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Positive Toeplitz operators on weighted Bergman spaces of a minimal bounded homogeneous domain

By Satoshi YAMAJI

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Abstract. We give criteria for the boundedness of positive Toeplitz operators on weighted Bergman spaces of a minimal bounded homogeneous domain in terms of the Berezin symbol or the averaging function of the symbol. Moreover, we estimate the essential norm of positive Toeplitz operators assuming that they are bounded. As an application of these estimates, we also give necessary and sufficient conditions for the positive Toeplitz operators to be compact.

1. Introduction.

In 1988, Zhu [**14**] obtained conditions in order that a positive Toeplitz operator is bounded or compact on weighted Bergman spaces of a bounded symmetric domain in its Harish-Chandra realization. He characterized the conditions in terms of Carleson type measures, the averaging function and the Berezin symbol. After, we consider positive Toeplitz operators on the non-weighted Bergman space of minimal bounded homogeneous domains [**11**]. In [**12**], we study Carleson type measures, the averaging function and the Berezin symbol on weighted Bergman spaces of a minimal bounded homogeneous domain. In this paper, we consider positive Toeplitz operators on weighted Bergman spaces of minimal bounded homogeneous domains. Moreover, we estimate the essential norm of bounded positive Toeplitz operators. The essential norm $\|T\|_e$ of a bounded operator T is defined by

$$\|T\|_e := \inf\{\|T - K\|\,; K \text{ is compact}\}.$$

It is easy to see that T is compact if and only if $\|T\|_e = 0$, so essential norm estimates enable us to show compactness conditions of operators.

Let $\mathcal{U} \subset \mathbb{C}^d$ be a minimal bounded homogeneous domain with center $t \in \mathcal{U}$

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(for the definition of the minimal domain, see [**7**], [**9**]). For example, the unit ball, bounded symmetric domain in its Harish-Chandra realization, and representative domain of bounded homogeneous domains are minimal bounded homogeneous domains with center 0. Let dV be the Lebesgue measure, $\mathcal{O}(\mathcal{U})$ the space of all holomorphic functions on $\mathcal{U}$, and $L^p_a(\mathcal{U}, dV)$ the Bergman space $L^p(\mathcal{U}, dV) \cap \mathcal{O}(\mathcal{U})$ of $\mathcal{U}$. We denote by $K_{\mathcal{U}}$ the Bergman kernel of $\mathcal{U}$, that is, the reproducing kernel of $L^2_a(\mathcal{U}, dV)$. For $\alpha \in \mathbb{R}$, let dV_α denote the measure on $\mathcal{U}$ given by $dV_\alpha(z) := K_{\mathcal{U}}(z,z)^{-\alpha} dV(z)$. Then there exists a constant $\varepsilon_{\min}$ such that the weighted Bergman space $L^p_a(\mathcal{U}, dV_\alpha) := L^p(\mathcal{U}, dV_\alpha) \cap \mathcal{O}(\mathcal{U})$ is non-trivial for all $0 < p < \infty$ if $\alpha > \varepsilon_{\min}$. Let $K^{(\alpha)}_{\mathcal{U}}$ be the reproducing kernel of $L^2_a(\mathcal{U}, dV_\alpha)$. It is known that

$$K^{(\alpha)}_{\mathcal{U}}(z,w) = C_\alpha K_{\mathcal{U}}(z,w)^{1+\alpha} \tag{1.1}$$

for some positive constant C_α. Moreover, we show that the Bergman kernel of a minimal bounded homogeneous domain satisfies a useful estimate (see [**7**, Theorem A]). This estimate and (1.1) tell us that the boundedness of positive Toeplitz operators on $L^2_a(\mathcal{U}, dV_\alpha)$ is also characterized by using Carleson type measures, the averaging function and the Berezin symbol (the definitions of them, see Section 2).

Let μ be a Borel measure on $\mathcal{U}$. For $f \in L^2_a(\mathcal{U}, dV_\alpha)$, the Toeplitz operator T_μ with symbol μ is defined by

$$T_\mu f(z) := \int_{\mathcal{U}} K^{(\alpha)}_{\mathcal{U}}(z,w) f(w)\, d\mu(w) \quad (z \in \mathcal{U}).$$

If $d\mu(w) = u(w) dV_\alpha(w)$ holds for some $u \in L^\infty(\mathcal{U})$, we have $T_\mu f = P_{\mathcal{U}}(uf)$, where $P_{\mathcal{U}}$ is the orthogonal projection from $L^2(\mathcal{U}, dV_\alpha)$ onto $L^2_a(\mathcal{U}, dV_\alpha)$. A Toeplitz operator is called positive if its symbol is positive. Throughout this paper, we assume that μ is a positive Borel measure on $\mathcal{U}$. We obtain the following theorem from the boundedness of the positive Bergman operator and Zhu's method (see [**14**] or [**15**]).

THEOREM A. *The following conditions are all equivalent.*

(a) *T_μ is a bounded operator on $L^2_a(\mathcal{U}, dV_\alpha)$.*
(b) *The Berezin symbol $\widetilde{\mu}$ of μ is a bounded function on $\mathcal{U}$.*
(c) *For all $p > 0$, μ is a Carleson measure for $L^p_a(\mathcal{U}, dV_\alpha)$.*
(d) *The averaging function $\widehat{\mu}$ of μ is bounded on $\mathcal{U}$.*

Theorem A is the same as [**14**, Theorem A] if $\mathcal{U}$ is a Harish-Chandra realization of bounded symmetric domain and as [**11**, Theorem 1.2] if $\alpha = 0$, that is, the case of the non-weighted Bergman space.

Next, we consider the compactness of T_μ. Zhu proved that the necessary and sufficient conditions for the positive Toeplitz operator on weighted Bergman spaces of a bounded symmetric domain in its Harish-Chandra realization to be compact are described in terms of the boundary value of the Berezin symbol or the averaging function. By using this technique, we also obtain the conditions for T_μ to be compact on $L^2_a(\mathcal{U}, dV_\alpha)$. However, we show them in terms of the essential norm estimates because essential norm estimates give us a further information. The essential norm of a bounded operator is the distance from the operator to the space of the compact operators. Essential norm estimates for Toeplitz operators with symbol in L^∞ are considered in [**13**], [**2**]. In 2007, Čučković and Zhao [**3**] gave estimates for the essential norms of weighted composition operators by using the Berezin symbol. We apply their methods for the case of essential norm estimates for positive Toeplitz operators.

Theorem B. *Assume that T_μ is a bounded operator on $L^2_a(\mathcal{U}, dV_\alpha)$. Then, one has*

$$\|T_\mu\|_e \sim \limsup_{z \to \partial\mathcal{U}} \widetilde{\mu}(z) \sim \limsup_{z \to \partial\mathcal{U}} \widehat{\mu}(z),$$

where the notation $\sim$ means that the ratios of the two terms are bounded below and above by constants and $z \to \partial\mathcal{U}$ means that the Euclidean distance of z and $\partial\mathcal{U}$ tends to 0.

Since T_μ is compact if and only $\|T_\mu\|_e = 0$, Theorem B yields the following theorem.

Theorem C. *Let μ be a finite positive Borel measure on $\mathcal{U}$. Then the following conditions are all equivalent.*

(a) *T_μ is a compact operator on $L^2_a(\mathcal{U}, dV_\alpha)$.*
(b) *$\widetilde{\mu}$ tends to* 0 *as $z \to \partial\mathcal{U}$.*
(c) *For all $p > 0$, μ is a vanishing Carleson measure for $L^p_a(\mathcal{U}, dV_\alpha)$.*
(d) *$\widehat{\mu}$ tends to* 0 *as $z \to \partial\mathcal{U}$.*

Let us explain the organization of this paper. Section 2 is preliminaries. In 2.1, we review properties of the weighted Bergman spaces of a minimal bounded homogeneous domain. Proposition 2.2 plays an important role in the lower estimate for $\|T_\mu\|_e$. In 2.2 and 2.3, we recall the definitions of the Berezin symbol, the averaging function, Carleson measures and vanishing Carleson measures. The equivalent conditions of these notions for $L^2(\mathcal{U}, dV_\alpha)$ are considered in [**12**]. In 2.4, we prove the boundedness of the positive Bergman operator $P^+_{\mathcal{U}}$ on $L^2(\mathcal{U}, dV_\alpha)$ by using the boundedness of $P^+_{\mathcal{D}}$, where $\mathcal{D}$ is a Siegel domain biholomorphic to $\mathcal{U}$. The boundedness of $P^+_{\mathcal{D}}$ is given by Békollé and Kagou [**1**]. In Section 3, we show necessary and sufficient conditions for positive Toeplitz operators on the weighted Bergman space of $\mathcal{U}$ to be bounded (Theorem A). By using the boundedness of $P^+_{\mathcal{U}}$, we prove that $T_\mu f$ is in $L^2_a(\mathcal{U}, dV_\alpha)$ for any $f \in L^2_a(\mathcal{U}, dV_\alpha)$ if μ is a Carleson measure (Proposition 3.1). Moreover, we also obtain in the same proposition that the inner product of $T_\mu f$ and g in $L^2_a(\mathcal{U}, dV_\alpha)$ is equal to the inner product of f and g in $L^2_a(\mathcal{U}, d\mu)$. In Section 4, we give upper and lower estimates for the essential norm of T_μ. The weak convergence of the sequence $\{k^{(\alpha)}_z\}_{z\to\partial\mathcal{U}}$ yields the lower estimate for $\|T_\mu\|_e$. On the other hand, $\|T_\mu\|_e$ is less than or equal to the operator norm of $T_\mu - K$ by the definition of the essential norm, where K is an arbitrarily compact operator. We take $K = T_\mu Q_n$, where Q_n is an operator defined from an orthonormal basis of $L^2_a(\mathcal{U}, dV_\alpha)$, so that we have the upper estimate for $\|T_\mu\|_e$ in 4.2.

Throughout the paper, C denotes a positive constant whose value may change from one occurrence to the next one.

2. Preliminaries.

2.1. Weighted Bergman spaces of a minimal bounded homogeneous domain.

It is known that a bounded domain $\mathcal{U} \subset \mathbb{C}^d$ is a minimal domain with center $t \in \mathcal{U}$ if and only if

$$K_{\mathcal{U}}(z, t) = \frac{1}{\mathrm{Vol}\,(\mathcal{U})}$$

for all $z \in \mathcal{U}$ (see [**9**, Theorem 3.1]). For $z \in \mathcal{U}$ and $r > 0$, let

$$B(z, r) := \{w \in \mathcal{U} \mid d_{\mathcal{U}}(z, w) \le r\}$$

be the Bergman metric disk with center z and radius r, where $d_{\mathcal{U}}(\cdot,\cdot)$ denotes the Bergman distance on $\mathcal{U}$.

For $f \in L^2_a(\mathcal{U}, dV_\alpha)$, we write

$$\|f\|_\alpha := \left(\int_{\mathcal{U}} |f(z)|^2 \, dV_\alpha(z)\right)^{1/2}$$

and write $\langle \cdot, \cdot \rangle_\alpha$ for the inner product of $L^2_a(\mathcal{U}, dV_\alpha)$. For any Borel set E in $\mathcal{U}$, we define

$$\mathrm{Vol}_\alpha(E) := \int_E dV_\alpha(w).$$

For $z \in \mathcal{U}$, we denote by $k_z^{(\alpha)}$ the normalized reproducing kernel of $L^2_a(\mathcal{U}, dV_\alpha)$, that is,

$$k_z^{(\alpha)}(w) := \frac{K_{\mathcal{U}}^{(\alpha)}(w, z)}{K_{\mathcal{U}}^{(\alpha)}(z, z)^{1/2}} = \sqrt{C_\alpha} \left(\frac{K_{\mathcal{U}}(w, z)}{K_{\mathcal{U}}(z, z)^{1/2}} \right)^{1+\alpha}.$$

To show the weak convergence of $\{k_z^{(\alpha)}\}$, we first prove the following lemma. Although the proof is the same as the one for the case of other domains (see [**3**], [**5**]), we write it here for the sake of completeness.

LEMMA 2.1. *A sequence of functions $\{f_n\}$ in $L^2_a(\mathcal{U}, dV_\alpha)$ converges to* 0 *weakly in $L^2_a(\mathcal{U}, dV_\alpha)$ if and only if $\{f_n\}$ is bounded in $L^2_a(\mathcal{U}, dV_\alpha)$ and converges to* 0 *uniformly on each compact set of $\mathcal{U}$.*

PROOF. Suppose $\{f_n\}$ converges to 0 weakly in $L^2_a(\mathcal{U}, dV_\alpha)$. It is known that $\{f_n\}$ is norm bounded in $L^2_a(\mathcal{U}, dV_\alpha)$ and converges to 0 pointwise. Take any subsequence of $\{f_n\}$. Then, there exists a subsubsequence of $\{f_n\}$ that converges to 0 uniformly on each compact set of $\mathcal{U}$ by Montel's theorem. Therefore, $\{f_n\}$ itself converges to 0 uniformly on each compact set of $\mathcal{U}$.

On the other hand, suppose $\{f_n\}$ is norm bounded and converges to 0 uniformly on each compact set of $\mathcal{U}$. Take any $\varepsilon > 0$. For any $g \in L^2_a(\mathcal{U}, dV_\alpha)$, there exists a compact set $K \subset \mathcal{U}$ such that

$$\int_{\mathcal{U} \setminus K} |g(z)|^2 \, dV_\alpha(z) < \varepsilon. \tag{2.1}$$

Hence, we have

$$\begin{aligned} |\langle f_n, g \rangle_\alpha| &\le \left| \int_K f_n(z) \overline{g(z)} \, dV_\alpha(z) \right| + \left| \int_{\mathcal{U} \setminus K} f_n(z) \overline{g(z)} \, dV_\alpha(z) \right| \\ &\le \|g\|_\alpha \sup_{z \in K} |f_n(z)| + \|f_n\|_\alpha \int_{\mathcal{U} \setminus K} |g(z)|^2 \, dV_\alpha(z). \end{aligned} \tag{2.2}$$

Since $\{f_n\}$ converges to 0 uniformly on K, there exists an $N \in \mathbb{N}$ such that the first term of (2.2) is less than or equal to $\|g\|_\alpha \varepsilon$ for any $n \geq N$. On the other hand, since $\{f_n\}$ is norm bounded, there exists an $M > 0$ such that $\|f_n\|_\alpha \leq M$. This together with (2.1) tells us that the second term of (2.2) is less than or equal to $M\varepsilon$. Hence, we obtain $\langle f_n, g\rangle_\alpha \to 0$ as $n \to \infty$. This means that $\{f_n\}$ converges to 0 weakly in $L_a^2(\mathcal{U}, dV_\alpha)$. □

PROPOSITION 2.2.

(i) *For all compact set $K \subset \mathcal{U}$, there exists a constant $C > 0$ such that $C^{-1} \leq |K_{\mathcal{U}}^{(\alpha)}(z,w)| \leq C$ for any $z \in \mathcal{U}$ and $w \in K$.*

(ii) *A sequence $\{k_z^{(\alpha)}\}$ converges to 0 uniformly on each compact set of $\mathcal{U}$ as $z \to \partial\mathcal{U}$.*

(iii) *A sequence $\{k_z^{(\alpha)}\}$ converges to 0 weakly in $L_a^2(\mathcal{U}, dV_\alpha)$ as $z \to \partial\mathcal{U}$.*

PROOF. Take a compact set $K \subset \mathcal{U}$. Then, we can find a positive constant ρ satisfying $K \subset B(t,\rho)$. By [**7**, Proposition 6.1], there exists a positive constant M_ρ such that $M_\rho^{-1} \leq |K_{\mathcal{U}}(z,w)| \leq M_\rho$ for any $z \in \mathcal{U}$ and $w \in B(t,\rho)$. This together with (1.1) implies (i). Moreover, there exists a constant $C > 0$ such that

$$\left|k_z^{(\alpha)}(w)\right| = \left|\frac{K_{\mathcal{U}}^{(\alpha)}(w,z)}{K_{\mathcal{U}}^{(\alpha)}(z,z)^{1/2}}\right| \leq \frac{C}{K_{\mathcal{U}}(z,z)^{(1+\alpha)/2}}$$

holds for all $w \in K$ and $z \in \mathcal{U}$ by (i). Since $K_{\mathcal{U}}(z,z) \to \infty$ as $z \to \partial\mathcal{U}$ (see [**8**, Proposition 5.2]), we obtain (ii). Lemma 2.1 tells us the equivalence of (ii) and (iii). □

2.2. The Berezin symbol and the averaging function.

We define a function $\widetilde{\mu}$ on $\mathcal{U}$ by

$$\widetilde{\mu}(z) := \int_{\mathcal{U}} \left|k_z^{(\alpha)}(w)\right|^2 d\mu(w) \quad (z \in \mathcal{U}).$$

The function $\widetilde{\mu}$ is called the Berezin symbol of the measure μ. Since $|K_{\mathcal{U}}(z,w)|$ is a bounded function on $B(t,\rho) \times \mathcal{U}$, $\widetilde{\mu}$ is a continuous function if μ is finite. Fixing $\rho > 0$ once and for all, we set

$$\widehat{\mu}(z) := \frac{\mu(B(z,\rho))}{\mathrm{Vol}_\alpha\,(B(z,\rho))} \quad (z \in \mathcal{U}).$$

We call $\widehat{\mu}$ the averaging function of the measure μ. The dependence of $\widehat{\mu}$ on ρ will not be considered in this paper.

2.3. Carleson measures and vanishing Carleson measures.

We say that μ is a Carleson measure for $L^p_a(\mathcal{U}, dV_\alpha)$ if there exists a constant $M > 0$ such that

$$\int_{\mathcal{U}} |f(z)|^p \, d\mu(z) \leq M \int_{\mathcal{U}} |f(z)|^p \, dV_\alpha(z)$$

for all $f \in L^p_a(\mathcal{U}, dV_\alpha)$. It is easy to see that μ is a Carleson measure for $L^p_a(\mathcal{U}, dV_\alpha)$ if and only if $L^p_a(\mathcal{U}, dV_\alpha) \subset L^p_a(\mathcal{U}, d\mu)$ and the inclusion map

$$i_p : L^p_a(\mathcal{U}, dV_\alpha) \longrightarrow L^p_a(\mathcal{U}, d\mu)$$

is bounded.

Suppose μ is a Carleson measure for $L^p_a(\mathcal{U}, dV_\alpha)$. We say that μ is a vanishing Carleson measure for $L^p_a(\mathcal{U}, dV_\alpha)$ if

$$\lim_{k\to\infty} \int_{\mathcal{U}} |f_k(w)|^p \; d\mu(w) = 0$$

whenever $\{f_k\}$ is a bounded sequence in $L^p_a(\mathcal{U}, dV_\alpha)$ that converges to 0 uniformly on each compact subset of $\mathcal{U}$.

The properties of being a Carleson measure and a vanishing Carleson measure for $L^p_a(\mathcal{U}, dV_\alpha)$ are independent of p (see [**12**, Theorems 3.2 and 3.3]).

2.4. Boundedness of the positive Bergman operator.

In order to prove Theorem A, we use the boundedness of the positive Bergman operator $P^+_{\mathcal{U}}$ on $L^2(\mathcal{U}, dV_\alpha)$ defined by

$$P^+_{\mathcal{U}} g(z) := \int_{\mathcal{U}} \left|K^{(\alpha)}_{\mathcal{U}}(z,w)\right| g(w) \, dV_\alpha(w) \quad (g \in L^2(\mathcal{U}, dV_\alpha)). \tag{2.3}$$

It is known that every bounded homogeneous domain is holomorphically equivalent to a homogeneous Siegel domain (see [**10**]). Let Φ be a biholomorphic map from $\mathcal{U}$ to a Siegel domain $\mathcal{D}$. We define a unitary map U_Φ from $L^2(\mathcal{U}, dV_\alpha)$ to $L^2(\mathcal{D}, K_{\mathcal{D}}(\zeta,\zeta)^{-\alpha} dV(\zeta))$ by

$$U_\Phi f(\zeta) := f(\Phi^{-1}(\zeta)) \left|\det J(\Phi^{-1}, \zeta)\right|^{1+\alpha} \quad (f \in L^2(\mathcal{U}, dV_\alpha)).$$

Then, we have

$$U_\Phi \circ P^+_{\mathcal{U}} = P^+_{\mathcal{D}} \circ U_\Phi.$$

This means that the boundedness of $P_{\mathcal{U}}^{+}$ on $L^2(\mathcal{U}, dV_\alpha)$ is equivalent to the boundedness of $P_{\mathcal{D}}^{+}$ on $L^2(\mathcal{D}, K_{\mathcal{D}}(\zeta,\zeta)^{-\alpha} dV(\zeta))$. On the other hand, Békollé and Kagou proved the boundedness of $P_{\mathcal{D}}^{+}$ ([**1**, Theorem II.7]). Therefore, we have the following lemma.

Lemma 2.3. *The operator $P_{\mathcal{U}}^{+}$ is bounded on $L^2(\mathcal{U}, dV_\alpha)$.*

3. Boundedness of the Toeplitz operator.

In this section, we prove Theorem A. First, we show the following proposition.

Proposition 3.1. *Assume that μ is a Carleson measure. Then $T_\mu f$ is in $L_a^2(\mathcal{U}, dV_\alpha)$ for all $f \in L_a^2(\mathcal{U}, dV_\alpha)$ and one has*

$$\langle T_\mu f, g \rangle_\alpha = \int_{\mathcal{U}} f(w)\overline{g(w)}\, d\mu(w) \tag{3.1}$$

for $f, g \in L_a^2(\mathcal{U}, dV_\alpha)$.

Proof. For $f \in L_a^2(\mathcal{U}, dV_\alpha)$, we have

$$\begin{aligned} \|T_\mu f\|_\alpha^2 &= \int_{\mathcal{U}} \left| \int_{\mathcal{U}} K_{\mathcal{U}}^{(\alpha)}(z,w) f(w)\, d\mu(w) \right|^2 dV_\alpha(z) \\ &\leq \int_{\mathcal{U}} \left(\int_{\mathcal{U}} \left|K_{\mathcal{U}}^{(\alpha)}(z,w)\right| |f(w)|\, d\mu(w) \right)^2 dV_\alpha(z). \end{aligned} \tag{3.2}$$

Since $K_{\mathcal{U}}^{(\alpha)}(\cdot, z) f$ is in $L_a^1(\mathcal{U}, dV_\alpha)$ and μ is a Carleson measure, there exists a positive constant M such that

$$\int_{\mathcal{U}} \left|K_{\mathcal{U}}^{(\alpha)}(z,w)\right| |f(w)|\, d\mu(w) \leq M \int_{\mathcal{U}} \left|K_{\mathcal{U}}^{(\alpha)}(z,w)\right| |f(w)|\, dV_\alpha(w). \tag{3.3}$$

Note that M is independent of z by the definition of the Carleson measure. Therefore, (3.2) and (3.3) yield

$$\|T_\mu f\|_\alpha^2 \leq M^2 \int_{\mathcal{U}} \left(\int_{\mathcal{U}} \left|K_{\mathcal{U}}^{(\alpha)}(z,w)\right| |f(w)|\, dV_\alpha(w) \right)^2 dV_\alpha(z). \tag{3.4}$$

Moreover, the right hand side of (3.4) is equal to $M^2 \left\| P_{\mathcal{U}}^{+} f^{+} \right\|_\alpha^2$, where $f^{+} := |f|$. Hence, the boundedness of $P_{\mathcal{U}}^{+}$ tells us that

$$\|T_\mu f\|_\alpha \le M \left\|P_{\mathcal{U}}^+ f^+\right\|_\alpha \le CM \|f\|_\alpha . \tag{3.5}$$

Clearly, this shows $T_\mu f \in L^2(\mathcal{U}, dV_\alpha)$. Next, we prove $T_\mu f \in \mathcal{O}(\mathcal{U})$. For $g \in L^2(\mathcal{U}, dV_\alpha)$, we have

$$\begin{aligned}\langle T_\mu f, g\rangle_\alpha &= \int_{\mathcal{U}} \left\{ \int_{\mathcal{U}} K_{\mathcal{U}}^{(\alpha)}(z,w) f(w)\, d\mu(w) \right\} \overline{g(z)}\, dV_\alpha(z) \\ &= \int_{\mathcal{U}} \overline{\left\{ \int_{\mathcal{U}} K_{\mathcal{U}}^{(\alpha)}(w,z) g(z)\, dV_\alpha(z) \right\}} f(w)\, d\mu(w) \\ &= \int_{\mathcal{U}} \overline{P_{\mathcal{U}} g(w)} f(w)\, d\mu(w). \end{aligned} \tag{3.6}$$

Note that since

$$\int_{\mathcal{U}} \int_{\mathcal{U}} \left|K_{\mathcal{U}}^{(\alpha)}(w,z) g(z) f(w)\right| d\mu(w) dV_\alpha(z) \le M \left\|P_{\mathcal{U}}^+\right\| \|f\|_\alpha \|g\|_\alpha < \infty, \tag{3.7}$$

the second equality of (3.6) follows from Fubini's theorem. Therefore, the last term of (3.6) is equal to 0 for any $g \in L_a^2(\mathcal{U}, dV_\alpha)^\perp = \ker P_{\mathcal{U}}$. This means that $T_\mu f$ is in $L_a^2(\mathcal{U}, dV_\alpha)$.

In view of (3.6) for the case that g is in $L_a^2(\mathcal{U}, dV_\alpha)$, we obtain (3.1). □

Theorem 3.2 (Theorem A). *Let μ be a positive Borel measure on $\mathcal{U}$. Then the following conditions are all equivalent.*

(a) *T_μ is a bounded operator on $L_a^2(\mathcal{U}, dV_\alpha)$.*
(b) *$\widetilde{\mu}$ is a bounded function on $\mathcal{U}$.*
(c) *For all $p > 0$, μ is a Carleson measure for $L_a^p(\mathcal{U}, dV_\alpha)$.*
(d) *$\widehat{\mu}$ is a bounded function on $\mathcal{U}$.*

Proof. The equivalence of (b), (c) and (d) follows from [**12**, Theorem 3.2]. Moreover, (3.5) yields (c) $\Longrightarrow$ (a). We prove (a) $\Longrightarrow$ (b). Since $T_\mu k_z^{(\alpha)}$ is in $L_a^2(\mathcal{U}, dV_\alpha)$ by (*a*), we have

$$\left\langle T_\mu k_z^{(\alpha)}, k_z^{(\alpha)} \right\rangle_\alpha = \frac{T_\mu k_z^{(\alpha)}(z)}{\sqrt{K_{\mathcal{U}}^{(\alpha)}(z,z)}} = \frac{1}{\sqrt{K_{\mathcal{U}}^{(\alpha)}(z,z)}} \int_{\mathcal{U}} K(z,w) k_z^{(\alpha)}(w)\, d\mu(w) = \widetilde{\mu}(z)$$

by the reproducing property. Hence, we obtain

$$|\widetilde{\mu}(z)| = \left|\left\langle T_\mu k_z^{(\alpha)}, k_z^{(\alpha)} \right\rangle_\alpha\right| \le \|T_\mu\| \|k_z^{(\alpha)}\|_\alpha^2 = \|T_\mu\| < \infty. \qquad \square$$

4. Essential norm estimates for the Toeplitz operator.

In this section, we prove Theorem B. By [**12**, (3.4)], there exists a constant $C > 0$ such that $\widehat{\mu}(z) \leq C\widetilde{\mu}(z)$ holds for all $z \in \mathcal{U}$. Therefore, it is enough to prove

$$\limsup_{z\to\partial\mathcal{U}} \widetilde{\mu}(z) \leq \|T_\mu\|_e \leq C \limsup_{z\to\partial\mathcal{U}} \widehat{\mu}(z).$$

4.1. A lower estimate for the essential norm.

Theorem 4.1. *If T_μ is bounded, one has*

$$\limsup_{z\to\partial\mathcal{U}} \widetilde{\mu}(z) \leq \|T_\mu\|_e .$$

Proof. Take a compact operator K on $L^2_a(\mathcal{U}, dV_\alpha)$ arbitrarily. Since $\{k_z^{(\alpha)}\}$ converges to 0 weakly in $L^2_a(\mathcal{U}, dV_\alpha)$ as $z \to \partial\mathcal{U}$ by Proposition 2.2, we have $\|Kk_z^{(\alpha)}\|_\alpha \to 0$ as $z \to \partial\mathcal{U}$. Hence, we obtain

$$\|T_\mu - K\| \geq \limsup_{z\to\partial\mathcal{U}} \left\|(T_\mu - K)k_z^{(\alpha)}\right\|_\alpha \geq \limsup_{z\to\partial\mathcal{U}} \left\|T_\mu k_z^{(\alpha)}\right\|_\alpha. \tag{4.1}$$

Since (4.1) holds for every compact operator K, it follows that

$$\|T_\mu\|_e \geq \limsup_{z\to\partial\mathcal{U}} \left\|T_\mu k_z^{(\alpha)}\right\|_\alpha. \tag{4.2}$$

On the other hand, since T_μ is bounded, we have

$$\widetilde{\mu}(z) = \left|\langle T_\mu k_z^{(\alpha)}, k_z^{(\alpha)}\rangle_\alpha\right| \leq \left\|T_\mu k_z^{(\alpha)}\right\|_\alpha. \tag{4.3}$$

From (4.2) and (4.3), we complete the proof. □

4.2. An upper estimate for the essential norm.

Suppose $\{e_n\}$ is a complete orthonormal system of $L^2_a(\mathcal{U}, dV_\alpha)$. For $n \in \mathbb{N}$, we define Q_n by

$$Q_n f := \sum_{j=1}^{n} \left\langle f, e_j\right\rangle_\alpha e_j \quad (f \in L^2_a(\mathcal{U}, dV_\alpha)).$$

The operator Q_n is compact on $L^2_a(\mathcal{U}, dV_\alpha)$. Let $R_n := I - Q_n$. It is easy to see that $R_n^* = R_n$ and $R_n^2 = R_n$. Moreover, we have

$$\lim_{n\to\infty} \|R_n f\|_\alpha = 0$$

for each $f \in L^2_a(\mathcal{U}, dV_\alpha)$.

For $r > 0$, let $\mathcal{U}_r := \mathcal{U}\backslash B(t,r)$ and $d\mu_r(z) := \chi_{\mathcal{U}_r}(z)d\mu(z)$, where $\chi_{\mathcal{U}_r}$ is the characteristic function on $\mathcal{U}_r$. Then, we have the following lemma.

LEMMA 4.2. *Assume that T_μ is bounded on $L^2_a(\mathcal{U}, dV_\alpha)$. For any $r > 0$, one has*

$$\lim_{n\to\infty} \sup_{\|f\|_\alpha=1} \|T_\mu R_n f\|_{L^2(d\mu)} \le C \sup_{z\in\mathcal{U}} \widehat{\mu_r}(z), \tag{4.4}$$

$$\lim_{n\to\infty} \sup_{\|f\|_\alpha=1} \|R_n f\|_{L^2(d\mu)} \le C \sup_{z\in\mathcal{U}} \widehat{\mu_r}(z). \tag{4.5}$$

PROOF. Since the proofs of (4.4) and (4.5) are almost the same, we only prove (4.4). First, we show

$$\lim_{n\to\infty} \sup_{\|f\|_\alpha=1} \int_{B(t,r)} |T_\mu R_n f(z)|^2 \, d\mu(z) = 0. \tag{4.6}$$

Since $T_\mu R_n f \in L^2_a(\mathcal{U}, dV_\alpha)$, we obtain

$$|T_\mu R_n f(z)| = \left|\left\langle T_\mu R_n f, K_z^{(\alpha)}\right\rangle_\alpha\right| = \left|\left\langle f, R_n T_\mu^* K_z^{(\alpha)}\right\rangle_\alpha\right| \le \|f\|_\alpha \left\|R_n T_\mu^* K_z^{(\alpha)}\right\|_\alpha,$$

where the first equality follows from the reproducing property. Hence, we have

$$\sup_{\|f\|_\alpha=1} \int_{B(t,r)} |T_\mu R_n f(z)|^2 \, d\mu(z) \le \int_{B(t,r)} \left\|R_n T_\mu^* K_z^{(\alpha)}\right\|_\alpha^2 d\mu(z).$$

Therefore, it is enough to prove

$$\lim_{n\to\infty} \int_{B(t,r)} \left\|R_n T_\mu^* K_z^{(\alpha)}\right\|_\alpha^2 d\mu(z) = 0.$$

This follows from Lebesgue's dominated convergence theorem because

$$\left\|R_n T_\mu^* K_z^{(\alpha)}\right\|_\alpha^2 \le \|T_\mu\|^2 \left\|K_z^{(\alpha)}\right\|_\alpha^2 = \|T_\mu\|^2 K_{\mathcal{U}}^{(\alpha)}(z,z)$$

and $K_{\mathcal{U}}^{(\alpha)}(z,z) \in L^\infty(B(t,r))$.

Next, we prove

$$\sup_{\|f\|_\alpha=1} \int_{\mathcal{U}_r} |T_\mu R_n f(z)|^2 \, d\mu(z) \le C \sup_{z\in\mathcal{U}} \widehat{\mu_r}(z). \tag{4.7}$$

This follows from

$$\begin{aligned}\int_{\mathcal{U}_r} |T_\mu R_n f(z)|^2 \, d\mu(z) &= \int_{\mathcal{U}} |T_\mu R_n f(z)|^2 \, d\mu_r(z) \\ &\le C \int_{\mathcal{U}} \widehat{\mu_r}(z) \, |T_\mu R_n f(z)|^2 \, dV_\alpha(z) \\ &\le C \sup_{z\in\mathcal{U}} \widehat{\mu_r}(z) \, \|T_\mu R_n f\|_\alpha^2 \\ &\le C \, \|T_\mu\|^2 \, \|f\|_\alpha^2 \sup_{z\in\mathcal{U}} \widehat{\mu_r}(z),\end{aligned}$$

where the second inequality follows from [**12**, Lemma 3.1].

We obtain (4.4) from (4.6) and (4.7). □

Referring to the case of a strongly pseudoconvex domain (see [**3**, Lemma 3.2]), we show a relation between $\widehat{\mu_r}$ and $\widehat{\mu}$.

Lemma 4.3. *Assume that T_μ is bounded on $L_a^2(\mathcal{U}, dV_\alpha)$. For any $r > \rho$, one has*

$$\sup_{z\in\mathcal{U}} \widehat{\mu_r}(z) \le C \sup_{z\in\mathcal{U}_{r-\rho}} \widehat{\mu}(z),$$

where C is a positive constant that is independent of r.

Proof. The definition of the averaging function and [**12**, Lemma 2.3] yield

$$\begin{aligned}\widehat{\mu_r}(z) &= \frac{1}{\mathrm{Vol}_\alpha\left(B(z,\rho)\right)} \int_{B(z,\rho)\cap\mathcal{U}_r} d\mu(w) \\ &\le C \int_{B(z,\rho)\cap\mathcal{U}_r} \left|k_z^{(\alpha)}(w)\right|^2 d\mu(w).\end{aligned} \tag{4.8}$$

By [**12**, Lemma 2.5], we have

$$\left|k_z^{(\alpha)}(w)\right|^2 \le \frac{C}{\mathrm{Vol}_\alpha\left(B(w,\rho)\right)} \int_{B(w,\rho)} \left|k_z^{(\alpha)}(u)\right|^2 dV_\alpha(u)$$

for any $w \in B(z,\rho)$. Therefore, the last term of (4.8) is less than or equal to

$$
\begin{aligned}
&C\int_{B(z,\rho)\cap\mathcal{U}_r}\int_{\mathcal{U}}\frac{\chi_{B(w,\rho)}(u)|k_z^{(\alpha)}(u)|^2}{\mathrm{Vol}_\alpha\,(B(w,\rho))}\,dV_\alpha(u)d\mu(w)\\
&= C\int_{\mathcal{U}}\left\{\int_{B(z,\rho)\cap\mathcal{U}_r}\frac{\chi_{B(u,\rho)}(w)}{\mathrm{Vol}_\alpha\,(B(w,\rho))}d\mu(w)\right\}\left|k_z^{(\alpha)}(u)\right|^2 dV_\alpha(u)\\
&\le C\sup_{u\in\mathcal{U}}\left\{\int_{B(z,\rho)\cap\mathcal{U}_r}\frac{\chi_{B(u,\rho)}(w)}{\mathrm{Vol}_\alpha\,(B(w,\rho))}d\mu(w)\right\}\int_{\mathcal{U}}\left|k_z^{(\alpha)}(u)\right|^2 dV_\alpha(u)\\
&= C\sup_{u\in\mathcal{U}}\left\{\int_{B(z,\rho)\cap B(u,\rho)\cap\mathcal{U}_r}\frac{1}{\mathrm{Vol}_\alpha\,(B(w,\rho))}d\mu(w)\right\}. \qquad (4.9)
\end{aligned}
$$

Now we remark that $B(z,\rho)\cap B(u,\rho)\cap\mathcal{U}_r=\emptyset$ for any $u\in B(t,r-\rho)$. Indeed, if $w\in B(z,\rho)\cap B(u,\rho)\cap\mathcal{U}_r$, we have $d_{\mathcal{U}}(w,u)\le\rho$ and $d_{\mathcal{U}}(t,w)>r$, so that

$$d_{\mathcal{U}}(t,u)\ge d_{\mathcal{U}}(t,w)-d_{\mathcal{U}}(u,w)>r-\rho,$$

which implies $u\notin B(t,r-\rho)$. Therefore, the last term of (4.9) is equal to

$$
\begin{aligned}
&C\sup_{u\notin B(t,r-\rho)}\left\{\int_{B(z,\rho)\cap B(u,\rho)\cap\mathcal{U}_r}\frac{1}{\mathrm{Vol}_\alpha\,(B(w,\rho))}d\mu(w)\right\}\\
&\le C\sup_{u\in\mathcal{U}_{r-\rho}}\frac{1}{\mathrm{Vol}_\alpha\,(B(u,\rho))}\left\{\int_{B(z,\rho)\cap B(u,\rho)\cap\mathcal{U}_r}d\mu(w)\right\}\\
&\le C\sup_{u\in\mathcal{U}_{r-\rho}}\widehat{\mu}(u).
\end{aligned}
$$

Hence, we complete the proof. □

Theorem 4.4. *If T_μ is bounded, one has*

$$\|T_\mu\|_e\le C\limsup_{z\to\partial\mathcal{U}}\widehat{\mu}(z).$$

Proof. Take any $n\in\mathbb{N}$. Since Q_n is compact, $T_\mu Q_n$ is also compact. Therefore, we have $\|T_\mu\|_e\le\|T_\mu-T_\mu Q_n\|=\|T_\mu R_n\|$. Proposition 3.1 yields

$$\|T_\mu R_n f\|_\alpha^2\le\|R_n f\|_{L^2(d\mu)}\|T_\mu R_n f\|_{L^2(d\mu)}.$$

Therefore, we have

$$\|T_\mu\|_e^2 \le \|T_\mu R_n\|^2 \le \sup_{\|f\|_\alpha=1} \|R_n f\|_{L^2(d\mu)} \sup_{\|f\|_\alpha=1} \|T_\mu R_n f\|_{L^2(d\mu)} .$$

Take $n \to \infty$. Then Lemmas 4.2 and 4.3 give us

$$\|T_\mu\|_e^2 \le C\left(\sup_{z\in\mathcal{U}} \widehat{\mu_r}(z)\right)^2 \le C\left(\sup_{z\in\mathcal{U}_{r-\rho}} \widehat{\mu}(z)\right)^2 .$$

Letting $r \to \infty$, we get

$$\|T_\mu\|_e \le C \limsup_{z\to\partial\mathcal{U}} \widehat{\mu}(z). \qquad \square$$

Theorem B follows from Theorems 4.1 and 4.4. Applying Theorem B, we obtain necessary and sufficient conditions for T_μ to be compact.

Corollary 4.5 (Theorem C). *Let μ be a finite positive Borel measure on $\mathcal{U}$. Then the following conditions are all equivalent.*

(a) *T_μ is a compact operator on $L^2_a(\mathcal{U}, dV_\alpha)$.*
(b) *$\widetilde{\mu}(z) \to 0$ as $z \to \partial\mathcal{U}$.*
(c) *For all $p > 0$, μ is a vanishing Carleson measure for $L^p_a(\mathcal{U}, dV_\alpha)$.*
(d) *$\widehat{\mu}(z) \to 0$ as $z \to \partial\mathcal{U}$.*

Proof. For the implication (b) $\iff$ (c) $\iff$ (d), see [**12**, Theorem 3.3]. Let us prove (a) $\iff$ (b). Assume that T_μ is compact, so that $\|T_\mu\|_e = 0$. Since T_μ is bounded, we can apply Theorem B to obtain $\limsup_{z\to\partial\mathcal{U}} \widetilde{\mu}(z) = 0$, which is equivalent to (b). On the contrary, if we assume (b), then $\widetilde{\mu}$ is bounded because $\widetilde{\mu}$ is continuous for the finite measure μ (Section 2.2). Thus T_μ is bounded by Theorem A. Therefore, Theorem B tells us the compactness of T_μ. $\square$

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Satoshi Yamaji
Graduate School of Mathematics
Nagoya University
Chikusa-ku
Nagoya 464-8602, Japan
E-mail: satoshi.yamaji@math.nagoya-u.ac.jp