rho_1,\dots,\rho_6)\in\mathbb{Z}_3^6$, are all inequivalent irreducible $(V_L^\tau)^{\otimes 6}(=V_{L^{(0,0)}}(0)^{\otimes 6})$-modules, there exists $(\rho'_1,\dots,\rho'_6)\in\mathbb{Z}_3^6$ such that

$$V_{L^{(c,j)}}^{\otimes 6}\cdot V_L^{T,k}(\tau)[\varepsilon]^{\otimes 6}\supset\bigotimes_{m=1}^{6}V_L^{T,k-j}(\tau)[\rho'_m]$$

by (4.23). By (4.22) and (4.24), we have

$$V_{L^{(c,j)}}^{\otimes 6}\cdot V_L^{T,k}(\tau)[\varepsilon]^{\otimes 6}=\bigoplus_{(\rho_1,\dots,\rho_6)\in\mathbb{Z}_3^6}\bigotimes_{m=1}^{6}V_L^{T,k-j}(\tau)[\rho_m]. \tag{4.25}$$

For $\rho=(\rho_1,\dots,\rho_6)\in\mathbb{Z}_3^6$, let $\mathrm{pr}_\rho\colon V^{T,\mathbf{0}}_{L_{C(c^6)\times D(1^6)}}(\tau)\to\bigotimes_{m=1}^6 V_L^{T,k-j}(\tau)[\rho_m]$ be a projection. For $u\in V_{L^{(c,j)}}^{\otimes 6}$, $v\in V_L^{T,k}(\tau)[\varepsilon]^{\otimes 6}$, set $f_\rho(u,x)v=\mathrm{pr}_\rho Y_{V^{T,\mathbf{0}}_{L_{C(c^6)\times D(1^6)}}(\tau)}(u,x)v$. Then $f_\rho(\cdot,x)$ is a nonzero intertwining operator

$$f_\rho(\cdot,x):V_{L^{(c,j)}}^{\otimes 6}\to\mathrm{Hom}_{\mathbb{C}}\left(V_L^{T,k}(\tau)[\varepsilon]^{\otimes 6},\bigotimes_{m=1}^{6}V_L^{T,k-j}(\tau)[\rho_m]\right)\{x\}$$

for $(V_L^\tau)^{\otimes 6}$ by (4.25). Thus,

$$V_{L^{(c,j)}}\times V_L^{T,k}(\tau)[\varepsilon]\ge\sum_{\rho=0}^{2}V_L^{T,k-j}(\tau)[\rho]$$

holds by [**11**, Proposition 11.9] and [**18**, Proposition 2.10] and hence (4.17) holds by (4.8). We can show (4.16) and (4.17) for $i=2$ similarly. □

Remark 4.6. We can show that the equalities hold in the last two formulas in (4.9) by using (4.2), (4.10), Proposition 4.5 and [**11**, Proposition 11.9].

5. Modules of $V^\tau_{L^{\oplus\ell}}$.

Let ℓ be a positive integer. In this section we discuss $V^\tau_{L^{\oplus\ell}}$-modules, namely the case $C=\{\mathbf{0}\}$ and $D=\{\mathbf{0}\}$. We shall determine some fusion rules for $V^\tau_{L^{\oplus\ell}}$.

In view of Proposition 4.5, we introduce a new index set $\tilde{\mathcal{K}}=\{0,1,2,a,b,c\}$ and define a new commutative binary operation on $\tilde{\mathcal{K}}$ by

$$
\begin{aligned}
&i+j=i+j \pmod 3 && \text{for } i,j=0,1,2,\\
&j+x=x && \text{for } j=0,1,2,\ x=a,b,c,\\
&x+x=0 && \text{for } x=a,b,c,\\
&a+b=c, \quad b+c=a, \quad c+a=b.
\end{aligned}
$$

Then, $\tilde{\mathcal{K}}$ contains $\mathbb{Z}_3$ and $\mathcal{K}$. Note that this binary operation is not associative. We use $\tilde{\mathcal{K}}$ to describe fusion rules for $(V_L^\tau)^{\otimes \ell}$ in (5.4). Define an action of τ on $\tilde{\mathcal{K}}$ by $\tau(a)=b$, $\tau(b)=c$, $\tau(c)=a$, and $\tau(j)=j$, $j=0,1,2$, which is compatible with the action of τ on $\mathbb{Z}_3$ and $\mathcal{K}$. This action of τ preserves the binary operation on $\tilde{\mathcal{K}}$. The set of τ-orbits on $\tilde{\mathcal{K}}$ is $\{0,1,2,c\}$. We consider the componentwise action of H_ℓ on $\tilde{\mathcal{K}}^\ell$ and the componentwise binary operation on $\tilde{\mathcal{K}}^\ell$. The symmetric group $\mathfrak{S}_\ell$ acts on $\tilde{\mathcal{K}}^\ell$ by permuting the components and so G_ℓ acts on $\tilde{\mathcal{K}}^\ell$ naturally. For $\lambda=(\lambda_1,\ldots,\lambda_\ell)\in\tilde{\mathcal{K}}^\ell$, its *support* is defined to be $\mathrm{supp}_{\tilde{\mathcal{K}}}(\lambda)=\{i \mid \lambda_i\in\{a,b,c\}\}$. The cardinality of $\mathrm{supp}_{\tilde{\mathcal{K}}}(\lambda)$ is called the *weight* of λ. We denote the weight of λ by $\mathrm{wt}_{\tilde{\mathcal{K}}}(\lambda)$. For $\lambda^1,\lambda^2\in\tilde{\mathcal{K}}^\ell$, we write $\lambda^1\equiv_\tau\lambda^2$ if λ^1 and λ^2 belong to the same orbit of $\tau=(\tau,\ldots,\tau)$ in $\tilde{\mathcal{K}}^\ell$. We denote by $(\tilde{\mathcal{K}^\ell})_{\equiv_\tau}$ the set of all orbits of τ in $\tilde{\mathcal{K}}^\ell$. For a τ-invariant subset P of $\tilde{\mathcal{K}}^\ell$, $P_{\equiv_\tau}$ denotes the set of all orbits of τ in P.

By Proposition 4.1 and [**18**, Proposition 2.7], $(V_L^\tau)^{\otimes\ell}$ is a rational and C_2-cofinite vertex operator algebra. Moreover,

$$\{U^1\otimes\cdots\otimes U^\ell \mid U^1,\ldots,U^\ell \text{ are irreducible } V_L^\tau\text{-modules}\} \tag{5.1}$$

is a complete list of irreducible $(V_L^\tau)^{\otimes\ell}$-modules up to isomorphism. Set

$$
\begin{aligned}
\mathcal{P}_0&=\{U^1\otimes\cdots\otimes U^\ell \mid U^1,\ldots,U^\ell\in\{V_{L^{(0,j)}}(\varepsilon),V_{L^{(c,j)}} \mid j,\varepsilon\in\mathbb{Z}_3\}\},\\
\mathcal{P}_1&=\big\{U^1\otimes\cdots\otimes U^\ell \mid U^1,\ldots,U^\ell\in\big\{V_{L^{\oplus\ell}}^{T,k}(\tau)[\varepsilon] \mid k,\varepsilon\in\mathbb{Z}_3\big\}\big\},\\
\mathcal{P}_2&=\big\{U^1\otimes\cdots\otimes U^\ell \mid U^1,\ldots,U^\ell\in\big\{V_{L^{\oplus\ell}}^{T,k}(\tau^2)[\varepsilon] \mid k,\varepsilon\in\mathbb{Z}_3\big\}\big\},\\
\mathcal{P}&=\mathcal{P}_0\cup\mathcal{P}_1\cup\mathcal{P}_2.
\end{aligned}
\tag{5.2}
$$

Set $\bar{H}_\ell=\{(\tau^{i_1},\ldots,\tau^{i_{\ell-1}},1)\in H_\ell \mid i_1,\ldots,i_{\ell-1}\in\mathbb{Z}\}$. Then $\bar{H}_\ell$ acts on $V_{L^{\oplus\ell}}^\tau$ naturally and $(V_{L^{\oplus\ell}}^\tau)^{\bar{H}_\ell}=(V_L^\tau)^{\otimes\ell}$.

For $i\in\tilde{\mathcal{K}}$ and $j=0,1,2$, set

$$X_{i,j}=\begin{cases} V_{L^{(0,j)}}(i) & \text{if } i=0,1,2,\\ V_{L^{(i,j)}} & \text{if } i=a,b,c.\end{cases}$$

For $\xi = (\xi_1, \ldots, \xi_\ell) \in \tilde{\mathcal{K}}^\ell$ and $\gamma = (\gamma_1, \ldots, \gamma_\ell) \in \mathbb{Z}_3^\ell$, set

$$X_{\xi,\gamma} = \bigotimes_{i=1}^{\ell} X_{\xi_i,\gamma_i}.$$

Then, for $\lambda \in \mathcal{K}^\ell$ and $\gamma \in \mathbb{Z}_3^\ell$ Lemma 2.13 implies that $V_{L_{(\lambda,\gamma)}} = \bigoplus_\xi X_{\xi,\gamma}$ and

$$V_{L_{(g(\lambda),\gamma)}} = \bigoplus_{\xi} X_{g(\xi),\gamma}, \tag{5.3}$$

where $g \in H_\ell$ and ξ runs over the set $\{\xi = (\xi_1, \ldots, \xi_\ell) \in \tilde{\mathcal{K}}^\ell \mid \xi_k = \lambda_k \text{ for all } k \in \operatorname{supp}_{\mathcal{K}}(\lambda)\}$. This observation will be used in the argument just after (5.36).

We have $X_{\xi,\gamma} \cong X_{g(\xi),\gamma}$ as $(V_L^\tau)^{\otimes \ell}$-modules for $g \in H_\ell$ since $V_{L_{(\tau^i(c),j)}} \cong V_{L_{(c,j)}}$ as V_L^τ-modules for $i, j \in \mathbb{Z}_3$. Thus, we can choose ξ to be an element of $\{0, 1, 2, c\}^\ell$ when we deal with $X_{\xi,\gamma}$ as $(V_L^\tau)^{\otimes \ell}$-modules. Using this notation, we can describe some fusion rules for $(V_L^\tau)^{\otimes \ell}$ by Proposition 4.5 and [**18**, Proposition 2.10] as follows:

$$X_{\rho,\gamma^1} \times X_{\xi,\gamma^2} = X_{\rho+\xi,\gamma^1+\gamma^2}, \tag{5.4}$$

for $\rho \in \mathbb{Z}_3^\ell$, $\xi \in \{0, 1, 2, c\}^\ell$, and $\gamma^1, \gamma^2 \in \mathbb{Z}_3^\ell$.

For any $\mathbf{0} \neq \lambda \in \mathcal{K}^\ell$, $\gamma \in \mathbb{Z}_3^\ell$, and $\varepsilon = 0, 1, 2$, set

$$\begin{aligned} P(V_{L_{(\mathbf{0},\gamma)}}(\varepsilon)) &= \left\{ \xi = (\xi_k) \in \mathbb{Z}_3^\ell \,\middle|\, \sum_{k=1}^{\ell} \xi_k \equiv \varepsilon \pmod 3 \right\}, \\ P(V_{L_{(\lambda,\gamma)}}) &= \{\xi \in \{0, 1, 2, c\}^\ell \mid \operatorname{supp}_{\tilde{\mathcal{K}}}(\xi) = \operatorname{supp}_{\mathcal{K}}(\lambda)\}. \end{aligned} \tag{5.5}$$

Then, Lemma 2.13 implies that

$$\begin{aligned} V_{L_{(\mathbf{0},\gamma)}}(\varepsilon) &\cong \bigoplus_{\xi \in P(V_{L_{(\mathbf{0},\gamma)}}(\varepsilon))} X_{\xi,\gamma}, \\ V_{L_{(\lambda,\gamma)}} &\cong \bigoplus_{\xi \in P(V_{L_{(\lambda,\gamma)}})} X_{\xi,\gamma} \end{aligned} \tag{5.6}$$

as $(V_L^\tau)^{\otimes \ell}$-modules. In particular, we have

$$V_{L^{\oplus\ell}}^{\tau} \cong \bigoplus_{\substack{\rho=(\rho_i)\in\mathbb{Z}_3^{\ell}\\ \rho_1+\cdots+\rho_\ell=0}} X_{\rho,\mathbf{0}} \tag{5.7}$$

as $(V_L^{\tau})^{\otimes\ell}$-modules.

We have already seen in (2.25) and Section 3 that for $\lambda\in\mathcal{K}^{\ell}$, $\gamma\in\mathbb{Z}_3^{\ell}$, $\eta\in\mathbb{Z}_3^{\ell}$, and $g\in H_\ell$

$$\begin{aligned} V_{L_{(\lambda,\gamma)}}\circ g &\cong g^{-1}\big(V_{L_{(\lambda,\gamma)}}\big) = V_{L_{(g^{-1}(\lambda),\gamma)}},\\ V_{L^{\oplus\ell}}^{T,\eta}(\tau^i)\circ g &\cong g^{-1}\big(V_{L^{\oplus\ell}}^{T,\eta}(\tau^i)\big) = V_{L^{\oplus\ell}}^{T,\eta}(\tau^i) \end{aligned} \tag{5.8}$$

as $V_{L^{\oplus\ell}}$-modules or τ^i-twisted $V_{L^{\oplus\ell}}$-modules. Hence for any $\mathbf{0}\neq\lambda\in\mathcal{K}^{\ell}$, $\gamma\in\mathbb{Z}_3^{\ell}$, $\varepsilon=0,1,2$, and $g\in\bar{H}_\ell$,

$$\begin{aligned} V_{L_{(\mathbf{0},\gamma)}}(\varepsilon)\circ g \cong V_{L_{(\mathbf{0},\gamma)}}(\varepsilon), \qquad & V_{L_{(\lambda,\gamma)}}\circ g\cong V_{L_{(g^{-1}(\lambda),\gamma)}},\\ V_{L^{\oplus\ell}}^{T,\eta}(\tau^i)[\varepsilon]\circ g &\cong V_{L^{\oplus\ell}}^{T,\eta}(\tau^i)[\varepsilon] \end{aligned} \tag{5.9}$$

as $V_{L^{\oplus\ell}}^{\tau}$-modules.

Lemma 5.1. *Let N be an $\mathbb{N}$-graded weak $V_{L^{\oplus\ell}}^{\tau}$-module. Then any irreducible $(V_L^{\tau})^{\otimes\ell}$-submodule of N is isomorphic to an element of $\mathcal{P}$ as defined in* (5.2).

Proof. Let U be an irreducible $(V_L^{\tau})^{\otimes\ell}$-submodule of N. By (5.1), there are irreducible V_L^{τ}-modules $U^1,\dots,U^{\ell}$ such that $U\cong U^1\otimes\cdots\otimes U^{\ell}$. Set $S=V_{L^{\oplus\ell}}^{\tau}\cdot U$. For the same reason as in [**36**, Proof of Lemma 5.2], S is an ordinary $V_{L^{\oplus\ell}}^{\tau}$-module. Moreover, $Y_N(v,x)U\neq 0$ for any nonzero $v\in V_{L^{\oplus\ell}}^{\tau}$.

Set

$$\begin{aligned} Q_0 &= \{i\in\{1,\dots,\ell\}\mid U^i\in\{V_{L^{(0,j)}}(\varepsilon), V_{L^{(c,j)}}\mid j,\varepsilon\in\mathbb{Z}_3\}\},\\ Q_1 &= \big\{i\in\{1,\dots,\ell\}\mid U^i\in\big\{V_{L^{\oplus\ell}}^{T,k}(\tau)[\varepsilon]\mid k,\varepsilon\in\mathbb{Z}_3\big\}\big\},\\ Q_2 &= \big\{i\in\{1,\dots,\ell\}\mid U^i\in\big\{V_{L^{\oplus\ell}}^{T,k}(\tau^2)[\varepsilon]\mid k,\varepsilon\in\mathbb{Z}_3\big\}\big\}. \end{aligned}$$

Let ω_L be the Virasoro element of V_L^{τ}. By [**36**, Section 4], the eigenvalues of $(\omega_L)_1$ on the top levels of irreducible V_L^{τ}-modules are

$$\begin{aligned} &0 \text{ for } V_{L^{(0,0)}}(0),\\ &1 \text{ for } V_{L^{(0,0)}}(\varepsilon), && \varepsilon=1,2, \end{aligned}$$

$$
\begin{aligned}
&2/3 \text{ for } V_{L^{(0,j)}}(\varepsilon), && j=1,2,\ \varepsilon=0,1,2,\\
&1/2 \text{ for } V_{L^{(c,0)}},\\
&1/6 \text{ for } V_{L^{(c,j)}}, && j=1,2,\\
&1/9 \text{ for } V_L^{T,0}(\tau^i)[0] \text{ and } V_L^{T,j}(\tau^i)[2], && i=1,2,\ j=1,2,\\
&4/9 \text{ for } V_L^{T,0}(\tau^i)[2] \text{ and } V_L^{T,j}(\tau^i)[1], && i=1,2,\ j=1,2,\\
&7/9 \text{ for } V_L^{T,0}(\tau^i)[1] \text{ and } V_L^{T,j}(\tau^i)[0], && i=1,2,\ j=1,2.
\end{aligned}
\tag{5.10}
$$

Let W^1 be an irreducible V_L^τ-module and let $W^{1,r}$, $r=0,1,2$ be the irreducible V_L^τ-module determined by the fusion rule $V_{L^{(0,0)}}(r) \times W^1 = W^{1,r}$ in Proposition 4.5. Let λ_1 and $\lambda_{1,r}$ be the eigenvalues of $(\omega_L)_1$ on the top levels of W^1 and $W^{1,r}$, respectively. By (5.10), we have

$$
\lambda_{1,r} - \lambda_1 \equiv \begin{cases} 0 & \text{if } W^1 \in \{V_{L^{(0,j)}}(\varepsilon), V_{L^{(c,j)}} \mid j,\varepsilon = 0,1,2\},\\ 2r/3 & \text{if } W^1 \in \big\{V_L^{T,j}(\tau)[\varepsilon] \mid j,\varepsilon=0,1,2\big\},\\ r/3 & \text{if } W^1 \in \big\{V_L^{T,j}(\tau^2)[\varepsilon] \mid j,\varepsilon=0,1,2\big\}. \end{cases} \pmod{\mathbb{Z}} \tag{5.11}
$$

Let ω be the Virasoro element of $V_{L^{\oplus\ell}}^\tau$. Assume that $Q_s, Q_t \neq \emptyset$, $s \neq t$. Take $i_s \in Q_s$ and $i_t \in Q_t$ and define $\rho = (\rho_i) \in \mathbb{Z}_3^\ell$ by

$$
\rho_i = \begin{cases} 1 & \text{if } i = i_s,\\ 2 & \text{if } i = i_t,\\ 0 & \text{otherwise.} \end{cases}
$$

By (5.7), $X_{\rho,\mathbf{0}}$ is an irreducible $(V_L^\tau)^{\otimes\ell}$-submodule of $V_{L^{\oplus\ell}}^\tau$. Using (5.11), Proposition 4.5, and [**18**, Proposition 2.10], one can show that S has a $(V_L^\tau)^{\otimes\ell}$-submodule W such that the difference of the minimal eigenvalues of ω_1 in W and in U is not an integer since $0 \neq X_{\rho,\mathbf{0}} \cdot U \subset S$. This is a contradiction. Hence the assertion holds. □

Lemma 5.2. *Let N be an $\mathbb{N}$-graded weak $V_{L^{\oplus\ell}}^\tau$-module. Let M be an irreducible $(V_L^\tau)^{\otimes\ell}$-submodule of N and N^1 the $V_{L^{\oplus\ell}}^\tau$-submodule of N generated by M. Then N^1 is isomorphic to one of the following inequivalent irreducible $V_{L^{\oplus\ell}}^\tau$-modules.*

(1) $V_{L_{(\mathbf{0},\gamma)}}(\varepsilon)$, $\gamma \in \mathbb{Z}_3^\ell$, $\varepsilon = 0,1,2$.
(2) $V_{L_{(\lambda,\gamma)}}$, $\mathbf{0} \neq \lambda \in (\mathcal{K}^\ell)_{\equiv_\tau}$, $\gamma \in \mathbb{Z}_3^\ell$.
(3) $V_{L^{\oplus\ell}}^{T,\eta}(\tau^i)[\varepsilon]$, $\eta \in \mathbb{Z}_3^\ell$, $i = 1,2$, $\varepsilon = 0,1,2$.

PROOF. By (5.8), we have

$$V_{L_{(\mathbf{0},\gamma)}} \circ \tau \cong V_{L_{(\mathbf{0},\gamma)}}, \qquad V_{L_{(\lambda,\gamma)}} \circ \tau \cong V_{L_{(\tau^{-1}(\lambda),\gamma)}} \ncong V_{L_{(\lambda,\gamma)}}$$

as $V_{L^{\oplus \ell}}$-modules for $\mathbf{0} \neq \lambda \in \mathcal{K}^\ell$ and $\gamma \in \mathbb{Z}_3^\ell$ and

$$V_{L^{\oplus \ell}}^{T,\eta}(\tau^i) \circ \tau \cong \tau^{-1}\big(V_{L^{\oplus \ell}}^{T,\eta}(\tau^i)\big) = V_{L^{\oplus \ell}}^{T,\eta}(\tau^i)$$

as τ^i-twisted $V_{L^{\oplus \ell}}$-modules for $\eta \in \mathbb{Z}_3^\ell$. It follows from [**34**, Theorem 2] that the $V_{L^{\oplus \ell}}^\tau$-modules in the above list are irreducible and inequivalent.

By Lemma 5.1, M is an element of $\mathcal{P}$ in (5.2). Suppose $M \in \mathcal{P}_0$, that is, $M \cong X_{\xi,\gamma}$, $\xi \in \{0,1,2,c\}^\ell$, $\gamma \in \mathbb{Z}_3^\ell$. Set $\Xi = \{\rho + \xi \in \{0,1,2,c\}^\ell \mid \rho = (\rho_i) \in \mathbb{Z}_3^\ell, \sum_{i=1}^\ell \rho_i = 0\}$. Since $(V_L^\tau)^{\otimes \ell}$ is a rational vertex operator algebra, N^1 is a direct sum of $(V_L^\tau)^{\otimes \ell}$-modules. By (5.4) and (5.7), we can write $N^1 = \oplus_{j \in \mathcal{J}} M^j$ where each M^j is isomorphic to $X_{\nu^j,\gamma}$, $\nu^j \in \Xi$. We can take $M^{j_1} = M$ for some $j_1 \in \mathcal{J}$. Let $\mathrm{pr}_j \colon N^1 \to M^j$, $j \in \mathcal{J}$ be projections. For any $j \in \mathcal{J}$, $u \in X_{\rho,\mathbf{0}} \subset V_{L^{\oplus \ell}}^\tau$, $v \in M$, define $f_j(u,x)v = \mathrm{pr}_j(Y_N(u,x)v)$. Then $f_j \in I_{(V_L^\tau)^{\otimes \ell}}\binom{M^j}{X_{\rho,\mathbf{0}}\ M}$. For each $\nu \in \Xi$, we see from (5.4) that there is at most one $j \in \mathcal{J}$ such that $M^j \cong X_{\nu,\gamma}$ (cf. [**36**, Proof of Lemma 5.6]).

Assume that $X_{\rho,\mathbf{0}} \cdot M = 0$ for $X_{\rho,\mathbf{0}} \subset V_{L^{\oplus \ell}}^\tau$. Then

$$\begin{aligned} 0 &= V_{L^{\oplus \ell}}^\tau \cdot (X_{\rho,\mathbf{0}} \cdot M) = \big(V_{L^{\oplus \ell}}^\tau \cdot X_{\rho,\mathbf{0}}\big) \cdot M \\ &= V_{L^{\oplus \ell}}^\tau \cdot M \supset M \end{aligned}$$

since $V_{L^{\oplus \ell}}^\tau$ is simple. This is a contradiction. Hence $0 \neq X_{\rho,\mathbf{0}} \cdot M$, and consequently $X_{\rho,\mathbf{0}} \cdot M \cong X_{\rho+\xi,\gamma}$ as $(V_L^\tau)^{\otimes \ell}$-modules. Therefore, we have

$$N^1 \cong \bigoplus_{\nu \in \Xi} X_{\nu,\gamma} \tag{5.12}$$

as $(V_L^\tau)^{\otimes \ell}$-modules. Applying the above arguments to $V_{L^{\oplus \ell}}^\tau$-module N^1, we conclude that N^1 is irreducible.

By [**20**, Theorem 6.14], if two irreducible $V_{L^{\oplus \ell}}^\tau$-modules W^1, W^2 have an isomorphic irreducible $(V_L^\tau)^{\otimes \ell}$-submodule, then there exists $g \in \bar{H}_\ell$ such that $W^1 \circ g \cong W^2$. Hence by (5.6) and (5.9), N^1 is isomorphic to $V_{L_{(\mathbf{0},\gamma)}}(\varepsilon)$, $\varepsilon \in \mathbb{Z}_3$ or $V_{L_{(\lambda,\gamma)}}$, $\mathbf{0} \neq \lambda \in \mathcal{K}^\ell$.

For $i = 1, 2$, we see from Theorem 3.13 that every irreducible $(V_L^\tau)^{\otimes \ell}$-module in $\mathcal{P}_i$ appears in the irreducible $V_{L^{\oplus \ell}}^\tau$-modules listed in (3). Hence one can show that if $M \in \mathcal{P}_i$, then $N^1 \cong V_{L^{\oplus \ell}}^{T,\eta}(\tau^i)[\varepsilon]$, $\eta \in \mathbb{Z}_3^\ell$, $\varepsilon \in \mathbb{Z}_3$ similarly. $\square$

Proposition 5.3. *$V_{L^{\oplus\ell}}^{\tau}$ is a simple, rational, C_2-cofinite, and CFT type vertex operator algebra. The following is a complete set of representatives of equivalence classes of irreducible $V_{L^{\oplus\ell}}^{\tau}$-modules.*

(1) $V_{L_{(\mathbf{0},\gamma)}}(\varepsilon)$, $\gamma \in \mathbb{Z}_3^{\ell}$, $\varepsilon = 0,1,2$.
(2) $V_{L_{(\lambda,\gamma)}}$, $\mathbf{0} \neq \lambda \in (\mathcal{K}^{\ell})_{\equiv_\tau}$, $\gamma \in \mathbb{Z}_3^{\ell}$.
(3) $V_{L^{\oplus\ell}}^{T,\eta}(\tau^i)[\varepsilon]$, $\eta \in \mathbb{Z}_3^{\ell}$, $i = 1,2$, $\varepsilon = 0,1,2$.

Proof. By (5.7) and [**5**], $V_{L^{\oplus\ell}}^{\tau}$ is a C_2-cofinite vertex operator algebra. The classification of irreducible $V_{L^{\oplus\ell}}^{\tau}$-modules follows from Lemma 5.2. Since $(V_L^{\tau})^{\otimes\ell}$ is rational, the rationality of $V_{L^{\oplus\ell}}^{\tau}$ follows from Lemma 5.2. □

The following lemma gives lower bounds for some fusion rules for $V_{L^{\oplus\ell}}^{\tau}$.

Lemma 5.4. *Let $\lambda, \lambda^1, \lambda^2$ be nonzero elements of $\mathcal{K}^{\ell}$ such that $\lambda^1 \not\equiv_\tau \lambda^2$, $\gamma, \gamma^1, \gamma^2$, $\eta \in \mathbb{Z}_3^{\ell}$, $i = 1,2$, and $\varepsilon, \varepsilon_1$, $\varepsilon_2 = 0,1,2$. Then*

$$V_{L_{(\mathbf{0},\gamma^1)}}(\varepsilon_1) \times V_{L_{(\mathbf{0},\gamma^2)}}(\varepsilon_2) \geq V_{L_{(\mathbf{0},\gamma^1+\gamma^2)}}(\varepsilon_1+\varepsilon_2), \tag{5.13}$$

$$V_{L_{(\mathbf{0},\gamma^1)}}(\varepsilon) \times V_{L_{(\lambda,\gamma^2)}} \geq V_{L_{(\lambda,\gamma^1+\gamma^2)}}, \tag{5.14}$$

$$V_{L_{(\lambda^1,\gamma^1)}} \times V_{L_{(\lambda^2,\gamma^2)}} \geq \sum_{j=0}^{2} V_{L_{(\lambda^1+\tau^j(\lambda^2),\gamma^1+\gamma^2)}}, \tag{5.15}$$

$$V_{L_{(\lambda,\gamma^1)}} \times V_{L_{(\lambda,\gamma^2)}} \geq \sum_{\rho=0}^{2} V_{L_{(\mathbf{0},\gamma^1+\gamma^2)}}(\rho) + 2V_{L_{(\lambda,\gamma^1+\gamma^2)}}, \tag{5.16}$$

$$V_{L_{(\mathbf{0},\gamma)}}(\varepsilon_1) \times V_{L^{\oplus\ell}}^{T,\eta}(\tau^i)[\varepsilon_2] \geq V_{L^{\oplus\ell}}^{T,\eta-i\gamma}(\tau^i)[i\varepsilon_1+\varepsilon_2], \tag{5.17}$$

$$V_{L_{(\lambda,\gamma)}} \times V_{L^{\oplus\ell}}^{T,\eta}(\tau^i)[\varepsilon] \geq \sum_{\rho=0}^{2} V_{L^{\oplus\ell}}^{T,\eta-i\gamma}(\tau^i)[\rho]. \tag{5.18}$$

Proof. Restricting intertwining operators for $V_{L^{\oplus\ell}}$ in Lemma 2.14 to $V_{L^{\oplus\ell}}^{\tau}$-modules, we have (5.13)–(5.16), where $\dim_{\mathbb{C}} I_{V_{L^{\oplus\ell}}^{\tau}}\binom{V_{L_{(\lambda,\gamma^1+\gamma^2)}}}{V_{L_{(\lambda,\gamma^1)}}\ V_{L_{(\lambda,\gamma^2)}}} \geq 2$ follows from the same arguments as in the proof of [**35**, Lemma 6 (2)].

We shall show (5.18) for $i = 1$. (5.17) and (5.18) for $i = 2$ can be proved by a similar argument. It is easy to see that

$$I_{V_{L^{\oplus\ell}}^{\tau}}\begin{pmatrix} M \\ V_{L_{(\lambda,\gamma)}}\ \ V_{L^{\oplus\ell}}^{T,\eta}(\tau)[\varepsilon]\end{pmatrix} = 0 \tag{5.19}$$

for all $M \not\cong V_{L^{\oplus\ell}}^{T,\eta-\gamma}(\tau)[r]$, $r = 0,1,2$, by Proposition 4.5 and [**18**, Proposition 2.10]. Set $\tilde{\lambda} = (\lambda,\lambda,\lambda,\lambda,\lambda,\lambda) \in \mathcal{K}^{6\ell}$ and $\tilde{\gamma} = (\gamma,\gamma,\gamma,\gamma,\gamma,\gamma) \in \mathbb{Z}_3^{6\ell}$. Recall that $C(\tilde{\lambda})$ is the $\mathcal{K}$-code generated by $\tilde{\lambda}$ and $\tau(\tilde{\lambda})$ and that $D(\tilde{\gamma})$ is the $\mathbb{Z}_3$-code generated by $\tilde{\gamma}$ (cf. Section 4). Lemma 2.6 implies $L_{C(\tilde{\lambda})\times D(\tilde{\gamma})}$ is a τ-invariant even lattice. To obtain (5.18), we use the lattice vertex operator algebra $V_{L_{C(\tilde{\lambda})\times D(\tilde{\gamma})}}$ instead of $V_{L_{(\lambda,\gamma)}}$ since the lattice $L_{(\lambda,\gamma)}$ is not even. Let $\eta \in \mathbb{Z}_3^{\ell}$ and set $\tilde{\eta} = (\eta,\eta,\eta,\eta,\eta,\eta) \in \mathbb{Z}_3^{6\ell}$. Consider a τ-twisted $V_{L_{C(\tilde{\lambda})\times D(\tilde{\gamma})}}$-module $V_{L_{C(\tilde{\lambda})\times D(\tilde{\gamma})}}^{T,\tilde{\eta}}(\tau)$. It follows from (3.27) that

$$V_{L_{C(\tilde{\lambda})\times D(\tilde{\gamma})}}^{T,\tilde{\eta}}(\tau) \cong \bigoplus_{j=0}^{2} V_{L^{\oplus 6\ell}}^{T,\tilde{\eta}-j\tilde{\gamma}}(\tau) \tag{5.20}$$

as τ-twisted $V_{L^{\oplus 6\ell}}$-modules. We have

$$V_{L_{C(\tilde{\lambda})\times D(\tilde{\gamma})}}^{T,\tilde{\eta}}(\tau) \cong \bigoplus_{j=0}^{2} \bigoplus_{\rho_1,\dots,\rho_6\in\mathbb{Z}_3} \bigotimes_{i=1}^{6} V_{L^{\oplus\ell}}^{T,\eta-j\gamma}(\tau)[\rho_i] \tag{5.21}$$

as $(V_{L^{\oplus\ell}}^{\tau})^{\otimes 6}$-modules by the same argument as was used in the proof of Theorem 3.13 by replacing V_L, $V_{L^{\oplus\ell}}$, C, D, and η by $V_{L^{\oplus\ell}}$, $V_{L^{\oplus 6\ell}}$, $C(\tilde{\lambda})$, $D(\tilde{\gamma})$, and $\tilde{\eta}$, respectively. Since $V_{L_{C(\tilde{\lambda})\times D(\tilde{\gamma})}}$ is simple, it follows from Lemma 2.13 and (4.14) that

$$\left(\bigotimes_{m=1}^{6} V_{L_{(\mathbf{0},\mathbf{0})}}(\nu_m)\right)\cdot V_{L_{(\lambda,\gamma)}}^{\otimes 6} = V_{L_{(\lambda,\gamma)}}^{\otimes 6}$$

in $V_{L_{C(\tilde{\lambda})\times D(\tilde{\gamma})}}$ for $\nu_1,\dots,\nu_6 \in \mathbb{Z}_3$. Therefore,

$$\begin{aligned} V_{L_{(\lambda,\gamma)}}^{\otimes 6}\cdot V_{L^{\oplus\ell}}^{T,\eta}(\tau)[\varepsilon]^{\otimes 6} &= \left(\left(\bigotimes_{m=1}^{6} V_{L_{(\mathbf{0},\mathbf{0})}}(\nu_m)\right)\cdot V_{L_{(\lambda,\gamma)}}^{\otimes 6}\right)\cdot V_{L^{\oplus\ell}}^{T,\eta}(\tau)[\varepsilon]^{\otimes 6} \\ &= \left(\bigotimes_{m=1}^{6} V_{L_{(\mathbf{0},\mathbf{0})}}(\nu_m)\right)\cdot\left(V_{L_{(\lambda,\gamma)}}^{\otimes 6}\cdot V_{L^{\oplus\ell}}^{T,\eta}(\tau)[\varepsilon]^{\otimes 6}\right) \end{aligned} \tag{5.22}$$

in $V_{L_{C(\tilde{\lambda})\times D(\tilde{\gamma})}}^{T,\tilde{\eta}}(\tau)$. By Proposition 4.5 and (5.21),

$$V_{L_{(\lambda,\gamma)}}^{\otimes 6}\cdot\left(V_{L^{\oplus\ell}}^{T,\eta}(\tau)[\varepsilon]\right)^{\otimes 6} \subset \bigoplus_{\rho_1,\dots,\rho_6\in\mathbb{Z}_3} \bigotimes_{m=1}^{6} V_{L^{\oplus\ell}}^{T,\eta-\gamma}(\tau)[\rho_m]$$

and for $\nu_1, \ldots, \nu_6, \rho_1, \ldots, \rho_6 \in \mathbb{Z}_3$,

$$\left(\bigotimes_{m=1}^{6} V_{L_{(\mathbf{0},\mathbf{0})}}(\nu_m)\right) \cdot \left(\bigotimes_{m=1}^{6} V_{L^{\oplus\ell}}^{T,\eta-\gamma}(\tau)[\rho_m]\right) \subset \bigotimes_{m=1}^{6} V_{L^{\oplus\ell}}^{T,\eta-\gamma}(\tau)[\nu_m+\rho_m] \quad (5.23)$$

in $V^{T,\tilde{\eta}}_{L_{C(\tilde{\lambda})\times D(\tilde{\gamma})}}(\tau)$. Since $V^{T,\tilde{\eta}}_{L_{C(\tilde{\lambda})\times D(\tilde{\gamma})}}(\tau)$ is a τ-twisted irreducible $V_{L_{C(\tilde{\lambda})\times D(\tilde{\gamma})}}$-module, $V_{L_{(\lambda,\gamma)}}^{\otimes 6} \cdot \big(V_{L^{\oplus\ell}}^{T,\eta}(\tau)[\varepsilon]\big)^{\otimes 6}$ is a nonzero $(V_{L^{\oplus\ell}}^{\tau})^{\otimes 6}$-module. Since $\bigotimes_{m=1}^{6} V_{L^{\oplus\ell}}^{T,\eta-\gamma}(\tau)[\rho_m]$, $(\rho_1,\ldots,\rho_6) \in \mathbb{Z}_3^6$, are all inequivalent irreducible $(V_{L^{\oplus\ell}}^{\tau})^{\otimes 6}$-modules, there exists $(\rho_1',\ldots,\rho_6') \in \mathbb{Z}_3^6$ such that

$$V_{L_{(\lambda,\gamma)}}^{\otimes 6} \cdot \big(V_{L^{\oplus\ell}}^{T,\eta}(\tau)[\varepsilon]\big)^{\otimes 6} \supset \bigotimes_{i=1}^{6} V_{L^{\oplus\ell}}^{T,\eta-\gamma}(\tau)[\rho_i']. \quad (5.24)$$

By (5.22)–(5.24), we have

$$V_{L_{(\lambda,\gamma)}}^{\otimes 6} \cdot V_{L^{\oplus\ell}}^{T,\eta}(\tau)[\varepsilon]^{\otimes 6} = \bigoplus_{\rho_1,\ldots,\rho_6\in\mathbb{Z}_3} \bigotimes_{i=1}^{6} V_{L^{\oplus\ell}}^{T,\eta-\gamma}(\tau)[\rho_i]. \quad (5.25)$$

Using the same argument as in the proof of (4.17), we have (5.18). □

We want to use the results in [**20**] and [**35**]. We follow the notation of [**20**]. Note that we can take all 2-cocyles in [**20**] to be trivial in our setting. Let $\mathcal{S}$ be a finite $\bar{H}_\ell$-stable set of irreducible $V_{L^{\oplus\ell}}^{\tau}$-modules (cf. Section 2.1). Set $\mathcal{M} = \oplus_{M\in\mathcal{S}} M$. Note that $\bar{H}_\ell$ acts on $\mathcal{M}$ by (2.25), (3.25), and (3.26). Define a vector space $\mathbb{C}\mathcal{S} = \oplus_{M\in\mathcal{S}} e(M)$ with formal basis $e(M)$, $M \in \mathcal{S}$. The space $\mathbb{C}\mathcal{S}$ is an associative algebra under the product $e(M)e(N) = \delta_{M,N} e(M)$. Define the vector space $\mathcal{A}(\bar{H}_\ell, \mathcal{S}) = \mathbb{C}[\bar{H}_\ell] \otimes \mathbb{C}\mathcal{S}$ with basis $g \otimes e(M)$ for $g \in \bar{H}_\ell$ and $M \in \mathcal{S}$, and a multiplication on it by:

$$g \otimes e(M) \cdot h \otimes e(N) = gh \otimes e(h^{-1}(M))e(N).$$

Then $\mathcal{A}(\bar{H}_\ell, \mathcal{S})$ is an associative algebra with the identity element $\sum_{M\in\mathcal{S}} 1 \otimes e(M)$. We define an action of $\mathcal{A}(\bar{H}_\ell, \mathcal{S})$ on $\mathcal{M}$ as follows: For $M, N \in \mathcal{S}, w \in N$ and $g \in \bar{H}_\ell$, we set

$$g \otimes e(M) \cdot w = \delta_{M,N} g w. \quad (5.26)$$

For $M \in \mathcal{S}$, define a subgroup $(\bar{H}_\ell)_M = \{g \in \bar{H}_\ell \mid g(M) = M\}$ of $\bar{H}_\ell$

and define subalgebras $s(M) = \mathrm{span}_{\mathbb{C}}\{g \otimes e(M) \mid g \in (\bar{H}_\ell)_M\}$ and $D(M) = \mathrm{span}_{\mathbb{C}}\{g \otimes e(M) \mid g \in \bar{H}_\ell\}$ of $\mathcal{A}(\bar{H}_\ell, \mathcal{S})$. Note that $s(M)$ is isomorphic to the group algebra of $(\bar{H}_\ell)_M$. Decompose $\mathcal{S}$ into a disjoint union of $\bar{H}_\ell$-orbits $\mathcal{S} = \cup_{j \in J} \mathcal{O}_j$. Let $M^{(j)}$ be a representative of $\mathcal{O}_j$.

We shall compute some fusion rules for $V_{L^{\oplus \ell}}^\tau$ in Proposition 5.7 by using [**35**, Theorem 2]. We need the following result which gives a complete set of representatives of isomorphism classes of irreducible $\mathcal{A}(\bar{H}_\ell, \mathcal{S})$-modules.

THEOREM 5.5 ([**20**, Theorem 3.6]). *$\mathcal{A}(\bar{H}_\ell, \mathcal{S})$ is semisimple and the irreducible $\mathcal{A}(\bar{H}_\ell, \mathcal{S})$-modules are precisely $D(M^{(j)}) \otimes_{s(M^{(j)})} U$, where U ranges over the irreducible $s(M^{(j)})$-modules and $j \in J$.*

Note that $\bar{H}_\ell$ acts on $(\mathcal{K}^\ell)_{\equiv_\tau}$. Let $\mathbf{0} \neq \lambda \in \{0, c\}^\ell$ and $\gamma \in \mathbb{Z}_3^\ell$ and set $\mathcal{R}_\lambda$ be the $\bar{H}_\ell$-orbit in $(\mathcal{K}^\ell)_{\equiv_\tau}$ containing λ. Then

$$\mathcal{S}_{\lambda,\gamma} = \{V_{L_{(\mu,\gamma)}} \mid \mu \in \mathcal{R}_\lambda\} \tag{5.27}$$

is an $\bar{H}_\ell$-stable set. We shall describe the irreducible $\mathcal{A}(\bar{H}_\ell, \mathcal{S}_{\lambda,\gamma})$-modules in Proposition 5.6. Theorem 5.5 implies that the irreducible $\mathcal{A}(\bar{H}_\ell, \mathcal{S}_{\lambda,\gamma})$-modules are obtained by the irreducible $s(V_{L_{(\lambda,\gamma)}})$-modules. In order to classify the irreducible $s(V_{L_{(\lambda,\gamma)}})$-modules, we first investigate the action of $s(V_{L_{(\lambda,\gamma)}})$ on $V_{L_{(\lambda,\gamma)}}$. We recall the decomposition $V_{L_{(\lambda,\gamma)}} = \oplus_{\xi \in P(V_{L_{(\lambda,\gamma)}})} X_{\xi,\gamma}$ in (5.6), where $P(V_{L_{(\lambda,\gamma)}})$ is given in (5.5). For $g \in \bar{H}_\ell$, g is an element in $(\bar{H}_\ell)_{V_{L_{(\lambda,\gamma)}}}$ if and only if $g\lambda \equiv_\tau \lambda$. Thus, $(\bar{H}_\ell)_{V_{L_{(\lambda,\gamma)}}}$ consists of the elements

$$(\tau^{j_1}, \ldots, \tau^{j_\ell})(\tau, \ldots, \tau)^{-j_\ell} \in \bar{H}_\ell \tag{5.28}$$

with $j_k = 0$ for all $k \in \mathrm{supp}_{\mathcal{K}}(\lambda)$. Note that $|(\bar{H}_\ell)_{V_{L_{(\lambda,\gamma)}}}| = |P(V_{L_{(\lambda,\gamma)}})| = 3^{\ell - \mathrm{wt}_{\mathcal{K}}(\lambda)}$. We have

$$\begin{aligned} gu &= (\tau^{j_1}, \ldots, \tau^{j_\ell})(\tau, \ldots, \tau)^{-j_\ell} u \\ &= \zeta_3^{\sum_k j_k \xi_k} (\tau, \ldots, \tau)^{-j_\ell} u \in (\tau, \ldots, \tau)^{-j_\ell}(X_{\xi,\gamma}) \end{aligned}$$

for $\xi = (\xi_i) \in P(V_{L_{(\lambda,\gamma)}})$, $u \in X_{\xi,\gamma}$ and $g \in (\bar{H}_\ell)_{V_{L_{(\lambda,\gamma)}}}$ of the form (5.28), where we define $0c = 0$ in the sum $\sum_k j_k \xi_k$. Note that the linear map $X_{\xi,\gamma} \ni u \mapsto (\tau, \ldots, \tau)^{-j_\ell} u \in (\tau, \ldots, \tau)^{-j_\ell}(X_{\xi,\gamma})$ is an isomorphism of $(V_L^\tau)^{\otimes \ell}$-modules induced by the isomorphism $(\tau, \ldots, \tau)^{-j_\ell} \colon V_{L_{(\lambda,\gamma)}} \to V_{L_{(\tau^{-j_\ell}(\lambda),\gamma)}}$ of $V_{L^{\oplus \ell}}^\tau$-modules.

For $\xi = (\xi_i) \in \{0, 1, 2, c\}^\ell$, $\mathbb{C}e(\xi)$ denotes a vector space with formal basis $e(\xi)$. In view of the above observation, we define an action of $s(V_{L_{(\lambda,\gamma)}})$ on $\mathbb{C}e(\xi)$

by setting

$$g \otimes e(V_{L_{(\lambda,\gamma)}}) \cdot e(\xi) = \zeta_3^{\sum_k j_k \xi_k} e(\xi)$$

for $g \in (\bar{H}_\ell)_{V_{L_{(\lambda,\gamma)}}}$ of the form (5.28) and $\xi \in P(V_{L_{(\lambda,\gamma)}})$, where we define $0c = 0$ in the sum $\sum_k j_k \xi_k$.

Let $\gamma \in \mathbb{Z}_3^\ell$ and $\varepsilon \in \mathbb{Z}_3$. Then $\{V_{L_{(\mathbf{0},\gamma)}}(\varepsilon)\}$ is an $\bar{H}_\ell$-stable set. For the same reason as in the case of $\mathcal{S}_{\lambda,\gamma}$ discussed above, we define an action of $s(V_{L_{(\mathbf{0},\gamma)}}(\varepsilon))$ on $\mathbb{C}e(\xi)$ by setting

$$g \otimes e(V_{L_{(\mathbf{0},\gamma)}}(\varepsilon)) \cdot e(\xi) = \zeta_3^{\sum_k j_k \xi_k} e(\xi)$$

for $g = (\tau^{j_1}, \dots, \tau^{j_{\ell-1}}, 1) \in (\bar{H}_\ell)_{V_{L_{(\mathbf{0},\gamma)}}(\varepsilon)} = \bar{H}_\ell$ and $\xi \in P(V_{L_{(\mathbf{0},\gamma)}}(\varepsilon))$.

We have the following result.

Lemma 5.6. *With the above notation, the following assertions hold.*

(1) $3^{\ell - \mathrm{wt}_{\mathcal{K}}(\lambda)}$ *inequivalent irreducible* $s(V_{L_{(\lambda,\gamma)}})$*-modules* $\mathbb{C}e(\xi)$, $\xi \in P(V_{L_{(\lambda,\gamma)}})$, *form a complete set of irreducible* $s(V_{L_{(\lambda,\gamma)}})$*-modules up to isomorphism and for nonzero* $u \in X_{\xi,\gamma}$, $\mathbb{C}u \cong \mathbb{C}e(\xi)$ *as* $s(V_{L_{(\lambda,\gamma)}})$*-modules. Moreover,*

$$\left\{ D(V_{L_{(\lambda,\gamma)}}) \otimes_{s(V_{L_{(\lambda,\gamma)}})} \mathbb{C}e(\xi) \mid \xi \in P(V_{L_{(\lambda,\gamma)}}) \right\} \tag{5.29}$$

is a complete set of irreducible $\mathcal{A}(\bar{H}_\ell, \mathcal{S}_{\lambda,\gamma})$*-modules up to isomorphism and*

$$\dim_{\mathbb{C}} D(V_{L_{(\lambda,\gamma)}}) \otimes_{s(V_{L_{(\lambda,\gamma)}})} \mathbb{C}e(\xi) = |\mathcal{R}_\lambda| = 3^{\mathrm{wt}_{\mathcal{K}}(\lambda) - 1} \tag{5.30}$$

for $\xi \in P(V_{L_{(\lambda,\gamma)}})$.

(2) $3^{\ell-1}$ *inequivalent irreducible* $s(V_{L_{(\mathbf{0},\gamma)}}(\varepsilon))$*-modules* $\mathbb{C}e(\xi)$, $\xi \in P(V_{L_{(\mathbf{0},\gamma)}}(\varepsilon))$, *form a complete set of irreducible* $s(V_{L_{(\mathbf{0},\gamma)}}(\varepsilon))$*-modules up to isomorphism and for nonzero* $u \in X_{\xi,\gamma}$, $\mathbb{C}u \cong \mathbb{C}e(\xi)$ *as* $s(V_{L_{(\mathbf{0},\gamma)}}(\varepsilon))$*-modules. Moreover,*

$$\left\{ D(V_{L_{(\mathbf{0},\gamma)}}(\varepsilon)) \otimes_{s(V_{L_{(\mathbf{0},\gamma)}}(\varepsilon))} \mathbb{C}e(\xi) \mid \xi \in P(V_{L_{(\mathbf{0},\gamma)}}(\varepsilon)) \right\} \tag{5.31}$$

is a complete set of irreducible $\mathcal{A}(\bar{H}_\ell, \{V_{L_{(\mathbf{0},\gamma)}}(\varepsilon)\})$*-modules up to isomorphism and*

$$D(V_{L_{(\mathbf{0},\gamma)}}(\varepsilon)) \otimes_{s(V_{L_{(\mathbf{0},\gamma)}}(\varepsilon))} \mathbb{C}e(\xi) \cong \mathbb{C}e(\xi) \tag{5.32}$$

as vector spaces for $\xi \in P(V_{L_{(\mathbf{0},\gamma)}}(\varepsilon))$.

PROOF. We show the first assertion. The argument just before the lemma shows that $\mathbb{C}u \cong \mathbb{C}e(\xi)$ as $s(V_{L_{(\lambda,\gamma)}})$-modules for nonzero $u \in X_{\xi,\gamma}$. It is clear that $|\mathcal{R}_\lambda| = 3^{\mathrm{wt}_\mathcal{K}(\lambda)-1}$. We have (5.30) since $|\bar{H}_\ell| = 3^{\ell-1}$ and $|(\bar{H}_\ell)_{V_{L_{(\lambda,\gamma)}}}| = 3^{\ell-\mathrm{wt}_\mathcal{K}(\lambda)}$. It follows from $|P(V_{L_{(\lambda,\gamma)}})| = 3^{\ell-\mathrm{wt}_\mathcal{K}(\lambda)}$ that $\{\mathbb{C}e(\xi) \mid \xi \in P(V_{L_{(\lambda,\gamma)}})\}$ is a complete set of irreducible $s(V_{L_{(\lambda,\gamma)}})$-modules up to isomorphism. It follows from Theorem 5.5 that (5.29) is a complete set of irreducible $\mathcal{A}(\bar{H}_\ell, \mathcal{S}_{\lambda,\gamma})$-modules up to isomorphism.

The second assertion can be obtained by a similar argument. □

We want to use the result [**35**, Theorem 2] in Proposition 5.7. Let $\lambda, \lambda^1, \lambda^2$ be nonzero elements of $\mathcal{K}^\ell$ such that $\lambda^1 \not\equiv_\tau \lambda^2$ and let γ^1, γ^2 be elements of $\mathbb{Z}_3^\ell$. Set $\gamma^3 = \gamma^1 + \gamma^2$, $\mathcal{R}_i = \mathcal{R}_{\lambda^i}$, and $\mathcal{S}_i = \mathcal{S}_{\lambda^i,\gamma^i}$ for $i = 1,2$ (cf. (5.27)). For $i = 1,2$, set $\xi^i = (\xi^i_j) \in \{0,c\}^\ell$ by

$$\xi^i_j = \begin{cases} 0 & \text{if } \lambda^i_j = 0, \\ c & \text{if } \lambda^i_j = a,b,c. \end{cases} \tag{5.33}$$

Note that $\xi^i \in \mathcal{R}_i$ and $\mathcal{R}_i = \{\mu \in (\mathcal{K}^\ell)_{\equiv_\tau} \mid \mathrm{supp}_\mathcal{K}(\mu) = \mathrm{supp}_\mathcal{K}(\lambda^i)\}$ for $i = 1,2$. Set $\mathcal{S}_3 = \{V_{L_{(\mu,\gamma^3)}}, V_{L_{(\mathbf{0},\gamma^3)}}(\varepsilon) \mid \mathbf{0} \neq \mu \in (\mathcal{K}^\ell)_{\equiv_\tau}, \varepsilon = 0,1,2\}$. For each $i = 1,2,3$, $\mathcal{S}_i$ is an $\bar{H}_\ell$-stable set. Set $\mathcal{T}_i = \{V_{L_{(\xi^i,\gamma^i)}}\}$, $i = 1,2$ and $\mathcal{T}_3 = \{V_{L_{(\mathbf{0},\gamma^3)}}(\varepsilon) \mid \varepsilon = 0,1,2\} \cup \{V_{L_{(\mu,\gamma^3)}} \mid \mathbf{0} \neq \mu \in \{0,c\}^\ell\}$. Then, $\mathcal{T}_i$ is a complete set of representatives of $\bar{H}_\ell$-orbits in $\mathcal{S}_i$ for $i = 1,2,3$. We simply write $P_i = P(V_{L_{(\xi^i,\gamma^i)}}), i = 1,2$ (cf. (5.5)). Let $P_3 = \{0,1,2,c\}^\ell$.

We note that

$$\bigcup_{\varepsilon=0}^{2} \big\{D(V_{L_{(\mathbf{0},\gamma^3)}(\varepsilon)}) \otimes_{s(V_{L_{(\mathbf{0},\gamma^3)}(\varepsilon)})} \mathbb{C}e(\xi) \mid \xi \in P(V_{L_{(\mathbf{0},\gamma^3)}}(\varepsilon))\big\}$$
$$\cup \bigcup_{\mathbf{0}\neq\lambda\in\{0,c\}^\ell} \big\{D(V_{L_{(\lambda,\gamma^3)}}) \otimes_{s(V_{L_{(\lambda,\gamma^3)}})} \mathbb{C}e(\xi) \mid \xi \in P(V_{L_{(\lambda,\gamma^3)}})\big\} \tag{5.34}$$

is a complete set of representatives of isomorphism classes of irreducible $\mathcal{A}(\bar{H}_\ell, \mathcal{S}_3)$-modules by Theorem 5.5 and

$$\{0,1,2,c\}^\ell = \bigcup_{\varepsilon=0}^{2} P\big(V_{L_{(\mathbf{0},\gamma^3)}}(\varepsilon)\big) \cup \bigcup_{\mathbf{0}\neq\lambda\in\{0,c\}^\ell} P\big(V_{L_{(\lambda,\gamma^3)}}\big); \quad \text{disjoint.}$$

Set $\mathcal{M}_i = \oplus_{M\in\mathcal{S}_i} M$ for $i = 1,2,3$. For $i = 1,2,3$, we write $W_{i,\xi}$ for an irreducible $\mathcal{A}(\bar{H}_\ell, \mathcal{S}_i)$-module $D(M) \otimes_{s(M)} \mathbb{C}e(\xi)$ in (5.29), (5.31), and (5.34) since they are parametrized by $\xi \in P_i$. In (5.30) and (5.32), we have already seen

$$\dim_{\mathbb{C}} W_{i,\xi} = 3^{\max\{0, \mathrm{wt}_{\tilde{\mathcal{K}}}(\xi)-1\}}. \tag{5.35}$$

For $i = 1, 2$ and $\xi \in P_i$, we note that $\mathrm{supp}_{\tilde{\mathcal{K}}}(\xi) = \mathrm{supp}_{\mathcal{K}}(\lambda^i)$ and $\dim_{\mathbb{C}} W_{i,\xi} = |\mathcal{R}_i|$. We have

$$\mathcal{M}_i = \bigoplus_{\xi \in P_i} W_{i,\xi} \otimes \mathrm{Hom}_{\mathcal{A}(\bar{H}_\ell, \mathcal{S}_i)}(W_{i,\xi}, \mathcal{M}_i)$$

as an $\mathcal{A}(\bar{H}_\ell, \mathcal{S}_i) \otimes_{\mathbb{C}} (V_L^\tau)^{\otimes \ell}$-module for $i = 1, 2, 3$. Then $\mathrm{Hom}_{\mathcal{A}(\bar{H}_\ell, \mathcal{S}_i)}(W_{i,\xi}, \mathcal{M}_i)$, $\xi \in P_i$ are nonzero inequivalent irreducible $(V_L^\tau)^{\otimes \ell}$-modules by [**20**, Theorem 6.14]. For any $V_{L^{\oplus \ell}}^\tau$-module M in $\mathcal{T}_i$ and any nonzero $u \in X_{\xi, \gamma^i}$, $\xi \in P_i$, in the decomposition (5.6) of M, the $\mathcal{A}(\bar{H}_\ell, \mathcal{S}_i)$-submodule of $\mathcal{M}_i$ generated by u is isomorphic to $W_{i,\xi}$ since $\mathbb{C}e(\xi) \cong \mathbb{C}u$ as $s(M)$-modules by Lemma 5.6. Hence there exists a unique $f_v \in \mathrm{Hom}_{\mathcal{A}(\bar{H}_\ell, \mathcal{S}_i)}(W_{i,\xi}, \mathcal{M}_i)$ such that $f_v(1 \otimes e(\xi)) = v$. In fact, the map $v \mapsto f_v$ is a linear isomorphism. Therefore we identify $\mathrm{Hom}_{\mathcal{A}(\bar{H}_\ell, \mathcal{S}_i)}(W_{i,\xi}, \mathcal{M}_i)$ with X_{ξ, γ^i} and we write

$$\mathcal{M}_i = \bigoplus_{\xi \in P_i} W_{i,\xi} \otimes X_{\xi, \gamma^i}. \tag{5.36}$$

For any $\xi \in P_i$ and any nonzero $v^\xi \in X_{\xi, \gamma^i}$, we can take a basis $\{w^{ij} \mid j = 1, \ldots, \dim_{\mathbb{C}} W_{i,\xi}\}$ of $W_{i,\xi}$ such that for $j = 1, \ldots, \dim_{\mathbb{C}} W_{i,\xi}$, $w^{ij} \otimes v^\xi$ is an element of an irreducible $V_{L^{\oplus \ell}}^\tau$-module in $\mathcal{S}_i$ and if $\dim_{\mathbb{C}} W_{i,\xi} \geq 2$, which implies $\xi \neq \mathbf{0}$, then for $j \neq k$, $w^{ij} \otimes v^\xi$ and $w^{ik} \otimes v^\xi$ belong to different irreducible $V_{L^{\oplus \ell}}^\tau$-modules by (5.3). For $i = 1, 2$, since $\dim_{\mathbb{C}} W_{i,\xi} = |\mathcal{R}_i|$ by (5.35), there exists a bijection $\{1, \ldots, \dim_{\mathbb{C}} W_{i,\xi}\} \ni j \mapsto \mu_j^i \in \mathcal{R}_i$ where μ_j^i is determined by $w^{ij} \otimes v^\xi \in V_{L_{(\mu_j^i, \gamma^i)}} \in \mathcal{S}_i$.

To see the above situation, we describe the case of $i = 1$ as an example. Let $\{h_1, \ldots, h_r\}$ be a complete set of coset representatives of $(\bar{H}_\ell)_{V_{L_{(\lambda^1, \gamma^1)}}} = \{g \in \bar{H}_\ell \mid g(\lambda^1) \equiv_\tau \lambda^1\}$ in $\bar{H}_\ell$ where $r = |\bar{H}_\ell / (\bar{H}_\ell)_{V_{L_{(\lambda^1, \gamma^1)}}}|$. We recall $r = |\mathcal{R}_1| = \dim_{\mathbb{C}} W_{1,\xi}$ for $\xi \in P_1$. By (5.3), we have

$$\mathcal{M}_1 = \bigoplus_{\mu \in \mathcal{R}_1} V_{L_{(\mu, \gamma^1)}} = \bigoplus_{j=1}^{r} V_{L_{(h_j(\lambda^1), \gamma^1)}}$$

$$= \bigoplus_{j=1}^{r} \bigoplus_{\xi\in P_1} X_{h_j(\xi),\gamma^1} = \bigoplus_{\xi\in P_1} \bigoplus_{j=1}^{r} X_{h_j(\xi),\gamma^1}$$
$$= \bigoplus_{\xi\in P_1} W_{1,\xi} \otimes X_{\xi,\gamma^1},$$

where we identify $\bigoplus_{j=1}^{r} X_{h_j(\xi),\gamma^1}$ with $W_{1,\xi}\otimes X_{\xi,\gamma^1}$. Let ξ be an element of P_1 and v^ξ a nonzero element of X_{ξ,γ^1}. For $j=1,\dots,r$, we can take $h_j(v^\xi) \in X_{h_j(\xi),\gamma^1} \subset V_{L_{(h_j(\lambda^1),\gamma^1)}}$ as $w^{1j}\otimes v^\xi$ in the argument above.

Set

$$\mathcal{I} = \bigoplus_{(M^1,M^2,M^3)\in\mathcal{S}_1\times\mathcal{S}_2\times\mathcal{S}_3} I_{V^\tau_{L\oplus\ell}}\binom{M^3}{M^1\ M^2} \otimes_{\mathbb{C}} M^1 \otimes_{\mathbb{C}} M^2. \tag{5.37}$$

Let $M^i \in \mathcal{S}_i$ for $i=1,2,3$. For $f \in I_{V^\tau_{L\oplus\ell}}\binom{M^3}{M^1\ M^2}$ and $g \in \bar{H}_\ell$, we define ${}_gf \in I_{V^\tau_{L\oplus\ell}}\binom{g(M^3)}{g(M^1)\ g(M^2)}$ as follows: For $u\in g(M^1), v\in g(M^2)$, set

$${}_gf(u,x)v = g(f(g^{-1}u,x)g^{-1}(v)).$$

We define an action of $\mathcal{A}(\bar{H}_\ell,\mathcal{S}_3)$ on $\mathcal{I}$ as follows: Let $M^i\in\mathcal{S}_i$ for $i=1,2,3$. For $g\otimes e(M)\in\mathcal{A}(\bar{H}_\ell,\mathcal{S}_3)$, $v\in M^1, w\in M^2$, and $f\in I_{V^\tau_{L\oplus\ell}}\binom{M^3}{M^1\ M^2}$, set

$$(g\otimes e(M))\cdot(f\otimes v\otimes w) = \delta_{M,M^3}\cdot {}_gf\otimes g(v)\otimes g(w)$$
$$\in I_{V^\tau_{L\oplus\ell}}\binom{g(M^3)}{g(M^1)\ g(M^2)} \otimes_{\mathbb{C}} g(M^1)\otimes_{\mathbb{C}} g(M^2).$$

Let $\xi^i\in P_i$ for $i=1,2$. Fix a nonzero $v^{i0}\in X_{\xi^i,\gamma^i}$. Set

$$\mathcal{I}(\xi^1,\xi^2) = \mathrm{span}_{\mathbb{C}}\left\{f\otimes w^1\otimes v^{10}\otimes w^2\otimes v^{20}\in\mathcal{I} \mid w^1\in W_{1,\xi^1}, w^2\in W_{2,\xi^2}\right\}, \tag{5.38}$$

which is an $\mathcal{A}(\bar{H}_\ell,\mathcal{S}_3)$-submodule of $\mathcal{I}$. It follows from the comments right after (5.36) that

$$\dim_{\mathbb{C}}\mathcal{I}(\xi^1,\xi^2) = \sum_{\mu^1\in\mathcal{R}_1,\mu^2\in\mathcal{R}_2}\sum_{M^3\in\mathcal{S}_3} \dim_{\mathbb{C}} I_{V^\tau_{L\oplus\ell}}\binom{M^3}{V_{L_{(\mu^1,\gamma^1)}}\ V_{L_{(\mu^2,\gamma^2)}}}. \tag{5.39}$$

We have the following decomposition of $\mathcal{I}(\xi^1,\xi^2)$ as an $\mathcal{A}(\bar{H}_\ell,\mathcal{S}_3)$-module.

$$\mathcal{I}(\xi^1,\xi^2)=\bigoplus_{\xi\in P_3}W_{3,\xi}\otimes\operatorname{Hom}_{\mathcal{A}(\bar{H}_\ell,\mathcal{S}_3)}(W_{3,\xi},\mathcal{I}(\xi^1,\xi^2)). \tag{5.40}$$

By [**35**, Theorem 2], we have

$$\dim_{\mathbb{C}}\operatorname{Hom}_{\mathcal{A}(\bar{H}_\ell,\mathcal{S}_3)}(W_{3,\xi},\mathcal{I}(\xi^1,\xi^2))\le\dim_{\mathbb{C}} I_{(V_L^\tau)^{\otimes\ell}}\binom{X_{\xi,\gamma^3}}{X_{\xi^1,\gamma^1}\ X_{\xi^2,\gamma^2}} \tag{5.41}$$

for $\xi\in P_3$.

Now we compute some fusion rules for $V_{L^{\oplus\ell}}^\tau$.

Proposition 5.7. *Let $\lambda,\lambda^1,\lambda^2$ be nonzero elements of $\mathcal{K}^\ell$ such that $\lambda^1\not\equiv_\tau\lambda^2$, $\gamma,\gamma^1,\gamma^2,\eta\in\mathbb{Z}_3^\ell$, $i=1,2$, and $\varepsilon,\varepsilon_1,\varepsilon_2=0,1,2$. Then*

$$V_{L_{(\mathbf{0},\gamma^1)}}(\varepsilon_1)\times V_{L_{(\mathbf{0},\gamma^2)}}(\varepsilon_2)=V_{L_{(\mathbf{0},\gamma^1+\gamma^2)}}(\varepsilon_1+\varepsilon_2), \tag{5.42}$$

$$V_{L_{(\mathbf{0},\gamma^1)}}(\varepsilon)\times V_{L_{(\lambda,\gamma^2)}}=V_{L_{(\lambda,\gamma^1+\gamma^2)}}, \tag{5.43}$$

$$V_{L_{(\lambda^1,\gamma^1)}}\times V_{L_{(\lambda^2,\gamma^2)}}=\sum_{j=0}^{2}V_{L_{(\lambda^1+\tau^j(\lambda^2),\gamma^1+\gamma^2)}}, \tag{5.44}$$

$$V_{L_{(\lambda,\gamma^1)}}\times V_{L_{(\lambda,\gamma^2)}}=\sum_{\rho=0}^{2}V_{L_{(\mathbf{0},\gamma^1+\gamma^2)}}(\rho)+2V_{L_{(\lambda,\gamma^1+\gamma^2)}}, \tag{5.45}$$

$$V_{L_{(\mathbf{0},\gamma)}}(\varepsilon_1)\times V_{L^{\oplus\ell}}^{T,\eta}(\tau^i)[\varepsilon_2]=V_{L^{\oplus\ell}}^{T,\eta-i\gamma}(\tau^i)[i\varepsilon_1+\varepsilon_2], \tag{5.46}$$

$$V_{L_{(\lambda,\gamma)}}\times V_{L^{\oplus\ell}}^{T,\eta}(\tau^i)[\varepsilon]=\sum_{\rho=0}^{2}V_{L^{\oplus\ell}}^{T,\eta-i\gamma}(\tau^i)[\rho]. \tag{5.47}$$

Proof. We shall show (5.44) and (5.45). We put $\lambda=\lambda^1$ in (5.45) to deal with (5.44) and (5.45) simultaneously. For $\lambda^i, i=1,2$, define $\xi^i=(\xi^i_j)\in\{0,c\}^\ell$ by (5.33). By (5.6), X_{ξ^i,γ^i} is a $(V_L^\tau)^{\otimes\ell}$-submodule of $V_{L_{(\lambda^i,\gamma^i)}}$ and

$$I_{(V_L^\tau)^{\otimes\ell}}\binom{V_{L_{(\mathbf{0},\gamma)}}(r)}{X_{\xi^1,\gamma^1}\ X_{\xi^2,\gamma^2}}\cong\bigoplus_{\substack{\xi=(\xi_j)\in\mathbb{Z}_3^\ell\\ \xi_1+\cdots+\xi_\ell=r}}I_{(V_L^\tau)^{\otimes\ell}}\binom{X_{\xi,\gamma}}{X_{\xi^1,\gamma^1}\ X_{\xi^2,\gamma^2}}$$

for $\gamma\in\mathbb{Z}_3^\ell$ and $r\in\mathbb{Z}_3$. For $\xi\in\mathbb{Z}_3^\ell$ and $\gamma\in\mathbb{Z}_3^\ell$ such that $\gamma^1+\gamma^2\neq\gamma$, it follows from Proposition 4.5 and [**18**, Proposition 2.10] that

$$I_{(V_L^\tau)^{\otimes\ell}}\begin{pmatrix} X_{\xi,\gamma} \\ X_{\xi^1,\gamma^1}\ X_{\xi^2,\gamma^2}\end{pmatrix}=0.$$

By [**11**, Proposition 11.9], we obtain

$$\begin{aligned}\dim_{\mathbb{C}} I_{V_{L^{\oplus\ell}}^\tau}\begin{pmatrix} V_{L_{(\mathbf{0},\gamma)}}(r) \\ V_{L_{(\lambda^1,\gamma^1)}}\ V_{L_{(\lambda^2,\gamma^2)}}\end{pmatrix} &\le \dim_{\mathbb{C}} I_{(V_L^\tau)^{\otimes\ell}}\begin{pmatrix} V_{L_{(\mathbf{0},\gamma)}}(r) \\ X_{\xi^1,\gamma^1}\ X_{\xi^2,\gamma^2}\end{pmatrix}\\ = \sum_{\substack{\xi=(\xi_j)\in\mathbb{Z}_3^\ell\\ \xi_1+\cdots+\xi_\ell=r}} \dim_{\mathbb{C}} I_{(V_L^\tau)^{\otimes\ell}}\begin{pmatrix} X_{\xi,\gamma} \\ X_{\xi^1,\gamma^1}\ X_{\xi^2,\gamma^2}\end{pmatrix} &= 0.\end{aligned}$$

For the same reason, we can show easily that

$$I_{V_{L^{\oplus\ell}}^\tau}\begin{pmatrix} M \\ V_{L_{(\lambda^1,\gamma^1)}}\ V_{L_{(\lambda^2,\gamma^2)}}\end{pmatrix}=0 \tag{5.48}$$

for $M\notin\{V_{L_{(\lambda,\gamma^1+\gamma^2)}}, V_{L_{(\mathbf{0},\gamma^1+\gamma^2)}}(r)\mid \mathbf{0}\neq\lambda\in(\mathcal{K}^\ell)_{\equiv_\tau}, r=0,1,2\}$.

From now on, we use the notation in the preparation just before this proposition. For example, $\gamma^3=\gamma^1+\gamma^2$, $\mathcal{R}_i=\{h(\lambda^i)\in(\mathcal{K}^\ell)_{\equiv_\tau}\mid h\in\bar{H}_\ell\}$, and $\mathcal{S}_i=\{V_{L_{(\mu^i,\gamma^i)}}\mid \mu^i\in\mathcal{R}_i\}$ for $i=1,2$. The following symbols are used to describe the fusion rules for $(V_L^\tau)^{\otimes\ell}$: Set

$$\Xi(\xi^1,\xi^2)=\left\{\xi=(\xi_j)\in\{0,1,2,c\}^\ell \,\middle|\, \begin{array}{l}\xi_j=\xi_j^1+\xi_j^2\\ \text{for all } j\notin\{k\mid \xi_k^1=\xi_k^2=c\}\end{array}\right\}$$

and

$$\Xi(\xi^1,\xi^2)_k=\{\xi=(\xi_j)\in\Xi(\xi^1,\xi^2)\mid |\{j\mid \xi_j^1=\xi_j^2=\xi_j=c\}|=k\}$$

for nonnegative integers k. For example, if $\xi^1=(0,c,1,c)$ and $\xi^2=(1,c,c,2)$ in $\{0,1,2,c\}^4$, then

$$\Xi(\xi^1,\xi^2)=\{(1,0,c,c),(1,1,c,c),(1,2,c,c),(1,c,c,c)\}$$

and $\Xi(\xi^1,\xi^2)_1=\{(1,c,c,c)\}$. Note that $|\Xi(\xi^1,\xi^2)_k|=\binom{|\{j\mid\xi_j^1=\xi_j^2=c\}|}{k}3^{|\{j\mid\xi_j^1=\xi_j^2=c\}|-k}$ and for $\xi\in\Xi(\xi^1,\xi^2)_k$,

$$\mathrm{wt}_{\tilde{\mathcal{K}}}(\xi)=\mathrm{wt}_{\tilde{\mathcal{K}}}(\xi^1)+\mathrm{wt}_{\tilde{\mathcal{K}}}(\xi^2)-2|\{j\mid \xi_j^1=\xi_j^2=c\}|+k. \tag{5.49}$$

By Proposition 4.5 and [**18**, Proposition 2.10], we have

$$\dim_{\mathbb{C}} I_{(V_L^\tau)^{\otimes\ell}}\binom{X_{\xi,\gamma^3}}{X_{\xi^1,\gamma^1}\ X_{\xi^2,\gamma^2}} = \begin{cases} 2^k & \text{if } \xi\in\Xi(\xi^1,\xi^2)_k, \\ 0 & \text{if } \xi\notin\Xi(\xi^1,\xi^2)\end{cases} \tag{5.50}$$

for $\xi\in\{0,1,2,c\}^\ell$.

By Lemma 5.4, we have

$$V_{L_{(\mu^1,\gamma^1)}}\times V_{L_{(\mu^2,\gamma^2)}} \geq \sum_{j=0}^{2} V_{L_{(\mu^1+\tau^j(\mu^2),\gamma^3)}}, \tag{5.51}$$

$$V_{L_{(\mu,\gamma^1)}}\times V_{L_{(\mu,\gamma^2)}} \geq \sum_{\rho=0}^{2} V_{L_{(\mathbf{0},\gamma^3)}}(\rho)+2V_{L_{(\mu,\gamma^3)}} \tag{5.52}$$

for $\mu,\mu_1\in\mathcal{R}_1$ and $\mu_2\in\mathcal{R}_2$. We shall compute the dimension of $\mathcal{I}(\xi^1,\xi^2)$ in two ways using (5.38) and (5.40).

Case 1: We deal with the case $\mathrm{supp}_{\mathcal{K}}(\lambda^1)\neq\mathrm{supp}_{\mathcal{K}}(\lambda^2)$. Note that $g_1(\lambda^1)\not\equiv_\tau g_2(\lambda^2)$ for all $g_1,g_2\in\bar{H}_\ell$ and $\mathrm{wt}_{\tilde{\mathcal{K}}}(\xi)>0$ for all $\xi\in\Xi(\xi^1,\xi^2)$. We recall that $\dim_{\mathbb{C}} W_{i,\xi^i}=|\mathcal{R}_i|=3^{\mathrm{wt}_{\tilde{\mathcal{K}}}(\xi^i)-1}$ for $i=1,2$ by (5.35). By (5.39) and (5.51), we have

$$\dim_{\mathbb{C}}\mathcal{I}(\xi^1,\xi^2)\geq 3|\mathcal{R}_1||\mathcal{R}_2| = 3^{\mathrm{wt}_{\tilde{\mathcal{K}}}(\xi^1)+\mathrm{wt}_{\tilde{\mathcal{K}}}(\xi^2)-1}. \tag{5.53}$$

On the other hand, we have

$$\begin{aligned}\dim_{\mathbb{C}}\mathcal{I}(\xi^1,\xi^2) &= \sum_{\xi\in P_3}\dim_{\mathbb{C}} W_{3,\xi}\dim_{\mathbb{C}}\mathrm{Hom}_{\mathcal{A}(\bar{H}_\ell,\mathcal{S}_3)}(W_{3,\xi},\mathcal{I}(\xi^1,\xi^2)) \\ &\leq \sum_{\xi\in\Xi(\xi^1,\xi^2)} 3^{\mathrm{wt}_{\tilde{\mathcal{K}}}(\xi)-1}\dim_{\mathbb{C}} I_{(V_L^\tau)^{\otimes\ell}}\binom{X_{\xi,\gamma^3}}{X_{\xi^1,\gamma^1}\ X_{\xi^2,\gamma^2}} \\ &= \sum_{k=0}^{|\{j|\xi^1_j=\xi^2_j=c\}|}\ \sum_{\xi\in\Xi(\xi^1,\xi^2)_k} 3^{\mathrm{wt}_{\tilde{\mathcal{K}}}(\xi^1)+\mathrm{wt}_{\tilde{\mathcal{K}}}(\xi^2)-2|\{j|\xi^1_j=\xi^2_j=c\}|+k-1}2^k \\ &= \sum_{k=0}^{|\{j|\xi^1_j=\xi^2_j=c\}|}\binom{|\{j\mid \xi^1_j=\xi^2_j=c\}|}{k}3^{|\{j|\xi^1_j=\xi^2_j=c\}|-k} \\ &\quad\times 3^{\mathrm{wt}_{\tilde{\mathcal{K}}}(\xi^1)+\mathrm{wt}_{\tilde{\mathcal{K}}}(\xi^2)-2|\{j|\xi^1_j=\xi^2_j=c\}|+k-1}2^k\end{aligned}$$

$$= \sum_{k=0}^{|\{j|\xi^1_j=\xi^2_j=c\}|} \binom{|\{j \mid \xi^1_j=\xi^2_j=c\}|}{k} 3^{\mathrm{wt}_{\tilde{\mathcal{K}}}(\xi^1)+\mathrm{wt}_{\tilde{\mathcal{K}}}(\xi^2)-|\{j|\xi^1_j=\xi^2_j=c\}|-1} 2^k$$
$$= 3^{\mathrm{wt}_{\tilde{\mathcal{K}}}(\xi^1)+\mathrm{wt}_{\tilde{\mathcal{K}}}(\xi^2)-1} \tag{5.54}$$

by (5.40), (5.41), (5.49) and (5.50). By (5.53) and (5.54), we have

$$\begin{aligned}\dim_{\mathbb{C}} \mathcal{I}(\xi^1,\xi^2) &= \sum_{M^3\in\mathcal{S}_3}\sum_{\mu^1\in\mathcal{R}_1,\mu^2\in\mathcal{R}_2} \dim_{\mathbb{C}} I_{V^{\tau}_{L^{\oplus\ell}}}\binom{M^3}{V_{L_{(\mu^1,\gamma^1)}}\ V_{L_{(\mu^2,\gamma^2)}}}\\ &= 3^{\mathrm{wt}_{\tilde{\mathcal{K}}}(\xi^1)+\mathrm{wt}_{\tilde{\mathcal{K}}}(\xi^2)-1},\end{aligned}$$

and thus it follows from (5.48) that the equality holds in (5.51). Setting $\mu^1=\lambda^1$ and $\mu^2=\lambda^2$ in (5.51), we have (5.44).

Case 2: We deal with the case $\mathrm{supp}_{\mathcal{K}}(\lambda^1)=\mathrm{supp}_{\mathcal{K}}(\lambda^2)$. Note that $\mathcal{R}_1=\mathcal{R}_2$, $|\{(\mu^1,\mu^2)\in\mathcal{R}_1\times\mathcal{R}_1 \mid \mu^1\not\equiv_\tau \mu^2\}| = |\mathcal{R}_1|(|\mathcal{R}_1|-1)$ and $\xi^1=\xi^2$ in this case. By (5.39), (5.51), and (5.52), we have

$$\begin{aligned}\dim_{\mathbb{C}} \mathcal{I}(\xi^1,\xi^1) &\geq 5|\mathcal{R}_1| + 3|\mathcal{R}_1|(|\mathcal{R}_1|-1)\\ &= 3^{2\,\mathrm{wt}_{\tilde{\mathcal{K}}}(\xi^1)-1} + 2\cdot 3^{\mathrm{wt}_{\tilde{\mathcal{K}}}(\xi^1)-1}.\end{aligned} \tag{5.55}$$

On the other hand, we have

$$\begin{aligned}\dim_{\mathbb{C}} \mathcal{I}(\xi^1,\xi^1) &= \sum_{\xi\in P_3} \dim_{\mathbb{C}} W_{3,\xi} \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{A}(\bar{H}_\ell,\mathcal{S}_3)}(W_{3,\xi},\mathcal{I}(\xi^1,\xi^1))\\
&\leq \sum_{\xi\in P_3} 3^{\max\{0,\mathrm{wt}_{\tilde{\mathcal{K}}}(\xi)-1\}} \dim_{\mathbb{C}} I_{(V_L^{\tau})^{\otimes\ell}}\binom{X_{\xi,\gamma^3}}{X_{\xi^1,\gamma^1}\ X_{\xi^1,\gamma^2}}\\
&= \sum_{\substack{\xi\in\Xi(\xi^1,\xi^1),\\ \mathrm{wt}_{\tilde{\mathcal{K}}}(\xi)\neq 0}} 3^{\mathrm{wt}_{\tilde{\mathcal{K}}}(\xi)-1} \dim_{\mathbb{C}} I_{(V_L^{\tau})^{\otimes\ell}}\binom{X_{\xi,\gamma^3}}{X_{\xi^1,\gamma^1}\ X_{\xi^1,\gamma^2}}\\
&\quad + \sum_{\substack{\xi\in\Xi(\xi^1,\xi^1),\\ \mathrm{wt}_{\tilde{\mathcal{K}}}(\xi)=0}} \dim_{\mathbb{C}} I_{(V_L^{\tau})^{\otimes\ell}}\binom{X_{\xi,\gamma^3}}{X_{\xi^1,\gamma^1}\ X_{\xi^1,\gamma^2}}\\
&= \sum_{k=1}^{\mathrm{wt}_{\tilde{\mathcal{K}}}(\xi^1)} \sum_{\xi\in\Xi(\xi^1,\xi^1)_k} 3^{k-1}2^k + 3^{\mathrm{wt}_{\tilde{\mathcal{K}}}(\xi^1)}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{\mathrm{wt}_{\tilde{\mathcal{K}}}(\xi^1)} \binom{\mathrm{wt}_{\tilde{\mathcal{K}}}(\xi^1)}{k} 3^{\mathrm{wt}_{\tilde{\mathcal{K}}}(\xi^1)-k} 3^{k-1} 2^k + 3^{\mathrm{wt}_{\tilde{\mathcal{K}}}(\xi^1)} \\
&= 3^{2\,\mathrm{wt}_{\tilde{\mathcal{K}}}(\xi^1)-1} + 2 \cdot 3^{\mathrm{wt}_{\tilde{\mathcal{K}}}(\xi^1)-1} \qquad (5.56)
\end{aligned}
$$

by (5.40), (5.41), (5.49), and (5.50). By (5.55) and (5.56), we have

$$
\begin{aligned}
\dim_{\mathbb{C}} \mathcal{I}(\xi^1, \xi^1) &= \sum_{M^3 \in \mathcal{S}_3} \sum_{\mu^1, \mu^2 \in \mathcal{R}_1} \dim_{\mathbb{C}} I_{V_{L^{\oplus \ell}}^{\tau}} \begin{pmatrix} & M^3 & \\ V_{L_{(\mu^1,\gamma^1)}} & & V_{L_{(\mu^2,\gamma^2)}} \end{pmatrix} \\
&= 3^{2\,\mathrm{wt}_{\tilde{\mathcal{K}}}(\xi^1)-1} + 2 \cdot 3^{\mathrm{wt}_{\tilde{\mathcal{K}}}(\xi^1)-1}
\end{aligned}
$$

and thus it follows from (5.48) that the equality holds in (5.51) and (5.52). Setting $\mu^1 = \lambda^1, \mu^2 = \lambda^2$ in (5.51) and $\mu = \lambda^1 = \lambda$ in (5.52), we have (5.44) and (5.45).

The same argument as above shows (5.47). We shall sketch the proof of (5.47) for $i = 1$. By Proposition 4.5, we can show easily that

$$
I_{V_{L^{\oplus \ell}}^{\tau}} \begin{pmatrix} & M & \\ V_{L_{(\lambda,\gamma)}} & & V_{L^{\oplus \ell}}^{T,\eta}(\tau)[\varepsilon] \end{pmatrix} = 0
$$

for $M \notin \{V_{L^{\oplus \ell}}^{T,\eta-\gamma}(\tau)[\rho] \mid \rho = 0,1,2\}$. Take $\bar{H}_\ell$-stable sets $\mathcal{S}_2^T = \{V_{L^{\oplus \ell}}^{T,\eta}(\tau)[\varepsilon]\}$ and $\mathcal{S}_3^T = \{V_{L^{\oplus \ell}}^{T,\eta-\gamma}(\tau)[\rho] \mid \rho = 0,1,2\}$ and set $\mathcal{M}_i^T = \oplus_{M \in \mathcal{S}_i^T} M, i = 2,3$. Note that for $M \in \mathcal{S}_i^T$, $(\bar{H}_\ell)_M = \{g \in \bar{H}_\ell \mid g(M) = M\}$ equals $\bar{H}_\ell$. For $\xi \in \{0,1,2\}^\ell$, define an action of $D(M)$, $M \in \mathcal{S}_i^T$ on $\mathbb{C}e(\xi)$ as follows: For $g = (\tau^{i_1}, \ldots, \tau^{i_{\ell-1}}, 1) \in \bar{H}_\ell$, set

$$
g \otimes e(M) \cdot e(\xi) = \zeta_3^{\langle (i_1,\ldots,i_{\ell-1},0),\xi \rangle_{\mathbb{Z}_3}} e(\xi).
$$

Denote the $D(M)$-module $\mathbb{C}e(\xi)$ by $W_{i,\xi}^T$. Set

$$
\begin{aligned}
P\big(V_{L^{\oplus \ell}}^{T,\eta}(\tau)[\rho]\big) &= P\big(V_{L^{\oplus \ell}}^{T,\eta-\gamma}(\tau)[\rho]\big) \\
&= \left\{ \xi = (\xi_k) \in \{0,1,2\}^\ell \;\middle|\; \sum_{k=1}^{\ell} \xi_k \equiv \rho \pmod 3 \right\}
\end{aligned}
$$

for $\rho = 0,1,2$. Note that

$$
\{0,1,2\}^\ell = \bigcup_{\rho=0}^{2} P\big(V_{L^{\oplus \ell}}^{T,\eta-\gamma}(\tau)[\rho]\big); \quad \text{disjoint}.
$$

Set $P_2^T = P(V_{L^{\oplus \ell}}^{T,\eta}(\tau)[\varepsilon])$ and $P_3^T = \{0,1,2\}^\ell$. Then $\{W_{i,\xi}^T \mid \xi \in P_i^T\}$ is a complete list of irreducible $\mathcal{A}(\bar{H}_\ell, \mathcal{S}_i^T)$-modules by Theorem 5.5 for $i = 2,3$. For $\xi = (\xi_k)$, $\gamma = (\gamma_k) \in \{0,1,2\}^\ell$, set

$$X_{\xi,\gamma}^T = \bigotimes_{k=1}^{\ell} V_L^{T,\gamma_k}(\tau)[\xi_k].$$

For the same reason as in the proof of (5.36), we have

$$\mathcal{M}_2^T = \bigoplus_{\xi \in P_2^T} W_{2,\xi}^T \otimes X_{\xi,\eta}^T, \qquad \mathcal{M}_3^T = \bigoplus_{\xi \in P_3^T} W_{3,\xi}^T \otimes X_{\xi,\eta-\gamma}^T$$

as an $\mathcal{A}(\bar{H}_\ell, \mathcal{S}_2^T)$- and $\mathcal{A}(\bar{H}_\ell, \mathcal{S}_3^T)$-module, respectively. Set

$$\mathcal{I}^T = \bigoplus_{(M^1,M^2,M^3) \in \mathcal{S}_1 \times \mathcal{S}_2^T \times \mathcal{S}_3^T} I_{V_{L^{\oplus \ell}}^\tau} \binom{M^3}{M^1 \ M^2} \otimes_{\mathbb{C}} M^1 \otimes_{\mathbb{C}} M^2.$$

Let $\xi^1 \in P_1 = P(V_{L_{(\lambda,\gamma)}})$ and $\xi^2 \in P_2^T$. Fix nonzero elements $v^{10} \in X_{\xi^1,\gamma}$ and $v^{T,20} \in X_{\xi^2,\eta}^T$. Set

$$\mathcal{I}^T(\xi^1,\xi^2) = \operatorname{span}_{\mathbb{C}} \{ f \otimes w^1 \otimes v^{10} \otimes w^2 \otimes v^{T,20} \in \mathcal{I}^T \mid w^1 \in W_{1,\xi^1}, w^2 \in W_{2,\xi^2}^T \}.$$

Applying the same arguments as in the case of (5.44) and (5.45), we have

$$\dim_{\mathbb{C}} \mathcal{I}^T(\xi^1,\xi^2) = 3^{\mathrm{wt}_{\bar{\mathcal{K}}}(\xi^1)}.$$

Therefore, (5.47) holds.

The other formulas can be proved similarly. □

6. Modules of $V_{L_{\mathbf{0}\times D}}^\tau$.

Let D be a self-orthogonal $\mathbb{Z}_3$-code of length ℓ. In this section we discuss $V_{L_{\mathbf{0}\times D}}^\tau$-modules. Note that $V_{L_{\mathbf{0}\times D}}^\tau = \oplus_{\gamma \in D} V_{L_{(\mathbf{0},\gamma)}}(0)$ as $V_{L^{\oplus \ell}}^\tau$-modules. Let $\gamma^{(1)}, \dots, \gamma^{(\ell)}$ be a basis of $\mathbb{Z}_3^\ell$ such that $\gamma^{(1)}, \dots, \gamma^{(d)}$ form a basis of D.

For $j = 1, \dots, \ell$, define a linear transformation χ_j on $V_{(L^\perp)^{\oplus \ell}} = \oplus_{\delta \in \mathbb{Z}_3^\ell} V_{L_{\mathcal{K}^\ell \times \delta}}$ by $\chi_j(u) = \zeta_3^{p_j} u$ for $\delta = \sum_{k=1}^\ell p_k \gamma^{(k)} \in \mathbb{Z}_3^\ell$ and $u \in V_{L_{\mathcal{K}^\ell \times \delta}}$. The restriction of χ_j to $V_{L_{\mathbf{0}\times D}}$ is an automorphism of $V_{L_{\mathbf{0}\times D}}$ for $j = 1, \dots, \ell$. Let Φ_D be the automorphism group of $V_{L_{\mathbf{0}\times D}}$ generated by $\chi_1, \dots, \chi_d$. Since τ commutes with Φ_D, Φ_D induces an automorphism group of $V_{L_{\mathbf{0}\times D}}^{\tau}$. Note that $(V_{L_{\mathbf{0}\times D}}^{\tau})^{\Phi_D} = V_{L_{(\mathbf{0},\mathbf{0})}}(0) = V_{L^{\oplus\ell}}^{\tau}$.

For $j = 1, \ldots, d$, $\lambda \in \mathcal{K}^{\ell}$, and $\gamma \in \mathbb{Z}_3^{\ell}$, $V_{L_{\lambda\times(\gamma+D)}}$ is χ_j-invariant and

$$V_{L_{\lambda\times(\gamma+D)}} = \bigoplus_{\delta\in D} V_{L_{(\lambda,\gamma+\delta)}} \tag{6.1}$$

is an eigenspace decomposition for Φ_D. We also have

$$\begin{aligned} V_{L_{\mathbf{0}\times(\gamma+D)}}(\varepsilon) &= \bigoplus_{\delta\in D} V_{L_{(\mathbf{0},\gamma+\delta)}}(\varepsilon), \quad \varepsilon \in \mathbb{Z}_3,\ \gamma \in \mathbb{Z}_3^{\ell}, \\ V_{L_{\lambda\times(\gamma+D)}} &= \bigoplus_{\delta\in D} V_{L_{(\lambda,\gamma+\delta)}}, \qquad \lambda \in \mathcal{K}^{\ell},\ \gamma \in \mathbb{Z}_3^{\ell} \end{aligned} \tag{6.2}$$

as $V_{L^{\oplus\ell}}^{\tau}$-modules.

For $j = 1, \ldots, d$, $\lambda \in \mathcal{K}^{\ell}$, $\gamma \in D^{\perp}$, $u \in V_{L_{\mathbf{0}\times D}}$, and $v \in V_{L_{\lambda\times(\gamma+D)}}$, we have

$$Y_{V_{L_{\lambda\times(\gamma+D)}}}(\chi_j u, x)\chi_j v = \chi_j\big(Y_{V_{L_{\lambda\times(\gamma+D)}}}(u, x)v\big).$$

Hence $V_{L_{\lambda\times(\gamma+D)}} \circ \chi_j \cong \chi_j^{-1}(V_{L_{\lambda\times(\gamma+D)}}) = V_{L_{\lambda\times(\gamma+D)}}$ as $V_{L_{\mathbf{0}\times D}}$-modules and

$$\begin{aligned} V_{L_{\mathbf{0}\times(\gamma+D)}}(\varepsilon) \circ \chi_j &\cong \chi_j^{-1}\big(V_{L_{\mathbf{0}\times(\gamma+D)}}(\varepsilon)\big) = V_{L_{\mathbf{0}\times(\gamma+D)}}(\varepsilon), \quad \gamma \in D^{\perp},\ \varepsilon \in \mathbb{Z}_3, \\ V_{L_{\lambda\times(\gamma+D)}} \circ \chi_j &\cong \chi_j^{-1}\big(V_{L_{\lambda\times(\gamma+D)}}\big) = V_{L_{\lambda\times(\gamma+D)}}, \qquad \mathbf{0} \neq \lambda \in \mathcal{K}^{\ell},\ \gamma \in D^{\perp} \end{aligned}$$

as $V_{L_{\mathbf{0}\times D}}^{\tau}$-modules.

It follows from (3.27) and the corresponding formula for τ^2-twisted modules that for $\eta \in D^{\perp}$, $i = 1, 2$, and $r \in \mathbb{Z}_3$,

$$V_{L_{\mathbf{0}\times D}}^{T,\eta}(\tau^i)[r] \cong \bigoplus_{\gamma\in D} V_{L^{\oplus\ell}}^{T,\eta-i\gamma}(\tau^i)[r] \tag{6.3}$$

as $V_{L^{\oplus\ell}}^{\tau}$-modules. Using (6.3), we define an action of χ_j on $V_{L_{\mathbf{0}\times D}}^{T,\eta}(\tau^i)[r]$ for $j = 1, \ldots, \ell$ by setting $\chi_j(v) = \zeta_3^{-ip_j} v$ for $\delta = \sum_{k=1}^{\ell} p_k \gamma^{(k)} \in \mathbb{Z}_3^{\ell}$ and $v \in V_{L^{\oplus\ell}}^{T,\delta}(\tau^i)[r]$ and extending χ_j for arbitrary $v \in V_{L_{\mathbf{0}\times D}}^{T,\eta}(\tau^i)[r]$ by (6.3) and linearity.

By Proposition 5.7, we have

$$Y_{V_{L_{\mathbf{0}\times D}}^{T,\eta}(\tau^i)[r]}(\chi_j u, x)\chi_j v = \chi_j\big(Y_{V_{L_{\mathbf{0}\times D}}^{T,\eta}(\tau^i)[r]}(u, x)v\big)$$

for $u \in V_{L_{\mathbf{0}\times D}}$ and $v \in V^{T,\eta}_{L_{\mathbf{0}\times D}}(\tau^i)[r]$. Hence

$$V^{T,\eta}_{L_{\mathbf{0}\times D}}(\tau^i)[r] \circ \chi_j \cong \chi_j^{-1}\big(V^{T,\eta}_{L_{\mathbf{0}\times D}}(\tau^i)[r]\big) = V^{T,\eta}_{L_{\mathbf{0}\times D}}(\tau^i)[r]$$

and so we can define an action of Φ_D on $V^{T,\eta}_{L_{\mathbf{0}\times D}}(\tau^i)[r]$. Then it is clear that (6.3) is also an eigenspace decomposition of $V^{T,\eta}_{L_{\mathbf{0}\times D}}(\tau^i)[r]$ for Φ_D.

Lemma 6.1. *Let N be an $\mathbb{N}$-graded weak $V^{\tau}_{L_{\mathbf{0}\times D}}$-module and let M be an irreducible $V^{\tau}_{L^{\oplus\ell}}$-submodule of N. If M is isomorphic to $V_{L_{(0,\gamma)}}(\varepsilon)$, $\gamma \in \mathbb{Z}_3^{\ell}$, $\varepsilon \in \mathbb{Z}_3$ or $V_{L_{(\lambda,\gamma)}}$, $\mathbf{0} \neq \lambda \in \mathcal{K}^{\ell}$, $\gamma \in \mathbb{Z}_3^{\ell}$, then $\gamma \in D^{\perp}$. If M is isomorphic to $V^{T,\eta}_{L^{\oplus\ell}}(\tau^i)[\varepsilon]$, $i = 1, 2$, $\varepsilon \in \mathbb{Z}_3$, $\eta \in \mathbb{Z}_3^{\ell}$, then $\eta \in D^{\perp}$.*

Proof. Let ω_L be the Virasoro element of V^{τ}_L. For $i = 1, 2$ and $j, k, \varepsilon \in \{0, 1, 2\}$, let (W^1, W^2) be one of $(V_{L^{(0,j)}}(\varepsilon), V_{L^{(0,j+k)}}(\varepsilon))$, $(V_{L^{(c,j)}}, V_{L^{(c,j+k)}})$, or $(V^{T,j}_L(\tau^i)[\varepsilon], V^{T,j+k}_L(\tau^i)[\varepsilon])$. Let λ_s be the eigenvalue of $(\omega_L)_1$ on the top level of W^s for $s = 1, 2$. Note that

$$\begin{aligned} V_{L^{(0,-k)}}(0) \times W^1 &= W^2 \quad \text{if } (W^1, W^2) = \big(V^{T,j}_L(\tau)[\varepsilon], V^{T,j+k}_L(\tau)[\varepsilon]\big),\\ V_{L^{(0,k)}}(0) \times W^1 &= W^2 \quad \text{otherwise} \end{aligned}$$

by Proposition 4.5 and that $\lambda_2 - \lambda_1 \equiv (jk + 2k^2)/3 \pmod{\mathbb{Z}}$ by (5.10). We have already obtained a decomposition of every irreducible $V^{\tau}_{L^{\oplus\ell}}$-module as a $(V^{\tau}_L)^{\otimes\ell}$-module in Theorem 3.13 and (5.6).

Now the proof is similar to that of Lemma 5.1 since $V_{L^{(0,\gamma_1)}}(0) \otimes \cdots \otimes V_{L^{(0,\gamma_\ell)}}(0) \subset V^{\tau}_{L_{\mathbf{0}\times D}}$ for $\gamma = (\gamma_s)_{s=1}^{\ell} \in D$ and D is self-orthogonal. □

Using the same arguments as in the proofs of Lemma 5.2 and Proposition 5.3, we can show the following theorem. Indeed, we argue for $V^{\tau}_{L_{\mathbf{0}\times D}}$, $V^{\tau}_{L^{\oplus\ell}}$, and Φ_D in place of $V^{\tau}_{L^{\oplus\ell}}$, $(V^{\tau}_L)^{\otimes\ell}$, and $\bar{H}_\ell$ in Section 5, respectively.

Theorem 6.2. *Let D be a self-orthogonal $\mathbb{Z}_3$-code of length ℓ. Then $V^{\tau}_{L_{\mathbf{0}\times D}}$ is a simple, rational, C_2-cofinite, and CFT type vertex operator algebra. Let $D^{\perp}/D = \bigcup_{j=1}^{m}(\rho^j + D)$ be a coset decomposition. The following is a complete set of representatives of equivalence classes of irreducible $V^{\tau}_{L_{\mathbf{0}\times D}}$-modules.*

(1) $V_{L_{\mathbf{0}\times(\rho^j+D)}}(\varepsilon)$, $j = 1, \ldots, m$, $\varepsilon = 0, 1, 2$.
(2) $V_{L_{\lambda\times(\rho^j+D)}}$, $\mathbf{0} \neq \lambda \in (\mathcal{K}^{\ell})_{\equiv_\tau}$, $j = 1, \ldots, m$.
(3) $V^{T,\rho^j}_{L_{\mathbf{0}\times D}}(\tau^i)[\varepsilon]$, $i = 1, 2$, $j = 1, \ldots, m$, $\varepsilon = 0, 1, 2$.

We compute some fusion rules for $V^{\tau}_{L_{\mathbf{0}\times D}}$ which will be used in Section 7.2.

Proposition 6.3. *Let $\lambda, \lambda^1, \lambda^2$ be nonzero elements of $\mathcal{K}^\ell$ such that $\lambda^1 \not\equiv_\tau \lambda^2$, $\gamma, \gamma^1, \gamma^2, \eta \in D^\perp$, $i = 1, 2$, and $\varepsilon, \varepsilon_1, \varepsilon_2 = 0, 1, 2$. Then*

$$V_{L_{\mathbf{0}\times(\gamma^1+D)}}(\varepsilon_1) \times V_{L_{\mathbf{0}\times(\gamma^2+D)}}(\varepsilon_2) = V_{L_{\mathbf{0}\times(\gamma^1+\gamma^2+D)}}(\varepsilon_1+\varepsilon_2), \tag{6.4}$$

$$V_{L_{\mathbf{0}\times(\gamma^1+D)}}(\varepsilon) \times V_{L_{\lambda\times(\gamma^2+D)}} = V_{L_{\lambda\times(\gamma^1+\gamma^2+D)}}, \tag{6.5}$$

$$V_{L_{\lambda^1\times(\gamma^1+D)}} \times V_{L_{\lambda^2\times(\gamma^2+D)}} = \sum_{j=0}^{2} V_{L_{(\lambda^1+\tau^j(\lambda^2))\times(\gamma^1+\gamma^2+D)}}, \tag{6.6}$$

$$V_{L_{\lambda\times(\gamma^1+D)}} \times V_{L_{\lambda\times(\gamma^2+D)}} = \sum_{\rho=0}^{2} V_{L_{\mathbf{0}\times(\gamma^1+\gamma^2+D)}}(\rho) + 2V_{L_{\lambda\times(\gamma^1+\gamma^2+D)}}, \tag{6.7}$$

$$V_{L_{\mathbf{0}\times(\gamma+D)}}(\varepsilon_1) \times V_{L_{\mathbf{0}\times D}}^{T,\eta}(\tau^i)[\varepsilon_2] = V_{L_{\mathbf{0}\times D}}^{T,\eta-i\gamma}(\tau^i)[i\varepsilon_1+\varepsilon_2], \tag{6.8}$$

$$V_{L_{\lambda\times(\gamma+D)}} \times V_{L_{\mathbf{0}\times D}}^{T,\eta}(\tau^i)[\varepsilon] = \sum_{\rho=0}^{2} V_{L_{\mathbf{0}\times D}}^{T,\eta-i\gamma}(\tau^i)[\rho]. \tag{6.9}$$

Proof. We shall show (6.6). Restricting intertwining operators for $V_{L_{\mathbf{0}\times D}}$ in Lemma 2.14 to $V_{L_{\mathbf{0}\times D}}^{\tau}$-modules, we have

$$V_{L_{\lambda^1\times(\gamma^1+D)}} \times V_{L_{\lambda^2\times(\gamma^2+D)}} \geq \sum_{j=0}^{2} V_{L_{(\lambda^1+\tau^j(\lambda^2))\times(\gamma^1+\gamma^2+D)}}. \tag{6.10}$$

For $k = 1, 2$, $r \in \mathbb{Z}_3$, $\mathbf{0} \neq \lambda^3 \in \mathcal{K}^\ell$, and $\gamma^3 \in \mathbb{Z}_3^\ell$,

$$I_{V_{L^{\oplus\ell}}^{\tau}}\binom{V_{L_{\mathbf{0}\times(\gamma^3+D)}}(r)}{V_{L_{(\lambda^1,\gamma^1)}}\ V_{L_{(\lambda^2,\gamma^2)}}} \cong \bigoplus_{\delta\in D} I_{V_{L^{\oplus\ell}}^{\tau}}\binom{V_{L_{(\mathbf{0},\gamma^3+\delta)}}(r)}{V_{L_{(\lambda^1,\gamma^1)}}\ V_{L_{(\lambda^2,\gamma^2)}}},$$

$$I_{V_{L^{\oplus\ell}}^{\tau}}\binom{V_{L_{\lambda^3\times(\gamma^3+D)}}}{V_{L_{(\lambda^1,\gamma^1)}}\ V_{L_{(\lambda^2,\gamma^2)}}} \cong \bigoplus_{\delta\in D} I_{V_{L^{\oplus\ell}}^{\tau}}\binom{V_{L_{(\lambda^3,\gamma^3+\delta)}}}{V_{L_{(\lambda^1,\gamma^1)}}\ V_{L_{(\lambda^2,\gamma^2)}}},$$

$$I_{V_{L^{\oplus\ell}}^{\tau}}\binom{V_{L_{\mathbf{0}\times D}}^{T,\gamma^3}(\tau^k)[r]}{V_{L_{(\lambda^1,\gamma^1)}}\ V_{L_{(\lambda^2,\gamma^2)}}} \cong \bigoplus_{\delta\in D} I_{V_{L^{\oplus\ell}}^{\tau}}\binom{V_{L^{\oplus\ell}}^{T,\gamma^3-\delta}(\tau^k)[r]}{V_{L_{(\lambda^1,\gamma^1)}}\ V_{L_{(\lambda^2,\gamma^2)}}}$$

as vector spaces by (6.2) and (6.3). By [**11**, Proposition 11.9] and Proposition 5.7,

$$\dim_{\mathbb{C}} I_{V^{\tau}_{L_{\mathbf{0}\times D}}}\binom{V_{L_{(\lambda^1+\tau^j(\lambda^2))\times(\gamma^1+\gamma^2+D)}}}{V_{L_{\lambda^1\times(\gamma^1+D)}}\ V_{L_{\lambda^2\times(\gamma^2+D)}}} \le \dim_{\mathbb{C}} I_{V^{\tau}_{L^{\oplus\ell}}}\binom{V_{L_{(\lambda^1+\tau^j(\lambda^2))\times(\gamma^1+\gamma^2+D)}}}{V_{L_{(\lambda^1,\gamma^1)}}\ V_{L_{(\lambda^2,\gamma^2)}}}$$
$$= \sum_{\delta\in D}\dim_{\mathbb{C}} I_{V^{\tau}_{L^{\oplus\ell}}}\binom{V_{L_{(\lambda^1+\tau^j(\lambda^2),\gamma^1+\gamma^2+\delta)}}}{V_{L_{(\lambda^1,\gamma^1)}}\ V_{L_{(\lambda^2,\gamma^2)}}} = 1 \tag{6.11}$$

for $j = 0, 1, 2$ and

$$\dim_{\mathbb{C}} I_{V^{\tau}_{L_{\mathbf{0}\times D}}}\binom{W}{V_{L_{\lambda^1\times(\gamma^1+D)}}\ V_{L_{\lambda^2\times(\gamma^2+D)}}}$$
$$\le \dim_{\mathbb{C}} I_{V^{\tau}_{L^{\oplus\ell}}}\binom{W}{V_{L_{(\lambda^1,\gamma^1)}}\ V_{L_{(\lambda^2,\gamma^2)}}} = 0 \tag{6.12}$$

for any irreducible $V^{\tau}_{L_{\mathbf{0}\times D}}$-module $W \not\cong V_{L_{(\lambda^1+\tau^j(\lambda^2))\times(\gamma^1+\gamma^2+D)}}$, $j = 0, 1, 2$. By (6.10)–(6.12), we obtain (6.6).

The other formulas can be proved similarly. □

7. Modules of $V^{\tau}_{L_{C\times D}}$.

In this section we shall study $V^{\tau}_{L_{C\times D}}$-modules for an arbitrary τ-invariant self-dual $\mathcal{K}$-code C with minimum weight at least 4 and an arbitrary self-dual $\mathbb{Z}_3$-code D.

Let N be an $\mathbb{N}$-graded weak $V^{\tau}_{L_{C\times D}}$-module. Since N is a $V^{\tau}_{L^{\oplus\ell}}$-module, N is a direct sum of irreducible $V^{\tau}_{L^{\oplus\ell}}$-modules listed in Proposition 5.3. If N contains an irreducible $V^{\tau}_{L^{\oplus\ell}}$-module which is isomorphic to $V_{L_{(\lambda,\gamma)}}$ for a nonzero $\lambda \in \mathcal{K}^{\ell}$ and $\gamma \in \mathbb{Z}_3^{\ell}$, then Theorem 6.2 implies that N also contains an irreducible $V^{\tau}_{L^{\oplus\ell}}$-module which is isomorphic to $V_{L_{(\lambda,\mathbf{0})}}$ since N is a $V^{\tau}_{L_{\mathbf{0}\times D}}$-module and D is self-dual. This observation is important in the proof of Proposition 7.8. Thus, it is necessary to assume D is self-dual.

Recall that for $\mu \in \mathcal{K}^{\ell}$, $C(\mu)$ is the $\mathcal{K}$-code generated by μ and $\tau(\mu)$ (cf. Section 4). If $\mu \in \mathcal{K}^{\ell}$ has positive even weight, then

$$V^{\tau}_{L_{C(\mu)\times\mathbf{0}}} \cong V^{\tau}_{L^{\oplus\ell}} \oplus V_{L_{(\mu,\mathbf{0})}} \tag{7.1}$$

as $V^{\tau}_{L^{\oplus\ell}}$-modules. Since N is also a $V^{\tau}_{L_{C(\mu)\times\mathbf{0}}}$-module for each $\mu \in C$, using [**39**, Theorem 2.1.2], (7.1), and the fusion rules for $V^{\tau}_{L^{\oplus\ell}}$ in Proposition 5.7, we can obtain information about irreducible $V^{\tau}_{L^{\oplus\ell}}$-modules contained in N (See Proposition 7.5 and Proposition 7.8 below). Thus, we first study $\mathbb{N}$-graded weak $V^{\tau}_{L_{C(\mu)\times\mathbf{0}}}$-modules with some conditions for $\mu \in \mathcal{K}^{\ell}$ of positive even weight in Section 7.1.

Next, we shall classify the irreducible $V_{L_{C\times D}}^{\tau}$-modules and establish the rationality of $V_{L_{C\times D}}^{\tau}$ in Theorem 7.10 in Section 7.2.

7.1. Properties of $V_{L_{C(\mu)\times\mathbf{0}}}^{\tau}$-modules.

Throughout this subsection, ℓ and m are fixed even positive integers with $2 \le m \le \ell$. In this subsection we study $\mathbb{N}$-graded weak $V_{L_{C(\mu)\times\mathbf{0}}}^{\tau}$-modules with some conditions for $\mu \in \mathcal{K}^{\ell}$ of positive even weight. We deal with the case $\mu = (c^m 0^{\ell-m}) = (c,\dots,c,0,\dots,0)$ until Lemma 7.4. We have

$$V_{L_{C(c^m 0^{\ell-m})\times\mathbf{0}}}^{\tau} \cong V_{L^{\oplus\ell}}^{\tau} \oplus V_{L_{((c^m 0^{\ell-m}),\mathbf{0})}} \tag{7.2}$$

as $V_{L^{\oplus\ell}}^{\tau}$-modules. We shall fix the following notation. Let $W^0 = \oplus_{i=0}^{\infty} W^0(i)$ be an $\mathbb{N}$-graded weak $V_{L_{C(c^m 0^{\ell-m})\times\mathbf{0}}}^{\tau}$-module. Let $M^0 = \oplus_{i=0}^{\infty} M^0(i)$ be a $V_{L^{\oplus\ell}}^{\tau}$-submodule of W^0 such that $M^0(0) \subset W^0(0)$. Assume that M^0 is isomorphic to $V_{L_{(\Delta,\mathbf{0})}}$ for some nonzero $\Delta = (\Delta_1,\dots,\Delta_\ell) \in \mathcal{K}^{\ell}$.

In Lemma 7.1 we shall describe the action of $o(u\circ v)$ (see [**39**, Definition 2.1.1]) on the top level of M^0 for some elements $u, v \in V_{L_{((c^m 0^{\ell-m}),\mathbf{0})}} \subset V_{L_{C(c^m 0^{\ell-m})\times\mathbf{0}}}^{\tau}$ in the decomposition (7.2). Since $o(u\circ v)w = 0$ for all elements w in the top level of M^0, we shall obtain the relations (7.5) and (7.6) below, which play an important role to get information about Δ in Lemma 7.4. With the help of the action of G_ℓ, Lemma 7.4 immediately induces Proposition 7.5.

For $S \subset \{1,\dots,\ell\}$, set

$$S^* = \{i \in \{1,\dots,m\} \mid i \notin S\}. \tag{7.3}$$

Recall that for each $x \in \mathcal{K}$ we assign $\beta(x) \in L^{\perp}$ by $\beta(0) = 0$, $\beta(a) = \beta_2/2$, $\beta(b) = \beta_0/2$, and $\beta(c) = \beta_1/2$. For $j \in \mathcal{K}$, we use $\beta^{(s)}(j)$ to denote the element $\beta(j) \in L^{\perp}$ in the s-th entry of $(L^{\perp})^{\oplus\ell}$. For $p = (p_i) \in \mathcal{K}^{\ell}$ and $\epsilon = (\epsilon_i) \in \{1,-1\}^{\ell}$, set $\beta(p;\epsilon) = \sum_{i=1}^{\ell} \epsilon_i \beta^{(i)}(p_i)$. For example,

$$\sum_{i=1}^{m} \epsilon_i \beta_1^{(i)}/2 = \sum_{i=1}^{m} \epsilon_i \beta^{(i)}(c) = \beta((c^m 0^{\ell-m});\epsilon).$$

We simply write $\beta(p)$ for $\beta(p;(1,\dots,1))$. For $\alpha \in (L^{\perp})^{\oplus\ell}$, set $\mathbf{e}(\alpha) = e^{\alpha}$. Set

$$S_j(\Delta) = \{i \in \{1,\dots,m\} \mid \Delta_i = j\}$$

for $j = a, b, c$.

For a formal Laurent series $p(x) = \sum_{n\in\mathbb{Z}} p_n x^n$ in one variable x and $i \in \mathbb{Z}$, set

$p(x)|_{x^i} = p_i$. For $p(x,y) = \sum_{n,m\in\mathbb{Z}} p_{nm}x^n y^m$ and $i,j \in \mathbb{Z}$, set $p(x,y)|_{x^i,y^j} = p_{ij}$ similarly. For homogeneous $u, v \in V$, we shall use the following expression:

$$o(u \circ v) = \sum_{r=0}^{\mathrm{wt}\, u} \binom{\mathrm{wt}\, u}{r} Y_M(Y(u,x)v,y)w\big|_{x^{1-r},y^{-\mathrm{wt}\, u - \mathrm{wt}\, v + r - 1}}. \tag{7.4}$$

The following lemma is the key result in this section.

LEMMA 7.1. (1) *Let S be a subset of $\{1,\dots,m\}$ with $1 \le |S| \le m/2$. If $\Delta \not\equiv_\tau (c^m 0^{\ell-m})$ or $|S| \le m/2 - 2$, then for $j = 0,1,2$ and $\epsilon = (\epsilon_i) \in \{1,-1\}^\ell$, we have*

$$\delta_{\langle \sum_{i\in S} \epsilon_i \beta_j^{(i)}/2, \beta(\Delta)\rangle, -|S|} \binom{\langle \sum_{i=1}^m \epsilon_i \beta_j^{(i)}/2, \beta(\Delta)\rangle + m/2}{m - 2|S| + 1} = 0. \tag{7.5}$$

(2) *Suppose $\Delta \not\equiv_\tau (c^m 0^{\ell-m})$ or $m \ge 4$. Then, for $\epsilon = (\epsilon_i) \in \{1,-1\}^\ell$ we have*

$$\sum_{j=0}^{2} \binom{\langle \sum_{i=1}^m \epsilon_i \beta_j^{(i)}/2, \beta(\Delta)\rangle + m/2}{m+1} = 0. \tag{7.6}$$

PROOF. Let S be a subset of $\{1,\dots,m\}$ and set $s = |S|$. Let

$$\mathbf{u} = \sum_{j=0}^{2} \mathbf{e}\Big(\sum_{i=1}^{m} \frac{\beta_j^{(i)}}{2}\Big), \qquad \mathbf{v} = \sum_{j=0}^{2} \mathbf{e}\Big(\sum_{i\in S} \frac{\beta_j^{(i)}}{2} + \sum_{i\in S^*} \frac{-\beta_j^{(i)}}{2}\Big).$$

Then

$$\mathbf{u} = \sum_{j=0}^{2} \tau^j \mathbf{e}\Big(\sum_{i=1}^{m} \frac{\beta_1^{(i)}}{2}\Big), \qquad \mathbf{v} = \sum_{j=0}^{2} \tau^j \mathbf{e}\Big(\sum_{i\in S} \frac{\beta_1^{(i)}}{2} + \sum_{i\in S^*} \frac{-\beta_1^{(i)}}{2}\Big)$$

by (2.15) and hence $\mathbf{u}$ and $\mathbf{v}$ are elements of $V^\tau_{L_{C(c^m 0^{\ell-m})\times\mathbf{0}}}$ of weight $m/2$. We shall describe the action of

$$o(\mathbf{u} \circ \mathbf{v}) = \sum_{r=0}^{m/2} \binom{m/2}{r} Y_{W^0}(Y(\mathbf{u},x)\mathbf{v},y)\big|_{x^{1-r},y^{-m+r-1}} \tag{7.7}$$

on the top level of M^0 (cf. (7.4)). For $j = 0,1,2$, set

$$
\begin{aligned}
\Omega_{1j} &= \zeta_{36}^{9m+18s} x^{-m+2s} \exp\left(\sum_{k=1}^{\infty} \frac{(\sum_{i=1}^{m} \beta_j^{(i)}/2)(-k)}{k} x^k\right) \mathbf{e}\left(\sum_{i\in S} \beta_j^{(i)}\right), \\
\Omega_{2j} &= \zeta_{36}^{9m+18s} x^{(m-2s)/2} \exp\left(\sum_{k=1}^{\infty} \frac{(\sum_{i=1}^{m} \beta_{j+1}^{(i)}/2)(-k)}{k} x^k\right) \\
&\quad \times \mathbf{e}\left(\sum_{i\in S} \frac{-\beta_j^{(i)}}{2} + \sum_{i\in S^*} \frac{\beta_{j+1}^{(i)} - \beta_{j+2}^{(i)}}{2}\right) \\
&\quad + \zeta_{36}^{18m} x^{(m-2s)/2} \exp\left(\sum_{k=1}^{\infty} \frac{(\sum_{i=1}^{m} \beta_{j+2}^{(i)}/2)(-k)}{k} x^k\right) \\
&\quad \times \mathbf{e}\left(\sum_{i\in S} \frac{-\beta_j^{(i)}}{2} + \sum_{i\in S^*} \frac{\beta_{j+2}^{(i)} - \beta_{j+1}^{(i)}}{2}\right).
\end{aligned} \tag{7.8}
$$

Using (2.11) we have

$$
Y(\mathbf{u}, x)\mathbf{v} = \sum_{j=0}^{2} (\Omega_{1j} + \Omega_{2j}) \tag{7.9}
$$

and hence

$$
\begin{aligned}
o(\mathbf{u} \circ \mathbf{v}) &= \sum_{j=0}^{2} \sum_{r=0}^{m/2} \binom{m/2}{r} Y_{W^0}\big(\Omega_{1j}, y\big)\big|_{x^{1-r}, y^{-m+r-1}} \\
&\quad + \sum_{j=0}^{2} \sum_{r=0}^{m/2} \binom{m/2}{r} Y_{W^0}\big(\Omega_{2j}, y\big)\big|_{x^{1-r}, y^{-m+r-1}}.
\end{aligned} \tag{7.10}
$$

In the decomposition (7.2) we have $\Omega_{10} + \Omega_{11} + \Omega_{12} \in V_{L\oplus\ell}^{\tau}((x))$ and $\Omega_{20} + \Omega_{21} + \Omega_{22} \in V_{L_{((c^m 0^{\ell-m}), \mathbf{0})}}((x))$ since $(\beta_{j+1} - \beta_{j+2})/2 = \beta_j/2 + \beta_{j+1}$, $j = 0, 1, 2$.

By [**36**, Section 4], we see that the top level of M^0 is spanned by $\{\mathbf{e}(\Delta; \epsilon) \mid \epsilon \in \{1, -1\}^{\ell}\}$. We shall compute $\sum_{r=0}^{m/2} \binom{m/2}{r} Y_{W^0}\big(\Omega_{1j}, y\big)\big|_{x^{1-r}, y^{-m+r-1}} \mathbf{e}(\Delta; \epsilon)$, $j = 0, 1, 2$. A similar computation as [**22**, (8.6.9)] shows the following formula:

$$
\begin{aligned}
&Y_{W^0}\big(\Omega_{1j},y\big)\mathbf{e}(\Delta;\epsilon)\\
&\quad=\zeta_{36}^{9m+18s}Y_{W^0}\bigg(x^{-m+2s}\exp\bigg(\sum_{k=1}^{\infty}\frac{(\sum_{i=1}^{m}\beta_j^{(i)}/2)(-k)}{k}x^k\bigg)\mathbf{e}\bigg(\sum_{i\in S}\beta_j^{(i)}\bigg),y\bigg)\mathbf{e}(\Delta;\epsilon)\\
&\quad=\zeta_{36}^{9m+18s}x^{-m+2s}y^{\langle\sum_{i\in S}\beta_j^{(i)},\beta(\Delta;\epsilon)\rangle}\left(1+\frac{x}{y}\right)^{\langle\sum_{i=1}^{m}\beta_j^{(i)}/2,\beta(\Delta;\epsilon)\rangle}\\
&\qquad\times\exp\left(\sum_{k=1}^{\infty}\frac{(\sum_{i\in S}\beta_j^{(i)})(-k)}{k}y^k\right)\\
&\qquad\times\exp\left(\sum_{k=1}^{\infty}\left(\left(\sum_{i=1}^{m}\frac{\beta_j^{(i)}}{2k}\right)(-k)(y+x)^k-\left(\sum_{i=1}^{m}\frac{\beta_j^{(i)}}{2k}\right)(-k)y^k\right)\right)\\
&\qquad\times\mathbf{e}\bigg(\sum_{i\in S}\beta_j^{(i)}\bigg)\mathbf{e}(\Delta;\epsilon).
\end{aligned}\tag{7.11}
$$

Setting

$$
\begin{aligned}
\Psi=\zeta_{36}^{9m+18s}&\exp\left(\sum_{m=1}^{\infty}\frac{(\sum_{i\in S}\beta_j^{(i)})(-m)}{m}y^m\right)\\
&\times\exp\left(\sum_{k=1}^{\infty}\left(\left(\sum_{i=1}^{m}\frac{\beta_j^{(i)}}{2k}\right)(-k)(y+x)^k-\left(\sum_{i=1}^{m}\frac{\beta_j^{(i)}}{2k}\right)(-k)y^j\right)\right),
\end{aligned}
$$

we have

$$
\begin{aligned}
&Y_{W^0}\big(\Omega_{1j},y\big)\mathbf{e}(\Delta;\epsilon)\\
&\quad=\sum_{t=0}^{\infty}\binom{\langle\sum_{i=1}^{m}\beta_j^{(i)}/2,\beta(\Delta;\epsilon)\rangle}{t}x^{-m+2s+t}y^{-t+\langle\sum_{i\in S}\beta_j^{(i)},\beta(\Delta;\epsilon)\rangle}\\
&\qquad\times\Psi\mathbf{e}\bigg(\sum_{i\in S}\beta_j^{(i)}\bigg)\mathbf{e}(\Delta;\epsilon).
\end{aligned}\tag{7.12}
$$

Let r be an integer with $0\le r\le m/2$. To describe the first term of (7.10), we need to investigate the coefficient of $x^{1-r}y^{-m+r-1}$ in (7.12). First, we shall discuss the case that there is a nonnegative integer t such that $1-r\ge -m+2s+t$ and $-m+r-1\ge -t+\langle\sum_{i\in S}\beta_j^{(i)},\beta(\Delta;\epsilon)\rangle$. Note that if no such t exists, then the coefficient of $x^{1-r}y^{-m+r-1}$ in (7.12) is equal to zero since $\Psi\mathbf{e}(\sum_{i\in S}\beta_j^{(i)})\mathbf{e}(\Delta;\epsilon)\in$

$W^0[[x,y]]$. Since $\langle \pm\beta_i/2, \pm\beta_j/2\rangle \in \{\pm 1, \pm 1/2\}$ for $0 \le i,j \le 2$, we have $2s + \langle \sum_{i\in S}\beta_j^{(i)}, \beta(\Delta;\epsilon)\rangle \ge 0$ and hence

$$\begin{aligned}
-m+r-1 &\ge -t + \Big\langle \sum_{i\in S}\beta_j^{(i)}, \beta(\Delta;\epsilon)\Big\rangle \\
&\ge -m+2s-1+r+\Big\langle \sum_{i\in S}\beta_j^{(i)}, \beta(\Delta;\epsilon)\Big\rangle \\
&\ge -m+r-1.
\end{aligned}$$

This implies that $2s + \langle \sum_{i\in S}\beta_j^{(i)}, \beta(\Delta;\epsilon)\rangle = 0$, $t = m-2s-r+1$, and the coefficient of $x^{1-r}y^{-m+r-1}$ in (7.12) is

$$\zeta_{36}^{9m+18s}\delta_{\langle \sum_{i\in S}\beta_j^{(i)}/2, \beta(\Delta;\epsilon)\rangle, -s}\binom{\langle \sum_{i=1}^m \beta_j^{(i)}/2, \beta(\Delta;\epsilon)\rangle}{m-2s-r+1}\mathbf{e}\Big(\sum_{i\in S}\beta_j^{(i)}\Big)\mathbf{e}(\Delta;\epsilon). \tag{7.13}$$

Next, we shall discuss the case that $1-r < -m+2s+t$ or $-m+r-1 < -t + \langle \sum_{i\in S}\beta_j^{(i)}, \beta(\Delta;\epsilon)\rangle$ for all nonnegative integer t. Since $\Psi\mathbf{e}(\sum_{i\in S}\beta_j^{(i)})\mathbf{e}(\Delta;\epsilon) \in W^0[[x,y]]$, the coefficient of $x^{1-r}y^{-m+r-1}$ in (7.12) is equal to 0. If $m-2s-r+1 \ge 0$, then by setting $t_0 = m-2s-r+1$, we have $1-r \ge -m+2s+t_0$ and hence

$$\begin{aligned}
-m+r-1 &< -t_0 + \Big\langle \sum_{i\in S}\beta_j^{(i)}, \beta(\Delta;\epsilon)\Big\rangle \\
&= -m+2s+r-1+\Big\langle \sum_{i\in S}\beta_j^{(i)}, \beta(\Delta;\epsilon)\Big\rangle.
\end{aligned}$$

Thus, in this case $2s + \langle \sum_{i\in S}\beta_j^{(i)}, \beta(\Delta;\epsilon)\rangle \ne 0$ and hence the coefficient of $x^{1-r}y^{-m+r-1}$ in (7.12) is also given by (7.13). By (7.13) and $\langle \beta(\lambda^1), \beta(\lambda^2;\epsilon)\rangle = \langle \beta(\lambda^1;\epsilon), \beta(\lambda^2)\rangle$ for $\lambda^1, \lambda^2 \in \mathcal{K}^m$, we have obtained

$$\begin{aligned}
&\sum_{r=0}^{m/2}\binom{m/2}{r} Y_{W^0}(\Omega_{1j}, y)|_{x^{1-r}, y^{-m+r-1}}\mathbf{e}(\Delta;\epsilon) \\
&\quad = \zeta_{36}^{9m+18s}\delta_{\langle \sum_{i\in S}\epsilon_i\beta_j^{(i)}, \beta(\Delta)\rangle, -s} \\
&\qquad \times \sum_{r=0}^{m/2}\binom{m/2}{r}\binom{\langle \sum_{i=1}^m \beta_j^{(i)}/2, \beta(\Delta;\epsilon)\rangle}{m-2s-r+1}\mathbf{e}\Big(\sum_{i\in S}\beta_j^{(i)}\Big)\mathbf{e}(\Delta;\epsilon)
\end{aligned}$$

$$= \zeta_{36}^{9m+18s}\delta_{\langle\sum_{i\in S}\epsilon_i\beta_j^{(i)},\beta(\Delta)\rangle,-s} \times \binom{\langle\sum_{i=1}^m \epsilon_i\beta_j^{(i)}/2,\beta(\Delta)\rangle + m/2}{m-2s+1}\mathbf{e}\Bigg(\sum_{i\in S}\beta_j^{(i)}\Bigg)\mathbf{e}(\Delta;\epsilon) \tag{7.14}$$

for $j = 0, 1, 2$.

We next investigate $\sum_{r=0}^{m/2}\binom{m/2}{r}Y_{W^0}(\Omega_{2j}, y)\big|_{x^{1-r},y^{-m+r-1}}\mathbf{e}(\Delta;\epsilon)$ for $j = 0, 1, 2$. We expand Ω_{2j} as

$$\begin{aligned}\Omega_{2j} = \zeta_{36}^{9m+18s}x^{(m-2s)/2}&\Bigg(1 + \Bigg(\sum_{i=1}^m \frac{\beta_{j+1}^{(i)}}{2}\Bigg)(-1)x + \cdots\Bigg)\\ &\times \mathbf{e}\Bigg(\sum_{i\in S}\frac{-\beta_j^{(i)}}{2} + \sum_{i\in S^*}\frac{\beta_{j+1}^{(i)} - \beta_{j+2}^{(i)}}{2}\Bigg)\\ &+ \zeta_{36}^{18m}x^{(m-2s)/2}\Bigg(1 + \Bigg(\sum_{i=1}^m \frac{\beta_{j+2}^{(i)}}{2}\Bigg)(-1)x + \cdots\Bigg)\\ &\times \mathbf{e}\Bigg(\sum_{i\in S}\frac{-\beta_j^{(i)}}{2} + \sum_{i\in S^*}\frac{\beta_{j+2}^{(i)} - \beta_{j+1}^{(i)}}{2}\Bigg).\end{aligned}$$

If $0 \le s \le m/2 - 2$, then $(m-2s)/2 \ge 2$ and hence $\Omega_{2j} \in x^2W[[x]]$. This tells us that

$$\sum_{r=0}^{m/2}\binom{m/2}{r}Y_{W^0}(\Omega_{2j}, y)\big|_{x^{1-r},y^{-m+r-1}}\mathbf{e}(\Delta;\epsilon) = 0. \tag{7.15}$$

In the case of $s = m/2 - 1$, $m/2$, we do not need explicit expressions of $\sum_{r=0}^{m/2}\binom{m/2}{r}Y_{W^0}(\Omega_{2j}, y)\big|_{x^{1-r},y^{-m+r-1}}\mathbf{e}(\Delta;\epsilon)$ to obtain (7.5) and (7.6).

Let $\mathrm{pr}_{M^0} : W^0 \to M^0$ be a projection. By (7.4), (7.9), (7.14), (7.15) and [**39**, Theorem 2.1.2], in the case of $0 \le s \le m/2 - 2$, we have

$$\begin{aligned}0 &= \mathrm{pr}_{M^0}\, o(\mathbf{u}\circ\mathbf{v})\mathbf{e}(\Delta;\epsilon)\\ &= \zeta_{36}^{9m+18s}\sum_{j=0}^{2}\delta_{\langle\sum_{i\in S}\epsilon_i\beta_j^{(i)}/2,\beta(\Delta)\rangle,-s}\binom{\langle\sum_{i=1}^m \epsilon_i\beta_j^{(i)}/2,\beta(\Delta)\rangle + m/2}{m-2s+1}\\ &\quad\times \mathbf{e}\Bigg(\sum_{i\in S}\beta_j^{(i)}\Bigg)\mathbf{e}(\Delta;\epsilon).\end{aligned} \tag{7.16}$$

In the case of $s = m/2 - 1$, we have

$$\begin{aligned}
0 &= \mathrm{pr}_{M^0}\, o(\mathbf{u} \circ \mathbf{v})\mathbf{e}(\Delta; \epsilon) \\
&= \zeta_{36}^{9m+18s} \sum_{j=0}^{2} \delta_{\langle \sum_{i \in S} \epsilon_i \beta_j^{(i)}/2, \beta(\Delta)\rangle, -m/2+1} \binom{\langle \sum_{i=1}^{m} \epsilon_i \beta_j^{(i)}/2, \beta(\Delta)\rangle + m/2}{3} \\
&\quad \times \mathbf{e}\Bigl(\sum_{i \in S} \beta_j^{(i)}\Bigr)\mathbf{e}(\Delta; \epsilon) \\
&\quad + \mathrm{pr}_{M^0} \sum_{j=0}^{2} \sum_{r=0}^{m/2} \binom{m/2}{r} Y_{W^0}(\Omega_{2j}, y)\big|_{x^{1-r}, y^{-m+r-1}} \mathbf{e}(\Delta; \epsilon).
\end{aligned} \tag{7.17}$$

In the case of $s = m/2$, we have

$$\begin{aligned}
0 &= \mathrm{pr}_{M^0}\, o(\mathbf{u} \circ \mathbf{v})\mathbf{e}(\Delta; \epsilon) \\
&= \zeta_{36}^{9m+18s} \sum_{j=0}^{2} \delta_{\langle \sum_{i \in S} \epsilon_i \beta_j^{(i)}/2, \beta(\Delta)\rangle, -m/2} \left(\left\langle \sum_{i=1}^{m} \frac{\epsilon_i \beta_j^{(i)}}{2}, \beta(\Delta) \right\rangle + m/2 \right) \\
&\quad \times \mathbf{e}\Bigl(\sum_{i \in S} \beta_j^{(i)}\Bigr)\mathbf{e}(\Delta; \epsilon) \\
&\quad + \mathrm{pr}_{M^0} \sum_{j=0}^{2} \sum_{r=0}^{m/2} \binom{m/2}{r} Y_{W^0}(\Omega_{2j}, y)\big|_{x^{1-r}, y^{-m+r-1}} \mathbf{e}(\Delta; \epsilon).
\end{aligned} \tag{7.18}$$

If $1 \le s \le m/2 - 2$, then (7.5) follows from (7.16) since $\mathbf{e}(\sum_{i \in S} \beta_0^{(i)})\mathbf{e}(\Delta; \epsilon)$, $\mathbf{e}(\sum_{i \in S} \beta_1^{(i)})\mathbf{e}(\Delta; \epsilon)$, $\mathbf{e}(\sum_{i \in S} \beta_2^{(i)})\mathbf{e}(\Delta; \epsilon)$ are linearly independent. If $m \ge 4$, then (7.6) follows by taking $S = \emptyset$ in (7.16).

The map $f(\cdot, x)$ defined by $f(u, x)w = \mathrm{pr}_{M^0}(Y_{W^0}(u, x)w)$ for $u \in V_{L_{((c^m 0^{\ell-m}), \mathbf{0})}}$ in (7.2) and $w \in V_{L_{(\Delta, \mathbf{0})}}$ is an element of $I_{V_{L^{\oplus \ell}}^{\tau}}\binom{V_{L_{(\Delta,\mathbf{0})}}}{V_{L_{((c^m 0^{\ell-m}),\mathbf{0})}}\ V_{L_{(\Delta,\mathbf{0})}}}$.

Suppose $\Delta \not\equiv_\tau (c^m 0^{\ell-m})$. Then, by (5.44) we have

$$\dim_{\mathbb{C}} I_{V_{L^{\oplus \ell}}^{\tau}}\binom{V_{L_{(\Delta,\mathbf{0})}}}{V_{L_{((c^m 0^{\ell-m}),\mathbf{0})}}\ V_{L_{(\Delta,\mathbf{0})}}} = 0$$

and hence in (7.17) and (7.18) the second terms are zero:

$$\mathrm{pr}_{M^0}\sum_{j=0}^{2}\sum_{r=0}^{m/2}\binom{m/2}{r}Y_{W^0}(\Omega_{2j},y)\big|_{x^{1-r},y^{-m+r-1}}\mathbf{e}(\Delta;\epsilon)=0.$$

Moreover, if $1\le s\le m/2$, then (7.5) follows from (7.16)–(7.18). Taking $S=\emptyset$ in (7.16)–(7.18), we have (7.6). □

REMARK 7.2. In the case of $\ell=m=2$, consider the vertex operator algebra $V_{L_{C((c,c))\times\mathbf{0}}}$. We see that $V_{L_{C((c,c))\times\mathbf{0}}}(1)=\{u\in V_{L_{C((c,c))\times\mathbf{0}}}\mid \tau u=\zeta_3 u\}$ is an irreducible $V^{\tau}_{L_{C((c,c))\times\mathbf{0}}}$-module and that

$$V_{L_{C((c,c))\times\mathbf{0}}}(1)\cong V_{L_{(\mathbf{0},\mathbf{0})}}(1)\oplus V_{L_{((c,c),\mathbf{0})}}$$

as $V^{\tau}_{L^{\oplus 2}}$-modules. Note that the top level of $V_{L_{((c,c),\mathbf{0})}}$ is a subspace of the top level of $V_{L_{C((c,c))\times\mathbf{0}}}(1)$ and is a subspaces of $(V_{L_{C((c,c))\times\mathbf{0}}})_1$. However, we have

$$\begin{aligned}
&\sum_{j=0}^{2}\binom{\langle\beta_j^{(1)}/2+\beta_j^{(2)}/2,\beta((c,c))\rangle+2/2}{2+1}\\
&=\binom{\langle\beta_1^{(1)}/2+\beta_1^{(2)}/2,\beta_1^{(1)}/2+\beta_1^{(2)}/2\rangle+1}{3}\\
&\quad+\sum_{j=0,2}\binom{\langle\beta_j^{(1)}/2+\beta_j^{(2)}/2,\beta_1^{(1)}/2+\beta_1^{(2)}/2\rangle+1}{3}\\
&=\binom{2+1}{3}+2\binom{-1+1}{3}\\
&=1\neq 0.
\end{aligned}$$

Hence formula (7.6) does not hold in this case.

LEMMA 7.3. *Assume that $\Delta\not\equiv_\tau(c^m0^{\ell-m})$. For $j=a,b,c$, the following assertions hold.*

(1) *If $m/2\le|S_j(\Delta)|\le m$, then $|S_j(\Delta)|=m/2$, $|S_k(\Delta)|=0$ for all $k\ne j$, and $|S_j(\Delta)|$ is an even integer. In particular, $\langle(k^m0^{\ell-m}),\Delta\rangle_{\mathcal{K}}=0$ for all $k=a,b,c$.*
(2) *If $1\le|S_j(\Delta)|\le m/2$, then $\sum_{k\in\{a,b,c\},k\ne j}|S_k(\Delta)|$ is an even integer. In particular, $\langle(j^m0^{\ell-m}),\Delta\rangle_{\mathcal{K}}=0$.*

PROOF. Suppose $m/2\le|S_j(\Delta)|\le m$. We use the notation defined just after (7.3). Take $S \subset S_j(\Delta)$ such that $|S| = m/2$ and set $\epsilon = (\epsilon_i) \in \{1, -1\}^\ell$ by

$$\epsilon_i = \begin{cases} -1 & \text{if } i \in S_j(\Delta), \\ 1 & \text{otherwise.} \end{cases} \tag{7.19}$$

Then, $\langle \sum_{i \in S} \epsilon_i \beta^{(i)}(j), \beta(\Delta)\rangle = -|S|$ and by (7.5),

$$\begin{aligned} 0 &= \binom{\langle \beta((j^m 0^{\ell-m}); \epsilon), \beta(\Delta)\rangle + m/2}{m - 2|S| + 1} \\ &= \binom{-|S_j(\Delta)| - \sum_{k \in \{a,b,c\}, k \neq j} |S_k(\Delta)|/2 + m/2}{m - 2 \cdot m/2 + 1} \\ &= -|S_j(\Delta)| - \sum_{k \in \{a,b,c\}, k \neq j} |S_k(\Delta)|/2 + m/2 \\ &\leq -m/2 - \sum_{k \in \{a,b,c\}, k \neq j} |S_k(\Delta)|/2 + m/2 \\ &= - \sum_{k \in \{a,b,c\}, k \neq j} |S_k(\Delta)|/2 \leq 0. \end{aligned}$$

Thus $|S_j(\Delta)| = m/2$ and $|S_k(\Delta)| = 0$ for all $k \in \{a, b, c\}$, $k \neq j$. By (7.6),

$$\begin{aligned} 0 &= \sum_{k=a,b,c} \binom{\langle \beta((k^m 0^{\ell-m}); \epsilon), \beta(\Delta)\rangle + m/2}{m+1} \\ &= \binom{-|S_j(\Delta)| + m/2}{m+1} + \sum_{k \in \{a,b,c\}, k \neq j} \binom{|S_j(\Delta)|/2 + m/2}{m+1} \\ &= \binom{0}{m+1} + 2\binom{m/4 + m/2}{m+1} \\ &= 2\binom{3m/4}{m+1}. \end{aligned}$$

Hence $|S_j(\Delta)| = m/2$ is even. In particular, $\langle (k^m 0^{\ell-m}), \Delta\rangle_{\mathcal{K}} = 0$ for $k = a, b, c$. Therefore, (1) holds.

Suppose $1 \leq |S_j(\Delta)| \leq m/2$. Set $\epsilon = (\epsilon_i) \in \{-1, 1\}^\ell$ by (7.19). Then $\langle \sum_{i \in S_j(\Delta)} \epsilon_i \beta^{(i)}(j), \beta(\Delta)\rangle = -|S_j(\Delta)|$. By (7.5), we have

$$
\begin{aligned}
0 &= \binom{\langle \beta((j^m 0^{\ell-m});\epsilon), \beta(\Delta)\rangle + m/2}{m - 2|S_j(\Delta)| + 1} \\
&= \binom{-|S_j(\Delta)| - \sum_{k\in\{a,b,c\},k\neq j} |S_k(\Delta)|/2 + m/2}{m - 2|S_j(\Delta)| + 1}.
\end{aligned}
$$

Since $|S_j(\Delta)| \leq m/2$, $\sum_{k\in\{a,b,c\},k\neq j} |S_k(\Delta)|$ is an even integer. Hence $\langle (j^m 0^{\ell-m}), \Delta\rangle_{\mathcal{K}} = 0$. This proves (2). $\square$

LEMMA 7.4. (1) $\langle (c^m 0^{\ell-m}), \Delta\rangle_{\mathcal{K}} = 0$.
(2) *If $m \geq 4$, then $|\operatorname{supp}_{\mathcal{K}}(\Delta) \cap \{1, \dots, m\}| < m$.*

PROOF. We may assume $\operatorname{supp}_{\mathcal{K}}(\Delta) \cap \{1, \dots, m\} \neq \emptyset$. First, we shall show that $\langle (c^m 0^{\ell-m}), \Delta\rangle_{\mathcal{K}} = 0$. If $\Delta \equiv_\tau (c^m 0^{\ell-m})$, then the assertion is clear from the definition of $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{K}}$. Assume that $\Delta \not\equiv_\tau (c^m 0^{\ell-m})$. We may also assume that $|S_c(\Delta)| = 0$ and $0 \leq |S_a(\Delta)|, |S_b(\Delta)| \leq m/2$ by Lemma 7.3. If $1 \leq |S_a(\Delta)|, |S_b(\Delta)|$, then $|S_b(\Delta)| = |S_b(\Delta)| + |S_c(\Delta)|$ and $|S_a(\Delta)| = |S_a(\Delta)| + |S_c(\Delta)|$ are even integers by Lemma 7.3 (2). Hence, we have $\langle (c^m 0^{\ell-m}), \Delta\rangle_{\mathcal{K}} = 0$ in this case. Suppose $|S_a(\Delta)| = 0$. Then $|S_b(\Delta)| > 0$ since $\operatorname{supp}_{\mathcal{K}}(\Delta) \cap \{1, \dots, m\} \neq \emptyset$. Set $\epsilon = (\epsilon_i) \in \{-1, 1\}^\ell$ by (7.19) with $j = b$. Note that $\binom{-|S_b(\Delta)|+m/2}{m+1} = 0$ since $1 \leq |S_b(\Delta)| \leq m/2$. By (7.6),

$$
\begin{aligned}
0 &= \sum_{j=a,b,c} \binom{\langle \beta((j^m 0^{\ell-m});\epsilon), \beta(\Delta)\rangle + m/2}{m+1} \\
&= \binom{-|S_b(\Delta)| + m/2}{m+1} + \sum_{j=a,c} \binom{|S_b(\Delta)|/2 + m/2}{m+1} \\
&= 2\binom{|S_b(\Delta)|/2 + m/2}{m+1}.
\end{aligned}
$$

Hence $|S_b(\Delta)|$ is an even integer. In particular, $\langle (c^m 0^{\ell-m}), \Delta\rangle_{\mathcal{K}} = 0$. In the case of $|S_b(\Delta)| = 0$, we can show that $\langle (c^m 0^{\ell-m}), \Delta\rangle_{\mathcal{K}} = 0$ similarly.

Next, we shall show that if $m \geq 4$ then $|\operatorname{supp}_{\mathcal{K}}(\Delta) \cap \{1, \dots, m\}| < m$. Suppose by contradiction that $|\operatorname{supp}_{\mathcal{K}}(\Delta) \cap \{1, \dots, m\}| = |S_a(\Delta)| + |S_b(\Delta)| + |S_c(\Delta)| = m$.

Case 1: Suppose $|S_k(\Delta)| = m$ for some $k \in \{a, b, c\}$. Setting $\epsilon = (1, \dots, 1)$ in (7.6), we have

$$
\begin{aligned}
0 &= \sum_{j=a,b,c} \binom{\langle \beta((j^m 0^{\ell-m})), \beta(\Delta)\rangle + m/2}{m+1} \\
&= \binom{\langle \beta((k^m 0^{\ell-m})), \beta(\Delta)\rangle + m/2}{m+1} + \sum_{j \neq k} \binom{\langle \beta((j^m 0^{\ell-m})), \beta(\Delta)\rangle + m/2}{m+1} \\
&= \binom{m + m/2}{m+1} + 2\binom{-m/2 + m/2}{m+1} \\
&= \binom{m + m/2}{m+1} \neq 0.
\end{aligned}
$$

This is a contradiction.

Case 2: Suppose $|S_k(\Delta)| < m$ for all $k = a, b, c$. Note that $(c^m 0^{\ell-m}) \not\equiv_\tau \Delta$ in this case. There exists $j \in \{a, b, c\}$ such that $1 \leq |S_j(\Delta)| \leq m/2$ since $\mathrm{supp}_{\mathcal{K}}(\Delta) \cap \{1, \ldots, m\} \neq \emptyset$. Set $\epsilon = (\epsilon_i) \in \{-1, 1\}^\ell$ by (7.19). Then $\langle \sum_{i \in S_j(\Delta)} \epsilon_i \beta^{(i)}(j), \beta(\Delta)\rangle = -|S_j(\Delta)|$. By (7.5), we have

$$
\begin{aligned}
0 &= \binom{\langle \beta((j^m 0^{\ell-m}); \epsilon), \beta(\Delta)\rangle + m/2}{m - 2|S_j(\Delta)| + 1} \\
&= \binom{-|S_j(\Delta)| - \sum_{k \in \{a,b,c\}, k \neq j} |S_k(\Delta)|/2 + m/2}{m - 2|S_j(\Delta)| + 1} \\
&= \binom{-|S_j(\Delta)| - (m - |S_j(\Delta)|)/2 + m/2}{m - 2|S_j(\Delta)| + 1} \\
&= \binom{-|S_j(\Delta)|/2}{m - 2|S_j(\Delta)| + 1} \neq 0.
\end{aligned}
$$

This is a contradiction. Therefore, we conclude that $\mathrm{wt}_{\mathcal{K}}(\Delta) < m$. □

PROPOSITION 7.5. *Let $\mu = (\mu_k)$ be a nonzero element of $\mathcal{K}^\ell$ such that $\mathrm{wt}_{\mathcal{K}}(\mu)$ is even and $\mathrm{wt}_{\mathcal{K}}(\mu) \geq 4$. Let $W = \oplus_{i=0}^{\infty} W(i)$ be an $\mathbb{N}$-graded weak $V^\tau_{L_{C(\mu)\times \mathbf{0}}}$-module. Let $M = \oplus_{i=0}^{\infty} M(i)$ be an irreducible $V^\tau_{L^{\oplus \ell}}$-submodule of W such that $M(0) \subset W(0)$. Assume that M is isomorphic to $V_{L_{(\Delta,\mathbf{0})}}$ for some nonzero $\Delta = (\Delta_k) \in \mathcal{K}^\ell$. Then $\langle \mu, \Delta\rangle_{\mathcal{K}} = 0$ and $|\mathrm{supp}_{\mathcal{K}}(\mu) \cap \mathrm{supp}_{\mathcal{K}}(\Delta)| < \mathrm{wt}_{\mathcal{K}}(\mu)$.*

PROOF. There exists $g \in G_\ell$ such that $g(\mu) = (c^m 0^{\ell-m})$, where $m = \mathrm{wt}_{\mathcal{K}}(\mu)$. Consider a vertex operator algebra $V^\tau_{L_{C(g(\mu))\times \mathbf{0}}}$ and a $V^\tau_{L_{C(g(\mu))\times \mathbf{0}}}$-module $W \circ g^{-1}$ defined by $W \circ g^{-1} = W$ as vector spaces and $Y_{W\circ g^{-1}}(u, x) = Y_W(g^{-1}u, x)$ for $u \in V^\tau_{L_{C(g(\mu))\times \mathbf{0}}}$. Note that $M \circ g^{-1}$ is a $V^\tau_{L^{\oplus \ell}}$-submodule of $W \circ g^{-1}$ which is isomorphic to $V_{L_{(g(\Delta),\mathbf{0})}}$. Since g is an automorphism of $\mathcal{K}^\ell$, it is sufficient to show that $\langle g(\mu), g(\Delta)\rangle_{\mathcal{K}} = \langle (c^m 0^{\ell-m}), g(\Delta)\rangle_{\mathcal{K}} = 0$ and $|\operatorname{supp}_{\mathcal{K}}(g(\Delta)) \cap \{1,\dots,m\}| < m$ for $V^\tau_{L_{g(C(\mu))\times\mathbf{0}}}$ and a $V^\tau_{L_{g(C(\mu))\times\mathbf{0}}}$-module $W\circ g^{-1}$. These results hold by Lemma 7.4. □

7.2. Modules of $V^\tau_{L_{C\times D}}$.

In this subsection we shall classify the irreducible $V^\tau_{L_{C\times D}}$-modules and establish the rationality of $V^\tau_{L_{C\times D}}$ for arbitrary τ-invariant self-dual $\mathcal{K}$-code C with minimum weight at least 4 and arbitrary self-dual $\mathbb{Z}_3$-code D.

For any nonzero $\mu \in \mathcal{K}^\ell$ of even weight and any self-orthogonal $\mathbb{Z}_3$-code D, we have

$$V^\tau_{L_{C(\mu)\times D}} \cong V_{L_{\mathbf{0}\times D}}(0) \oplus V_{L_{\mu\times D}} \tag{7.20}$$

as $V^\tau_{L_{\mathbf{0}\times D}}$-modules. The following lemma will be used in Lemma 7.7 and Proposition 7.9.

Lemma 7.6. *For any nonzero $\mu \in \mathcal{K}^\ell$ of even weight and any self-orthogonal $\mathbb{Z}_3$-code D, we have $V_{L_{\mu\times D}} \cdot V_{L_{\mu\times D}} = V^\tau_{L_{C(\mu)\times D}}$ in (7.20).*

Proof. We may assume that $\mu = (c^m 0^{\ell-m})$, $m > 0$ by the action of G_ℓ (see Proof of Proposition 7.5). Then the assertion follows from (7.9). □

For the remainder of this paper, C is a τ-invariant self-dual $\mathcal{K}$-code of length ℓ with minimum weight at least 4 and D is a self-dual $\mathbb{Z}_3$-code of the same length. Let $C_{\equiv_\tau}$ be the set of all orbits of τ in C. Note that

$$V_{L_{C\times D}}(\varepsilon) \cong V_{L_{\mathbf{0}\times D}}(\varepsilon) \oplus \bigoplus_{\mathbf{0}\neq\lambda\in C_{\equiv_\tau}} V_{L_{\lambda\times D}}, \quad \varepsilon = 0,1,2$$

as $V^\tau_{L_{\mathbf{0}\times D}}$-modules by Lemma 2.13.

By Proposition 6.3 and Lemma 7.6, the same argument as in the proof of [**26**, Theorem 5.4] shows the following lemma.

Lemma 7.7. *Let (N^1, Y^1) and (N^2, Y^2) be irreducible $V^\tau_{L_{C\times D}}$-modules and let $\varepsilon \in \mathbb{Z}_3$. Suppose for each $i = 1, 2$, there is a $V^\tau_{L^{\oplus\ell}}$-submodule of N^i which is isomorphic to $V_{L_{(\mathbf{0},\mathbf{0})}}(\varepsilon)$. Then, N^1 and N^2 are isomorphic $V^\tau_{L_{C\times D}}$-modules.*

As it was mentioned at the beginning of this section, we need to assume that D is self-dual to show the following proposition.

Proposition 7.8. *Let N be an $\mathbb{N}$-graded weak $V^\tau_{L_{C\times D}}$-module which has a*

$V^{\tau}_{L^{\oplus\ell}}$-submodule isomorphic to $V_{L_{(\lambda,\gamma)}}$ for some nonzero $\lambda \in \mathcal{K}^{\ell}$ and $\gamma \in \mathbb{Z}_3^{\ell}$. Then there exists a $V^{\tau}_{L^{\oplus\ell}}$-submodule M of N which is isomorphic to $V_{L_{(\mathbf{0},\mathbf{0})}}(\varepsilon)$ for some $\varepsilon \in \mathbb{Z}_3$. Consequently, there exists a $V^{\tau}_{L_{\mathbf{0}\times D}}$-submodule of N which is isomorphic to $V_{L_{\mathbf{0}\times D}}(\varepsilon)$. The $V^{\tau}_{L_{C\times D}}$-submodule of N generated by M is isomorphic to $V_{L_{C\times D}}(\varepsilon)$.

PROOF. Let W^1 be an irreducible $V^{\tau}_{L^{\oplus\ell}}$-submodule of N which is isomorphic to $V_{L_{(\lambda,\gamma)}}$ for a nonzero $\lambda \in \mathcal{K}^{\ell}$ and $\gamma \in \mathbb{Z}_3^{\ell}$. Since N is a $V^{\tau}_{L_{\mathbf{0}\times D}}$-module, $\gamma \in D^{\perp} = D$ by Theorem 6.2 and consequently, there exists a $V^{\tau}_{L^{\oplus\ell}}$-submodule W^2 of N which is isomorphic to $V_{L_{(\lambda,\mathbf{0})}}$.

Suppose for any $\varepsilon \in \mathbb{Z}_3$, there is no $V^{\tau}_{L^{\oplus\ell}}$-submodule of N which is isomorphic to $V_{L_{(\mathbf{0},\mathbf{0})}}(\varepsilon)$. Let $N^1 = \oplus_{n=0}^{\infty} N^1(n)$ be the $V^{\tau}_{L_{C\times\mathbf{0}}}$-submodule of N generated by W^2. Note that every irreducible $V^{\tau}_{L^{\oplus\ell}}$-submodule of N^1 is isomorphic to $V_{L_{(\lambda^1,\mathbf{0})}}$ for a nonzero $\lambda^1 \in \mathcal{K}^{\ell}$ by Proposition 6.3 and the assumption. Let $M = \oplus_{n=0}^{\infty} M(n)$ be an irreducible $V^{\tau}_{L^{\oplus\ell}}$-submodule of N^1 such that $M(0) \subset N^1(0)$. There exists a nonzero $\Delta \in \mathcal{K}^{\ell}$ such that M is isomorphic to $V_{L_{(\Delta,\mathbf{0})}}$ as $V^{\tau}_{L^{\oplus\ell}}$-modules. Since N^1 is a $V^{\tau}_{L_{C(\mu)\times\mathbf{0}}}$-module for all $\mu \in C$, we have $\langle \mu, \Delta\rangle_{\mathcal{K}} = 0$ by Proposition 7.5 and hence $\Delta \in C^{\perp} = C$. By Proposition 7.5 again, $\mathrm{wt}_{\mathcal{K}}(\Delta) = |\operatorname{supp}_{\mathcal{K}}(\Delta) \cap \operatorname{supp}_{\mathcal{K}}(\Delta)| < \mathrm{wt}_{\mathcal{K}}(\Delta)$. This is a contradiction. Thus, there exists an irreducible $V^{\tau}_{L^{\oplus\ell}}$-module M isomorphic to $V_{L_{(\mathbf{0},\mathbf{0})}}(\varepsilon)$ for some $\varepsilon \in \mathbb{Z}_3$. By Proposition 5.3 and Theorem 6.2, the $V^{\tau}_{L_{\mathbf{0}\times D}}$-submodule of N generated by M is isomorphic to $V_{L_{\mathbf{0}\times D}}(\varepsilon)$.

Let N^2 be the $V^{\tau}_{L_{C\times D}}$-submodule of N generated by M. By Proposition 6.3,

$$N^2 \cong V_{L_{\mathbf{0}\times D}}(\varepsilon) \oplus \bigoplus_{\mathbf{0}\neq\lambda\in C_{\equiv_\tau}} V_{L_{\lambda\times D}}$$

as $V^{\tau}_{L_{\mathbf{0}\times D}}$-modules (cf. Proof of Lemma 5.2). Since any nonzero $V^{\tau}_{L_{C\times D}}$-submodule of N^2 must contain $V_{L_{\mathbf{0}\times D}}(\varepsilon)$ by the argument above, N^2 is irreducible. By Lemma 7.7, N^2 is isomorphic to $V_{L_{C\times D}}(\varepsilon)$ as $V^{\tau}_{L_{C\times D}}$-modules. □

PROPOSITION 7.9. *Let N be an $\mathbb{N}$-graded weak $V^{\tau}_{L_{C\times D}}$-module. Suppose N has a $V^{\tau}_{L^{\oplus\ell}}$-submodule M which is isomorphic to $V^{T,\eta}_{L^{\oplus\ell}}(\tau^i)[\varepsilon]$ for some $\eta \in \mathbb{Z}_3^{\ell}$ and $\varepsilon \in \mathbb{Z}_3$. Then M is a $V^{\tau}_{L_{C\times D}}$-submodule of N which is isomorphic to $V^{T,\mathbf{0}}_{L_{C\times D}}(\tau^i)[\varepsilon]$.*

PROOF. Note that the $V^{\tau}_{L_{\mathbf{0}\times D}}$-submodule of N generated by M is isomorphic to $V^{T,\mathbf{0}}_{L_{\mathbf{0}\times D}}(\tau^i)[\varepsilon]$ by Theorem 6.2. Take any nonzero $\lambda \in C$ and consider a vertex operator subalgebra $V^{\tau}_{L_{C(\lambda)\times D}}$ of $V^{\tau}_{L_{C\times D}}$. Let N^1 be the $V^{\tau}_{L_{C(\lambda)\times D}}$-submodule of N generated by M. Note that for $\varepsilon_1 \in \mathbb{Z}_3$ with $\varepsilon_1 \neq \varepsilon$, the difference of the minimal eigenvalues of ω_1 in $V^{T,\mathbf{0}}_{L_{\mathbf{0}\times D}}(\tau^i)[\varepsilon_1]$ and in $V^{T,\mathbf{0}}_{L_{\mathbf{0}\times D}}(\tau^i)[\varepsilon]$ is not an integer, where ω is the Virasoro element of $V^{\tau}_{L_{C(\lambda)\times D}}$. By Theorem 6.2 and Proposition 6.3, N^1 is a direct sum of $V^{\tau}_{L_{\mathbf{0}\times D}}$-modules, each of which is isomorphic to $V^{T,\mathbf{0}}_{L_{\mathbf{0}\times D}}(\tau^i)[\varepsilon]$. We write $N^1 = \oplus_{j\in\mathcal{J}} M^j$, $M^j \cong V_{L_{\mathbf{0}\times D}}^{T,\mathbf{0}}(\tau^i)[\varepsilon]$. We can take $M^{j_1} = M$ for some $j_1 \in \mathcal{J}$. For each $j \in \mathcal{J}$, let $\varphi_j \colon M^j \to V_{L_{\mathbf{0}\times D}}^{T,\mathbf{0}}(\tau^i)[\varepsilon]$ be an isomorphism of $V_{L_{\mathbf{0}\times D}}^{\tau}$-modules and let $\mathrm{pr}_j \colon N^1 \to M^j$ be a projection.

We want to show that $|\mathcal{J}| = 1$. Suppose $\mathcal{J}$ contains at least two elements and take $j_2 \in \mathcal{J}$, $j_2 \neq j_1$. For any $j \in \mathcal{J}$, $v \in V_{L_{\lambda\times D}}$, and $w \in M$, define $f_j(v,x)w = \varphi_j(\mathrm{pr}_j(Y_N(v,x)w))$. Then, $f_j \in I_{V_{L_{\mathbf{0}\times D}}^{\tau}}\binom{V_{L_{\mathbf{0}\times D}}^{T,\mathbf{0}}(\tau^i)[\varepsilon]}{V_{L_{\lambda\times D}}\ V_{L_{\mathbf{0}\times D}}^{T,\mathbf{0}}(\tau^i)[\varepsilon]}$. Note that at most one f_j is not zero since $\dim_{\mathbb{C}} I_{V_{L_{\mathbf{0}\times D}}^{\tau}}\binom{V_{L_{\mathbf{0}\times D}}^{T,\mathbf{0}}(\tau^i)[\varepsilon]}{V_{L_{\lambda\times D}}\ V_{L_{\mathbf{0}\times D}}^{T,\mathbf{0}}(\tau^i)[\varepsilon]} = 1$ (cf. [**36**, Proof of Lemma 5.6]). Since N^1 is generated by M, we have $f_{j_2} \neq 0$. Consequently, $\mathcal{J} = \{j_1, j_2\}$ and $f_{j_1} = 0$. Namely,

$$N^1 = M^{j_1} \oplus M^{j_2}$$

and $V_{L_{\lambda\times D}} \cdot M^{j_1} = M^{j_2}$. For any $k = 1,2$, $v \in V_{L_{\lambda\times D}}$, and $w \in M^{j_2}$, define $f_{2,j_k}(v,x)w = \varphi_{j_k}(\mathrm{pr}_{j_k}(Y_N(v,x)w))$. Then, $f_{2,j_k} \in I_{V_{L_{\mathbf{0}\times D}}^{\tau}}\binom{V_{L_{\mathbf{0}\times D}}^{T,\mathbf{0}}(\tau^i)[\varepsilon]}{V_{L_{\lambda\times D}}\ V_{L_{\mathbf{0}\times D}}^{T,\mathbf{0}}(\tau^i)[\varepsilon]}$. By Lemma 7.6, we have

$$\begin{aligned} V_{L_{\lambda\times D}} \cdot M^{j_2} &= V_{L_{\lambda\times D}} \cdot (V_{L_{\lambda\times D}} \cdot M^{j_1}) = (V_{L_{\lambda\times D}} \cdot V_{L_{\lambda\times D}}) \cdot M^{j_1} \\ &= (V_{L_{\mathbf{0}\times D}}^{\tau} \oplus V_{L_{\lambda\times D}}) \cdot M^{j_1} = M^{j_1} \oplus M^{j_2}. \end{aligned}$$

Hence f_{2,j_1} and f_{2,j_2} are linearly independent (cf. [**36**, Proof of Lemma 5.6]). This contradicts the fact that $\dim_{\mathbb{C}} I_{V_{L_{\mathbf{0}\times D}}^{\tau}}\binom{V_{L_{\mathbf{0}\times D}}^{T,\mathbf{0}}(\tau^i)[\varepsilon]}{V_{L_{\lambda\times D}}\ V_{L_{\mathbf{0}\times D}}^{T,\mathbf{0}}(\tau^i)[\varepsilon]} = 1$.

Therefore, M is a $V_{L_{C\times D}}^{\tau}$-submodule of N. By Theorems 3.13 and 6.2, $V_{L_{C\times D}}^{T,\mathbf{0}}(\tau^i)[\varepsilon] \cong V_{L_{\mathbf{0}\times D}}^{T,\mathbf{0}}(\tau^i)[\varepsilon]$ as $V_{L_{\mathbf{0}\times D}}^{\tau}$-modules. The same arguments as in [**28**, Lemma C.2] can show that any irreducible $V_{L_{C\times D}}^{\tau}$-module which is isomorphic to $V_{L_{\mathbf{0}\times D}}^{T,\mathbf{0}}(\tau^i)[\varepsilon]$ as $V_{L_{\mathbf{0}\times D}}^{\tau}$-modules must be isomorphic to $V_{L_{C\times D}}^{T,\mathbf{0}}(\tau^i)[\varepsilon]$. Hence the assertion holds. □

Theorem 7.10. *Let C be a τ-invariant self-dual $\mathcal{K}$-code of length ℓ with minimum weight at least 4 and let D be a self-dual $\mathbb{Z}_3$-code of the same length. Then $V_{L_{C\times D}}^{\tau}$ is a simple, rational, C_2-cofinite, and CFT type vertex operator algebra. There are exactly 9 equivalence classes of irreducible $V_{L_{C\times D}}^{\tau}$-modules which are represented by the following ones.*

(1) $V_{L_{C\times D}}(\varepsilon)$, $\varepsilon = 0,1,2$.
(2) $V_{L_{C\times D}}^{T,\mathbf{0}}(\tau^i)[\varepsilon]$, $i = 1,2, \varepsilon = 0,1,2$.

Proof. The simplicity of $V_{L_{C\times D}}^{\tau}$ is a consequence of [**17**, Theorem 4.4]. Since $V_{L_{C\times D}}^{\tau}$ is a direct sum of finitely many irreducible $V_{L_{\mathbf{0}\times D}}^{\tau}$-modules, $V_{L_{C\times D}}^{\tau}$ is C_2-cofinite by [**5**]. The classification of irreducible $V_{L_{C\times D}}^{\tau}$-modules follows from Propositions 7.8 and 7.9.

We shall show that $V_{L_{C\times D}}^{\tau}$ is rational. Let N be an $\mathbb{N}$-graded weak $V_{L_{C\times D}}^{\tau}$-module. Let M be the sum of irreducible $V_{L_{\mathbf{0}\times D}}^{\tau}$-submodules of N, each of which is isomorphic to any of $V_{L_{\mathbf{0}\times D}}(\varepsilon)$, $V_{L_{\mathbf{0}\times D}}^{T,\mathbf{0}}(\tau^i)[\varepsilon]$, $\varepsilon \in \mathbb{Z}_3$, $i = 1, 2$. We denote by W the $V_{L_{C\times D}}^{\tau}$-submodule of N generated by M. By Propositions 7.8 and 7.9, W is a completely reducible $V_{L_{C\times D}}^{\tau}$-module. If the $V_{L_{C\times D}}^{\tau}$-module N/W is not zero, then N/W has a $V_{L_{\mathbf{0}\times D}}^{\tau}$-submodule isomorphic to one of $V_{L_{\mathbf{0}\times D}}(\varepsilon)$, $V_{L_{\mathbf{0}\times D}}^{T,\mathbf{0}}(\tau^i)[\varepsilon]$, $\varepsilon \in \mathbb{Z}_3$, $i = 1, 2$ by Propositions 7.8 and 7.9. This contradicts our choice of W. Hence $N = W$. This implies that $V_{L_{C\times D}}^{\tau}$ is rational. $\square$

Remark 7.11. In [**24**], it is shown that there exist a $\mathcal{K}$-code C of length 12 and a $\mathbb{Z}_3$-code D of the same length, which satisfy the conditions in Theorem 7.10, and such that $L_{C\times D}$ is isomorphic to the Leech lattice Λ. In this case τ corresponds to a unique fixed-point-free isometry of Λ of order 3 up to conjugacy (cf. [**6**]). Hence, as a special case of Theorem 7.10, we obtain the classification of irreducible modules, the rationality, and the C_2-cofiniteness for V_{Λ}^{τ}.

Remark 7.12. For $\ell = 4$, let C and D be a $\mathcal{K}$-code and a $\mathbb{Z}_3$-code with generating matrices

$$\begin{pmatrix} a & a & 0 & 0 \\ b & b & 0 & 0 \\ 0 & 0 & a & a \\ 0 & 0 & b & b \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix},$$

respectively. It is clear that C is τ-invariant self-dual and D is self-dual. The lattice $L_{\mathbf{0}\times D}$ is a $\sqrt{2}\,(E_8\text{-lattice})$ and $L_{C\times D}$ is an E_8-lattice. Note that D is the $[4, 2, 3]$ ternary tetra code.

We can not apply Theorem 7.10 to $V_{L_{C\times D}}$ since the minimum weight of C equals 2.

8. List of Notations.

ζ_n	$\exp(2\pi\sqrt{-1}/n)$.
$\langle\,\cdot\,,\,\cdot\,\rangle$	the ordinary inner product of the Euclidean space $\mathbb{R}^{\ell}$.
L	$\sqrt{2}(A_2\text{-lattice})$.
$L^{\perp}$	the dual lattice of L.
β_1, β_2	a $\mathbb{Z}$-basis of L such that $\langle\beta_1, \beta_1\rangle = \langle\beta_2, \beta_2\rangle = 4$ and $\langle\beta_1, \beta_2\rangle = -2$.

β_0	$\beta_0 = -\beta_1 - \beta_2$.
$\tilde{\beta}_1, \tilde{\beta}_2$	the basis of $L^\perp$ defined by $\tilde{\beta}_1 = \beta_1/2$ and $\tilde{\beta}_2 = (\beta_1 - \beta_2)/6$.
τ	an isometry of L induced by the permutation $\beta_1 \mapsto \beta_2 \mapsto \beta_0 \mapsto \beta_1$.
H_ℓ	the direct product of ℓ copies of the group $\langle\tau\rangle$ generated by τ.
τ	$\tau = (\tau, \dots, \tau) \in H_\ell$ (For simplicity of notation, we denote $(\tau, \dots, \tau)$ by τ also).
$\bar{H}_\ell$	$\{(\tau^{i_1}, \dots, \tau^{i_{\ell-1}}, 1) \in H_\ell \mid i_1, \dots, i_{\ell-1} \in \mathbb{Z}\}$.
G_ℓ	$H_\ell \rtimes \mathfrak{S}_\ell$, where $\mathfrak{S}_\ell$ is the symmetric group of degree ℓ.
$\mathcal{K}$	$\mathcal{K} = \{0, a, b, c\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is the Klein's four-group.
C	a code over $\mathcal{K}$.
D	a code over $\mathbb{Z}_3$.
$\mathrm{supp}_{\mathcal{K}}(\lambda)$	$\mathrm{supp}_{\mathcal{K}}(\lambda) = \{i \mid \lambda_i \neq 0\}$ where $\lambda = (\lambda_1, \dots, \lambda_\ell) \in \mathcal{K}^\ell$.
$\mathrm{supp}_{\mathbb{Z}_3}(\gamma)$	$\mathrm{supp}_{\mathbb{Z}_3}(\gamma) = \{i \mid \gamma_i \neq 0\}$ where $\gamma = (\gamma_1, \dots, \gamma_\ell) \in \mathbb{Z}_3^\ell$.
$\mathrm{wt}_{\mathcal{K}}(\lambda)$	the cardinality of $\mathrm{supp}_{\mathcal{K}}(\lambda)$.
$\mathrm{wt}_{\mathbb{Z}_3}(\gamma)$	the cardinality of $\mathrm{supp}_{\mathbb{Z}_3}(\gamma)$.
$\langle\lambda, \mu\rangle_{\mathcal{K}}$	$\langle\lambda, \mu\rangle_{\mathcal{K}} = \sum_{i=1}^\ell \lambda_i\mu_i \in \mathcal{K}$ where $\lambda = (\lambda_i), \mu = (\mu_i) \in \mathcal{K}^\ell$.
$\langle\gamma, \delta\rangle_{\mathbb{Z}_3}$	$\langle\gamma, \delta\rangle_{\mathbb{Z}_3} = \sum_{i=1}^\ell \gamma_i\delta_i \in \mathbb{Z}_3$ where $\gamma = (\gamma_i), \delta = (\delta_i) \in \mathbb{Z}_3^\ell$.
$C(\lambda)$	the $\mathcal{K}$-code generated by $\lambda \in \mathcal{K}^\ell$ and $\tau(\lambda)$.
$D(\gamma)$	the $\mathbb{Z}_3$-code generated by $\gamma \in \mathbb{Z}_3^\ell$.
$\beta(x)$	$\beta(0) = 0, \beta(a) = \beta_2/2, \beta(b) = \beta_0/2, \beta(c) = \beta_1/2$.
$L^{(x,i)}$	$L^{(x,i)} = \beta(x) + i(-\beta_1 + \beta_2)/3 + L$ where $x \in \mathcal{K}$ and $i \in \mathbb{Z}_3$.
$L_{(\lambda,\gamma)}$	$L_{(\lambda,\gamma)} = L^{(\lambda_1,\gamma_1)} \oplus \cdots \oplus L^{(\lambda_\ell,\gamma_\ell)} \subset (L^\perp)^{\oplus\ell}$ where $\lambda \in \mathcal{K}^\ell$ and $\gamma \in \mathbb{Z}_3^\ell$.
$L_{P\times Q}$	$L_{P\times Q} = \cup_{\lambda\in P,\gamma\in Q} L_{(\lambda,\gamma)}$.
T	a subgroup in the center of $(\widehat{L^\perp})^\ell$ generated by $\kappa_{36}^{(r)}(\kappa_{36}^{(s)})^{-1}$, $1 \leq r, s \leq \ell$, where $\kappa_{36}^{(s)}$ denotes κ_{36} in the s-th entry of $(\widehat{L^\perp})^\ell$.
K_0	$K_0 = \{a \times_\tau \tau(a)^{-1} \mid a \in \hat{L}_{C\times\mathbf{0},\tau}\}$.
K	$K = \{a \times_\tau \tau(a)^{-1} \mid a \in \hat{L}_{C\times D,\tau}\}$.
$V_{L_{C\times D}}^{T,\eta}(\tau^i)$	irreducible τ^i-twisted $V_{L_{C\times D}}$-module where $\eta \in D^\perp$ and $i = 1, 2$.
$\tilde{\mathcal{K}}$	$\tilde{\mathcal{K}} = \{0, 1, 2, a, b, c\}$ (cf. Section 5).
$\mathrm{supp}_{\tilde{\mathcal{K}}}(\lambda)$	$\mathrm{supp}_{\tilde{\mathcal{K}}}(\lambda) = \{i \mid \lambda_i \in \{a, b, c\}\}$ where $\lambda = (\lambda_1, \dots, \lambda_\ell) \in \tilde{\mathcal{K}}^\ell$.
$\mathrm{wt}_{\tilde{\mathcal{K}}}(\lambda)$	the cardinality of $\mathrm{supp}_{\tilde{\mathcal{K}}}(\lambda)$.
$X_{i,j}$	$X_{i,j} = \begin{cases} V_{L^{(0,j)}}(i) & \text{if } i = 0, 1, 2, \\ V_{L^{(i,j)}} & \text{if } i = a, b, c. \end{cases}$
$X_{\xi,\gamma}$	$X_{\xi,\gamma} = \otimes_{i=1}^\ell X_{\xi_i,\gamma_i}$ where $\xi = (\xi_1, \dots, \xi_\ell) \in \tilde{\mathcal{K}}^\ell$ and $\gamma = (\gamma_1, \dots, \gamma_\ell) \in \mathbb{Z}_3^\ell$.
$P(V_{L_{(\mathbf{0},\gamma)}}(\varepsilon))$	$P(V_{L_{(\mathbf{0},\gamma)}}(\varepsilon)) = \{\xi = (\xi_k) \in \mathbb{Z}_3^\ell \mid \sum_{k=1}^\ell \xi_k \equiv \varepsilon \pmod 3\}$ where $\gamma \in \mathbb{Z}_3^\ell$ and $\varepsilon \in \mathbb{Z}_3$.

$P(V_{L_{(\lambda,\gamma)}})$	$P(V_{L_{(\lambda,\gamma)}}) = \{\xi \in \{0,1,2,c\}^\ell \mid \mathrm{supp}_{\tilde{\mathcal{K}}}(\xi) = \mathrm{supp}_{\mathcal{K}}(\lambda)\}$ where $0 \neq \lambda \in \mathcal{K}^\ell$ and $\gamma \in \mathbb{Z}_3^\ell$.
$\beta^{(i)}(j)$	$\beta^{(i)}(j)$ denotes $\beta(j) \in L^\perp$ in the i-th entry of $(L^\perp)^\ell$ where $j = a, b, c$.
$\beta(p;\epsilon)$	$\beta(p;\epsilon) = \sum_{i=1}^{\ell} \epsilon_i \beta^{(i)}(p_i)$ where $p = (p_i) \in \mathcal{K}^\ell$ and $\epsilon = (\epsilon_i) \in \{1,-1\}^\ell$.
$\beta(p)$	$\beta(p) = \beta(p;(1,\ldots,1))$.
$\mathbf{e}(\alpha)$	$\mathbf{e}(\alpha) = e^\alpha$ where $\alpha \in (L^\perp)^{\oplus \ell}$.
S^*	$S^* = \{i \in \{1,\ldots,m\} \mid i \notin S\}$ where S is a subset of $\{1,\ldots,\ell\}$.
$S_j(\lambda)$	$S_j(\lambda) = \{k \in \{1,\ldots,m\} \mid \lambda_k = j\}$ where $\lambda \in \mathcal{K}^\ell$ and $j = a, b, c$.

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