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Resolvent estimates in amalgam spaces and asymptotic expansions for Schrödinger equations

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Abstract. We consider Schrödinger equations $i\partial_t u = (-\Delta + V)u$ in $\mathbb{R}^3$ with a real potential V such that, for an integer $k \geq 0$, $\langle x \rangle^k V(x)$ belongs to an amalgam space $\ell^p(L^q)$ for some $1 \leq p < 3/2 < q \leq \infty$, where $\langle x \rangle = (1+|x|^2)^{1/2}$. Let $H = -\Delta + V$ and let P_{ac} be the projector onto the absolutely continuous subspace of $L^2(\mathbb{R}^3)$ for H. Assuming that zero is not an eigenvalue nor a resonance of H, we show that solutions $u(t) = \exp(-itH)P_{ac}\varphi$ admit asymptotic expansions as $t \to \infty$ of the form

$$\left\| \langle x \rangle^{-k-\varepsilon} \left(u(t) - \sum_{j=0}^{[k/2]} t^{-\frac{3}{2}-j} P_j \varphi \right) \right\|_\infty \leq C|t|^{-\frac{k+3+\varepsilon}{2}} \left\| \langle x \rangle^{k+\varepsilon} \varphi \right\|_1$$

for $0 < \varepsilon < 3(1/p - 2/3)$, where $P_0, \ldots, P_{[k/2]}$ are operators of finite rank and $[k/2]$ is the integral part of $k/2$. The proof is based upon estimates of boundary values on the reals of the resolvent $(-\Delta - \lambda^2)^{-1}$ as an operator-valued function between certain weighted amalgam spaces.

1. Introduction.

We consider the large time behavior of solutions of Schrödinger equations in $\mathbb{R}^3$:

$$i\partial_t u = Hu, \quad H = -\Delta + V(x). \tag{1.1}$$

We assume that V is real valued and belongs to the amalgam space

$$V \in \ell^p(L^q) \ \text{ for some } 1 \leq p < 3/2 < q \leq \infty. \tag{1.2}$$

Here, $\ell^r(L^s)$ for $1 \leq r,\ s \leq \infty$ is the space of functions which behave locally like L^s and globally like L^r-functions, and is defined by

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$$\ell^r(L^s) = \left\{ u \colon \|u\|_{\ell^r(L^s)} \equiv \left(\sum_{j\in\mathbb{Z}^3} \|\chi_{Q_j} u\|_s^r \right)^{1/r} < \infty \right\},$$

where Q_j is the unit cube centered at $j \in \mathbb{Z}^3$, χ_{Q_j} its characteristic function and $\|\cdot\|_s$ is the norm of Lebesgues space $L^s(\mathbb{R}^3)$: $\|u\|_s = \left(\int_{\mathbb{R}^3} |u(x)|^s dx \right)^{1/s}$.

Amalgam spaces are Banach spaces and satisfy inclusion relations

$$\ell^{r_1}(L^{s_1}) \subset \ell^{r_2}(L^{s_2}) \quad \text{for } r_1 \le r_2,\ s_1 \ge s_2. \tag{1.3}$$

It follows that $\ell^r(L^s) \subset L^r(\mathbb{R}^3) \cap L^s(\mathbb{R}^3)$ if $1 \le r \le s \le \infty$ and, we have

$$V \in L^p(\mathbb{R}^3) \cap L^q(\mathbb{R}^3) \quad \text{for some } 1 \le p < 3/2 < q \le \infty \tag{1.4}$$

under the assumption (1.2). Then, it is well known that:

(1) The operator $H = -\Delta + V$ is selfadjoint in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3)$ with domain $D(H) = \{u \in L^2 \colon Vu \in L^1_{\text{loc}}, -\Delta u + Vu \in L^2\}$ and the solution $u(t)$ in $\mathcal{H}$ of (1.1) with initial condition $u(0) = \varphi \in \mathcal{H}$ is uniquely given by $u(t) = e^{-itH}\varphi$.

(2) The spectrum of H consists of a finite number of non-positive eigenvalues of finite multiplicities and a purely absolutely continuous part $[0,\infty)$. Embedded positive eigenvalues are absent ([**10**]).

Definition 1.1. We say that H is of *generic type* if 0 is neither eigenvalue nor resonance of H, viz. there are no non-trivial solutions of $-\Delta u(x) + V(x)u(x) = 0$ which satisfy $u \in L^s(\mathbb{R}^3)$ for all $3 < s \le \infty$; and it is of *exceptional type* if otherwise.

In this paper, we show the following theorem. For $a \ge 0$, $[a]$ is the integral part of a; $\langle x\rangle = (1+|x|^2)^{1/2}$; P_{ac} is the orthogonal projection onto the absolutely continuous subspace of $\mathcal{H}$ for H.

Theorem 1.2. *Suppose that $V(x)$ satisfy for an integer $k \ge 0$ and some $1 \le p < 3/2 < q \le \infty$ that*

$$\langle x\rangle^k V \in \ell^p(L^q), \tag{1.5}$$

and that H is of generic type. Then, for any $0 \le \varepsilon < 3(1/p - 2/3)$, $e^{-itH}P_{ac}$ admits an expansion as $t \to \infty$ of the following form:

$$\left\| \langle x\rangle^{-k-\varepsilon} \left(e^{-itH} P_{ac} - \sum_{j=0}^{[k/2]} t^{-3/2-j} P_j \right) \varphi \right\|_\infty \le C|t|^{-(k+3+\varepsilon)/2} \left\| \langle x\rangle^{k+\varepsilon} \varphi \right\|_1, \quad (1.6)$$

where $P_0, \dots, P_{[k/2]}$ are finite rank operators which may be expressed via derivatives of $(H-\lambda^2)^{-1}$ at $\lambda = 0$, and C is a positive constant independent of t and φ. Similar statement holds as $t \to -\infty$.

Remark 1.3. (1) The case when V satisfies (1.5) with non-integral k is partly covered by the theorem: If V satisfies $\langle x\rangle^{k+\sigma} V \in L^p(L^q)$ for $0 < \sigma < 1$, then $\langle x\rangle^k V \in \ell^{\tilde p}(L^q)$ for any $\tilde p \ge 1$ such that $1/\tilde p < (1/p) + (\sigma/3)$, and the range of allowed ε in the theorem is widened by σ to $0 \le \varepsilon < \min(1, 3(1/p - 2/3) + \sigma)$.

(2) For the free Schrödinger operator $H = -\Delta$, (1.6) is well known and the decay rate $C|t|^{-(k+3+\varepsilon)/2}$ for the weight $\langle x\rangle^{k+\varepsilon}$ cannot be improved.

When V satisfies (1.4) and H is of generic type, Goldberg [**7**] has recently proved that solutions $u(t) = e^{-itH} P_{ac}\varphi$ of (1.1) satisfy dispersive estimate:

$$\left\| e^{-itH} P_{ac}\varphi \right\|_\infty \le C|t|^{-3/2} \|\varphi\|_1, \quad \varphi \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3). \quad (1.7)$$

Theorem 1.2 singles out the leading terms in (1.7) of $u(t,x)$ as $t \to \infty$ under a slightly stronger condition. We refer to [**12**], [**21**], [**18**], [**8**], [**3**], [**22**], [**4**], [**5**] for earlier works on dispersive estimates in three dimensions.

Theorem 1.2 is an extension of the well-known results of Rauch ([**17**]), Jensen-Kato ([**11**]), Murata ([**16**]) and etc. on asymptotic expansions in the weighted L^2 spaces, when spatial dimension is three and H is of generic type. However, results of these authors are not only for generic H but also for H of exceptional type, and for general dimensions $d \ge 1$. We plan to study these cases in a future investigation. See also [**6**] for corresponding results for time periodic potentials. We mention that, in one dimension, the expansion of the form (1.6) has recently been obtained by H. Mizutani ([**15**]) when V satisfies $\langle x\rangle^{-k} V \in L^1(\mathbb{R}^1)$ for an integer $k \ge 2$.

The rest of the paper is devoted to the proof of Theorem 1.2 and *we always assume that H is of generic type.* We write $R_0(z) = (H_0 - z)^{-1}$ and $R(z) = (H-z)^{-1}$ for resolvents and set $G_0(\lambda) = R_0(\lambda^2)$ and $G(\lambda) = R(\lambda^2)$ for $\lambda \in \overline{\mathbb{C}}^+$, $\overline{\mathbb{C}}^+ = \{\lambda \in \mathbb{C}\colon \Im\lambda \ge 0\}$ being the closed upper half plane. For boundary values on the reals, we have

$$G_0(\pm\lambda) = R_0(\lambda^2 \pm i0), \quad G(\pm\lambda) = R(\lambda^2 \pm i0), \quad \lambda \ge 0.$$

The proof is a modification of Goldberg's argument for proving the dispersive estimate (1.7) which heavily relies on the special feature in three dimensions that the resolvent $G_0(\lambda)$ has a simple convolution kernel

$$G_0(\lambda)u(x) = \frac{1}{4\pi}\int \frac{e^{i\lambda|x-y|}}{|x-y|}u(y)dy. \tag{1.8}$$

The modification requires in particular new estimates on boundary values on the reals of the weighted free resolvent $\langle x\rangle^{\varepsilon}G_0(\lambda)\langle x\rangle^{-\varepsilon}$, $\varepsilon>0$, and its derivatives with respect to λ as an operator-valued function between certain amalgam spaces which improve well known L^p-estimates by Kenig-Ruiz-Sogge ([**13**]) and Goldberg-Schlag ([**9**]). We formulate and prove them in Section 2.

We explain here the basic strategy of the proof for the case $k=0$, introducing some notation and pointing out what modifications are necessary to Goldberg's argument. Weighted L^r spaces are denoted by $\langle x\rangle^{-s}L^r = \{\langle x\rangle^{-s}u \colon u\in L^r\}$. The coupling of u in a function space and v in its dual space, i.e. for $u\in\mathcal{S}$ and $v\in\mathcal{S}'$, is indiscriminately denoted by $\langle u,v\rangle$:

$$\langle u,v\rangle = \int_{\mathbb{R}^3} u(x)\overline{v(x)}dx.$$

We reserve notation χ for a cut off function $\chi\in C_0^\infty(\mathbb{R})$ such that

$$\chi(\lambda)=\chi(-\lambda);\ \chi(\lambda)=\begin{cases}1 \text{ for } |\lambda|\le 1,\\ 0 \text{ for } |\lambda|\ge 2\end{cases},\quad \sum_{n=0}^{\infty}\chi(\lambda-3n)=1 \tag{1.9}$$

and, for $L>0$, $\chi_L(\lambda)=\chi(\lambda/L)$. We use resolvent equations in two ways:

$$G(\lambda) = (1+G_0(\lambda)V)^{-1}G_0(\lambda) = G_0(\lambda)(1+VG_0(\lambda))^{-1}. \tag{1.10}$$

By virtue of the assumption that H is of generic type, we may write e^{-itH} by using boundary values of $G(\lambda)$ on the reals. We then apply integration by parts with respect to λ and obtain for $u,v\in\mathcal{S}(\mathbb{R}^3)$ that

$$\begin{aligned}\left(e^{-itH}P_{ac}u,v\right) &= \lim_{L\to\infty}\frac{1}{i\pi}\int_{\mathbb{R}} e^{-it\lambda^2}(G(\lambda)u,v)\lambda\chi_L(\lambda)d\lambda \\ &= -\frac{1}{2t\pi}\lim_{L\to\infty}\left(\int_{\mathbb{R}} e^{-it\lambda^2}(G'(\lambda)u,v)\chi_L(\lambda)d\lambda \right.\\ &\qquad\left. + \int_{\mathbb{R}} e^{-it\lambda^2}(G(\lambda)u,v)\chi_L'(\lambda)d\lambda\right)\end{aligned}$$

$$\equiv \lim_{L\to\infty} \big((U_{1,L}(t)u,v) + (U_{2,L}(t)u,v)\big). \tag{1.11}$$

It suffices to show that $U_{1,L}(t)$ satisfies (1.6) and that $U_{2,L}(t)$ does

$$\big\|\langle x\rangle^{-k-\varepsilon} U_{2,L}\varphi\big\|_\infty \le C|t|^{-(k+3+\varepsilon)/2}\big\|\langle x\rangle^{k+\varepsilon}\varphi\big\|_1 \tag{1.12}$$

with $C>0$ independent of $L\ge R$, $R>0$ being a large constant. We explain here how to do this only for $U_{1,L}(t)$. We write, following Goldberg ([**7**]),

$$G'(\lambda) = 2\lambda G(\lambda)^2 = (1+G_0(\lambda)V)^{-1}G_0'(\lambda)(1+VG_0(\lambda))^{-1} \tag{1.13}$$

and define for $w\in\mathcal{S}(\mathbb{R}^3)$ and $L\ge 1$:

$$w_L(\lambda,\cdot) = \chi(\lambda/2L)(1+VG_0(\lambda))^{-1}w(x). \tag{1.14}$$

Since $((1+G_0(\lambda)V)^{-1})^* = (1+VG_0(-\lambda))^{-1}$, we have for $u,v\in\mathcal{S}(\mathbb{R}^3)$ that

$$(U_{1,L}(t)u,v) = -\frac{1}{2t\pi}\int_{\mathbb{R}} e^{-it\lambda^2}\big\langle \chi(\lambda/L)G_0'(\lambda)u_L(\lambda), v_L(-\lambda)\big\rangle d\lambda. \tag{1.15}$$

For the partial Fourier transform $\hat u_L(\rho,x) = (\mathcal{F}_{\lambda\to\rho}u_L)(\rho,x)$, it is proven in Theorem 4 of [**7**] that $\|\hat u_L\|_{L^1(\mathbb{R}^4_{\lambda,x})} \le C\|u\|_{L^1(\mathbb{R}^3)}$ which immediately implies dispersive estimates (as can be seen below). We improve that estimate to the following weighted ones:

Lemma 1.4. *Let ε be as in Theorem* 1.2. *Then, $\hat u_L(\rho,x)$ satisfies*

$$\big\|(\langle\rho\rangle^\varepsilon + \langle x\rangle^\varepsilon)\hat u_L(\rho,x)\big\|_{L^1(\mathbb{R}^4)} \le C\big\|\langle x\rangle^\varepsilon u\big\|_{L^1(\mathbb{R}^3)}, \tag{1.16}$$

where the constant C is independent of $u\in\langle x\rangle^{-\varepsilon}L^1(\mathbb{R}^3)$ and $L\ge 1$.

Theorem 1.2 for $k=0$ will follow from Lemma 1.4 as follows. We define

$$F_L(\sigma) = \mathcal{F}_{\lambda\to\sigma}\big(\langle\chi(\lambda/L)G_0'(\lambda)u_L(\lambda), v_L(-\lambda)\rangle\big)(\sigma).$$

Since $G_0'(\lambda)$ has the kernel $ie^{i\lambda|x-y|}/4\pi$, we may compute as

$$F_L(\sigma) = \frac{i}{2(2\pi)^{3/2}}\iint L\hat\chi(L(\sigma-|x-y|-\mu-\rho))\hat u_L(\mu,y)\overline{\hat v_L(\rho,x)}d\mu d\rho dx dy.$$

It follows from elementary Lemma 3.5 below and Lemma 1.4 above that

$$
\begin{aligned}
&\int_{\mathbb{R}^1} \langle\sigma\rangle^{\varepsilon} |F_L(\sigma)| d\sigma \\
&\quad \le C \iint_{\mathbb{R}^8} \langle |x-y| + \mu + \rho\rangle^{\varepsilon} |\hat{u}_L(\mu, y)\hat{v}_L(\rho, x)| d\mu d\rho dx dy \\
&\quad \le C \big\| (\langle\mu\rangle^{\varepsilon} + \langle y\rangle^{\varepsilon})\hat{u}_L \big\|_1 \big\| (\langle\rho\rangle^{\varepsilon} + \langle x\rangle^{\varepsilon})\hat{v}_L \big\|_1 \le C \|\langle x\rangle^{\varepsilon} u\|_1 \|\langle x\rangle^{\varepsilon} v\|_1. \qquad (1.17)
\end{aligned}
$$

The Parseval identity and the identity $e^{i\sigma^2/4t} = 1 + (e^{i\sigma^2/4t} - 1)$ imply

$$
\langle U_{1,L}(t)u, v\rangle = \frac{e^{\mp 3i\pi/4}}{(2|t|)^{3/2} i\pi} \bigg(\int_{\mathbb{R}} F_L(\sigma) d\sigma + \int_{\mathbb{R}} \big(e^{i\sigma^2/4t} - 1\big) F_L(\sigma) d\sigma \bigg). \qquad (1.18)
$$

The Fourier inversion formula shows that the first term on the right yields the leading term

$$
\frac{\sqrt{2}e^{\mp 3i\pi/4}}{(2|t|)^{3/2} i\sqrt{\pi}} \langle G_0'(0)u_L(0), v_L(0)\rangle = \frac{\sqrt{2}e^{\mp 3i\pi/4}}{(2|t|)^{3/2} i\sqrt{\pi}} \langle G'(0)u, v\rangle \qquad (1.19)
$$

and (1.17) implies that the second is a remainder:

$$
\frac{2^{1-\varepsilon}}{(2|t|)^{(3/2)+(\varepsilon/2)}\pi} \int_{\mathbb{R}} |\sigma|^{\varepsilon} |F_L(\sigma)| d\sigma \le \frac{C}{|t|^{(3/2)+(\varepsilon/2)}} \|\langle x\rangle^{\varepsilon} u\|_1 \|\langle x\rangle^{\varepsilon} v\|_1
$$

with a constant C independent of L. We prove Lemma 1.4 by splitting $u_L(\lambda)$ into the large and the small energy parts:

$$
u_{L,high}(\lambda) = \chi_{\ge}(\lambda/\lambda_0) u_L(\lambda), \quad u_{L,low}(\lambda) = \chi(\lambda/\lambda_0) u_L(\lambda), \qquad (1.20)
$$

where $\chi_{\ge}(\lambda) = 1 - \chi(\lambda)$ and $\lambda_0 > 0$ is a large parameter. We prove a generalization of Lemma 1.4 for derivatives $u_{L,high}^{(j)}(\lambda)$ and $u_{L,low}^{(j)}(\lambda)$ in Sections 3 and 4 respectively by modifying the argument of Goldberg via estimates of Section 2. Here and hereafter $f^{(j)}(\lambda)$ is the j-th derivative of $f(\lambda)$. For Banach spaces X and Y, $\mathbf{B}(X, Y)$ is the Banach space of bounded operators from X to Y. $\mathbf{B}(X) = \mathbf{B}(X, X)$.

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2. Resolvent estimates in weighted amalgam spaces.

In this section, we improve the well-known L^p estimates in [**13**] and [**9**] on boundary values on the reals of the free resolvent $G_0(\lambda)$ to the weighted estimates in amalgam spaces. We write q' for the dual exponent of q: $1/q + 1/q' = 1$ and $1 \le q,\ q' \le \infty$. We recall the complex interpolation for amalgam spaces: For $0 < \theta < 1$ we have

$$[\ell^{r_0}(L^{s_0}), \ell^{r_1}(L^{s_1})]_\theta = \ell^{r_\theta}(L^{s_\theta}), \quad \frac{1}{r_\theta} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}, \quad \frac{1}{s_\theta} = \frac{1-\theta}{s_0} + \frac{\theta}{s_1}.$$

Weak spaces $\ell^r_w(L^s)$, $\ell^r(L^s_w)$ and $\ell^r_w(L^s_w)$ are likewise defined. If we denote Lorentz spaces by $L^{p,q}$, $L^q_w = L^{q,\infty}$ and it is the dual space of $L^{q',1}$. The real interpolation theorems apply to $L^{p,q}$ and, hence, to amalgam spaces built over them. For $1 < p_j, q_j < \infty$, $j = 1,2,3$ such that $1/p_1 + 1/p_2 = 1 + 1/p_3$ and $1/q_1 + 1/q_2 = 1 + 1/q_3$ we have weak Young's inequality for amalgam spaces:

$$\ell^{p_1}_w(L^{q_1}_w) * \ell^{p_2}(L^{q_2}) \subset \ell^{p_3}(L^{q_3}). \tag{2.1}$$

We recall that the norm of the space L^r_w for $1 < r \le \infty$ may be defined by

$$\|u\|_{r,w} = \sup_A \frac{1}{|A|^{1/r'}} \int_A |u(x)|dx \tag{2.2}$$

and L^r_w becomes a Banach space, where A runs over measurable subsets with finite measures (see p. 99 of [**14**]).

For $0 \le \varepsilon \le 1$ and $\lambda \in \overline{\mathbb{C}}^+$, we define the weighted resolvent:

$$G_{0,\pm\varepsilon}(\lambda) = \langle x\rangle^{\pm\varepsilon} G_0(\lambda) \langle x\rangle^{\mp\varepsilon}$$

and we denote its integral kernel by

$$G_{0,\pm\varepsilon}(\lambda, x, y) = \frac{1}{4\pi} \frac{\langle x\rangle^{\pm\varepsilon} e^{i\lambda|x-y|} \langle y\rangle^{\mp\varepsilon}}{|x-y|}. \tag{2.3}$$

In the $(1/r, 1/s)$-plane $\Omega = (0,1) \times (0,1)$ is the *open* unit cube. We set

$$L(\mu) = \{(1/r, 1/s) \in \Omega\colon 1/r - 1/s = (2+\mu)/3\},$$

for $0 \le \mu \le 1$. The Hardy-Littlewood-Sobolev inequality, which is a special case of (2.1), implies

$$\||x|^{-1+\mu} * u\|_s \le C\|u\|_r, \quad (1/r, 1/s) \in L(\mu). \tag{2.4}$$

For $\sigma > 0$, $\Delta^\sigma f(\lambda) = f(\lambda+\sigma) - f(\lambda)$ is the difference operator.

Lemma 2.1. *Let $0 \le \varepsilon \le 1$ and $0 \le \gamma \le 1-\varepsilon$. Then $G_{0,\pm\varepsilon}(\lambda)$ satisfies following properties as an operator valued function of $\lambda \in \mathbb{C}^+$, where $\ell^{1,1}$ should be replaced by ℓ^1 whenever it appears*:

(1) *For $(1/r, 1/s) \in L(0)$ and $(1/\tilde r, 1/\tilde s) \in L(\varepsilon+\gamma)$, it is analytic and bounded as a $\mathbf{B}(\ell^{\tilde r}(L^r), \ell^{\tilde s}(L^s))$-valued function. If $\gamma > 0$, it extends to $\overline{\mathbb{C}}^+$ as a Hölder continuous function of order γ, and if $\gamma = 0$, it does so as a strongly continuous bounded function. There exists a constant $C > 0$ such that, for $\lambda \in \overline{\mathbb{C}}^+$, $\sigma \in \mathbb{R}\setminus\{0\}$ and $u \in \ell^{\tilde r}(L^r)$,*

$$\|G_{0,\pm\varepsilon}(\lambda)u\|_{\ell^{\tilde s}(L^s)} + |\sigma|^{-\gamma}\|(\Delta^\sigma G_{0,\pm\varepsilon})(\lambda)u\|_{\ell^{\tilde s}(L^s)} \le C\|u\|_{\ell^{\tilde r}(L^r)}. \tag{2.5}$$

(2) *If $(1/r, 1/s) \in \overline{L(0)}$ and $(1/\tilde r, 1/\tilde s) \in \overline{L(\gamma+\varepsilon)}$ are both at the ends of respective line segments, then it satisfies the properties of* (1) *as a function with values in any of the following spaces, where strong continuity should be replaced by weak continuity if $\varepsilon = \gamma = 0$ and if the space is one of those in the first line* (2.6) :

$$\mathbf{B}(L^1, \ell_w^{3/(1-\varepsilon-\gamma)}(L^3_w)), \qquad \mathbf{B}(\ell^1(L^{3/2,1}), \ell_w^{3/(1-\varepsilon-\gamma)}(L^\infty)), \tag{2.6}$$

$$\mathbf{B}(\ell^{3/(2+\varepsilon+\gamma),1}(L^{3/2,1}), L^\infty), \qquad \mathbf{B}(\ell^{3/(2+\varepsilon+\gamma),1}(L^1), \ell^\infty(L^3_w)). \tag{2.7}$$

(3) *If one and only one of $(1/r, 1/s) \in \overline{L(0)}$ or $(1/\tilde r, 1/\tilde s) \in \overline{L(\gamma+\varepsilon)}$ is at the ends, then, it satisfies the properties of* (1) *as a function with values in any of the following spaces, where the same comment as in* (2) *applies if $\varepsilon = \gamma = 0$ and if the space is the first of* (2.8) :

$$\mathbf{B}(\ell^1(L^r), \ell_w^{3/(1-\varepsilon-\gamma)}(L^s)), \qquad \mathbf{B}(\ell^{3/(2+\varepsilon+\gamma),1}(L^r), \ell^\infty(L^s)), \tag{2.8}$$

$$\mathbf{B}(\ell^{\tilde r}(L^{3/2,1}), L^{\tilde s}(L^\infty)), \qquad \mathbf{B}(\ell^{\tilde r}(L^1), \ell^{\tilde s}(L^3_w)). \tag{2.9}$$

Proof. In this proof we often omit the variable λ from $G_{0,\pm\varepsilon}$ and etc. The kernel $G_{0,\pm\varepsilon}(\lambda, x, y)$ is entire with respect to λ for $x \ne y$ and it satisfies along with derivatives that

$$\left|\partial_\lambda^j G_{0,\pm\varepsilon}(\lambda, x, y)\right| \le C_j \langle x-y\rangle^\varepsilon |x-y|^{j-1} e^{-\Im\lambda|x-y|}, \quad j = 0, 1, \ldots. \tag{2.10}$$

Hence, the stated analyticity of $G_{0,\pm\varepsilon}(\lambda)$ follows by weak-Young's inequality (2.1). We next prove the estimate (2.5) corresponding to the cases in (2), viz. we assume that $(1/r, 1/s)$ and $(1/\tilde{r}, 1/\tilde{s})$ are end points and prove (2.5) with $\ell^{\tilde{r}}(L^r)$ and $\ell^{\tilde{s}}(L^s)$ being appropriately replaced by the function spaces in (2.6) and (2.7). We set

$$\begin{aligned} G_{0,\pm\varepsilon,\le}(\lambda,x,y) &= \chi(|x-y|)G_{0,\pm\varepsilon}(\lambda,x,y),\\ G_{0,\pm\varepsilon,\ge}(\lambda,x,y) &= (1-\chi(|x-y|))G_{0,\pm\varepsilon,\lambda}(x,y) \end{aligned}$$

and denote by $G_{0,\pm\varepsilon,\le}(\lambda)$ and $G_{0,\pm\varepsilon,\ge}(\lambda)$ the operators with respective integral kernels. We have

$$|G_{\lambda,\pm\varepsilon,\le}(x,y)| \le C\chi(|x-y|)/|x-y|. \tag{2.11}$$

Then, the characterization (2.2) of L^r_w spaces and the duality argument imply

$$\|G_{0,\pm\varepsilon,\le}(\lambda)u\|_{L^3_w} \le C\|u\|_{L^1}, \quad \|G_{0,\pm\varepsilon,\le}(\lambda)u\|_{L^\infty} \le C\|u\|_{L^{3/2,1}}. \tag{2.12}$$

It follows from the first of (2.12) and the inclusion relation (1.3) that

$$\|G_{0,\pm\varepsilon,\le}u\|_{\ell_w^{3/(1-\varepsilon)}(L^3_w)} \le C\|G_{0,\pm\varepsilon,\le}u\|_{L^3_w} \le C\|u\|_1. \tag{2.13}$$

Since $G_{\lambda,\pm\varepsilon,\le}(x,y)$ is supported by $\{|x-y|\le 2\}$, the second of (2.12) implies

$$\sum_{j\in\mathbb{Z}^3}\|G_{0,\pm\varepsilon,\le}u\|_{L^\infty(Q_j)} \le C\sum_{j\in\mathbb{Z}^3}\|u\|_{L^{3/2,1}(5Q_j)} \le 125C\sum_{j\in\mathbb{Z}^3}\|u\|_{L^{3/2,1}(Q_j)} \tag{2.14}$$

where $5Q_j$ is the cube of sides 5 concentric with Q_j, hence, from (1.3),

$$\|G_{0,\pm\varepsilon,\le}u\|_{\ell^{3/(1-\varepsilon)}(L^\infty)} \le C\|u\|_{\ell^1(L^{3/2,1})}. \tag{2.15}$$

The kernel $G_{0,\pm\varepsilon,\ge}(\lambda,x,y)$ is smooth with respect to (x,y) and satisfies

$$\sup_{x,y\in\mathbb{R}^3}\left|\chi_{Q_j}(x)G_{0,\pm\varepsilon,\ge}(\lambda,x,y)\chi_{Q_k}(y)\right| \le C\langle j-k\rangle^{-1+\varepsilon}. \tag{2.16}$$

Hence, if $F_{j,k}$ is the operator with kernel $\chi_{Q_j}(x)G_{0,\pm\varepsilon,\ge}(\lambda,x,y)\chi_{Q_k}(y)$,

$$\|G_{0,\pm\varepsilon,\ge}(\lambda)u\|_{L^3_w(Q_j)} \le \sum_k\|F_{j,k}\chi_{Q_k}u\|_{L^3(Q_j)} \le \sum_k\langle j-k\rangle^{-1+\varepsilon}\|u\|_{L^1(Q_k)},$$

$$\|G_{0,\pm\varepsilon,\geq}(\lambda)u\|_{L^\infty(Q_j)} \leq \sum_k \|F_{j,k}\chi_{Q_k}u\|_{L^\infty(Q_j)} \leq \sum_k \langle j-k\rangle^{-1+\varepsilon}\|u\|_{L^{3/2,1}(Q_k)}.$$

It follows by virtue of the discrete version of (2.2) that

$$\|G_{0,\pm\varepsilon,\geq}(\lambda)u\|_{\ell_w^{3/(1-\varepsilon)}(L_w^3)} \leq C\|u\|_1, \tag{2.17}$$

$$\|G_{0,\pm\varepsilon,\geq}(\lambda)u\|_{\ell_w^{3/(1-\varepsilon)}(L^\infty)} \leq C\|u\|_{\ell^1(L^{3/2,1})}. \tag{2.18}$$

The combination of (2.13) with (2.17) (resp. (2.15) with (2.18)) and the inclusion relation (1.3) of amalgam spaces imply that $G_{0,\pm\varepsilon}(\lambda)$ is a bounded function of $\lambda \in \overline{\mathbb{C}}^+$ with values in the first (resp. second) space of (2.6). Then, the duality implies that $G_{0,\pm\varepsilon}(\lambda)$ satisfies the same as a function with values in spaces in (2.7). Recall that $\ell_w^\infty = \ell^\infty = (\ell^1)^*$.

The integral kernels of $\Delta^\sigma G_{0,\pm\varepsilon,\leq}(\lambda)$ and $\Delta^\sigma G_{0,\pm\varepsilon,\geq}$ satisfy

$$|\Delta^\sigma G_{0,\pm\varepsilon,\leq}(\lambda,x,y)| \leq C|\sigma|^\gamma|x-y|^{-1}\chi(|x-y|/2),$$
$$\sup_{x,y\in\mathbb{R}^3}|\chi_{Q_j}(x)\Delta^\sigma G_{0,\pm\varepsilon,\geq}(\lambda,x,y)\chi_{Q_k}(y)| \leq C|\sigma|^\gamma\langle j-k\rangle^{-1+\varepsilon+\gamma}.$$

Thus, the argument above for $G_{0,\pm\varepsilon}(\lambda)$ implies that $|\sigma|^{-\gamma}\Delta^\sigma G_{0,\pm\varepsilon}(\lambda)$ is bounded with respect to $(\lambda,\sigma) \in \overline{\mathbb{C}}^+ \times \mathbb{R}$ as a function with values in any of the spaces of (2.6) and of (2.7). This proves the estimate (2.5) for the cases in the statement (2). Then, interpolation theory implies the same for statements (3) and (1). We next prove the strong continuity in the statement (1) for the case $\gamma = 0$. If $u \in C_0^\infty(\mathbb{R}^3)$, then $G_{0,\pm\varepsilon}(\lambda)u(x)$ is smooth with respect to $(\lambda, x) \in \overline{\mathbb{C}}^+ \times \mathbb{R}^3$, the derivatives are bounded in every compact sets, and we have that

$$|G_{0,\pm\varepsilon}(\lambda)u(x)| \leq C\langle x\rangle^{-1+\varepsilon}, \quad \lambda \in \overline{\mathbb{C}}^+.$$

Since $\langle x\rangle^{-1+\varepsilon} \in \ell^{\tilde{s}}(L^s)$ for $1 \leq s \leq \infty$ and $\tilde{s} > 3/(1-\varepsilon)$, Lebesgue's dominated convergence theorem implies that $G_{0,\pm\varepsilon}(\lambda)u$ is an $\ell^{\tilde{s}}(L^s)$ valued continuous function of $\lambda \in \overline{\mathbb{C}}^+$. Since $C_0^\infty(\mathbb{R}^3)$ is dense in $\ell^r(L^s)$ if $1 \leq r,\ s < \infty$, the desired strong continuity of $G_{0,\pm\varepsilon}(\lambda)$ follows. The strong or the weak continuities in other statements may be proved similarly. □

The following is the weighted version of the well-known L^p-estimates of Kenig-Ruiz-Sogge (Theorem 2.3 in [**13**]) and the proof patterns after their argument.

Lemma 2.2. *For $3/2 \le \rho \le 2$, let*

$$r(\rho) = \frac{2\rho}{\rho+1}, \quad s(\rho) = \frac{2\rho}{\rho-1} \quad \text{and} \quad 0 \le \varepsilon \le \frac{2}{\rho} - 1. \tag{2.19}$$

Then, $G_{0,\pm\varepsilon}(\lambda)$ is bounded from $L^{r(\rho)}$ to $L^{s(\rho)}$ for any $\lambda \in \overline{\mathbb{C}}^+ \setminus \{0\}$. For any $0 < c$, there exists a constant $C_{c,\rho}$ such that for $\lambda \in \overline{\mathbb{C}}^+$ with $|\lambda| \ge c > 0$,

$$\|G_{0,\pm\varepsilon}(\lambda)u\|_{s(\rho)} \le C_{c,\rho}|\lambda|^{-(2-(3/\rho))}\|u\|_{r(\rho)}. \tag{2.20}$$

Proof. We prove $+\varepsilon$ case only. Since $1/r(\rho) + 1/s(\rho) = 1$ and $G_0(\lambda)^* = G_0(-\overline{\lambda})$, the other case follows by the duality. We define closed strip S by

$$S = \{z \in \mathbb{C}\colon\ -2 \le \Re z \le 0\}$$

and, for $\lambda \in \mathbb{C}^+$, consider analytic family of operators $T(z) = m_z(-i\partial_x)$ defined for $z \in S$ via the Fourier multipliers

$$m_z(\xi) = \frac{e^{z^2}(\xi^2 - \lambda^2)^z}{\Gamma(z + (3/2))}$$

so that $G_0(\lambda) = \Gamma(1/2)e^{-1}T(-1)$. For $u, v \in \mathcal{S}(\mathbb{R}^3)$, $\langle T(z)u, v\rangle$ is continuous for $z \in S$, analytic in the interior of S and

$$|\langle T(i\tau)u, v\rangle| \le C\|u\|_2\|v\|_2, \quad \tau \in \mathbb{R}.$$

The Fourier transform of m_z is given by

$$\hat{m}_z(x) = \frac{e^{z^2}2^{z+1}}{\Gamma(-z)\Gamma((3/2)+z)}\left(\frac{\lambda}{i|x|}\right)^{(3/2)+z} K_{(3/2)+z}(-i\lambda|x|),$$

where $K_\nu(-is) = (i\pi/2)e^{i\nu\pi/2}H_\nu^{(1)}(s)$, $H_\nu^{(1)}(s)$ being Hankel function of the first kind, and, as is shown in (2.23) and (2.26) in page 339 of [**13**], we have

$$|e^{\nu^2}\nu K_\nu(w)| \le C|w|^{-|\Re\nu|}, \qquad |w| \le 1, \ \ \Re w \ge 0; \tag{2.21}$$

$$|K_\nu(w)| \le C_{\Re\nu}e^{-\Re w}|w|^{-1/2}, \quad |w| \ge 1, \ \ \Re w \ge 0. \tag{2.22}$$

This implies for $\lambda \in \mathbb{C}^+$ and for $-2 \le \Re z = -\rho \le -3/2$ that

$$|\hat{m}_z(x)| \leq C|\lambda|^{3-2\rho} \begin{Bmatrix} 1, & |\lambda||x| \leq 1, \\ (|\lambda||x|)^{\rho-2}, & |\lambda||x| \geq 1 \end{Bmatrix} \leq C|\lambda|^{3-2\rho}\langle|\lambda|x\rangle^{\rho-2}. \tag{2.23}$$

Thus, $\|T(z)u\|_\infty \leq C|\lambda|^{3-2\rho}\|u\|_1$ for $-2 \leq \Re z = -\rho \leq -3/2$ and Stein's theorem of complex interpolation implies

$$\|G_0(\lambda)u\|_{s(\rho)} = \Gamma(1/2)e^{-1}\|T(-1)u\|_{s(\rho)} \leq C|\lambda|^{(3/\rho)-2}\|u\|_{r(\rho)}. \tag{2.24}$$

We then fix $\rho \in [3/2, 2]$ and define another analytic family of operators:

$$\tilde{T}_\rho(z) = \langle|\lambda|x\rangle^{-z((2/\rho)-1)}T(z)\langle|\lambda|x\rangle^{z((2/\rho)-1)}.$$

By virtue of (2.23), it satisfies

$$\left\|\tilde{T}_\rho(z)u\right\|_2 \leq C\|u\|_2, \quad \Re z = 0; \quad \left\|\tilde{T}_\rho(z)u\right\|_\infty \leq C|\lambda|^{3-2\rho}\|u\|_1, \quad \Re z = -\rho.$$

It follows again by Stein's analytic interpolation theorem that

$$\|\langle|\lambda|x\rangle^{((2/\rho)-1)}G_0(\lambda)u\|_{s(\rho)} \leq C|\lambda|^{(3/\rho)-2}\|\langle|\lambda|x\rangle^{((2/\rho)-1)}u\|_{r(\rho)}. \tag{2.25}$$

Write $\varepsilon_* = (2/\rho) - 1$. We have $\langle x\rangle \leq 2^{1/2}|\lambda|^{-1}\langle|\lambda|x\rangle$ if $|x| \geq 1$, and $\langle x\rangle \geq 2^{-1/2}|\lambda|^{-1}\langle|\lambda|x\rangle$ if $|\lambda||x| \geq 1$. Hence, for $|\lambda| \geq c > 0$, (2.24) and (2.25) imply the following with $r(\rho) = r$ and $s(\rho) = s$:

$$\begin{aligned}
&\|\langle x\rangle^{\varepsilon_*}G_0(\lambda)u\|_s \\
&\quad \leq 2^{\varepsilon_*/2}\|G_0(\lambda)u\|_s + 2^{\varepsilon_*/2}|\lambda|^{-\varepsilon_*}\|\langle|\lambda||x|\rangle^{\varepsilon_*}G_0(\lambda)u\|_s \\
&\quad \leq C|\lambda|^{(3/\rho)-2}(\|u\|_r + |\lambda|^{-\varepsilon_*}\|\langle|\lambda|x\rangle^{\varepsilon_*}u\|_r) \\
&\quad \leq C|\lambda|^{(3/\rho)-2}\{(1+|\lambda|^{-\varepsilon_*})\|u\|_r + |\lambda|^{-\varepsilon_*}\|(1-\chi(|\lambda|x))\langle|\lambda|x\rangle^{\varepsilon_*}u\|_r\} \\
&\quad \leq C\{(1+|\lambda|^{-\varepsilon_*})|\lambda|^{(3/\rho)-2}\|u\|_r + \|\langle x\rangle^{\varepsilon_*}u\|_r\} \leq C(1+c^{-\varepsilon_*})|\lambda|^{(3/\rho)-2}\|\langle x\rangle^{\varepsilon_*}u\|_r.
\end{aligned}$$

The last estimate is (2.20) with $\varepsilon = \varepsilon_*$ and (2.24) is that with $\varepsilon = 0$. Thus, the desired estimate (2.20) follows by interpolation. □

We now interpolate estimates on $G_{0,\pm\varepsilon}(\lambda)$ obtained in the last two lemmas. There are various ways to interpolate them and results are not necessarily comparable. We have chosen the following which we think fits best for our purpose. We arbitrarily fix $3/2 \leq \rho \leq 2$ so that we have by (2.20):

$$\|\langle x\rangle^{\pm\varepsilon}G_0(\lambda)u\|_{s(\rho)} \le C_{\kappa,\rho}|\lambda|^{-\delta_*}\|\langle x\rangle^{\pm\varepsilon}u\|_{r(\rho)}, \quad 0\le\varepsilon\le\varepsilon_*(\rho) \tag{2.26}$$

with $r(\rho)=2\rho/(\rho+1)$ and $s(\rho)=2\rho/(\rho-1)$. Here and hereafter we define

$$\varepsilon_*(\rho)=\frac{2}{\rho}-1, \quad \delta_*(\rho)=2-\frac{3}{\rho}. \tag{2.27}$$

When ρ increases from $3/2$ to 2, $\varepsilon_*(\rho)$ decreases from $1/3$ to 0 whereas $\delta_*(\rho)$ increases from 0 to $1/2$.

For $0\le\varepsilon,\gamma$ such that $0\le\kappa=\varepsilon+\gamma\le 1$ we define two closed triangles

$$\mathcal{D}_\rho \equiv \triangle P_\rho QR\setminus\{Q,R\} \quad \text{and} \quad \mathcal{D}_\rho(\varepsilon,\gamma)\equiv\triangle P_\rho Q_\kappa R_\kappa\setminus\{Q_\kappa,R_\kappa\}$$

with two vertices removed in the $(1/r,1/s)$ plane, where

$$P_\rho=\left(\frac{1}{r(\rho)},\frac{1}{s(\rho)}\right), \quad Q=\left(\frac{2}{3},0\right), \quad R=\left(1,\frac{1}{3}\right),$$
$$Q_\kappa=\left(\frac{2+\kappa}{3},0\right), \quad R_\kappa=\left(1,\frac{1-\kappa}{3}\right).$$

Lines QR and $Q_\kappa R_\kappa$ are parallel to each other and $QR=\overline{L(0)}$ and $Q_\kappa R_\kappa=\overline{L(\kappa)}$ in the previous notation. We note that P_ρ is on the line $(1/r)-(1/s)=1/\rho$ and define for $0\le\theta\le 1$ two lines

$$L_\theta(\rho)=\mathcal{D}_\rho\cap\left\{\left(\frac{1}{r},\frac{1}{s}\right): \frac{1}{r}-\frac{1}{s}=\frac{1-\theta}{\rho}+\frac{2\theta}{3}\right\},$$
$$L_{\varepsilon,\gamma,\theta}(\rho)=\mathcal{D}_\rho(\varepsilon,\gamma)\cap\left\{\left(\frac{1}{r},\frac{1}{s}\right): \frac{1}{r}-\frac{1}{s}=\frac{1-\theta}{\rho}+\frac{(2+\kappa)\theta}{3}\right\}, \quad \kappa=\varepsilon+\gamma$$

which are respectively parallel to QR and $Q_\kappa R_\kappa$ and divide triangles $\mathcal{D}_\rho$ and $\mathcal{D}_\rho(\varepsilon,\gamma)$ into the ratio θ to $1-\theta$.

We remark that all estimates in what follows remain valid when $\varepsilon_*(\rho)$ is replaced by any ε_1 such that $0\le\varepsilon_1\le\varepsilon_*(\rho)$ because this is true for Theorem 2.3 and because estimates in what follows are obtained by interpolating those of Theorem 2.3 and those which are independent of $\varepsilon_*(\rho)$. We denote

$$\varepsilon(\rho,\theta)=\theta\varepsilon+(1-\theta)\varepsilon_*(\rho).$$

THEOREM 2.3. *Let $3/2\le\rho\le 2$, $0\le\varepsilon\le 1$, $0<\theta<1$ and let $0\le\gamma$ be such*

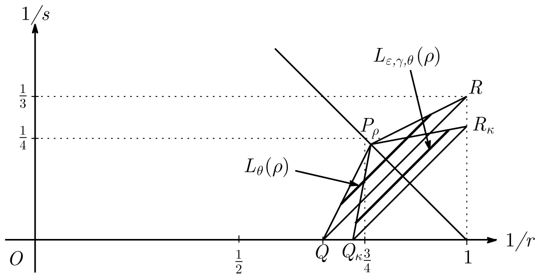

Figure 1.

that $\varepsilon+\gamma\le 1$. *Let* $(1/r,1/s)\in L_\theta(\rho)$ *and* $(1/\tilde r,1/\tilde s)\in L_{\varepsilon,\gamma,\theta}(\rho)$. *Then, we have the following statements*:

(1) *For every* $\lambda\in\mathbb{C}^+$, $G_{0,\pm\varepsilon(\rho,\theta)}(\lambda)$ *is bounded from* $\ell^{\tilde r}(L^r)$ *to* $\ell^{\tilde s}(L^s)$. *For* $c>0$, *there exists a constant* $C>0$ *such that*

$$\|G_{0,\pm\varepsilon(\rho,\theta)}(\lambda)u\|_{\mathbf{B}(\ell^{\tilde r}(L^r),\ell^{\tilde s}(L^s))}\le C|\lambda|^{-(1-\theta)\delta_*(\rho)} \tag{2.28}$$

for $\lambda\in\overline{\mathbb{C}}^+\setminus\{0\}$ *with* $|\lambda|\ge c$.

(2) *As a* $\mathbf{B}(\ell^{\tilde r}(L^r),\ell^{\tilde s}(L^s))$*-valued function of* λ, *it is analytic in* $\mathbb{C}^+$ *and it extends to* $\overline{\mathbb{C}}^+\setminus\{0\}$ *as a strongly continuous function. If* $\gamma>0$, *then it is a locally Hölder function of* $\lambda\in\overline{\mathbb{C}}^+\setminus\{0\}$ *of order* $\theta\gamma$. *For* $c>0$, *there exists a constant* $C>0$ *such that*

$$\|\Delta^\sigma G_{0,\pm\varepsilon(\rho,\theta)}(\lambda)u\|_{\mathbf{B}(\ell^{\tilde r}(L^r),\ell^{\tilde s}(L^s))}\le C|\lambda|^{-(1-\theta)\delta_*(\rho)}|\sigma|^{\theta\gamma} \tag{2.29}$$

for $|\lambda|>c$ *and* $|\sigma|<|\lambda|/2$.

Proof. We interpolate estimates in Lemma 2.1 and that in Lemma 2.2 with $\varepsilon=\varepsilon_*(\rho)$ by applying real or/and complex interpolation theories ([**2**]). This immediately produces (2.28) and proves statement (1).

We omit the proof of the analyticity and the strong continuity in statement (2) which is similar to that of the corresponding part of Lemma 2.1. For proving estimate (2.29), we interpolate

$$\|(\Delta^\rho G_{0,\pm\varepsilon_*(\rho)})(\lambda)\|_{s(\rho)}\le C_c|\lambda|^{-\delta_*(\rho)}\|u\|_{r(\rho)}, \tag{2.30}$$

which is valid for $|\lambda|\ge c$ and which trivially follows from (2.20), and the estimate (2.5):

$$\|(\Delta^\sigma G_{0,\pm\varepsilon})(\lambda)u\|_{\ell^{\tilde{s}}(L^s)} \le C|\sigma|^\gamma \|u\|_{\ell^{\tilde{r}}(L^r)} \tag{2.31}$$

for $(1/r, 1/s) \in L(0)$ and $(1/\tilde{r}, 1/\tilde{s}) \in L(\gamma+\varepsilon)$ and its modification at end points. Then, (2.29) follows as in the proof of (2.28). □

We next prove that $G_{0,\pm\varepsilon(\rho,\theta)}(\lambda)$ becomes continuous upto $\lambda = 0$ if the weights on both sides are suitably increased. In the following proposition we omit the variable λ from $G_{0,\pm\varepsilon(\rho,\theta)}(\lambda)$.

Proposition 2.4. *Let $\rho, \varepsilon, \gamma$ and θ be as in Theorem 2.3. Let $(1/r, 1/s) \in L_\theta(\rho)$ and $(1/\tilde{r}, 1/\tilde{s}) \in L_{\varepsilon,\gamma,\theta}(\rho)$. Suppose that $\mu > 1/2(1-(1/\rho))$. Then, there exists a constant $C > 0$ such that, for $\lambda \in \overline{\mathbb{C}}^+$ and $0 < |\sigma| \le 1$,*

$$\begin{aligned} \big\|\langle x\rangle^{-\mu} G_{0,\pm\varepsilon(\rho,\theta)}\langle x\rangle^{-\mu}u\big\|_{\ell^{\tilde{s}}(L^s)} + |\sigma|^{-\gamma\theta}\big\|\langle x\rangle^{-\mu}\Delta^\sigma G_{0,\pm\varepsilon(\rho,\theta)}\langle x\rangle^{-\mu}u\big\|_{\ell^{\tilde{s}}(L^s)} \\ \le C\langle\lambda\rangle^{-(1-\theta)\delta_*(\rho)}\|u\|_{\ell^{\tilde{r}}(L^r)}. \end{aligned} \tag{2.32}$$

Proof. In view of interpolation, it suffices to show (2.32) for $\theta = 0$ and $\theta = 1$. If $\theta = 1$, then (2.32) a fortiori holds since it is already true when $\mu = 0$ by virtue of Lemma 2.1. For $\theta = 0$, it suffices to show

$$\big\|\langle x\rangle^{-\mu} G_{0,\pm\varepsilon}(\lambda)\langle x\rangle^{-\mu}u\big\|_{s(\rho)} \le C\langle\lambda\rangle^{-\delta_*(\rho)}\|u\|_{r(\rho)} \tag{2.33}$$

for all $0 \le \varepsilon \le \varepsilon_*$. This is evident from (2.20) for $|\lambda| \ge 1$ and we may assume $|\lambda| \le 1$. Decompose the integral kernel of $\langle x\rangle^{-\mu}G_{0,\pm\varepsilon}(\lambda)\langle x\rangle^{-\mu}$ as

$$\langle x\rangle^{-\mu\pm\varepsilon}\left(\frac{e^{i\lambda|x-y|}\chi(|x-y|)}{4\pi|x-y|} + \frac{e^{i\lambda|x-y|}(1-\chi(|x-y|))}{4\pi|x-y|}\right)\langle y\rangle^{-\mu\mp\varepsilon}$$

and denote the operators with respective kernels by $T_\le(\lambda)$ and $T_\ge(\lambda)$. Then,

$$\left|\frac{\langle x\rangle^{-\mu\pm\varepsilon}e^{i\lambda|x-y|}\langle y\rangle^{-\mu\mp\varepsilon}\chi(|x-y|)}{4\pi|x-y|}\right| \le C\frac{\chi(|x-y|)}{|x-y|}$$

and $\|T_\le(\lambda)\|_{\mathbf{B}(L^{r(\rho)},L^{s(\rho)})} \le C$ by Young's inequality. For $T_\ge(\lambda)$, we have

$$\left|\frac{\langle x\rangle^{-\mu\pm\varepsilon}e^{i\lambda|x-y|}\langle y\rangle^{-\mu\mp\varepsilon}(1-\chi(|x-y|))}{4\pi|x-y|}\right| \le C\langle x\rangle^{-\mu}\langle x-y\rangle^{\varepsilon-1}\langle y\rangle^{-\mu}$$

and $(\mu/3)+((1-\varepsilon)/3)+(\mu/3) \geq 1-(1/\rho) = 1+((\rho-1)/2\rho)-((\rho+1)/2\rho)$ since $\varepsilon \leq \varepsilon_*(\rho)$. It follows again by virtue of Young's inequality that $\|T_{\geq}(\lambda)\|_{\mathbf{B}(L^{r(\rho)},L^{s(\rho)})} \leq C$ for $|\lambda| \leq 1$. This completes the proof. □

Recall that $\varepsilon(\rho,\theta) = \theta\varepsilon + (1-\theta)\varepsilon_*(\rho)$. To simplify notation we write

$$K_{l,\pm\varepsilon}(\lambda) = \langle x\rangle^{-l} G_{0,\pm\varepsilon}(\lambda)\langle x\rangle^{-l}, \quad l = 0,1,\ldots.$$

Theorem 2.5. *Let ρ, ε, γ and θ be as in Theorem* 2.3. *Let $(1/r,1/s) \in L_\theta(\rho)$, $(1/\tilde r, 1/\tilde s) \in L_{\varepsilon,\gamma,\theta}(\rho)$ and $l = 1,2,\ldots$. Then, $K_{l,\pm\varepsilon(\rho,\theta)}(\lambda)$ is analytic as a $\mathbf{B}(\ell^{\tilde r}(L^r), \ell^{\tilde s}(L^s))$-valued function of $\lambda \in \mathbb{C}^+$, and it extends to $\lambda \in \overline{\mathbb{C}}^+$ as a function of class C^l. There exists a constant $C > 0$ independent of u and $\lambda \in \overline{\mathbb{C}}^+$ such that*

$$\sum_{j=1}^{l} \big\|K^{(j)}_{l,\pm\varepsilon(\rho,\theta)}(\lambda)u\big\|_{\ell^{\tilde s}(L^s)} + \sup_{\sigma\neq 0} |\sigma|^{-\gamma\theta} \big\|\Delta^\sigma K^{(l)}_{l,\pm\varepsilon(\rho,\theta)}(\lambda)u\big\|_{\ell^{\tilde s}(L^s)} \leq C\langle\lambda\rangle^{-(1-\theta)\delta_*(\rho)}\|u\|_{\ell^{\tilde r}(L^r)}. \quad (2.34)$$

Proof. Differentiating under the sign of integration, we obtain for simple functions u that

$$\partial_\lambda^j G_0(\lambda)u(x) = \frac{i^j}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i\lambda|x-y|}|x-y|^j}{|x-y|} u(y)dy, \quad (2.35)$$

and the integral kernels of $K^{(j)}_{l,\pm\varepsilon}$ for $j = 1,\ldots,l$ and that of $\Delta^\sigma K^{(l)}_{l,\pm\varepsilon}$ are bounded respectively by

$$C\langle x\rangle^{-1\pm\varepsilon}\langle y\rangle^{-1\mp\varepsilon} \leq C\langle x-y\rangle^{-1+\varepsilon}, \quad (2.36)$$

$$C|\sigma|^\gamma\langle x\rangle^{-1\pm\varepsilon}\langle y\rangle^{-1\mp\varepsilon}|x-y|^\gamma \leq C|\sigma|^\gamma\langle x-y\rangle^{-1+\varepsilon+\gamma} \quad (2.37)$$

for any $0 \leq \gamma \leq 1$. It follows by the argument of the proof of Lemma 2.1 that $K_{l,\pm\varepsilon(\rho,\theta)}(\lambda)$ satisfies (2.34) when $\theta = 1$. If $\theta = 0$, using (2.36) and Hölder's inequality, we have for $0 \leq \varepsilon \leq \varepsilon_*$ that

$$\big\|K^{(j)}_{l,\pm\varepsilon}(\lambda)u\big\|_{s(\rho)} \leq C\big\|\langle x\rangle^{-1+\varepsilon}\|^2_{s(\rho)}\|u\big\|_{r(\rho)} \leq C\|u\|_{r(\rho)}, \quad j = 1,\ldots,l,$$

since $(1-\varepsilon)s(\rho) \geq (1-\varepsilon_*(\rho))s(\rho) = 4$. Thus, in view of the interpolation theory, it suffices to prove for all $0 \leq \varepsilon \leq \varepsilon_*$ and for $|\lambda| > 10$ that

$$\left\| K_{l,\pm\varepsilon}^{(j)}(\lambda)u \right\|_{s(\rho)} \le C\langle\lambda\rangle^{-\delta_*(\rho)}\|u\|_{r(\rho)}, \quad j=1,\dots,l. \tag{2.38}$$

Since $(x-y)^{2m}$ is a sum of monomials $x^\alpha y^\beta$, $|\alpha+\beta|=2m$, we have

$$\partial_\lambda^{2m+k} G_0(\lambda)u(x) = \sum_{|\alpha+\beta|=2m} C_{\alpha\beta} x^\alpha \partial_\lambda^k G_0(\lambda)(x^\beta u) \tag{2.39}$$

and (2.38) for even $j=2m$ follows immediately from (2.28). Thus, by virtue of (2.39) we have only to prove (2.38) for $l=1$. We deal with $K_{1,\varepsilon}^{(1)}(\lambda)$ only. The other may be dealt with similarly. We write $\lambda=\mu+i\kappa$ with $\kappa>0$, $|\lambda|\ge 10$. Since the case $|\kappa|\ge|\lambda|/10$ is much easier to dealt with (see below), we assume $|\kappa|\le|\lambda|/10$ and we further assume $\mu>0$ since the case $-\mu<0$ may be treated similarly. Write $u_\varepsilon(x)=\langle x\rangle^{-1+\varepsilon}u(x)$ and $v_{-\varepsilon}(x)=\langle x\rangle^{-1-\varepsilon}v(x)$. Then,

$$\begin{aligned}
&\frac{d}{d\lambda}\langle G_0(\lambda)u_\varepsilon, v_{-\varepsilon}\rangle \\
&= \frac{d}{d\lambda}\int \frac{\hat u_\varepsilon(\xi)\overline{\hat v_{-\varepsilon}(\xi)}}{|\xi|^2-\lambda^2}d\xi = \int \frac{2\lambda \hat u_\varepsilon(\xi)\overline{\hat v_{-\varepsilon}(\xi)}}{(|\xi|^2-\lambda^2)^2}d\xi \\
&= \frac{1}{\lambda}\left[\frac{d}{d\theta}\int \frac{\hat u_\varepsilon(\xi)\overline{\hat v_{-\varepsilon}(\xi)}}{|\xi|^2-e^{2\theta}\lambda^2}d\xi\right]\Bigg|_{\theta=0} = \frac{1}{\lambda}\left[\frac{d}{d\theta}\int \frac{e^\theta \hat u_\varepsilon(e^\theta\xi)\overline{\hat v_{-\varepsilon}(e^\theta\xi)}}{|\xi|^2-\lambda^2}d\xi\right]\Bigg|_{\theta=0} \\
&= \frac{1}{\lambda}\int \frac{\hat u_\varepsilon(\xi)\overline{\hat v_{-\varepsilon}(\xi)}}{|\xi|^2-\lambda^2}d\xi + \frac{1}{\lambda}\int \frac{\xi\cdot\nabla_\xi(\hat u_\varepsilon(\xi)\overline{\hat v_{-\varepsilon}(\xi)})}{|\xi|^2-\lambda^2}d\xi.
\end{aligned} \tag{2.40}$$

The first term on the right is equal to $\lambda^{-1}\langle G_0(\lambda)u_\varepsilon, v_{-\varepsilon}\rangle$ and, by virtue of (2.20), it is bounded in modulus by $C|\lambda|^{-1-\delta_*(\rho)}\|u\|_{r(\rho)}\|v\|_{r(\rho)}$. We take $\varphi\in C_0^\infty(\mathbb{R}^3)$ such that $\varphi(\xi)=1$ when $1/2\le|\xi|\le 2$ and $\varphi(\xi)=0$ outside $1/4\le|\xi|\le 4$ and decompose the second term of (2.40) as

$$\begin{aligned}
&\frac{\mu}{\lambda}\sum_{j=1}^3\int \frac{\varphi(\xi/\mu)(\xi_j/\mu)\cdot(\partial_{\xi_j}\hat u_\varepsilon(\xi)\overline{\hat v_{-\varepsilon}(\xi)} + \hat u_\varepsilon(\xi)\overline{\partial_{\xi_j}\hat v_{-\varepsilon}(\xi)})}{|\xi|^2-\lambda^2}d\xi \\
&\qquad + \frac{1}{\lambda}\sum_{j=1}^3\int \frac{\varphi_{\ge}(\xi/\mu)\xi_j\cdot\partial_{\xi_j}(\hat u_\varepsilon(\xi)\overline{\hat v_{-\varepsilon}(\xi)})}{|\xi|^2-\lambda^2}d\xi
\end{aligned} \tag{2.41}$$

where $\varphi_{\ge}(\xi)=1-\varphi(\xi)$. For $j=1,2,3$, we may write $\mathcal{F}\{\varphi(\xi/\mu)(\xi_j/\mu)\}(x)=\mu^3h_j(\mu x)$ with $h_j\in\mathcal{S}(\mathbb{R}^3)$. Define $h_{j\mu}(x)\equiv\mu^3h_j(\mu x)$. Then, for any $\tau\in\mathbb{R}$,

$$\langle x\rangle^{\tau}\mu^3|h_j(\mu(x-y))|\langle y\rangle^{-\tau} \le C\mu^3|h_j(\mu(x-y))|\langle\mu(x-y)\rangle^{\tau}$$

for $\mu > 1$ and the operator of convolution with $h_{j\mu}(x)$, $\mu \ge 1$ is uniformly bounded in $\langle x\rangle^{\tau}L^r(\mathbb{R}^3)$. The first term of (2.41) may be written in the form

$$\frac{\mu}{\lambda}\sum_{j=1}^{3}\big\{(G_0(\lambda)(h_{j\mu} * (x_j u_\varepsilon)), v_{-\varepsilon}) - (G_0(\lambda)u_\varepsilon, h_{j\mu} * (x_j v_{-\varepsilon}))\big\}$$

and it is bounded by $C|\lambda|^{-\delta_*(\rho)}\|u\|_{r(\rho)}\|v\|_{r(\rho)}$ by virtue of (2.20). Integration by parts implies that the second term of (2.41) is a sum over $j = 1, 2, 3$ of

$$-\frac{1}{\lambda}\int \frac{\{(\partial/\partial\xi_j)(\varphi_{\ge}(\xi/\mu)\xi_j)\}\hat{u}_\varepsilon(\xi)\overline{\hat{v}_\varepsilon(\xi)}}{|\xi|^2-\lambda}d\xi + \frac{1}{\lambda}\int \frac{2\varphi_{\ge}(\xi/\mu)\xi_j^2}{|\xi|^2-\lambda}\frac{\hat{u}_\varepsilon(\xi)\overline{\hat{v}_\varepsilon(\xi)}}{|\xi|^2-\lambda^2}d\xi.$$

In the last two integrals, both Fourier multipliers

$$\frac{\partial}{\partial\xi_j}\left(\varphi_{\ge}\left(\frac{\xi}{\mu}\right)\xi_j\right) = \frac{\partial\varphi_{\ge}}{\partial\xi_j}\left(\frac{\xi}{\mu}\right)\frac{\xi_j}{\mu} + \varphi_{\ge}\left(\frac{\xi}{\mu}\right), \quad \frac{2\varphi_{\ge}(\xi/\mu)\xi_j^2}{|\xi|^2-\lambda}$$

satisfy the Mikhlin-Hörmander condition uniformly with respect to $|\mu| > |\lambda|/2$, $|\lambda| > 10$. Hence the second term is estimated as in the previous case by $C|\lambda|^{-1-\delta_*(\rho)}\|u\|_{r(\rho)}\|v\|_{r(\rho)}$. This completes the proof. □

For $0 \le \varepsilon, \gamma$ such that $0 \le (\varepsilon+\gamma)/3 < (1/p) - (2/3) \le 1/3$, we define p_* by

$$\frac{1}{p_*} = 1 - \left(\frac{1}{p} - \frac{2+\varepsilon+\gamma}{3}\right). \tag{2.42}$$

We have that $1 < p_* \le 3/(2+\varepsilon+\gamma)$.

Lemma 2.6. *Suppose that* $V \in \ell^p(L^q)$ *for* $1 \le p < 3/2 < q \le \infty$. *Let* $0 \le \varepsilon, \gamma$ *be such that* $(\varepsilon+\gamma) < 3((1/p) - (2/3))$. *Then*:

(1) *Let* $1 \le r < 3q/(q+3)$. *Then, for* $l = 0, \ldots,$ *we have*

$$\sum_{j=0}^{l}\left\|VK_{l,\pm\varepsilon}^{(j)}(\lambda)u\right\|_{L^r} + \sup_{\sigma\ne0}|\sigma|^{-\gamma}\left\|\Delta^\sigma VK_{l,\pm\varepsilon}^{(l)}(\lambda)u\right\|_{L^r} \le C\|V\|_{\ell^p(L^q)}\|u\|_{L^1}, \tag{2.43}$$

where the constant $C > 0$ *is independent of* $\lambda \in \overline{\mathbb{C}}^{+}$, u *and* V.

(2) *Let $1 \le r \le p_*$. Then, for $l = 0, \dots,$ we have*

$$\sum_{j=0}^{l} \left\|VK^{(j)}_{l,\pm\varepsilon}(\lambda)u\right\|_{L^1} + \sup_{\sigma\neq 0} |\sigma|^{-\gamma} \left\|\Delta^\sigma VK^{(l)}_{l,\pm\varepsilon}(\lambda)u\right\|_{L^1} \le C\|V\|_{\ell^p(L^q)}\|u\|_{L^r}, \tag{2.44}$$

where the constant $C > 0$ is independent of $\lambda \in \overline{\mathbb{C}}^+$, u and V.

(3) *Let $3/2 \le \rho \le 2$. Define $r_-(\theta)$ and $r_+(\theta)$ for $0 \le \theta \le 1$ by*

$$\frac{1}{r_-(\theta)} = \frac{\rho+1}{2\rho}(1-\theta)+\theta, \quad \frac{1}{r_+(\theta)} = \frac{\rho+1}{2\rho}(1-\theta)+\frac{2+\varepsilon+\gamma}{3}\theta.$$

Let $\theta_0 = 0$ if $q \ge \rho$ and θ_0 be such that $1/q = ((1-\theta_0)/\rho) + (2\theta_0/3)$ if $q < \rho$. Then, for $\theta_0 \le \theta \le 1$, the following is satisfied for $r_-(\theta) \le r \le r_+(\theta)$:

(a) *For any $c > 0$, there exists a constant C such that*

$$\left\|VG_{0,\pm\varepsilon(\rho,\theta)}(\lambda)u\right\|_r + |\sigma|^{-\gamma\theta}\left\|\Delta^\sigma G_{0,\pm\varepsilon(\rho,\theta)}(\lambda)u\right\|_r \le C|\lambda|^{-(1-\theta)\delta_*(\rho)}\|u\|_r \tag{2.45}$$

for $|\lambda| \ge c$ and $|\sigma| \le |\lambda|/2$. If $\theta = 1$, then (2.45) holds for all $\lambda \in \overline{\mathbb{C}}^+$ and $\sigma \in \mathbb{R}\setminus\{0\}$.

(b) *For $l = 1, \dots, k$, we have for a constant $C > 0$ that*

$$\sum_{j=0}^{l} \left\|K^{(j)}_{l,\pm\varepsilon(\rho,\theta)}(\lambda)u\right\|_r + |\sigma|^{-\gamma\theta}\left\|\Delta^\sigma VK^{(l)}_{l,\pm\varepsilon(\rho,\theta)}(\lambda)u\right\|_r \le C\langle\lambda\rangle^{-(1-\theta)\delta_*(\rho)}\|u\|_r \tag{2.46}$$

for $\lambda \in \mathbb{R}$ and $\sigma \neq 0$.

Proof. We prove statements and estimates for the case $l = 0$ only. The proof for the case $l \ge 1$ may be given similarly by using Theorem 2.5 in place of Lemma 2.1 or Theorem 2.3. The proof of statements (1) and (2) uses only Lemma 2.1 and Hölder's inequality.

(1) By virtue of Lemma 2.1, for any $\tilde{s}$ such that $0 \le 1/\tilde{s} < (1-(\varepsilon+\gamma))/3$, $G_{0,\pm\varepsilon}(\lambda)$ maps L^1 continuously into $\ell^{\tilde{s}}(L^3_w)$ and

$$\|G_{0,\pm\varepsilon}(\lambda)u\|_{\ell^{\tilde{s}}(L^3_w)} + \sup_{\sigma\in\mathbb{R}} |\sigma|^{-\gamma}\|\Delta^\sigma G_{0,\pm\varepsilon}(\lambda)u\|_{\ell^{\tilde{s}}(L^3_w)} \le C\|u\|_{L^1}. \tag{2.47}$$

By Hölder's inequality we have $L^q(Q)\cdot L^3_w(Q)\subset L^r(Q)$ whenever $(1/3)+(1/q)<1/r$. Likewise, $\ell^p\cdot\ell^{\tilde s}\subset\ell^r$ if $(1/p)+(1/\tilde s)\ge 1/r$ and such an $\tilde s$ with $0\le 1/\tilde s<(1-(\varepsilon+\gamma))/3$ may be found since $(1/p)+((1-(\varepsilon+\gamma))/3)>1$. This with (2.47) implies (2.43).

(2) We may assume $r>1$ as the case $r=1$ is proven in (1). Recall that $1/p_*>(2+\varepsilon+\gamma)/3$. Then, $G_{0,\pm\varepsilon}(\lambda)$ maps L^r, $1<r\le p_*$, continuously into $\ell^{\tilde s}(L^s)$ for $1/s=(1/r)-(2/3)$ and $1/\tilde s=(1/r)-((2+\varepsilon+\gamma)/3)$ and

$$\|G_{0,\pm\varepsilon}(\lambda)u\|_{\ell^{\tilde s}(L^s)}+\sup_{\sigma\in\mathbb{R}}|\sigma|^{-\gamma}\|\Delta^s G_{0,\pm\varepsilon}(\lambda)u\|_{\ell^{\tilde s}(L^s)}\le C\|u\|_{L^r}. \tag{2.48}$$

Since $(1/r)-(2/3)+(1/q)<1$ and $r\le p_*$ implies

$$\frac{1}{r}+\frac{1}{p}-\frac{2+\varepsilon+\gamma}{3}\ge\frac{1}{p_*}+\frac{1}{p}-\frac{2+\varepsilon+\gamma}{3}=1,$$

Hölder's inequality and the inclusion relation for amalgam spaces produce the desired estimate (2.44).

(3) For $0\le\theta<1$, $(1/r,1/s)\in L_\theta(\rho)$ and $(1/\tilde r,1/\tilde s)\in L_{\varepsilon,\gamma,\theta}(\rho)$, we have

$$\begin{aligned}\|G_{0,\pm\varepsilon(\rho,\theta)}u\|_{\ell^{\tilde s}(L^s)}&+|\sigma|^{-\theta\gamma}\|\Delta^\sigma G_{0,\pm\varepsilon(\rho,\theta)}u\|_{\ell^{\tilde s}(L^s)}\\&\le C_c|\lambda|^{-(1-\theta)\delta_*(\rho)}\|u\|_{\ell^{\tilde r}(L^r)}\end{aligned} \tag{2.49}$$

for $|\lambda|>c$ and $|\sigma|<|\lambda|/2$ by virtue of Theorem 2.3; if $\theta=1$, then this holds for all $\lambda\in\overline{\mathbb{C}}^+$ and $\sigma\in\mathbb{R}\setminus\{0\}$ by virtue of Lemma 2.1. Note that $1/r_+(\theta)$ is the common $1/r$-coordinates of the right ends of $L_\theta(\rho)$ and $L_{\varepsilon,\gamma,\theta}(\rho)$, and $1/r_-(\theta)$ is the one of the left end of $L_{\varepsilon,\gamma,\theta}(\rho)$ which is larger than that of $L_\theta(\rho)$. Hence, $(1/r,1/s)\in L_\theta(\rho)$ and $(1/r,1/\tilde s)\in L_{\varepsilon,\gamma,\theta}(\rho)$ simultaneously for some $1/s$ and $1/\tilde s$ if and only if $r_-(\theta)\le r\le r_+(\theta)$. Then, we have

$$\frac{1}{s}+\frac{1}{q}=\frac{1}{r}-\frac{1-\theta}{\rho}-\frac{2\theta}{3}+\frac{1}{q}\le\frac{1}{r}$$

for $\theta\ge\theta_0$. On the other hand, since $1/\rho\le 2/3$ and, by assumption, $(2+\varepsilon+\gamma)/3<1/p$, we have for any $0\le\theta\le1$ that

$$\frac{1}{\tilde s}+\frac{1}{p}=\frac{1}{r}-\frac{1-\theta}{\rho}-\frac{(2+\varepsilon+\gamma)\theta}{3}+\frac{1}{p}\ge\frac{1}{r}-\frac{2+\varepsilon+\gamma}{3}+\frac{1}{p}>\frac{1}{r}.$$

Thus, (a) follows from (2.49) via the help of Hölder's inequality as previously. $\square$

In view of Lemma 2.6, we define

$$r_{\max} = \min\left(p_*, \frac{3q}{q+3}\right) \quad \text{and} \quad \kappa_{\max} = 3\left(\frac{1}{p} - \frac{2}{3}\right). \tag{2.50}$$

PROPOSITION 2.7. *Let $0 \le \varepsilon, \gamma$ be such that $\varepsilon + \gamma < \kappa_{\max}$ and $3/2 \le \rho \le 2$. Let $1 < r < r_{\max}$. Then, there exists $0 < \theta_* = \theta_*(\rho, \varepsilon, \gamma, r) < 1$ such that, for all $\theta_* \le \theta \le 1$, estimates* (2.43), (2.44), (2.45) *and* (2.46) *of Lemma* 2.6 *are all satisfied with $\varepsilon(\rho, \theta)$ being replaced by ε.*

PROOF. We recall that all estimates which contain $\varepsilon_*(\rho)$ are valid when it is replaced by smaller non-negative numbers. Note that

$$r_-(\theta) \to 1, \quad r_+(\theta) \to 3/(2 + \varepsilon + \gamma) \ge p_* \quad (\theta \to 1). \tag{2.51}$$

Thus, the proposition is obvious if $\varepsilon_*(\rho) \ge \varepsilon$. If $\varepsilon_*(\rho) < \varepsilon$, we take an $\varepsilon' > \varepsilon$ such that $\varepsilon' + \gamma < \kappa_{\max}$ and $r < 3/(2+\varepsilon'+\gamma)$ and apply Lemma 2.6 (3) with ε' in place of ε. We let $r'_+(\theta)$ be the $r_+(\theta)$ for the triplet $(\rho, \varepsilon', \gamma)$. Then, by virtue of (2.51), we can find $\theta_0 \le \theta_* < 1$ such that $(1-\theta)\varepsilon' + \theta\varepsilon_*(\rho) > \varepsilon$ and $r_-(\theta) < r < r'_+(\theta)$ for all $\theta_* \le \theta \le 1$. This completes the proof. □

REMARK 2.8. If $V \in \ell^p(L^q)$ for some $1 \le p < 3/2 < q \le \infty$, then the proof of statements (1) and (2) of Lemma 2.6 shows that integral operators K whose kernels satisfy

$$|K(x, y)| \le C|V(x)| \frac{\langle x - y \rangle^\rho}{|x - y|} \tag{2.52}$$

for $\rho < \kappa_{\max}$ are bounded in $L^1(\mathbb{R}^3)$. We shall use this fact in the proof of Proposition 3.3.

LEMMA 2.9. *Suppose that $V \in \ell^p(L^q)$ for some $1 \le p < 3/2 < q \le \infty$. Let $0 \le \varepsilon < \kappa_{\max}$ and $1 \le r < r_{\max}$. Then, $VG_{0,\pm\varepsilon}(\lambda) \in \mathbf{B}(L^r)$ is compact and is norm continuous with respect to $\lambda \in \overline{\mathbb{C}}^+$. For $l = 1, \dots$ and $j = 0, \dots, l$, $\langle x \rangle^{-l} VG^{(j)}_{0,\pm\varepsilon}(\lambda)\langle x \rangle^{-l} \in \mathbf{B}(L^r)$ satisfies the same property.*

PROOF. We prove the lemma for $l = 0$ only by the same reason as before. We set $\theta = 1$ in (a) of Lemma 2.6 (3). Then, (2.45) with $\theta = 1$ is satisfied for all $1 < r \le r_{\max}$. This holds also for $r = 1$ as, then, it is the same as (2.43) or/and (2.44). Since the constant C on the right of (2.45) is bounded by a constant times $\|V\|_{\ell^p(L^q)}$ and $C_0^\infty(\mathbb{R}^3)$ is dense in $\ell^p(L^q)$, it suffices to prove the lemma when

$V \in C_0^\infty(\mathbb{R}^3)$. Suppose, then, that $V(x) = 0$ outside $B_R = \{x \in \mathbb{R}^3 \colon |x| < R\}$. Then, $VG_{0,\pm\varepsilon}(\lambda)$ maps $L^r(\mathbb{R}^3)$ continuously into $W^{2,r}(B_R)$ and Rellich's theorem implies that $VG_{0,\pm\varepsilon}(\lambda)$ is a compact operator in $L^r(\mathbb{R}^3)$. The norm continuity is obvious since we still may take $\gamma > 0$ such that $\varepsilon + \gamma < \kappa_{\max}$. Then, it is locally Hölder continuous of order γ. This completes the proof. $\square$

Lemma 2.10. *Suppose that $\langle x\rangle^k V \in \ell^p(L^q)$ for an integer $k \ge 0$ and that H is of generic type. Let $0 \le \varepsilon, \gamma$ satisfy $\varepsilon + \gamma < \kappa_{\max}$ and $1 \le r < r_{\max}$. Then, the following statements are satisfied for $l = 0, 1, \ldots, k$:*

(1) *The inverse $(1 + VG_0(\lambda))^{-1}$ exists in $\langle x\rangle^{-l\pm\varepsilon} L^r$ for all $\lambda \in \mathbb{R}$ and is norm continuous.*
(2) *Let $u \in \langle x\rangle^{-l\pm\varepsilon} L^r$. Then, $u(\lambda) = (1 + VG_0(\lambda))^{-1} u$ is an $\langle x\rangle^{\pm\varepsilon} L^r$ valued function of $\lambda \in \mathbb{R}$ of class C^l.*
(3) *There exits a constant C such that for $\lambda \in \mathbb{R}$*

$$\sum_{j=0}^{l} \|\langle x\rangle^{\pm\varepsilon} u^{(j)}(\lambda)\|_r + \sup_{0<|\sigma|\le 1} |\sigma|^{-\gamma} \|\langle x\rangle^{\pm\varepsilon} \Delta^\sigma u^{(l)}(\lambda)\|_r \le C \|\langle x\rangle^{l\pm\varepsilon} u\|_r. \tag{2.53}$$

For the proof we need the following lemma. It probably is known, but we give a proof as we were not able to find a reference. In the proof we write ${}_X\langle u, x_*\rangle_{X^*}$ for $x_*(u)$ and etc. when $u \in X$ and $x_* \in X^*$.

Lemma 2.11. *Let X and Y are Banach spaces. Suppose that there exists a linear topological space $\mathcal{X}$ such that $X \subset \mathcal{X}$ and $Y \subset \mathcal{X}$ and that $X \cap Y \subset X$ is a dense and continuous embedding, where $X \cap Y$ is the Banach space with the norm $\|u\|_X + \|u\|_Y$. Suppose that an operator A is compact both in X and Y. Then,* $\mathrm{Ker}_X(1 + A) \ne \{0\}$ *implies* $\mathrm{Ker}_Y(1 + A) \ne \{0\}$.

Proof. We write A_X and etc. when the operator A is considered as the one from X to X. With respect to the coupling

$$\langle u, x_* + y_*\rangle = {}_X\langle u, x_*\rangle_{X^*} + {}_Y\langle u, y_*\rangle_{Y^*},$$

we have $(X \cap Y)^* = X^* + Y^*$, the right being the sum space of Banach spaces X^* and Y^*, and $X^* \subset X^* + Y^*$ is a dense continuous embedding since so is $X \cap Y \subset X$. By the assumption $A_{X\cap Y}$ is compact in $X \cap Y$. It follows by virtue of Banach's theorem that $A^*_{X\cap Y}$ is also a compact operator in $X^* + Y^*$. Suppose that $\mathrm{Ker}_X(1 + A) \ne \{0\}$. Then, $\mathrm{Ker}(1 + A^*_X) \ne \{0\}$. Let $u_* \in X^* \setminus \{0\}$ satisfy $u_* + A^*_X u_* = 0$. Then, with the left most and the right most couplings being understood as the ones between $X \cap Y$ and $X^* + Y^*$, we have for any $u \in X \cap Y$

that

$$\langle A_{X\cap Y}u, u_*\rangle = {}_X\langle A_X u, u_*\rangle_{X^*} = {}_X\langle u, A_X^* u_*\rangle_{X^*} = -{}_X\langle u, u_*\rangle_{X^*} = -\langle u, u_*\rangle.$$

This implies $u^* + (A_{X\cap Y})^* u^* = 0$ and, as $u^* \neq 0$ in $X^* + Y^*$ as well, $-1 \in \sigma((A_{X\cap Y})^*) = \sigma(A_{X\cap Y})$. Thus, $\mathrm{Ker}_Y(1+A) \neq \{0\}$. □

PROOF OF LEMMA 2.10. By Lemma 2.9 $VG_0(\lambda) = \langle x\rangle^{-l}(\langle x\rangle^l V)G_0(\lambda) \in \mathbf{B}(\langle x\rangle^{-l\pm\varepsilon}L^r)$ is compact for any $l = 0, \dots, k$. It follows that $(1 + VG_0(\lambda))$ is invertible in $\langle x\rangle^{-l\pm\varepsilon}L^r$ if and only if $\mathrm{Ker}(1 + VG_0(\lambda)) = \{0\}$. Goldberg-Schlag [**9**] and Goldberg [**7**] have proven that, if $\mathrm{Ker}_{L^r}(1 + VG_0(\lambda)) \neq \{0\}$ for $\lambda \in \mathbb{R} \setminus \{0\}$, then $\lambda^2 > 0$ is an eigenvalue of H. It follows by virtue of the absence of positive eigenvalues for H proven by Ionescu-Jerison [**10**] and by the assumption that H is of generic type that $(1 + VG_0(\lambda))^{-1}$ exists in $\mathbf{B}(L^r)$ for every $\lambda \in \mathbb{R}$. Then, the same is true for $\mathrm{Ker}_{\langle x\rangle^{-l\pm\varepsilon}L^r}(1 + VG_0(\lambda))$ by virtue of Lemma 2.11 and $(1 + VG_0(\lambda))^{-1}$ exists in $\mathbf{B}(\langle x\rangle^{-l\pm\varepsilon}L^r)$ for every $\lambda \in \mathbb{R}$. It satisfies the stated norm continuity and differentiability properties by virtue of Lemma 2.9. For proving (3), we take $\theta_* \leq \theta < 1$ of Proposition 2.7. Then, if $r > 1$, (2.53) follows from (2.45) and (2.46). If $r = 1$, we take $r_-(\theta) < r_* < r_+(\theta)$. Then, we may estimate $\|(VG_0(\lambda))^3\|_{\mathbf{B}(\langle x\rangle^{-l\pm\varepsilon}L^1)}$ by

$$\begin{aligned}\|VG_0(\lambda)\|_{\mathbf{B}(\langle x\rangle^{-l\pm\varepsilon}L^1, \langle x\rangle^{-l\pm\varepsilon}L^{r_*})}&\|VG_0(\lambda)\|_{\mathbf{B}(\langle x\rangle^{-l\pm\varepsilon}L^{r_*})} \\ &\times \|VG_0(\lambda)\|_{\mathbf{B}(\langle x\rangle^{-l\pm\varepsilon}L^{r_*}, \langle x\rangle^{-l\pm\varepsilon}L^1)} \leq C\langle\lambda\rangle^{-(1-\theta)\delta_*(\rho)} \qquad (2.54)\end{aligned}$$

and, for large λ, we may write $(1 + VG_0(\lambda))^{-1}$ in the form

$$(1 + VG_0(\lambda))^{-1} = (1 - VG_0(\lambda) + (VG_0(\lambda))^2)(1 + VG_0(\lambda))^{-3}.$$

Then, the estimate (2.53) follows from Lemma 2.6. □

3. High energy estimates.

In this section we study the high energy part of propagator $e^{-itH}P_{ac}(H)$. We set, for large $\lambda_0 \leq L$,

$$h_{L,\lambda_0}(\lambda) = \chi_{\geq}(\lambda/\lambda_0)\chi(\lambda/L) \qquad (3.1)$$

and prove the following theorem. In what follows we always assume $\langle x\rangle^k V \in \ell^p(L^q)$ for $1 \leq p < 3/2 < q \leq \infty$, and define $\kappa_{\max} = 3(1/p - 2/3)$ and $r_{\max} =$

$\min(p_*, 3q/(q+3))$ as previously.

Theorem 3.1. *There exists $\lambda_0 > 0$ such that, for any $0 < \varepsilon < \kappa_{\max}$,*

$$\left|\left(e^{-itH} h_{L,\lambda_0}(\sqrt{H}) P_{ac} u, v\right)\right| \le C|t|^{-k-(3/2)-(\varepsilon/2)} \|\langle x\rangle^{k+\varepsilon} u\|_1 \|\langle x\rangle^{k+\varepsilon} v\|_1 \qquad (3.2)$$

for $|t| \ge 1$ with $C > 0$ independent of $L \ge 10\lambda_0$ and $u, v \in \langle x\rangle^{k+\varepsilon} L^1(\mathbb{R}^3)$.

The proof consists of several steps.

3.1. Stationary representation.

It suffices to prove Theorem 3.1 for $u, v \in \mathcal{S}(\mathbb{R}^3)$ which we assume in what follows. Define $u_{L,high}$ by

$$u_{L,high}(\lambda) = h_{2L,\lambda_0/2}(\lambda)(1 + VG_0(\lambda))^{-1} u \qquad (3.3)$$

and $v_{L,high}$ by the same formula with v replacing u. These are $\langle x\rangle^{\varepsilon} L^1(\mathbb{R}^3)$ valued functions of $\lambda \in \mathbb{R}$ of class $C^{k+\gamma}$, $\varepsilon + \gamma < \kappa_{\max}$ by virtue of Lemma 2.10. Using that $G_0'(\lambda)$ is weakly-$*$ continuous from L^1 to L^∞, we then define

$$(U_{1,L,high}(t)u, v) = -\frac{1}{2t\pi} \int_{\mathbb{R}} e^{-it\lambda^2} h_{L,\lambda_0}(\lambda) \langle G_0'(\lambda) u_{L,high}(\lambda), v_{L,high}(-\lambda)\rangle d\lambda; \qquad (3.4)$$

$$(U_{2,L,high}(t)u, v) = -\frac{1}{2t\pi} \int_{\mathbb{R}} e^{-it\lambda^2} h'_{L,\lambda_0}(\lambda) \langle G(\lambda) u, v\rangle d\lambda. \qquad (3.5)$$

Lemma 3.2. *We have*

$$\left(e^{-itH} h_{L,\lambda_0}(\sqrt{H}) P_{ac} u, v\right) = (U_{1,L,high}(t)u, v) + (U_{2,L,high}(t)u, v).$$

Proof. The well-known Stone formula implies

$$\left(e^{-itH} h_{L,\lambda_0}(\sqrt{H}) P_{ac} u, v\right) = \frac{1}{i\pi} \lim_{\eta \downarrow 0} \int_{\mathbb{R}} h_{L,\lambda_0}(\lambda) e^{-it\lambda^2} (G(\lambda + i\eta) u, v) \lambda d\lambda.$$

We apply integration by parts to the right hand side and take the limit $\eta \downarrow 0$. For $\Im\lambda > 0$ we have as in (1.13)

$$\langle G'(\lambda) u, v\rangle = \left\langle G_0'(\lambda)(1 + VG_0(\lambda))^{-1} u, (1 + VG_0(-\lambda))^{-1} v\right\rangle.$$

This extends to real λ by the weak-$*$ continuities of $G_0'(\lambda) \in \mathbf{B}(L^1, L^\infty)$ and the smoothness property of $(1+VG_0(\lambda))^{-1}u$ and $(1+VG_0(-\lambda))^{-1}v$. Lemma easily follows from this. □

We often write $h(\lambda)$ for $h_{L,\lambda_0}(\lambda)$, suppressing indices L and λ_0. We prove

$$|(U_{j,L,high}(t)u, v)| \le Ct^{-k-(3/2)-(\varepsilon/2)} \|\langle x\rangle^{k+\varepsilon} u\|_1 \|\langle x\rangle^{k+\varepsilon} v\|_1 \tag{3.6}$$

for $j=1$ and $j=2$.

3.2. Key lemma.

The following proposition, which is an improvement of Goldberg's estimate ([**7**]) and of Lemma 1.4 in the introduction, plays an essential role in what follows. $\mathcal{F}$ is the partial Fourier transform with respect to the λ variable.

Proposition 3.3. *Let $0 \le \tau, \varepsilon$ satisfy $\tau + \varepsilon < \kappa_{\max}$. Then, there exists $\Lambda > 0$ such that whenever $\lambda_0 \ge \Lambda$, $\mathcal{F}u_{L,high}(\rho, x)$ satisfies that*

$$\sum_{l=0}^{k} \iint_{\mathbb{R}^2} \langle\sigma\rangle^\tau \langle x\rangle^{k-l+\varepsilon} \big|\mathcal{F}_{\lambda\to\sigma} u^{(l)}_{L,high}(\sigma, x)\big| dx d\sigma \le C\|\langle x\rangle^{k+\varepsilon} u\|_{L^1(\mathbb{R}^3)}, \tag{3.7}$$

where the constant C is independent of u, $\lambda_0 \ge \Lambda$ and $L > \lambda_0$.

We prove the proposition by modifying the argument in [**7**]. In the sequel, we choose and fix $1 \le r < r_{\max}$ and $\theta_* < \theta < 1$ as in Proposition 2.7. Then, for $l = 0, \dots, k$, we may estimate $\|(VG_0(\lambda))^3\|_{\mathbf{B}(\langle x\rangle^{-l\pm\varepsilon}L^1)}$ as in (2.54) and, if λ_0 is large enough, for λ in the support of h_{L,λ_0} we may expand as

$$(1+VG_0(\lambda))^{-1} = \sum_{n=0}^{\infty} (-1)^n G_n(\lambda), \quad G_n(\lambda) \equiv (VG_0(\lambda))^n$$

in the space $\mathbf{B}(\langle x\rangle^{-l\pm\varepsilon}L^1)$. The proof of the proposition consists of the following two lemmas.

Lemma 3.4. *Let $0 \le \varepsilon \le 1$ and $\tau \ge 0$ satisfy $\varepsilon + \tau < \kappa_{\max}$. Then, there exists Λ such that the following statement is satisfied: For a constant C independent of $0 \le \alpha + \beta \le l \le k$, n and $L > \lambda_0 \ge \Lambda$, we have*

$$\int_{\mathbb{R}^4} \langle\sigma\rangle^\tau \langle x\rangle^{k-l+\varepsilon} \big|\mathcal{F}(h^{(\alpha)}_{L,\lambda_0}(\lambda) G_n^{(\beta)}(\lambda) u)(\sigma, x)\big| d\sigma dx \le C^n \|\langle x\rangle^{k+\varepsilon} u\|_1. \tag{3.8}$$

Proof. Following Rodnianski and Schlag ([**18**]), we write $G_n(\lambda)u(x)$ as an integral over $\mathbb{R}^{3n}$ and, differentiate under the sign of integration. We have that, with $x_0 = y$, $x = x_n$ and $\Sigma = \sum_{j=1}^n |x_j - x_{j-1}|$,

$$G_n^{(\beta)}(\lambda)u(x) = \int_{\mathbb{R}^{3n}} \frac{(i\Sigma)^\beta e^{i\lambda\Sigma}\prod_{j=1}^n V(x_j)}{\prod_{j=1}^n 4\pi|x_j - x_{j-1}|} u(y)dydx_1\cdots dx_{n-1}. \tag{3.9}$$

Here the integral on the right is absolutely convergent as the proof of Lemma 2.6 shows, and using Fubini's theorem, we have

$$\begin{aligned}&\mathcal{F}(h^{(\alpha)}(\lambda)G_n^{(\beta)}(\lambda)u)(\sigma, x)\\ &\quad= \int_{\mathbb{R}^{3n}} \frac{(i\Sigma)^\beta(\mathcal{F}h^{(\alpha)})(\sigma-\Sigma)\prod_{j=1}^n V(x_j)}{\prod_{j=1}^n 4\pi|x_j - x_{j-1}|} u(y)dydx_1\cdots dx_{n-1}. \end{aligned}\tag{3.10}$$

We use the following lemma (see Proposition 8 of [**7**]):

Lemma 3.5. *With a constant independent of $L \ge 1$, we have for $\rho \ge 0$ that*

$$\int \langle\sigma\rangle^\rho L|(\mathcal{F}\chi^{(\alpha)})(L(\sigma-\Sigma))|d\sigma \le C_\rho\langle\Sigma\rangle^\rho. \tag{3.11}$$

Proof. Since $\langle\sigma+\Sigma\rangle^\rho \le C_\rho(\langle\sigma\rangle^\rho + \langle\Sigma\rangle^\rho)$ and $\langle\sigma/L\rangle \le \langle\sigma\rangle$ for $L \ge 1$, we have

$$\begin{aligned}&\int \langle\sigma\rangle^\rho \big|L(\mathcal{F}\chi^{(\alpha)})(L(\sigma-\Sigma))\big|d\sigma\\ &\quad= \int \langle\sigma+\Sigma\rangle^\rho\big|L(\mathcal{F}\chi^{(\alpha)})(L\sigma)\big|d\sigma\\ &\quad\le C_\rho\bigg(\langle\Sigma\rangle^\rho + \int\langle\sigma/L\rangle^\rho\big|(\mathcal{F}\chi^{(\alpha)})(\sigma)\big|d\sigma\bigg) \le C(1+\langle\Sigma\rangle^\rho).\end{aligned}$$

This proves the lemma. $\square$

Continuation of the proof of Lemma 3.4. Since $\langle\sigma\rangle \le C\langle\mu\rangle\langle\sigma-\mu\rangle$, we see from (3.11) that, for $\lambda_0 \ge 2$,

$$\int_{\mathbb{R}}\langle\sigma\rangle^\tau\bigg(\int_{\mathbb{R}}\lambda_0\big|\widehat{\chi^{(\beta)}}(\lambda_0\mu)L\widehat{\chi^{(\alpha)}}(L(\sigma-\mu-\Sigma))\big|d\mu\bigg)d\sigma \le C\langle\Sigma\rangle^\tau. \tag{3.12}$$

This implies that

$$\int \langle\sigma\rangle^{\tau}\big|\mathcal{F}(h^{(\alpha)}(\lambda)G_n^{(\beta)}(\lambda)u)(\sigma,x)\big|d\sigma$$

$$\leq C_{\lambda_0}\int_{\mathbb{R}^{3n}}\frac{|\Sigma|^{\beta}\langle\Sigma\rangle^{\tau}\prod_{j=1}^{n}|V(x_j)|}{\prod_{j=1}^{n}4\pi|x_j-x_{j-1}|}|u(y)|dydx_1\cdots dx_{n-1}. \tag{3.13}$$

We estimate $|\Sigma|^{\beta}\langle\Sigma\rangle^{\tau}\leq\langle\Sigma\rangle^{\beta+\tau}$ by $C\langle n\rangle^{\beta+\tau}$ times

$$\sum_{j=1}^{n}\langle x_j-x_{j-1}\rangle^{\beta+\tau}\leq C\sum_{j=1}^{n}\langle x_j\rangle^{\beta}\langle x_{j-1}\rangle^{\beta}\langle x_j-x_{j-1}\rangle^{\tau} \tag{3.14}$$

and replace $|\Sigma|^{\beta}\langle\Sigma\rangle^{\tau}$ on the right of (3.13) by the right of (3.14). This produces following bound for (3.13):

$$C_{\lambda_0}\langle n\rangle^{\beta+\tau}\bigg(\big(K_1^{n-1}K_4|u|\big)(x)+\sum_{j=0}^{n-2}\big(K_1^{j}K_3K_2K_1^{n-j-2}|u|\big)(x)\bigg), \tag{3.15}$$

where K_1, K_2, K_3 and K_4 respectively are integral operators with the kernels

$$K_1(x,y)=\frac{V(x)}{4\pi|x-y|},\qquad K_2(x,y)=\frac{V(x)\langle x\rangle^{\beta}}{4\pi|x-y|},$$

$$K_3(x,y)=\frac{V(x)\langle x\rangle^{\beta}\langle x-y\rangle^{\tau}}{4\pi|x-y|},\qquad K_4(x,y)=\frac{V(x)\langle x\rangle^{\beta}\langle x-y\rangle^{\tau}\langle y\rangle^{\beta}}{4\pi|x-y|}.$$

Since $\langle x\rangle^{k-l}\langle x\rangle^{\beta}V\in\ell^p(L^q)$ for any $|\beta|\leq l$, Remark 2.8 implies that operators

$$\langle x\rangle^{k}K_1,\quad K_2,\quad \langle x\rangle^{k-l}K_3,\quad K_4\langle x\rangle^{-l},$$

are all bounded in $\langle x\rangle^{\varepsilon}L^1$ for $\varepsilon+\tau<\kappa_{\max}$. Estimating as $n^{\alpha}\leq C^n$ for $n\geq 1$ with a suitable constant C, we obtain the lemma. □

LEMMA 3.6. *Let $0\leq\varepsilon+\gamma<\kappa_{\max}$ and $3/2\leq\rho\leq 2$. Then, there exists $0<\theta_*<1$ such that the following statement is satisfied for any $\theta_*\leq\theta<1$: There exists a constant C independent of u, $n\geq 1$, $0\leq\alpha+\beta\leq l\leq k$ and $1\leq\lambda_0\leq L$ such that*

$$\sup_{\sigma\in\mathbb{R}} \|\langle x\rangle^{k-l+\varepsilon}\langle\sigma\rangle^{\theta\gamma}\mathcal{F}(h^{(\alpha)}_{L,\lambda_0}(\lambda)G^{(\beta)}_n(\lambda)u)(x,\sigma)\|_1 \\ \le C^n\lambda_0^{1-(n-2)(1-\theta)\delta_*(\rho)}\|\langle x\rangle^{k+\varepsilon}u\|_{L^1}. \quad (3.16)$$

Proof. We arbitrarily take $1<r<r_{\max}$ and then choose $\theta_*=\theta(\rho,\varepsilon,\gamma,r)$ as in Proposition 2.7. Using Leibniz' law, we write $G^{(\beta)}_n(\lambda)$ as a linear combination of $VG_0^{(j_1)}(\lambda)\cdots VG_0^{(j_n)}(\lambda)$ with $j_1+\cdots+j_n=\beta$. We apply (2.43) and (2.44) to the right most and to the left most factors respectively and (2.45) and (2.46) to the remaining $n-2$ factors in between. Noticing that $l\ge j_1$ and $k-j_{i-1}\ge j_i$ for $i=2,\dots,n-1$, we obtain

$$\begin{aligned}&\|\langle x\rangle^{k-l+\varepsilon}VG_0^{(j_1)}(\lambda)\cdots VG_0^{(j_n)}(\lambda)u\|_{L^1}\\ &\quad\le \|\langle x\rangle^{k-l+\varepsilon}VG_0^{(j_1)}\langle x\rangle^{-j_1-\varepsilon}\|_{\mathbf{B}(L^r,L^1)}\prod_{i=2}^{n-1}\|\langle x\rangle^{j_{i-1}+\varepsilon}VG_0^{(j_i)}\langle x\rangle^{-j_i-\varepsilon}\|_{\mathbf{B}(L^r)}\\ &\qquad\times\|\langle x\rangle^{j_n+\varepsilon}VG_0^{(j_n)}(\lambda)u\|_{L^1}\\ &\quad\le C(C/\lambda)^{(n-2)(1-\theta\delta_*(\rho))}\|\langle x\rangle^{-k-\varepsilon}u\|_1,\end{aligned} \quad (3.17)$$

and likewise estimate for difference quotient with respect to λ:

$$\sup_{0<|\sigma|\le 1}|\sigma|^{-\theta\gamma}\|\Delta^\sigma\langle x\rangle^{k-l+\varepsilon}VG_0^{(j_1)}(\lambda)\cdots VG_0^{(j_n)}(\lambda)u\|_{L^1} \\ \le Cn(C/\lambda)^{(n-2)(1-\theta\delta_*(\rho))}\|\langle x\rangle^{-k-\varepsilon}u\|_1. \quad (3.18)$$

We have for any integrable functions that

$$\int_{-\infty}^{\infty}e^{i\lambda\sigma}f(\lambda)d\lambda=\frac{1}{2}\int_{-\infty}^{\infty}e^{i\lambda\sigma}(f(\lambda)-f(\lambda-\pi/\sigma))d\lambda. \quad (3.19)$$

We integrate $e^{-i\lambda\sigma}h^{(\alpha)}(\lambda)G_n^{(\beta)}(\lambda)u$ with respect $\lambda\in R$ and apply (3.17) and (3.18). We obtain the lemma by using (3.19) and the fact that $h^{(\alpha)}(\lambda)$ is uniformly Lipschitz continuous with respect to $1\le\lambda_0<L$. □

Proof of Proposition 3.3. For given $\varepsilon,\tau\ge 0$ such that $\varepsilon+\tau<\kappa_{\max}$ we take $\gamma>0$ such that $\varepsilon+(\tau+\gamma)<\kappa_{\max}$. Take Λ and θ_* as in Lemma 3.4 and in Lemma 3.6 respectively with τ being placed by $\tau+\gamma$ and let $\theta_*<\theta<1$. Let $\langle x\rangle^{k+\varepsilon}u\in L^1$. Then, (3.17) implies that

$$\langle x\rangle^{k-l+\varepsilon}(1+VG_0(\lambda))^{-1}u=\sum_{n=0}^{\infty}(-1)^n\langle x\rangle^{k-l+\varepsilon}(-1)^nG_n(\lambda)u$$

converges in $L^1(\mathbb{R}^3)$ uniformly with respect to $\lambda\geq\lambda_0$ and that it is l-times termwise differentiable in $L^1(\mathbb{R}^3)$. We write, for $l=0,\ldots,k$,

$$f_{l,n}(\sigma)=\left\|\langle x\rangle^{k-l+\varepsilon}\mathcal{F}((h(\lambda)\cdot G_n(\lambda))^{(l)}u)(\sigma,x)\right\|_{L^1(\mathbb{R}^3_x)}. \tag{3.20}$$

Summing up (3.8) with τ being replaced by $\tau+\gamma$ over (α,β) with $\alpha+\beta=l$, we obtain that

$$\left\|\langle\sigma\rangle^{\tau+\gamma}f_{l,n}\right\|_1\leq C^n\|\langle x\rangle^{k+\varepsilon}u\|_1, \tag{3.21}$$

and likewise from (3.16) that

$$\left\|\langle\sigma\rangle^{\theta\gamma}f_{l,n}\right\|_\infty\leq C^n\lambda_0^{1-(n-2)\omega}\|\langle x\rangle^{k+\varepsilon}u\|_{L^1} \tag{3.22}$$

with $\omega=(1-\theta)\delta_*(\rho)>0$. We choose $\mu<\gamma$ close enough to γ such that

$$\frac{\mu(\tau+\gamma-\gamma\theta)}{\gamma-\mu}>1$$

and set

$$r=\frac{\tau+\gamma-\gamma\theta}{\tau+\mu-\gamma\theta}.$$

Then, $r(\tau+\mu)=\tau+\gamma+(r-1)\theta\gamma$, $r>1$ and we have $s\mu>1$ for $s=r/(r-1)$. It follows by Hölder's inequality, (3.21) and (3.22) that

$$\begin{aligned}\int_{\mathbb{R}}\langle\sigma\rangle^{\tau}f_{l,n}(\sigma)d\sigma&\leq\left(\int_{\mathbb{R}}f_{l,n}(\sigma)^r\langle\sigma\rangle^{r(\tau+\mu)}d\sigma\right)^{1/r}\left(\int_{\mathbb{R}}\langle\sigma\rangle^{-s\mu}d\sigma\right)^{1/s}\\&\leq C\left\|\langle\sigma\rangle^{\tau+\gamma}f_{l,n}\right\|_1^{1/r}\left\|\langle\sigma\rangle^{\theta\gamma}f_{l,n}\right\|_\infty^{(r-1)/r}\\&\leq C^n\lambda_0^{(1-(n-2)\omega)(r-1)/r}\|\langle x\rangle^{k+\varepsilon}u\|_{L^1}.\end{aligned} \tag{3.23}$$

If $\lambda_0>(2C)^{r/\omega(r-1)}$, then $C^n\lambda_0^{-n\omega(r-1)/r}\leq 2^{-n}$ produces an exponential fall-off as $n\to\infty$ and we may sum up (3.23) for $n=0,1,\ldots$, producing the desired estimate (3.7). This proves the proposition. □

PROOF OF THEOREM 3.1. Using that $(-2it\lambda)^{-1}\partial_\lambda e^{-it\lambda^2} = e^{-it\lambda^2}$, we apply integration by parts k-times with respect to λ to the integral (3.4). We obtain

$$\begin{aligned}(U_{1,L,high}(t)u,v) &= -\frac{1}{2t\pi}\int_{\mathbb{R}} e^{-it\lambda^2}h(\lambda)\big\langle G_0'(\lambda)u_{L,high}(\lambda), v_{L,high}(-\lambda)\big\rangle d\lambda \\ &= -\frac{1}{(2t)^{k+1}i^k\pi}\int_{\mathbb{R}} e^{-it\lambda^2}\left(\frac{\partial}{\partial\lambda}\cdot\frac{1}{\lambda}\right)^k h(\lambda) \\ &\quad\times \big\langle G_0'(\lambda)u_{L,high}(\lambda), v_{L,high}(-\lambda)\big\rangle d\lambda. \end{aligned} \tag{3.24}$$

It is elementary to check that, with suitable constants C_{kl},

$$\left(\frac{\partial}{\partial\lambda}\cdot\frac{1}{\lambda}\right)^k = \sum_{l=0}^{k}\left(\frac{\partial}{\partial\lambda}\right)^l \frac{C_{kl}}{\lambda^{2k-l}}, \quad k = 0, 1, \ldots. \tag{3.25}$$

Define $h_l(\lambda) = -(C_{kl}/i^k 2^{k+1}\lambda_0^{2k-l}\pi)\chi(\lambda/L)(\lambda/\lambda_0)^{l-2k}\chi_{\geq}(2\lambda/\lambda_0)$ for $0 \leq l \leq k$ and write the right of (3.24) as a sum over $l = 0, \ldots, k$ of

$$F_{l,L}(t) = t^{-k-1}\int_{\mathbb{R}} e^{-it\lambda^2}\partial_\lambda^l\big\{h_l(\lambda)\big\langle G_0'(\lambda)u_{L,high}(\lambda), v_{L,high}(-\lambda)\big\rangle\big\}d\lambda. \tag{3.26}$$

Note that $\lambda^{l-2k}\chi_{\geq}(\lambda)$ is smooth and $|\lambda^{l-2k}\chi_{\geq}(\lambda)| \leq C\langle\lambda\rangle^{-k}$.

LEMMA 3.7. *Let $0 \leq \varepsilon < \kappa_{\max}$. Suppose that, for every integer $0 \leq l \leq k$, $\langle x\rangle^{k-l+\varepsilon}u(\lambda,\cdot)$ is an $L^1(\mathbb{R}^3)$-valued function of compact support of $\lambda \in \mathbb{R}$ of class C^l and that it satisfies*

$$\int_{\mathbb{R}} \big\|(\langle\sigma\rangle^\varepsilon + \langle x\rangle^\varepsilon)\langle x\rangle^{k-l}(\mathcal{F}u^{(l)})(\sigma,x)\big\|_1 d\sigma \leq M(u). \tag{3.27}$$

Suppose that $v(\lambda,\cdot)$ satisfies the same property with v in place of u. Define

$$g(\lambda) = a(\lambda)\big\langle G_0'(\lambda)u(\lambda,\cdot), v(\lambda,\cdot)\big\rangle$$

for $a(\lambda) \in C_0^\infty(\mathbb{R})$. Then, $g(\lambda)$ is a function of class C^k and it satisfies

$$\int_{\mathbb{R}} \langle\sigma\rangle^\varepsilon|(\mathcal{F}g^{(l)})(\sigma)| \leq CC_l(a)M(u)M(v), \quad l = 0, 1, \ldots, k, \tag{3.28}$$

where $C_l(a) = \|\langle\sigma\rangle^{l+\varepsilon}\hat{a}(\sigma)\|_1$ and $C > 0$ is a constant independent of u, v and a.

Proof. By Leibniz' rule $g^{(l)}(\lambda)$ is a linear combination of

$$g_{ijmn}(\lambda) = a^{(i)}(\lambda)\big\langle G_0^{(j+1)}(\lambda)u^{(m)}(\lambda,\cdot), v^{(n)}(\lambda,\cdot)\big\rangle, \quad i+j+m+n = l.$$

Since the kernel of $G_0^{(j+1)}(\lambda)$ equals $i^{j+1}e^{i\lambda|x-y|}|x-y|^j/4\pi$, the Fourier transform $\hat{g}_{ijmn}(\sigma)$ may be given by a constant times

$$\iint \widehat{a^{(i)}}(\sigma-|x-y|-\mu-\rho)|x-y|^j\widehat{u^{(m)}}(\mu,y)\overline{\widehat{v^{(n)}}(\rho,x)}dydxd\mu d\rho.$$

Since $\int_{\mathbb{R}}\langle\sigma\rangle^{\varepsilon}|\widehat{a^{(i)}}(\sigma-\mu)|d\sigma \le CC_l(a)\langle\mu\rangle^{\varepsilon}$ and

$$\langle|x-y|+\mu+\rho\rangle^{\varepsilon} \le C_{\varepsilon}(\langle x-y\rangle^{\varepsilon}+\langle\mu\rangle^{\varepsilon}+\langle\rho\rangle^{\varepsilon}),$$

it follows that $\int_{\mathbb{R}}\langle\sigma\rangle^{\varepsilon}|\hat{g}_{ijmn}(\sigma)|d\sigma$ is bounded by $CC_l(h)$ times

$$\iint (\langle x-y\rangle^{\varepsilon}+\langle\mu\rangle^{\varepsilon}+\langle\rho\rangle^{\varepsilon})|x-y|^j\big|\widehat{u^{(m)}}(\mu,y)\widehat{v^{(n)}}(\rho,x)\big|dydxd\mu d\rho.$$

Estimating as $|x-y|^j \le (\langle x\rangle^j+\langle y\rangle^j)$ and remembering that $j \le \min(k-m,k-n)$, we obtain

$$\begin{aligned}\int_{\mathbb{R}}\langle\sigma\rangle^{\varepsilon}|\hat{g}_{ijmn}(\sigma)|d\sigma &\le CC_l(a)\bigg(\int \big\|(\langle\mu\rangle^{\varepsilon}+\langle x\rangle^{\varepsilon})\langle x\rangle^{k-m}\widehat{u^{(m)}}(\mu,x)\big\|_1 d\mu\bigg) \\ &\quad\times\bigg(\int \big\|(\langle\rho\rangle^{\varepsilon}+\langle x\rangle^{\varepsilon})\langle x\rangle^{k-n}\widehat{v^{(n)}}(\rho,x)\big\|_1 d\rho\bigg) \\ &\le CC_l(a)M(u)M(v). \end{aligned} \tag{3.29}$$

Lemma follows by summing (3.29) over (i,j,m,n) with $i+j+m+n=l$. □

Proposition 3.8. *Let $F_{l,L}(t)$, $l=0,1,\dots,k$ be defined by* (3.26). *Then for any $0 \le \varepsilon < \kappa_{\max}$, there exists a constant $C>0$ independent of $\lambda_0 \ge \Lambda$, $L \ge \lambda_0$ and u,v such that*

$$|F_{l,L}(t)| \le Ct^{-k-(3/2)-(\varepsilon/2)}\|\langle x\rangle^{k+\varepsilon}u\|\|\langle x\rangle^{k+\varepsilon}v\|. \tag{3.30}$$

Proof. If we define $g(\lambda) = a(\lambda)\langle G_0'(\lambda)u(\lambda), v(\lambda)\rangle$ as in Lemma 3.7 with

$$u(\lambda,\cdot)=u_{L,high}(\lambda,\cdot),\quad v(\lambda,\cdot)=v_{L,high}(-\lambda,\cdot),\quad \text{and}\quad a(\lambda)=h_l(\lambda)$$

then, we have the representation

$$F_{l,L}(t)=t^{-k-1}\int_{\mathbb{R}} e^{-it\lambda^2} g^{(l)}(\lambda)d\lambda.$$

Proposition 3.3 implies that $u(\lambda,\cdot)$ and $v(\lambda,\cdot)$ satisfy conditions of Lemma 3.7 with $M(u)=C\|\langle x\rangle^{k+\varepsilon}u\|$ and $M(v)=C\|\langle x\rangle^{k+\varepsilon}v\|$ and, by virtue of (3.11) and (3.12), $C_l(h_l)\le C$ with a constant C independent of $L>\lambda_0\ge\Lambda$. It follows that g satisfies (3.28) and, by definition, $g^{(l)}(0)=0$. Hence, by virtue of Fourier-Parseval identity as in (1.18), we obtain from (3.28) that

$$\begin{aligned}|F_{l,L}(t)| &\le Ct^{-k-(3/2)}\left|\int_{\mathbb{R}}\left(e^{-i\sigma^2/4t}-1\right)\widehat{g^{(l)}}(\sigma)d\sigma\right|\\ &\le Ct^{-k-(3/2)-(\varepsilon/2)}\left\|\langle\sigma\rangle^{\varepsilon}\widehat{g^{(l)}}(\sigma)\right\|_1\\ &\le Ct^{-k-(3/2)-(\varepsilon/2)}\|\langle x\rangle^{k+\varepsilon}u\|_1\|\langle x\rangle^{k+\varepsilon}v\|_1. \end{aligned}\tag{3.31}$$

This completes the proof of Proposition 3.8. □

Completion of the proof of Theorem 3.1. We still have to estimate $(U_{2,L,high}(t)u,v)$. We rewrite it as follows (cf. [**7**]): We have $h'(-\lambda)=-h'(\lambda)$ and $G(\lambda)-G(-\lambda)=(1+G_0(-\lambda)V)^{-1}\{G_0(\lambda)-G_0(-\lambda)\}(1+VG_0(\lambda))^{-1}$ by the resolvent equation. It follows by writing $\tilde{G}_0'(\lambda)$ for $(G_0(\lambda)-G_0(-\lambda))$

$$\begin{aligned}&(U_{2,L,high}(t)u,v)\\ &\quad=-\frac{1}{4t\pi}\int_{\mathbb{R}} e^{-it\lambda^2}\tilde{h}'(\lambda)\langle G(\lambda)-G(-\lambda)u,v\rangle d\lambda\\ &\quad=-\frac{1}{4t\pi}\int_{\mathbb{R}} e^{-it\lambda^2}\tilde{h}'(\lambda)\left\langle\tilde{G}_0'(\lambda)(1+VG_0(\lambda))^{-1}u,(1+VG_0(\lambda))^{-1}v\right\rangle d\lambda.\end{aligned}$$

We then insert $h_{2L,\lambda_0/2}(\lambda)^2$ which is 1 on the support of $\tilde{h}'(\lambda)$ and write as

$$(U_{2,L,high}(t)u,v)=-\frac{1}{4t\pi}\int_{\mathbb{R}} e^{-it\lambda^2}\tilde{h}'(\lambda)\left\langle\tilde{G}_0'(\lambda)u_{L,high}(\lambda),v_{L,high}(\lambda)\right\rangle d\lambda.\tag{3.32}$$

Here $\tilde{G}_0'(\lambda)u$ may be written as

$$\tilde{G}_0'(\lambda)u(x) = \frac{i}{2\pi}\int_{\mathbb{R}} \frac{\sin\lambda|x-y|}{|x-y|}u(y)dy. \tag{3.33}$$

We note that for $L \gg \lambda_0$ $h'(\lambda) = \lambda^{-1}\mu_L(\lambda) + \tau(\lambda)$ where

$$\mu_L(\lambda) = (\lambda/L)\chi'(\lambda/L)\chi(\lambda/\lambda_0), \quad \tau(\lambda) = (1/\lambda_0)\chi'(\lambda/\lambda_0)$$

and, substituting this for $h'(\lambda)$, we write $(U_{2,L,high}(t)u, v)$ as the sum

$$-\frac{1}{4t\pi}\int_{\mathbb{R}} e^{-it\lambda^2}\mu_L(\lambda)\big\langle\lambda^{-1}\tilde{G}_0'(\lambda)u_{L,high}(\lambda), v_{L,high}(\lambda)\big\rangle d\lambda \tag{3.34}$$

$$-\frac{1}{4t\pi}\int_{\mathbb{R}} e^{-it\lambda^2}\lambda\tau(\lambda)\big\langle\lambda^{-1}\tilde{G}_0'(\lambda)u_{L,high}(\lambda), v_{L,high}(\lambda)\big\rangle d\lambda. \tag{3.35}$$

Here the integral kernel of $\lambda^{-1}\tilde{G}_0'(\lambda)$ is given by

$$G_1(\lambda,x,y) = \frac{i\sin\lambda|x-y|}{2\pi\lambda|x-y|} = \frac{1}{4\pi i}\int_{-1}^{1} e^{i\theta\lambda|x-y|}d\theta$$

and in (3.35) $\lambda\tau(\lambda)$ is L-independent compactly supported smooth function supported far outside a neighborhood of 0. Thus, the argument used for studying $(U_{1,L,high}(t)u, v)$ applies to (3.34) and (3.35) with $e^{i\lambda(\theta|x-y|)}$ replacing the role of $e^{i\lambda|x-y|}$ and produces the estimate

$$|(U_{2,L,high}(t)u, v)| \le Ct^{-k-(3/2)-(\varepsilon/2)}\|\langle x\rangle^{k+\varepsilon}u\|_1\|\langle x\rangle^{k+\varepsilon}v\|_1. \tag{3.36}$$

This completes the proof of the theorem. □

4. Low energy estimate.

We now analyze the low energy part $e^{-itH}\chi(\sqrt{H}/\lambda_0)P_{ac}$. We define

$$u_{low}(\lambda) = \chi(\lambda/2\lambda_0)(1+VG_0(\lambda))^{-1}u \tag{4.1}$$

and, as in the previous section, express $(e^{-itH}\chi(\sqrt{H}/\lambda_0)u, v)$ as

$$e^{-itH}\chi\big(\sqrt{H}/\lambda_0\big)u = (U_{1,low}(t)u, v) + (U_{2,low}(t)u, v),$$

where, with the notation $\tilde{G}_0'(\lambda)$ of the previous section,

$$(U_{1,low}(t)u,v) = -\frac{1}{2t\pi}\int_{\mathbb{R}} e^{-it\lambda^2}\chi(\lambda/\lambda_0)\big\langle G_0'(\lambda)u_{low}(\lambda), v_{low}(-\lambda)\big\rangle d\lambda; \tag{4.2}$$

$$(U_{2,low}(t)u,v) = -\frac{1}{4t\pi}\int_{\mathbb{R}} e^{-it\lambda^2}\lambda_0^{-1}\chi'(\lambda/\lambda_0)\big\langle \tilde{G}_0'(\lambda)u_{low}(\lambda), v_{low}(\lambda)\big\rangle d\lambda. \tag{4.3}$$

The following proposition will play for the low energy part what Proposition 3.3 did for the high energy part.

Proposition 4.1. *Let $0 < \tau, \varepsilon$ be such that $\tau + \varepsilon < \kappa_{\max}$. Then, the partial Fourier transform $\hat{u}_{low}(\sigma, x) = (\mathcal{F}_{\lambda\to\rho}u_{low})(\sigma, x)$ satisfies*

$$\iint_{\mathbb{R}^4} \langle\sigma\rangle^\tau \langle x\rangle^{k-\ell+\varepsilon}\big|\hat{u}_{low}^{(\ell)}(\sigma,x)\big| dx d\sigma \le C\|\langle x\rangle^{k+\varepsilon}u\|_{L^1(\mathbb{R}^3)} \tag{4.4}$$

for every $0 \le \ell \le k$, where the constant C is independent of u.

For the proof of Proposition 4.1, we again borrow the basic strategy from [**7**]. Define $B(\lambda, \mu) = G_0(\lambda) - G_0(\mu)$. Next lemma may be found in [**7**]:

Lemma 4.2. *Let $0 \le \varepsilon + \theta < \kappa_{\max}$. Then, there exists a constant $C > 0$ independent of $\lambda, \mu \in \mathbb{R}$ such that $\|\langle x\rangle^{k+\varepsilon} VB(\lambda,\mu)\langle x\rangle^{-\varepsilon}\|_{\mathbf{B}(L^1)} \le C|\lambda-\mu|^\theta$.*

Proof. The integral kernel of $\langle x\rangle^\varepsilon B(\lambda,\mu)\langle x\rangle^{-\varepsilon}$ is bounded by

$$\left|\frac{\langle x\rangle^\varepsilon(e^{\lambda|x-y|} - e^{\mu|x-y|})\langle y\rangle^{-\varepsilon}}{4\pi|x-y|}\right| \le C|\lambda-\mu|^\theta \frac{\langle x-y\rangle^{\varepsilon+\theta}}{|x-y|}$$

and Remark 2.8 implies the lemma. □

For $\ell = 0, 1, \dots, k$, $(1 + VG_0(\lambda))^{-1}$ exists in $\mathbf{B}(\langle x\rangle^{-\ell-\varepsilon}L^1)$ and is norm continuous with respect to λ. Write $S(\mu) = (1+VG_0(\mu))^{-1}$ so that

$$S(\lambda) = (1 + VG_0(\lambda))^{-1} = (1 + S(\mu)VB(\lambda,\mu))^{-1}S(\mu). \tag{4.5}$$

Lemma 4.2 guarantees that we can find $d_0 > 0$ such that, for $l = 0, \dots, k$, and $|\lambda|, |\mu| \le 2\lambda_0$ with $|\lambda - \mu| < 4d_0$, we have

$$\|S(\mu)VB(\lambda,\mu)\|_{\mathbf{B}(\langle x\rangle^{-l-\varepsilon}L^1)} < 1/2. \tag{4.6}$$

For $0 < d < d_0$ and $\mu_j = 3jd$, $j = -n, \dots, n$, $n = [2(\lambda_0 + d)/3d] + 1$, we have

$$\chi(\lambda/2\lambda_0) = \sum_{j=-n}^{n} \chi_{j,d}(\lambda), \quad \chi_{j,d}(\lambda) \equiv \chi((\lambda-\mu_j)/d)\chi(\lambda/2\lambda_0). \tag{4.7}$$

Using this, we decompose as

$$u_{low}(\lambda) = \sum_{j=1}^{n} \chi_{j,d}(\lambda) S(\lambda) u \equiv \sum_{j=1}^{n} u_{low,j}\,. \tag{4.8}$$

We have for λ such that $\chi_{j,d}(\lambda) \neq 0$ that

$$S(\lambda) = \big(1 + \chi((\lambda-\mu_j)/2d) S(\mu_j) V B(\lambda,\mu_j)\big)^{-1} S(\mu_j)$$

and, by virtue of (4.6), we may expand the right hand side as

$$S(\lambda) = \sum_{m=0}^{\infty} (-1)^m \big(\chi((\lambda-\mu_j)/2d) S(\mu_j) V B(\lambda,\mu_j)\big)^m S(\mu_j) \tag{4.9}$$

in the space $\mathbf{B}(\langle x\rangle^{-l-\varepsilon}L^1)$, $0 \le l \le k$ and $0 \le \varepsilon < \kappa_{\max}$, uniformly on the support of $\chi_{j,d}(\lambda)$. We define for $|\mu| \le 2\lambda_0$

$$T_\mu(\lambda) = \chi((\lambda-\mu)/2d) S(\mu) V B(\lambda,\mu).$$

Lemma 4.3. *Let $0 \le \varepsilon+\tau < \kappa_{\max}$ and let $0 \le \theta < \kappa_{\max} - (\varepsilon+\tau)$. Then, for $0 < d < 1$, we have*

$$\int_{\mathbb{R}} \langle\sigma\rangle^{\tau} \big\| \langle x\rangle^{k-\ell+\varepsilon} \mathcal{F}(T_\mu^{(\ell)}(\lambda)u)(\sigma,x) \big\|_1 d\sigma \le C d^{-\ell+\theta} \|\langle x\rangle^{\ell+\varepsilon} u\|_1 \tag{4.10}$$

for $\ell = 0,\ldots,k$, where C does not depend on $|\mu| \le 2\lambda_0$ or $0 < d < 1$.

For the proof we need the following lemma.

Lemma 4.4. *Let $0 \le \varepsilon,\ \theta \le 1$ be such that $0 \le \theta+\varepsilon \le 1$. Then, there exists a constant $C > 0$ such that for $0 < a \le 1$ and $0 < \rho < \infty$ we have*

$$a \int \langle\sigma\rangle^{\varepsilon} |\hat{\chi}(a(\sigma-\rho)) - \hat{\chi}(a\sigma)| d\sigma \le C a^{\theta} \langle\rho\rangle^{\theta+\varepsilon}. \tag{4.11}$$

When $\varepsilon = 0$, we may replace $\langle\rho\rangle^{\theta}$ on the right by $|\rho|^{\theta}$.

Proof. We split the domain of integration into $|\sigma| \le 10\rho$ and $|\sigma| > 10\rho$. The integral over $|\sigma| \le 10\rho$ is bounded by both of

$$\langle 10\rho\rangle^{\varepsilon} a \int_{|\sigma|\le 10\rho} |\hat{\chi}(a(\sigma-\rho)) - \hat{\chi}(a\sigma)|d\sigma \le 2\langle 10\rho\rangle^{\varepsilon}\|\hat{\chi}\|_1, \tag{4.12}$$

$$2a\|\hat{\chi}\|_\infty \int_{|\sigma|<10\rho} \langle\sigma\rangle^{\varepsilon} d\sigma \le Ca\langle 10\rho\rangle^{1+\varepsilon}\|\hat{\chi}\|_\infty, \tag{4.13}$$

and, hence, by the right of (4.11). When $|\sigma| > 10\rho$, we apply mean value theorem and obtain that, for any $0 \le N < \infty$,

$$|\hat{\chi}(a\sigma - a\rho) - \hat{\chi}(a\sigma)| = a\rho|\hat{\chi}'(a\sigma - a\xi\rho)| \le C_N a\rho\langle a\sigma\rangle^{-N}, \quad 0<\xi<1.$$

It follows by taking $N = 2-\theta$ that the integral over $|\sigma| > 10\rho$ may be bounded by a constant times

$$a\rho\int_{|\sigma|\ge 10a\rho} \left\langle \frac{\sigma}{a}\right\rangle^{\varepsilon} \langle\sigma\rangle^{-2+\theta}d\sigma \le Ca\rho\left(1+\frac{1}{a^{\varepsilon}}\right)\langle a\rho\rangle^{\theta+\varepsilon-1} \le Ca^{1-\varepsilon}\rho\langle a\rho\rangle^{\theta+\varepsilon-1}.$$

The right hand side is bounded again by $Ca^{\theta}\langle\rho\rangle^{\theta+\varepsilon}$ because

$$\left(\frac{a\rho}{\langle a\rho\rangle}\right)^{1-\varepsilon-\theta} < 1 < \left(\frac{\langle\rho\rangle}{\rho}\right)^{\varepsilon+\theta}.$$

For $\varepsilon = 0$ the statement follows easily via the mean value theorem. □

Proof of Lemma 4.3. Since $S(\mu) \in \mathbf{B}(\langle x\rangle^{-l-\varepsilon}L^1)$ for $l = 0, \dots, k$ and it is independent of λ, it suffices to prove (4.10) for $\chi((\lambda-\mu)/2d)VB(\lambda,\mu)$ in place of $T_\mu(\lambda)$. We have

$$\mathcal{F}_{\lambda\to\sigma}\{\chi((\lambda-\mu)/2d)VB(\lambda,\mu)u\}(\sigma,x) = \int_{\mathbb{R}^3} K(\sigma,x,y)u(y)dy,$$

$$K(\sigma,x,y) = 2de^{-i\mu(\sigma-|x-y|)}V(x)\left(\frac{\hat{\chi}(2d(\sigma-|x-y|)) - \hat{\chi}(2d\sigma)}{4\pi|x-y|}\right).$$

Lemma 4.4 and $\langle x\rangle^{\varepsilon}\langle y\rangle^{-\varepsilon} \le C_\varepsilon\langle x-y\rangle^{\varepsilon}$ imply that, for any $0 \le \tau+\theta \le 1$,

$$\int \langle\sigma\rangle^{\tau}|\langle x\rangle^{k+\varepsilon}K(\sigma,x,y)\langle y\rangle^{-\varepsilon}|d\sigma \le C|\langle x\rangle^{k}V(x)|d^{\theta}\frac{\langle x-y\rangle^{\tau+\varepsilon+\theta}}{|x-y|}. \tag{4.14}$$

Since $\langle x\rangle^k V \in \ell^p(L^q)$ and $\tau+\varepsilon+\theta<\kappa_{\max}$, Remark 2.8 implies that the operator with integral kernel given by the right side of (4.14) is bounded in $L^1(\mathbb{R}^3)$ with norm bounded by d^θ. This proves (4.10) for $\ell=0$. We now let $\ell\ge 1$. By Leibniz' rule, we have

$$\begin{aligned}(\chi((\lambda-\mu)/2d)VB(\lambda,\mu))^{(\ell)}u \\ &= (1/2d)^\ell\chi^{(\ell)}((\lambda-\mu)/2d)VB(\lambda,\mu)u \\ &\quad + \sum_{l=1}^{\ell}(1/2d)^{(\ell-l)}\chi^{(\ell-l)}((\lambda-\mu)/2d)VG_0^{(l)}(\lambda)u. \end{aligned} \tag{4.15}$$

The argument for the case $\ell=0$ applies to the first term on the right, and we conclude that it satisfies the estimate (4.10). We have by Fubini's theorem that

$$\begin{aligned}&\mathcal{F}_{\lambda\to\sigma}\{(2d)^{l-\ell}\chi^{(\ell-l)}((\lambda-\mu)/2d)VG_0^{(l)}(\lambda)u(x)\}(\sigma) \\ &= \int_{\mathbb{R}^3}(2d)^{l-\ell+1}i^l e^{-i\mu(\sigma-|x-y|)}V(x)\frac{\widehat{\chi^{(\ell-l)}}(2d(\sigma-|x-y|))}{4\pi|x-y|^{1-l}}u(y)dy.\end{aligned}$$

We denote the integral kernel on the right by $K_{\ell l}(\sigma,x,y)$. We have that

$$\begin{aligned}&2d\int_{\mathbb{R}}\langle\sigma\rangle^\tau\left|\widehat{\chi^{(\ell-l)}}(2d(\sigma-|x-y|))\right|d\sigma \\ &\le C\left(\langle x-y\rangle^\tau+\int_{\mathbb{R}}\left(1+\left|\frac{\sigma}{d}\right|^\tau\right)\left|\widehat{\chi^{(\ell-l)}}(\sigma)\right|d\sigma\right)\le Cd^{-\tau}\langle x-y\rangle^\tau\end{aligned}$$

for $0<d<1$. It follows that

$$\int\langle\sigma\rangle^\tau\langle x\rangle^{k-\ell+\varepsilon}|K_{\ell,l}(\sigma,x,y)|\langle y\rangle^{-\ell-\varepsilon}d\sigma\le Cd^{l-\ell-\tau}\frac{\langle x\rangle^k|V(x)|\langle x-y\rangle^{\varepsilon+\tau}}{|x-y|}$$

and, as previously, the right side is the kernel of a bounded operator in L^1 with the norm bounded by $Cd^{l-\ell-\tau}\le Cd^{\theta-\ell}$, $l=1,\ldots,k$. This completes the proof. □

We continue to write $\hat f(\sigma,x)=(\mathcal{F}_{\lambda\to\sigma}f)(\sigma,x)$.

Lemma 4.5. *Let $0\le\varepsilon,\tau$ and θ satisfy $0\le\varepsilon+\tau+\theta<\kappa_{\max}$. Let $0<d<1$. Suppose that $S_1(\lambda),\ldots,S_n(\lambda)$ are $\mathbf{B}(L^1(\mathbb{R}^3))$-valued functions of $\lambda\in\mathbb{R}$ which are strongly C^ℓ class and are compactly supported. Write for $u\in L^1(\mathbb{R}^3)$*

$$(S_l u)(\lambda, x) = S_l(\lambda)u(x) \in L^1\big(\mathbb{R}^4_{(\lambda,x)}\big).$$

Suppose further that derivatives $S_l^{(\ell)} u$, $\ell = 0, \dots, k$, *satisfy*

$$\int_{\mathbb{R}} \langle\sigma\rangle^\tau \Big\| \langle x\rangle^{k-\ell+\varepsilon} \widehat{S_l^{(\ell)} u}(\sigma, x)\Big\|_1 d\sigma \le C_k d^{-\ell+\theta} \|\langle x\rangle^{\ell+\varepsilon} u\|_1. \tag{4.16}$$

Then, for $\ell = 0, \dots, k$, $T_n(\lambda) \equiv S_n(\lambda)\cdots S_1(\lambda)$ *satisfies*

$$\int_{\mathbb{R}} \langle\sigma\rangle^\tau \Big\| \langle x\rangle^{k-\ell+\varepsilon} \widehat{T_n^{(\ell)} u}(\sigma, x)\Big\|_1 d\sigma \le C^n \left(\prod_{k=1}^n C_k\right) d^{-\ell+n\theta} \|\langle x\rangle^{\ell+\varepsilon} u\|_1, \tag{4.17}$$

where C *is independent of* n, $C_1, \dots, C_n$, d *and* u.

Proof. We prove the lemma by induction. For $n = 1$, (4.17) holds trivially. We assume that it holds upto $n-1$ and prove that it holds also for n. Define for $u \in L^1(\mathbb{R}^3)$

$$(T_{n,\sigma}u)(\lambda, x) = \{S_n(\lambda)\hat{u}_{n-1}(\sigma)\}(x) \quad \hat{u}_{n-1}(\sigma, x) = \widehat{T_{n-1}u}(\sigma, x).$$

Since the Fourier inversion formula is satisfied for L^1-valued functions $f(\lambda)$ such that $f, \hat{f} \in L^1(\mathbb{R}, L^1(\mathbb{R}^3))$, we have

$$T_{n-1}u(\lambda, x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\lambda\sigma} \hat{u}_{n-1}(\sigma, x) d\sigma.$$

It follows after applying Fubini's theorem that

$$\begin{aligned} \widehat{T_n u}(\rho, x) &= \frac{1}{2\pi} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{-i\lambda(\rho-\sigma)} \{S_n(\lambda)\hat{u}_{n-1}(\sigma)\}(x) d\lambda \right) d\sigma \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{T_{n,\sigma}u}(\rho-\sigma, x) d\sigma. \end{aligned} \tag{4.18}$$

Thus, Minkowski's inequality and the assumption on $S_n(\lambda)$ imply

$$\begin{aligned} &\int_{\mathbb{R}} \langle\rho\rangle^\tau \Big\| \langle x\rangle^{k+\varepsilon} \widehat{T_n u}(\rho)\Big\|_1 d\rho \\ &\le \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \langle\rho\rangle^\tau \Big\| \langle x\rangle^{k+\varepsilon} \widehat{T_{n,\sigma}u}(\rho-\sigma)\Big\|_1 d\rho \right) d\sigma \end{aligned}$$

$$\le \frac{C_\tau}{\sqrt{2\pi}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\langle\rho\rangle^\tau\left\|\langle x\rangle^{k+\varepsilon}\widehat{T_{n,\sigma}u}(\rho)\right\|_1 d\rho+\langle\sigma\rangle^\tau\int_{\mathbb{R}}\left\|\langle x\rangle^{k+\varepsilon}\widehat{T_{n,\sigma}u}(\rho)\right\|_1 d\rho\right)d\sigma$$
$$\le \frac{C_\varepsilon C_n d^\theta}{\sqrt{2\pi}}\left(\int_{\mathbb{R}}\|\langle x\rangle^\varepsilon\hat{u}_{n-1}(\sigma)\|_1 d\sigma+\int_{\mathbb{R}}\langle\sigma\rangle^\tau\|\langle x\rangle^\varepsilon\hat{u}_{n-1}(\sigma)\|_1 d\sigma\right).$$

Then, the induction hypothesis implies that (4.17) holds if $\ell=0$. For derivatives we proceed similarly and argument goes almost in parallel. In view of Leibniz's rule it suffices to prove (4.17) for

$$T_n^{(\ell,l)}(\lambda)\equiv S_n^{(l)}(\lambda)T_{n-1}^{(\ell-l)}(\lambda),\quad l=0,\dots,\ell$$

in place of $T_n^{(\ell)}(\lambda)$. If we define $T_{n,\sigma}^{(\ell,l)}u(\lambda)=S_n^{(l)}(\lambda)\widehat{T_{n-1}^{(\ell-l)}u}(\sigma)$, then

$$\widehat{T_n^{(\ell,l)}}u(\rho)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\widehat{T_{n,\sigma}^{(\ell,l)}}u(\rho-\sigma)d\sigma \tag{4.19}$$

as in (4.18), and Minkowski's inequality and the assumption (4.16) imply

$$\int_{\mathbb{R}}\langle\rho\rangle^\tau\left\|\langle x\rangle^{k-\ell+\varepsilon}\widehat{T_n^{(\ell,l)}}u(\rho)\right\|_1 d\rho$$
$$\le\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\langle\rho\rangle^\tau\left\|\langle x\rangle^{k-l+\varepsilon}\widehat{T_{n,\sigma}^{(\ell,l)}}u(\rho-\sigma)\right\|_1 d\rho\right)d\sigma$$
$$\le C_\varepsilon C_n d^{\theta-l}\left(\int_{\mathbb{R}}\left\|\langle x\rangle^{l+\varepsilon}\widehat{T_{n-1}^{(\ell-l)}}u(\sigma)\right\|d\sigma+\int_{\mathbb{R}}\langle\sigma\rangle^\tau\left\|\langle x\rangle^{l+\varepsilon}\widehat{T_{n-1}^{(\ell-l)}}u(\sigma)\right\|_1 d\sigma\right)$$

as previously. Since $l\le k-(\ell-l)$, the induction hypothesis implies that the right side is bounded by

$$C_\varepsilon C_n d^{\theta-l}\cdot C^{n-1}\left(\prod_{j=1}^{n-1}C_j\right)d^{-(\ell-l)+(n-1)\theta}\|\langle x\rangle^{\ell-l+\varepsilon}u\|_1.$$

This implies (4.17) for n and completes the proof. □

Proof of Proposition 4.1. We apply Lemma 4.5 to

$$S_1(\lambda)=\cdots=S_m(\lambda)\equiv T_{\mu_j}(\lambda),\quad \mu_j=3jd,\quad j=-n,\dots,n.$$

Lemma 4.3 implies that $S_1(\lambda),\cdots,S_m(\lambda)$ satisfy the assumption of Lemma 4.5

and we have

$$\int_{\mathbb{R}} \langle\sigma\rangle^{\tau} \big\| \langle x\rangle^{k-\ell+\varepsilon} \mathcal{F}(\{T_{\mu_j}(\lambda)^m u\}^{(\ell)})(\sigma,x)\big\|_1 d\sigma \le C^m d^{m\theta-\ell} \|\langle x\rangle^{\ell+\varepsilon} u\|_1 \tag{4.20}$$

for $\ell = 0,\dots,k$, where C is independent of $0<d<1$, j and u. Choose d so that $Cd^{\theta} < 1/2$ and we sum up both sides of (4.20) for $m = 0,1,\dots$. It follows by virtue of (4.9) that all terms $u_{low,j}$ in the right of (4.8) satisfy

$$\int_{\mathbb{R}} \langle\sigma\rangle^{\tau} \big\| \langle x\rangle^{k-\ell+\varepsilon} \mathcal{F}(u^{(\ell)}_{low,j})(\sigma,x)\big\|_1 d\sigma \le C_\ell \|\langle x\rangle^{\ell+\varepsilon} u\|_1, \quad \ell = 0,\dots,k. \tag{4.21}$$

Hence, by summing up (4.21) over $j=-n,\dots,n$, we see that the same is true for u_{low}. This proves Proposition 4.1. □

5. Proof of Theorem 1.2.

By virtue of Proposition 4.1, the entirely the same argument used for proving Theorem 3.1 implies that, for any $0 \le \varepsilon < \kappa_{\max}$,

$$|(U_{2,low}(t)u,v)| \le Ct^{-k-(3/2)-(\varepsilon/2)} \|\langle x\rangle^{k+\varepsilon}u\| \|\langle x\rangle^{k+\varepsilon}v\|. \tag{5.1}$$

Thus, we have only to deal with $(U_{1,low}(t)u,v)$. We replace u_{low} in (4.2) by the right of (4.8) and v_{low} by the corresponding formula. This produces

$$(U_{1,low}(t)u,v) = -\sum_{a,b=-n}^{n} \frac{1}{2t\pi} \int_{\mathbb{R}} e^{-it\lambda^2} \chi(\lambda/\lambda_0) \big\langle G_0'(\lambda) u_{low,a}(\lambda), v_{low,b}(-\lambda)\big\rangle d\lambda. \tag{5.2}$$

Then, unless $a=b=0$, the integrand vanishes in a neighborhood of $\lambda=0$. Hence all terms except the one with $a=b=0$ in (5.2) are bounded by $Ct^{-k-(3/2)-(\varepsilon/2)}\|\langle x\rangle^{k+\varepsilon}u\|\|\langle x\rangle^{k+\varepsilon}v\|$ and we may put them into the remainder. We are left with

$$\frac{-1}{2t\pi} \int_{\mathbb{R}} e^{-it\lambda^2} \chi(\lambda/\lambda_0) \big\langle G_0'(\lambda) u_{low,0}(\lambda), v_{low,0}(-\lambda)\big\rangle d\lambda$$

which we write, using the Parseval formula, in the following form:

$$\frac{e^{\mp 3\pi/4}}{(2|t|)^{3/2} i\pi} \int_{\mathbb{R}} e^{i\sigma^2/4t} \mathcal{F}\big\{\chi(\lambda/\lambda_0)\big\langle G_0'(\lambda) u_{low,0}(\lambda), v_{low,0}(-\lambda)\big\rangle\big\}(\sigma) d\sigma. \tag{5.3}$$

We write $f(\lambda) = \chi(\lambda/\lambda_0)\langle G_0'(\lambda)u_{low,0}(\lambda), v_{low,0}(-\lambda)\rangle$ for shortening the formula. We expand $e^{i\sigma^2/4t}$ via Taylor's formula

$$\sum_{s=0}^{[k/2]} \frac{1}{s!}\left(i\frac{\sigma^2}{4t}\right)^s + \frac{1}{([k/2]-1)!}\left(i\frac{\sigma^2}{4t}\right)^{[k/2]} \int_0^1 (1-\theta)^{[k/2]-1}\big(e^{i\theta(\sigma^2/4t)} - 1\big)d\theta. \tag{5.4}$$

Inserting the remainder term of (5.4) into (5.3) yields

$$\frac{C}{|t|^{3/2+[k/2]}} \int_0^1 (1-\theta)^{[k/2]-1}\left(\int_{\mathbb{R}} \sigma^{2[k/2]}\big(e^{i\theta(\sigma^2/4t)} - 1\big)\hat{f}(\sigma)d\sigma\right)d\theta. \tag{5.5}$$

We estimate $|e^{i\theta(\sigma^2/4t)} - 1|$ by $C|\sigma^2/4t|^{\varepsilon/2}$ if k is even and by $C|\sigma^2/4t|^{(1+\varepsilon)/2}$ if k is odd and estimate the modulus of (5.5) by

$$\frac{C}{|t|^{(k+3+\varepsilon)/2}} \int_{\mathbb{R}} |\sigma|^{k+\varepsilon}|\hat{f}(\sigma)|d\sigma \le \frac{C}{|t|^{(k+3+\varepsilon)/2}} \int_{\mathbb{R}} \langle\sigma\rangle^{\varepsilon}\big|\widehat{f^{(k)}}(\sigma)\big|d\sigma.$$

Then, Lemma 3.7 and Proposition 4.1 imply that the right side is bounded by $C|t|^{(k+3+\varepsilon)/2}\|\langle x\rangle^{k+\varepsilon}u\|_1\|\langle x\rangle^{k+\varepsilon}v\|_1$ and the contribution from the remainder term (5.5) may also be put into the remainder. Replacing $e^{i\sigma^2/4t}$ in (5.3) by the first term of (5.4) produces

$$\begin{aligned}
&\sum_{s=0}^{[k/2]} \frac{1}{s!}\frac{\sqrt{2}e^{\mp 3\pi/4}i^s}{(2|t|)^{3/2}(4t)^s i\sqrt{\pi}}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \sigma^{2s}\hat{f}(\sigma)d\sigma \\
&\quad = \sum_{s=0}^{[k/2]} \frac{\sqrt{2}e^{\mp 3\pi/4}i^s}{(2|t|)^{3/2}(4t)^s i\sqrt{\pi}s!}\big((-\partial_\lambda^2)^s f\big)(0).
\end{aligned}$$

Recalling that $(1 + G_0(\lambda)V)^{-1}G_0'(\lambda)(1 + VG_0(\lambda))^{-1} = G'(\lambda)$, we see that $((-\partial_\lambda^2)^s f)(0) = (-1)^s\langle G^{(2s+1)}(0)u, v\rangle$ in the right hand side and the leading terms in the expansion are expressed in the form

$$\sum_{s=0}^{[k/2]} \frac{\sqrt{2}e^{\mp 3\pi/4}(-i)^s}{(2|t|)^{3/2}(4t)^s i\sqrt{\pi}s!}\langle G^{(2s+1)}(0)u, v\rangle.$$

Thus, the proof of Theorem 1.2 is completed by the following lemma.

Lemma 5.1. *When j is even, operator $G^{(j+1)}(0)$ is of finite rank.*

Proof. Write $Q(\lambda) = (1 + G_0(\lambda)V)^{-1}$. We have $(1 + VG_0(\lambda)V)^{-1} = Q(-\lambda)^*$. Differentiating (1.13) β times by using Leibniz' rule, we have

$$G^{(j+1)}(0) = \sum_{\alpha+\beta+\gamma=j} \frac{(-1)^\gamma l!}{\alpha!\beta!\gamma!} Q^{(\alpha)}(0) G_0^{(\beta+1)}(0) Q(0)^{*(\gamma)}.$$

If β is even, then $G_0^{(\beta+1)}(0)$ is of finite rank. If β is odd, then either α or γ is odd. If α is odd, $Q^{(\alpha)}(0)$ is of finite rank, since $Q^{(\alpha)}(0)$ is a linear combination of

$$QG_0^{(l_1)}VQG_0^{(l_2)}VQ \cdots G_0^{(l_n)}VQ, \quad l_1 + \cdots + l_n = \alpha,$$

where the variable $\lambda = 0$ is omitted, and at least one of l_j is odd. Likewise, $Q^{*(\gamma)}(0)$ is of finite rank if γ is odd. The lemma follows. $\square$

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