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Geometric intersection of curves on punctured disks

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Abstract. We give a recipe to compute the geometric intersection number of an integral lamination with a particular type of integral lamination on an n-times punctured disk. This provides a way to find the geometric intersection number of two arbitrary integral laminations when combined with an algorithm of Dynnikov and Wiest.

1. Introduction.

Given a surface M of genus g with s boundary components, a well known way of giving coordinates to integral laminations (i.e. a disjoint union of finitely many essential simple closed curves on M modulo isotopy) and measured foliations is to use either the Dehn-Thurston coordinates or train track coordinates. See [**10**] for details.

An alternative way to coordinatize integral laminations and measured foliations on an n-times punctured disk D_n is achieved by the *Dynnikov coordinate system*. That is, Dynnikov coordinate system provides an explicit bijection between the set of integral laminations $\mathcal{L}_n$ on D_n and $\mathbb{Z}^{2n-4} \setminus \{0\}$; and the set of measured foliations up to isotopy and Whitehead equivalence on D_n and $\mathbb{R}^{2n-4} \setminus \{0\}$.

Isotopy classes of orientation preserving homeomorphisms on punctured disks are described by elements of Artin's braid groups B_n [**1**], [**2**] and the action of B_n on $\mathcal{L}_n$ in terms of Dynnikov coordinates is described by the *update rules* [**5**], [**9**], [**8**].

The Dynnikov coordinate system together with the Dynnikov formulae (update rules) was introduced in [**5**]. Then, it was studied in [**3**], [**4**] as an efficient method for a solution of the word problem of B_n and in [**9**], [**7**], [**8**] for computing the topological entropy of braids.

In this paper, we shall use the Dynnikov coordinate system to study the geometric intersection number of two integral laminations on an n-times punctured disk. In particular, we shall give Theorem 11 which gives a recipe to compute the geometric intersection number of an integral lamination with a particular type of integral lamination, known as a *relaxed integral lamination*. This provides a way

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to find the geometric intersection number of two arbitrary integral laminations when combined with an algorithm of Dynnikov and Wiest [**6**], see Remark 10.

2. Dynnikov coordinates.

The aim of this section is to describe the Dynnikov coordinate system for the set of integral laminations $\mathcal{L}_n$ and prove that there is an explicit bijection between $\mathcal{L}_n$ and $\mathbb{Z}^{2n-4} \setminus \{0\}$. We shall begin with the triangle coordinates which describe each integral lamination by an element of $\mathbb{Z}^{3n-5}$ using its geometric intersection number with given $3n-5$ embedded arcs in D_n. *Dynnikov coordinates* [**5**] are certain linear combinations of these integers and yield a one-to-one correspondence between $\mathcal{L}_n$ and $\mathcal{C}_n = \mathbb{Z}^{2n-4} \setminus \{0\}$. This will be proved by Theorem 7 which gives the inversion of Dynnikov coordinates.

Let $\mathcal{A}_n$ be the set of arcs in D_n which have each endpoint either on the boundary or at a puncture. The arcs $\alpha_i \in \mathcal{A}_n$ $(1 \leq i \leq 2n-4)$ and $\beta_i \in \mathcal{A}_n$ $(1 \leq i \leq n-1)$ are as depicted in Figure 1: the arcs α_{2i-3} and α_{2i-2} (for $2 \leq i \leq n-1$) join the i^{th} puncture to the boundary, while the arc β_i has both endpoints on the boundary and passes between the i^{th} and $i+1^{\text{th}}$ punctures.

Observe that the arcs divide the disk into $2n-2$ (closed) regions and $2n-4$ of these are *triangular*: Identifying the outer boundary of the disk with a point, each region on the left and right side of the i^{th} puncture for $2 \leq i \leq n-1$ is a triangle since it is bounded by three arcs.

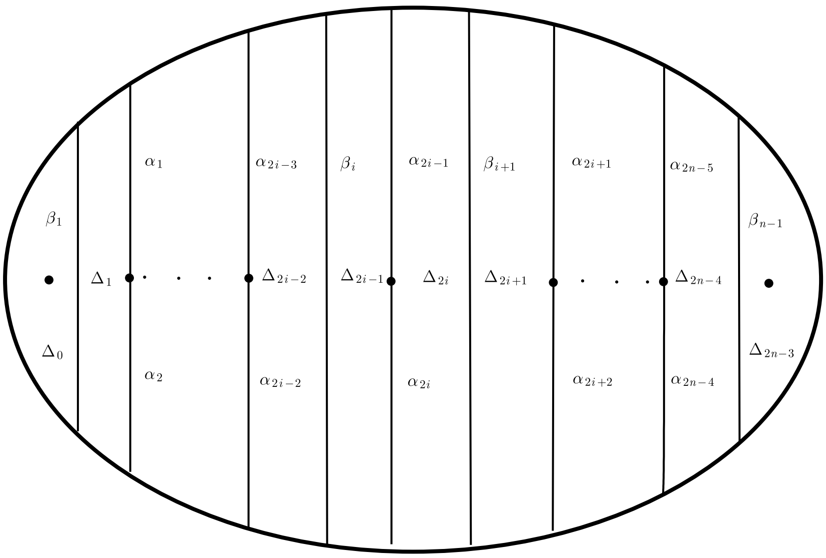

Figure 1. The arcs α_i, β_i and triangular regions Δ_i.

The two triangles Δ_{2i-3} and Δ_{2i-2} on the left and right side of the i^{th} puncture are defined by the arcs α_{2i-3}, α_{2i-2}, β_{i-1} and α_{2i-3}, α_{2i-2}, β_i respectively and the two end regions Δ_0 and Δ_{2n-3} are bounded by β_1 and β_{n-1} respectively. See Figure 1.

A naive way to describe integral laminations is achieved by triangle coordinates: Given $[\alpha]$ (the isotopy class of an arc $\alpha \in \mathcal{A}_n$ under isotopies through $\mathcal{A}_n$) and an integral lamination $\mathcal{L}$, we shall write α for the geometric intersection number of $\mathcal{L} \in \mathcal{L}_n$ with the arc $\alpha \in \mathcal{A}_n$: it will be clear from the context whether we mean the arc or the geometric intersection number assigned on the arc.

We also note that if $\mathcal{L} \in \mathcal{L}_n$ there is some curve system $L \in \mathcal{L}$ which is *taut* (has minimum number of intersections in its homotopy class with each α_i and β_i). We fix a taut representative L of a given integral lamination $\mathcal{L} \in \mathcal{L}_n$ throughout.

For each i with $1 \leq i \leq n-2$, define $S_i = \Delta_{2i-1} \cup \Delta_{2i}$ (see Figure 1). A *path component* of L in S_i is a component of $L \cap S_i$. There are four types of path components in S_i. An *above component* has end points on β_i and β_{i+1} and passes across α_{2i-1}. A *below component* has end points on β_i and β_{i+1} and passes across α_{2i}. A *left loop component* has both end points on β_{i+1} and a *right loop component* has both end points on β_i.

The solid lines in Figure 2 depict the above and below components. Left and right loop components are depicted by dashed lines. Note that there is one type of path component in the end regions: *left loop components* in region Δ_0 and *right loop components* in region Δ_{2n-3}.

Remark 1. We note that there could only be one of the two types of loop components (i.e. right or left) in each S_i since the curves in L are mutually disjoint.

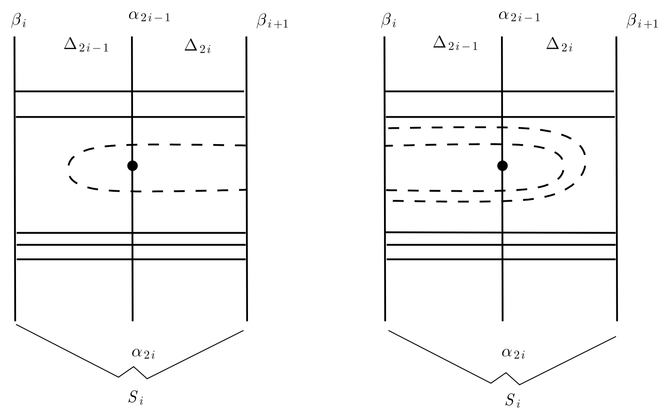

Figure 2. Above, below, left loop and right loop components.

For each $1 \leq i \leq n-2$ we define

$$b_i = \frac{\beta_i - \beta_{i+1}}{2}. \tag{1}$$

Then $|b_i|$ gives the number of loop components in S_i and $\epsilon_i = \operatorname{sgn}(b_i)$ tells whether the loop components are left or right. That is, when $b_i > 0$ the loop components are right and when $b_i < 0$ the loop components are left. See Figure 2: on the left, $\beta_{i+1} = \beta_i + 2$ (so $b_i = -1$) and the additional two intersections of L with β_{i+1} yield one left loop component. Similarly, on the right $\beta_i = \beta_{i+1} + 4$ (so $b_i = 2$) and the additional four intersections of L with β_i yield two right loop components.

The following Lemma is obvious since each above and below component intersects α_{2i-1} and α_{2i} respectively.

LEMMA 2. *The numbers of above and below components in region S_i are given by $\alpha_{2i-1} - |b_i|$ and $\alpha_{2i} - |b_i|$ respectively.*

Similarly, the next Lemma is obvious from Figure 3 and Figure 4.

LEMMA 3. *There are equalities for each S_i :*
When there are left loop components ($b_i < 0$),

$$\alpha_{2i} + \alpha_{2i-1} = \beta_{i+1} \tag{2}$$

$$\alpha_{2i} + \alpha_{2i-1} - \beta_i = 2|b_i|, \tag{3}$$

when there are right loop components ($b_i > 0$),

$$\alpha_{2i} + \alpha_{2i-1} = \beta_i \tag{4}$$

$$\alpha_{2i} + \alpha_{2i-1} - \beta_{i+1} = 2|b_i|, \tag{5}$$

and when there are no loop components ($b_i = 0$),

$$\alpha_{2i} + \alpha_{2i-1} = \beta_i = \beta_{i+1}. \tag{6}$$

Note that Lemma 3 implies that some coordinates are redundant.

The triangle coordinate function $\tau : \mathcal{L}_n \to \mathbb{Z}^{3n-5}_{\geq 0}$ is defined by

$$\tau(\mathcal{L}) = (\alpha_1, \ldots, \alpha_{2n-4}, \beta_1, \ldots, \beta_{n-1}).$$

$\tau : \mathcal{L}_n \to \mathbb{Z}^{3n-5}_{\geq 0}$ is injective: working in each region S_i, we can determine

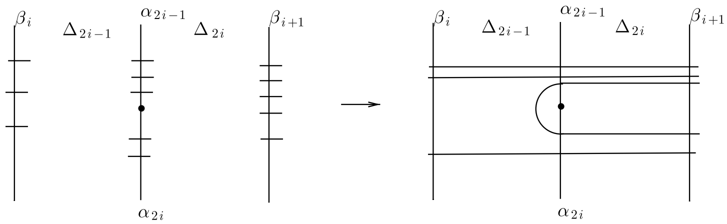

Figure 3. Left loop components and the case is $b_i \leq 0$.

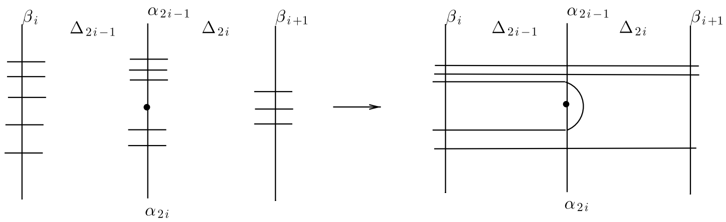

Figure 4. Right loop components and the case is $b_i \geq 0$.

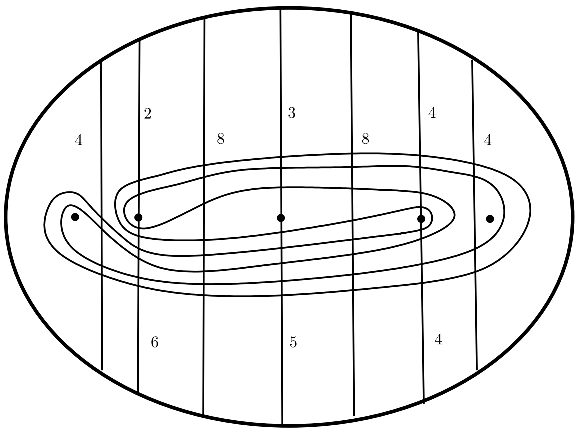

Figure 5. $\tau(\mathcal{L}) = (2, 6, 3, 5, 4, 4; 4, 8, 8, 4)$.

the number of above, below and right/left loop components. Therefore, the path components in each S_i are connected in a unique way up to isotopy and hence $\mathcal{L}$ is determined uniquely.

However, it is not always possible to construct an integral lamination from given triangle coordinates. Namely, $\tau : \mathcal{L}_n \to \mathbb{Z}_{\geq 0}^{3n-5}$ is not surjective since $\tau(\mathcal{L})$ must satisfy the triangle inequality in each of the strips of Figure 1, as well as additional conditions such as the equalities in Lemma 3. Next, we shall discuss what properties an integral lamination $\mathcal{L} \in \mathcal{L}_n$ satisfies in terms of its triangle coordinates and construct a new coordinate system from the triangle coordinates which describes integral laminations in a unique way. Namely, we shall describe the *Dynnikov coordinate system* [**5**].

Given a taut representative L of $\mathcal{L} \in \mathcal{L}_n$ one can initially observe the following:

REMARKS 4.

(i) Every component of L intersects each β_i an even number of times. Also recall that $b_i = (\beta_i - \beta_{i+1})/2$ and $|b_i|$ gives the number of loop components in S_i. When $b_i > 0$ the loop components are right and when $b_i < 0$ the loop components are left (Figure 6).

(ii) Set $x_i = |\alpha_{2i} - \alpha_{2i-1}|$ and $m_i = \min\{\alpha_{2i-1} - |b_i|, \alpha_{2i} - |b_i|\}$; $1 \leq i \leq n-2$. Then x_i gives the difference between the number of above and below components in S_i, and m_i gives the smaller of these two numbers by Lemma 2 (Figure 6). We note that x_i is even since each simple closed curve in L intersects $\alpha_{2i} \cup \alpha_{2i-1}$ an even number of times.

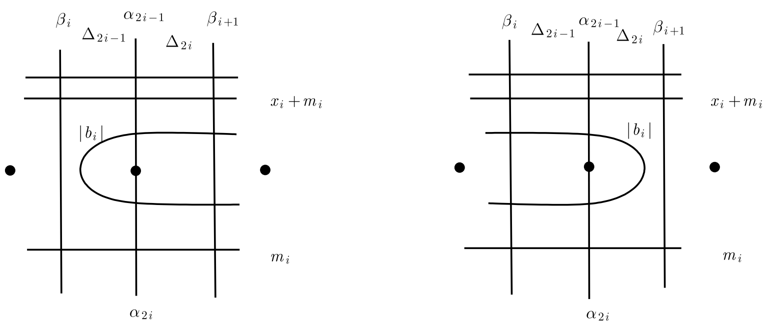

Figure 6. Number of above and below components in S_i.

(iii) Set $2a_i = \alpha_{2i} - \alpha_{2i-1}$; $1 \leq i \leq n-2$, (a_i is an integer since $|a_i| = x_i/2$). Assume that $b_i \geq 0$. Then, $\beta_i = \alpha_{2i} + \alpha_{2i-1}$ by Lemma 3. Since $2a_i = \alpha_{2i} - \alpha_{2i-1}$ it follows that

$$\alpha_{2i} = a_i + \frac{\beta_i}{2}; \quad \text{and} \quad \alpha_{2i-1} = -a_i + \frac{\beta_i}{2}.$$

A similar calculation for $b_i \leq 0$ gives

$$\alpha_{2i} = a_i + \frac{\beta_{i+1}}{2}; \quad \text{and} \quad \alpha_{2i-1} = -a_i + \frac{\beta_{i+1}}{2}.$$

That is to say:

$$\alpha_i = \begin{cases} (-1)^i a_{\lceil i/2 \rceil} + \dfrac{\beta_{\lceil i/2 \rceil}}{2} & \text{if } b_{\lceil i/2 \rceil} \geq 0; \\ (-1)^i a_{\lceil i/2 \rceil} + \dfrac{\beta_{1+\lceil i/2 \rceil}}{2} & \text{if } b_{\lceil i/2 \rceil} \leq 0 \end{cases}$$

where $\lceil x \rceil$ denotes the smallest integer which is not less than x.

(iv) It is straightforward to compute β_i; $1 \leq i \leq n-1$ from item (ii) and item (iii).

$$\beta_i = \begin{cases} 2m_i + 2|a_i| & \text{if } b_i \leq 0; \\ 2m_i + 2|a_i| + 2b_i & \text{if } b_i \geq 0. \end{cases} \tag{7}$$

That is,

$$\beta_i = 2[|a_i| + \max(b_i, 0) + m_i].$$

Since $\beta_i = \beta_1 - 2\sum_{j=1}^{i-1} b_j$ by (1),

$$\beta_1 = 2\left[|a_i| + \max(b_i, 0) + m_i + \sum_{j=1}^{i-1} b_j\right] \quad \text{for} \quad 1 \leq i \leq n-2.$$

(v) A crucial observation is that $m_i = 0$ for some $1 \leq i \leq n-1$ since otherwise there would be both above and below components in each S_i and hence the integral lamination would have a curve parallel to ∂D_n. Then,

When $m_i = 0$;

$$\beta_1 = 2\left[|a_i| + \max(b_i, 0) + \sum_{j=1}^{i-1} b_j\right].$$

When $m_i > 0$;

$$\beta_1 > 2\left[|a_i| + \max(b_i, 0) + \sum_{j=1}^{i-1} b_j\right].$$

Therefore,

$$\beta_1 = \max_{1 \le k \le n-2} 2\left[|a_k| + \max(b_k, 0) + \sum_{j=1}^{k-1} b_j \right].$$

We have seen that α_i and β_i have been recovered from a_i and b_i where

$$a_i = \frac{\alpha_{2i} - \alpha_{2i-1}}{2} \quad \text{and} \quad b_i = \frac{\beta_i - \beta_{i+1}}{2}.$$

Now, we are ready to define the *Dynnikov coordinate system* which has the advantage to coordinatize $\mathcal{L}_n$ bijectively and with the least number of coordinates.

DEFINITION 5. The *Dynnikov coordinate function* $\rho : \mathcal{L}_n \to \mathbb{Z}^{2n-4} \setminus \{0\}$ is defined by

$$\rho(\mathcal{L}) = (a, b) = (a_1, \ldots, a_{n-2},\, b_1, \ldots, b_{n-2}),$$

where for $1 \le i \le n-2$

$$a_i = \frac{\alpha_{2i} - \alpha_{2i-1}}{2} \quad \text{and} \quad b_i = \frac{\beta_i - \beta_{i+1}}{2}. \tag{8}$$

Let $\mathcal{C}_n = \mathbb{Z}^{2n-4} \setminus \{0\}$ denote the space of Dynnikov coordinates of integral laminations on D_n.

EXAMPLE 6. The integral lamination $\mathcal{L}$ in Figure 5 has Dynnikov coordinates $\rho(\mathcal{L}) = (2, 1, 0, -2, 0, 2)$. We have,

$$\begin{aligned}
&\alpha_1 = 2, && \beta_1 = 4, && a_1 = \frac{\alpha_2 - \alpha_1}{2} = \frac{6-2}{2} = 2\\
&\alpha_2 = 6, && \beta_2 = 8, && a_2 = \frac{\alpha_4 - \alpha_3}{2} = \frac{5-3}{2} = 1\\
&\alpha_3 = 3, && \beta_3 = 8, && a_3 = \frac{\alpha_5 - \alpha_6}{2} = \frac{4-4}{2} = 0\\
&\alpha_4 = 5, && \beta_4 = 4, && b_1 = \frac{\beta_1 - \beta_2}{2} = \frac{4-8}{2} = -2\\
&\alpha_5 = 4, && && b_2 = \frac{\beta_2 - \beta_3}{2} = \frac{8-8}{2} = 0\\
&\alpha_6 = 4, && && b_3 = \frac{\beta_3 - \beta_4}{2} = \frac{8-4}{2} = 2.
\end{aligned}$$

Note that b_i can easily be read off from a picture of the lamination by counting the number of loop components and checking whether they are left or right. For example, there are two left loop components in S_1, therefore b_1 should be -2.

THEOREM 7 (Inversion of Dynnikov coordinates). *Let $(a,b) \in \mathcal{C}_n$. Then (a,b) is the Dynnikov coordinate of exactly one element $\mathcal{L}$ of $\mathcal{L}_n$, which has*

$$\beta_i = 2 \max_{1\le k\le n-2} \left[|a_k| + \max(b_k, 0) + \sum_{j=1}^{k-1} b_j \right] - 2\sum_{j=1}^{i-1} b_j \tag{9}$$

$$\alpha_i = \begin{cases} (-1)^i a_{\lceil i/2\rceil} + \dfrac{\beta_{\lceil i/2\rceil}}{2} & \text{if } b_{\lceil i/2\rceil} \ge 0; \\ (-1)^i a_{\lceil i/2\rceil} + \dfrac{\beta_{1+\lceil i/2\rceil}}{2} & \text{if } b_{\lceil i/2\rceil} \le 0 \end{cases} \tag{10}$$

where $\lceil x\rceil$ denotes the smallest integer which is not less than x.

PROOF. ρ is injective: Let $\mathcal{L} \in \mathcal{L}_n$, with $\tau(\mathcal{L}) = (\alpha,\beta)$ and $\rho(\mathcal{L}) = (a,b)$. We showed in Remarks 4 that (α,β) must be given by (9) and (10). Hence there is no other $\mathcal{L}' \in \mathcal{L}_n$ with $\rho(\mathcal{L}') = (a,b)$ since the triangle coordinate function is injective.

ρ is surjective: Let $(a,b) \in \mathcal{C}_n$. We will show that (α,β) defined by (9) and (10) are the triangle coordinates of some $\mathcal{L} \in \mathcal{L}_n$ which has $\rho(\mathcal{L}) = (a,b)$. It is clear that if there is some $\mathcal{L}$ with $\tau(\mathcal{L}) = (\alpha,\beta)$, then $\rho(\mathcal{L}) = (a,b)$. By the construction in Remarks 4, it is possible to draw in each S_i, $1 \le i \le n-2$ some non-intersecting path components which intersect $\beta_i, \alpha_{2i-1}, \alpha_{2i}$, and β_{i+1} the number of times given by (α,β). Joining these components (and completing in the only way in the two end regions) gives a system of mutually disjoint simple closed curves in D_n. There are no curves that bound punctures as every path component of a curve system has the property that its intersection with each S_i is of one of the four types by construction, so in particular there can't be a curve that bounds a puncture. There are no curves parallel to ∂D_n as some m_i is equal to zero. Hence this is an integral lamination which has triangle coordinates (α,β) as required. □

In the next section we shall give a formula to compute the geometric intersection number of a given integral lamination $\mathcal{L} \in \mathcal{L}_n$ with a given *relaxed curve* [**6**] C_{ij} in D_n in terms of triangle coordinates. Furthermore, the formula can be given in terms of Dynnikov coordinates by Theorem 7.

3. Geometric intersection of integral laminations with relaxed curves.

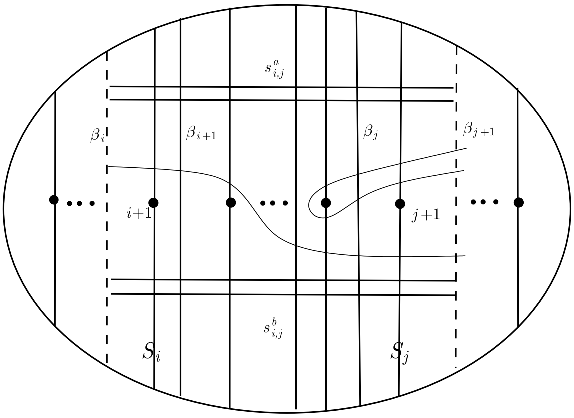

Figure 7. $s^a_{i,j}$ and $s^b_{i,j}$.

Let $S_{i,j} = \bigcup_{i\le k\le j} S_k$. A *path component* of L in $S_{i,j}$ is a component of $L \cap S_{i,j}$. An *above component* in $S_{i,j}$ has end points on β_i and β_{j+1} and does not intersect any α_{2k} with $i \le k \le j$. A *below component* in $S_{i,j}$ has end points on β_i and β_{j+1} and does not intersect any α_{2k-1} with $i \le k \le j$ (Figure 7). Using Lemma 2 one can compute the number of above and below components in $S_{i,j}$.

LEMMA 8. *The number of above and below components in $S_{i,j}$ is given by*

$$s^a_{i,j} = \min_{i\le k\le j}\{\alpha_{2k-1} - |b_k|\} \quad \textit{and} \quad s^b_{i,j} = \min_{i\le k\le j}\{\alpha_{2k} - |b_k|\}$$

respectively. Therefore the sum $s_{i,j} = s^a_{i,j} + s^b_{i,j}$ gives the number of above and below components in $S_{i,j}$.

PROOF. For each $1 \le k \le n-2$, $s^a_k = \alpha_{2k-1} - |b_k|$ and $s^b_k = \alpha_{2k} - |b_k|$ by Lemma 2.

Then $s^b_{i,j} = \min_{i\le k\le j}\{s^b_k\}$ and $s^a_{i,j} = \min_{i\le k\le j}\{s^a_k\}$. Hence,

$$s_{i,j} = \min_{i\le k\le j}\{s^a_k\} + \min_{i\le k\le j}\{s^b_k\}. \qquad \square$$

REMARK 9. Notice that the number of path components in $S_{i,j}$ which are not simple closed curves is given by $(\beta_i + \beta_{j+1})/2$ (Figure 7).

Given an essential simple closed curve C in D_n, $\|C\|$ denotes the minimum number of intersections of C with the x-axis. Then, given $\mathcal{L} \in \mathcal{L}_n$, the *norm* of $\mathcal{L}$ is defined as

$$\|\mathcal{L}\| = \sum_i \|C_i\|$$

where $\{C_i\}$ are connected components of $\mathcal{L}$. We say that C_i is *relaxed* if $\|C_i\| = 2$. Then, $\mathcal{L}$ is *relaxed* if each of its connected components C_i is relaxed [**6**].

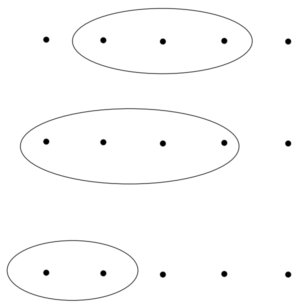

Figure 8. Relaxed curves C_{24}, C_{14}, C_{12} in D_5 from top to bottom.

For $1 \leq i < j < n$ or $1 < i < j \leq n$, $C_{ij} \in \mathcal{L}_n$ denotes the isotopy class of relaxed curves in D_n which bound a disk containing the set of punctures $\{i, i+1, \ldots, j\}$.

Hence, we observe that

$$\rho(C_{ij}) = (0, \ldots, 0, b_1, \ldots, b_{n-2})$$

where $b_{i-1} = -1$ if $i > 1$, $b_{j-1} = 1$ if $j < n$ and $b_k = 0$ for all other cases. Figure 8 shows some examples of relaxed curves.

Remark 10. It is always possible to turn a non-relaxed integral lamination $\mathcal{L} \in \mathcal{L}_n$ into one which is relaxed. That is to say, for any $\mathcal{L} \in \mathcal{L}_n$ there exists a braid $\beta \in B_n$ such that $\beta(\mathcal{L})$ is relaxed. An algorithm to accomplish this is given in [**6**].

Given $\mathcal{L}_1 \in \mathcal{L}_n$ and $\mathcal{L}_2 \in \mathcal{L}_n$ which are not relaxed, the geometric intersection number $i(\mathcal{L}_1, \mathcal{L}_2)$ can be computed by first relaxing one of the integral laminations with an n-braid β by the algorithm described in [**6**] and then computing $i(\beta(\mathcal{L}_1), \beta(\mathcal{L}_2))$ (note that $i(\mathcal{L}_1, \mathcal{L}_2) = i(\beta(\mathcal{L}_1), \beta(\mathcal{L}_2))$ since geometric intersection number is preserved under homeomorphisms). Hence, to compute $i(\mathcal{L}_1, \mathcal{L}_2)$, it is sufficient to find a formula that gives $i(C_{ij}, \mathcal{L})$ for a given $\mathcal{L} \in \mathcal{L}_n$.

THEOREM 11. *Given an integral lamination $\mathcal{L} \in \mathcal{L}_n$ with triangle coordinates (α, β) and $C_{ij} \in \mathcal{L}_n$, $i(\mathcal{L}, C_{ij})$ is given by,*

$$i(\mathcal{L}, C_{ij}) = \beta_{i-1} + \beta_j - 2s_{i-1,j-1} \tag{11}$$

where $s_{i,j}$ is defined as in Lemma 8.

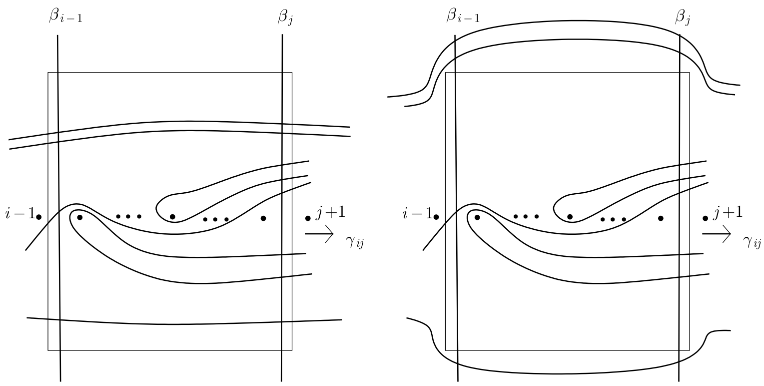

Figure 9. Proof of Theorem 11.

PROOF. Take a taut representative $L \in \mathcal{L}$ and a representative γ_{ij} of C_{ij} which is composed of subarcs of β_{i-1} and β_j and horizontal arcs which are such that the disk bounded by γ_{ij} contains all of the path components of L in $S_{i-1,j-1}$. The number of intersections of γ_{ij} with the path components of L in $S_{i-1,j-1}$ is given by $\beta_{i-1}+\beta_j$ (See Remark 9). This number can be minimized by subtracting from it the number of path components which can be isotoped so that they do not intersect γ_{ij} any more. Such path components can only be above and below components in $S_{i-1,j-1}$ (Figure 9). Since, each above and below component intersects γ_{ij} twice, we have that

$$i(\mathcal{L}, C_{ij}) = \beta_{i-1} + \beta_j - 2s_{i-1,j-1}. \qquad \square$$

Notice that the formulae given above can be written using Dynnikov coordinates since one can write each α_i and β_i in terms of a_i and b_i by Theorem 7.

EXAMPLE 12. Let $\rho(\mathcal{L}) = (2, 1, 0, -2, 0, 2)$ (Figure 5). We want to find $i(C_{24}, \mathcal{L})$. Using the formula (11) we get,

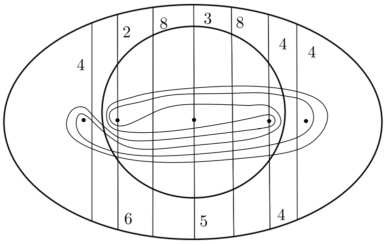

Figure 10. $i(\mathcal{L}, C_{ij}) = 4$.

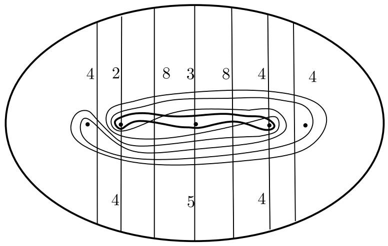

Figure 11. $i(\mathcal{L}, C_{ij}) = 4$.

$$i(\mathcal{L}, C_{24}) = \beta_1 + \beta_4 - 2s_{1,3}.$$

From Theorem 7, we know that

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6; \beta_1, \beta_2, \beta_3, \beta_4) = (2, 6, 3, 5, 4, 4; 4, 8, 8, 4).$$

From Lemma 8 we have,

$$s^a_{1,3} = \min_{1 \le k \le 3} \{\alpha_{2k-1} - |b_k|\} \quad \text{and} \quad s^b_{1,3} = \min_{1 \le k \le 3} \{\alpha_{2k} - |b_k|\}.$$

Therefore,

$$\begin{aligned} s^a_{1,3} &= \min\{\alpha_1 - |b_1|, \alpha_3 - |b_2|, \alpha_5 - |b_3|\} \\ &= \min\{2 - |-2|, 3 - 0, 4 - 2\} = 0 \end{aligned}$$

and

$$\begin{aligned} s^b_{1,3} &= \min\{\alpha_2 - |b_1|, \alpha_4 - |b_2|, \alpha_6 - |b_3|\} \\ &= \min\{6 - |-2|, 5 - 0, 4 - 2\} = 2. \end{aligned}$$

So the number of above and below components in $S_{1,3}$ equals $s^a_{1,3} + s^b_{1,3} = 2$.

Therefore,

$$i(\mathcal{L}, C_{24}) = \beta_1 + \beta_4 - 2s_{1,3} = 4 + 4 - 2 \times 2 = 4.$$

See Figure 10 and Figure 11.

REMARK 13. Observe that if $\mathcal{L}_1 = \bigcup C_{ij} \in \mathcal{L}_n$ and $\mathcal{L}_2 \in \mathcal{L}_n$, then

$$i(\mathcal{L}_1, \mathcal{L}_2) = \sum i(C_{ij}, \mathcal{L}_2)$$

since the above construction can be carried out for each C_{ij} in turn, working from the inside out.

The next result gives the geometric intersection number of two arbitrary integral laminations on a 3-times punctured disk using Theorem 11. Again, we note that the formula can be given in terms of Dynnikov coordinates by Theorem 7.

COROLLARY 14. *Let $\mathcal{L}_1 \in \mathcal{L}_3$ and $\mathcal{L}_2 \in \mathcal{L}_3$ have triangle coordinates (α^1, β^1) and (α^2, β^2) with Dynnikov coordinates (a^1, b^1) and (a^2, b^2) respectively. Then, the geometric intersection number $i(\mathcal{L}_1, \mathcal{L}_2)$ is given by*

$$i(\mathcal{L}_1, \mathcal{L}_2) = \begin{cases} \alpha_2^1\alpha_1^2 + \alpha_1^1\alpha_2^2 \ ; & \text{if } \epsilon^1\epsilon^2 = -1 \\ \left|\alpha_2^1\alpha_1^2 - \alpha_1^1\alpha_2^2\right|; & \text{if } \epsilon^1\epsilon^2 = +1 \end{cases} \tag{12}$$

where $\epsilon^1 = \operatorname{sgn}(b_1^1)$ and $\epsilon^2 = \operatorname{sgn}(b_1^2)$.

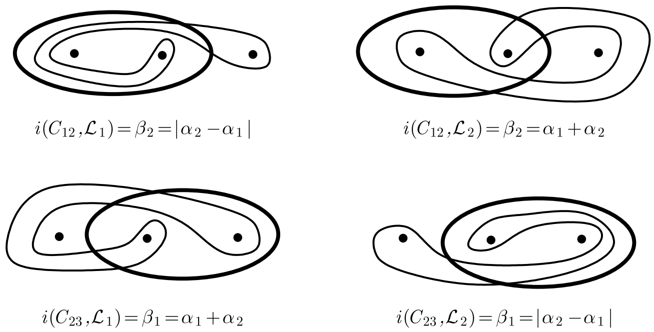

Figure 12. $i(C_{ij}, \mathcal{L})$ on D_3.

PROOF. We first observe that the only relaxed curves in D_3 are C_{12} and C_{23}. We also note that, given $\mathcal{L} \in \mathcal{L}_3$, $i(\mathcal{L}, C_{12}) = \beta_2$ and $i(\mathcal{L}, C_{23}) = \beta_1$. Hence the formula (12) is verified for $i(\mathcal{L}, C_{ij})$ by Lemma 2. See Figure 12.

For the general case, we recall that B_3 acts on both $\mathcal{L}$ and C_{ij} and there exists $\beta \in B_3$ such that $\beta(\mathcal{L})$ is either C_{12} or C_{23}. Since the geometric intersection number is preserved under homeomorphisms, it follows that the formula (12) is verified for $i(\mathcal{L}_1, \mathcal{L}_2)$ for any $\mathcal{L}_1 \in \mathcal{L}_3$ and $\mathcal{L}_2 \in \mathcal{L}_3$. □

Example 15. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be the integral laminations depicted in Figure 13 and; (α^1, β^1) and (α^2, β^2) be their triangle coordinates respectively. We observe that $(\alpha_1^1, \alpha_2^1) = (3, 1)$ and $(\alpha_1^2, \alpha_2^2) = (4, 2)$. Since $\mathcal{L}_1$ has right loop components and $\mathcal{L}_2$ has left loop components, $\epsilon^1\epsilon^2 = -1$ and hence by Corollary 14, $i(\mathcal{L}_1, \mathcal{L}_2)$ is given by

$$i(\mathcal{L}_1, \mathcal{L}_2) = \alpha_2^1\alpha_1^2 + \alpha_1^1\alpha_2^2 = 1 \times 4 + 3 \times 2 = 10.$$

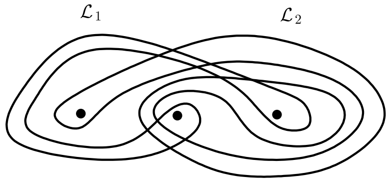

Figure 13. $i(\mathcal{L}_1, \mathcal{L}_2) = 10$.

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