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Multilinear version of reversed Hölder inequality and its applications to multilinear Calderón-Zygmund operators

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Abstract. In this paper, we give a natural, and generalized reverse Hölder inequality, which says that if $\omega_i \in A_\infty$, then for every cube Q,

$$\int_Q \prod_{i=1}^{m} {\omega_i}^{\theta_i} \geq \prod_{i=1}^{m} \left(\frac{\int_Q \omega_i}{[\omega_i]_{A_\infty}} \right)^{\theta_i}, \tag{0.1}$$

where $\sum_{i=1}^{m} \theta_i = 1$, $0 \leq \theta_i \leq 1$.

As a consequence, we get a more general inequality, which can be viewed as an extension of the reverse Jensen inequality in the theory of weighted inequalities. Based on this inequality (0.1), we then give some results concerning multilinear Calderón-Zygmund operators and maximal operators on weighted Hardy spaces, which improve some known results significantly.

1. Introduction and main results.

The theory of the reverse Jensen inequality (see [**1**], [**5**]) says that $\omega \in A_\infty$ if and only if there exists a constant C such that for every cube Q,

$$\frac{1}{|Q|} \int_Q w \leq C \exp\left(\frac{1}{|Q|} \int_Q \log w \right). \tag{1.1}$$

The sharp constant C is defined to be $[\omega]_{A_\infty}$. Meanwhile, as a result of the Jensen inequality and the fact that $(t^\theta - 1)/\theta \downarrow \log t$ as $\theta \downarrow 0$, we know that (see [**6**, p. 11])

$$\exp\left(\frac{1}{|Q|} \int_Q \log w \right) = \lim_{\theta \to 0} \left(\frac{1}{|Q|} \int_Q \omega^\theta \right)^{1/\theta}. \tag{1.2}$$

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Combine (1.1), (1.2) and the monotonic decrease of $((1/|Q|)\int_Q \omega^\theta)^{1/\theta}$ as $\theta \downarrow 0$, we can get $\omega \in A_\infty$ if and only if there exists a constant C such that

$$\frac{1}{|Q|}\int_Q \omega \le C\Big(\frac{1}{|Q|}\int_Q \omega^\theta\Big)^{1/\theta} \tag{1.3}$$

for every cube Q and every $\theta > 0$.

Being different from the classical reverse Hölder inequality (see [**3**] and the references therein), which states that if $\omega \in A_p$ for some p, $1 \le p < \infty$, then there exists constants C and γ such that for every cube Q,

$$\left(\frac{\int_Q \omega^{1+\gamma}}{|Q|}\right)^{1/(1+\gamma)} \le \frac{C}{|Q|}\int_Q \omega, \tag{1.4}$$

it can be seen that (1.3) actually gives the opposite inequality of (1.4). Obviously, it is impossible to deduce the reverse Hölder inequality by using (1.3) directly. And what seems interesting is that we also cannot deduce (1.3) with the reverse Hölder inequality. This can be clarified by taking $\omega = \textit{constant}$ when $[\omega]_{A_\infty} = 1$ and C in (1.4) is bigger than 1.

In this paper, we obtain an inequality which is quite useful as follows:

THEOREM 1.1. *If $\omega_i \in A_\infty$, then for every cube Q, we have*

$$\int_Q \prod_{i=1}^m {\omega_i}^{\theta_i} \ge \prod_{i=1}^m \left(\frac{\int_Q \omega_i}{[\omega_i]_{A_\infty}}\right)^{\theta_i}, \tag{1.5}$$

where $\sum_{i=1}^m \theta_i = 1$, $0 \le \theta_i \le 1$.

As a consequence of Theorem 1.1, we get a more general inequality, which can be viewed as an extension of the reverse Jensen inequality (1.3).

COROLLARY 1.1. *If $\omega_i \in A_\infty$ and $0 \le \sum_{i=1}^m \theta_i \le 1$, then for every cube Q, we have*

$$\int_Q \prod_{i=1}^m {\omega_i}^{\theta_i} \ge |Q|^{1-\sum_i \theta_i} \prod_{i=1}^m \left(\frac{\int_Q \omega_i}{[\omega_i]_{A_\infty}}\right)^{\theta_i}, \tag{1.6}$$

where $0 \le \theta_i \le 1$.

REMARK 1.1. When $m = 1$, (1.6) coincides with (1.3), thus it can be viewed as an extension of the reverse Jensen inequality (1.3). Moreover, basing on (1.3), the following inequality can be induced by definition which seems to be new:

If $\omega \in A_\infty$, and $\omega^{-p'/p} \in A_\infty$, then

$$[\omega]_{A_p} \leq \left[\omega^{-p'/p}\right]_{A_\infty} [\omega]_{A_\infty}. \tag{1.7}$$

Remark 1.2. Although the inequalities (1.5) and (1.6) are much connected with the reverse Jensen inequality, and do not coincide with the well-known reverse Hölder inequality, we would still like to call them generalized reverse Hölder inequalities as they are indeed reverse versions of generalized Hölder inequalities.

Inequality (1.5) turns out to be very suitable to solve some problems arising in the study of multilinear Calderón-Zygmund operators (and maximal operators) with multiple weights. Before we state another theorem, we firstly recall some definitions and backgrounds on the multilinear Calderón-Zygmund operator, as well as its maximal operator.

Definition 1.1 (Multilinear Calderón-Zygmund operators). Let T be a multilinear operator initially defined on the m-fold product of Schwartz spaces and taking values in the space of tempered distributions,

$$T : \mathscr{S}(\boldsymbol{R}^n) \times \cdots \times \mathscr{S}(\boldsymbol{R}^n) \longrightarrow \mathscr{S}'(\boldsymbol{R}^n).$$

Following [**6**], we say that T is an m-linear Calderón-Zygmund operator if for some $1 \leq q_j < \infty$, it extends to a bounded multilinear operator from $L^{q_1} \times \cdots \times L^{q_m}$ to L^q, where

$$\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}, \tag{1.8}$$

and if there exists a function K, defined off the diagonal $x = y_1 = \cdots = y_m$ in $(\boldsymbol{R}^n)^{m+1}$, satisfying

$$T(f_1, \ldots, f_m)(x) = \int_{(\boldsymbol{R}^n)^m} K(x, y_1, \ldots, y_m) f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m$$

for all $x \notin \bigcap_{j=1}^m \operatorname{supp} f_j$;

$$\left|K(y_0, y_1, \ldots, y_m)\right| \leq \frac{A}{\left(\sum_{k,l=0}^m |y_k - y_l|\right)^{mn}}; \tag{1.9}$$

and

$$\left|K(y_0,\dots,y_j,\dots,y_m)-K(y_0,\dots,y_j',\dots,y_m)\right| \le \frac{A\left|y_j-y_j'\right|^{\varepsilon}}{\left(\sum_{k,l=0}^m |y_k-y_l|\right)^{mn+\varepsilon}}, \quad (1.10)$$

for some $\varepsilon > 0$ and all $0\le j \le m$, whenever $|y_j - y_j'| \le 1/2 \max_{0\le k\le m} |y_j - y_k|$.

As in the linear theory, a certain amount of extra smoothness is required for these operators to have such boundedness properties on Hardy spaces. We will assume that $K(y_0, y_1, \dots, y_m)$ satisfies the following estimates

$$\left|\partial_{y_0}^{\alpha_0}\cdots\partial_{y_m}^{\alpha_m} K(y_0,y_1,\dots,y_m)\right| \le \frac{A_\alpha}{\left(\sum_{k,l=0}^m |y_k-y_l|\right)^{mn+|\alpha|}}, \quad (1.11)$$

for all $|\alpha| \le N$, where $\alpha = (\alpha_0, \dots, \alpha_m)$ is an ordered set of m-tuples of nonnegative integers, $|\alpha| = |\alpha_0| + \cdots + |\alpha_m|$, where $|\alpha_j|$ is the order of each multiindex α_j, and N is a large integer to be determined later.

The corresponding maximal operator T_* (see [**9**]) for the m-linear Calderón-Zygmund operator T is given by

$$\begin{aligned} T_*(\vec{f})(x) &= T_*(f_1,\dots,f_m)(x) \\ &= \sup_{\delta>0}\left|\int_{|x-y_1|^2+\cdots+|x-y_m|^2>\delta^2} K(x,y_1,\dots,y_m) f_1(y_1)\cdots f_m(y_m)\, d\vec{y}\right|. \end{aligned} \quad (1.12)$$

In [**8**], the authors proved that every m-linear Calderón-Zygmund operator can extend its boundedness on all other products of Lebesgue spaces with exponents $1 < q_j \le \infty$ satisfying (1.8). Similar results were obtained in [**9**] for T_* by proving a Coltlar's inequality for the maximal operator. Moreover, there are endpoint weak-type estimates when some of the exponents q_j are equal to one. In particular,

$$T : L^1 \times \cdots \times L^1 \to L^{1/m,\infty}. \quad (1.13)$$

In 2001, Grafakos and Kalton [**7**] proved the boundedness for multilinear Calderón-Zygmund operators and its maximal operator on products of Hardy spaces as follows:

THEOREM A ([**7**]). *Let $1 < q_1, \dots, q_m,\ q < \infty$ be fixed indices satisfying*

$$\frac{1}{q_1}+\cdots+\frac{1}{q_m}=\frac{1}{q}$$

and let $0<p_1,\dots,p_m,\ p\le 1$ *be real numbers satisfying*

$$\frac{1}{p_1}+\cdots+\frac{1}{p_m}=\frac{1}{p}.$$

Suppose that K *satisfies* (1.9)–(1.11) *with* $N=[n(1/p)-1]$*. Let* T *be related to* K *and assume that* T *admits an extension that maps* $L^{q_1}\times\cdots\times L^{q_m}$ *into* L^q *with norm* B*. Then both* T *and* T_* *can be extended to a bounded operator from* $H^{p_1}\times\cdots\times H^{p_m}$ *into* L^p*.*

The results for T and T_* on weighted Hardy Spaces with multiple weights were given in [**11**] and [**12**] respectively. Before we state these results, we need to mention the work on multiple weights by Lerner, Ombrosi, Pérez, Torres, Trujillo-González [**10**].

Definition 1.2 ([**10**]) (Multilinear $A_{\vec{p}}$ condition). Let $1\le p_1,\dots,p_m<\infty$. Given $\vec{\omega}=(\omega_1,\dots,\omega_m)$, set

$$\nu_{\vec{\omega}}=\prod_{j=1}^{m}\omega_j^{p/p_j}.$$

We say that $\vec{\omega}$ satisfies the $A_{\vec{p}}$ condition if

$$\sup_Q\left(\frac{1}{|Q|}\int_Q\prod_{i=1}^{m}{\omega_i}^{p/p_i}\right)^{1/p}\prod_{i=1}^{m}\left(\frac{1}{|Q|}\int_Q{\omega_i}^{1-p_i'}\right)^{1/p_i'}<\infty. \tag{1.14}$$

When $p_i=1$, $((1/|Q|)\int_Q{\omega_i}^{1-p_i'})^{1/p_i'}$ is understood as $(\inf_Q\omega_i)^{-1}$.

The $A_{\vec{p}}$ condition turns out to be able to characterize the strong-type inequalities for a more refined multilinear maximal function $\mathscr{M}$ with multiple weights defined by

$$\mathscr{M}(\vec{f})(x)=\sup_{Q\ni x}\prod_{i=1}^{m}\frac{1}{|Q|}\int_Q|f_i(y_i)|\,dy_i.$$

Theorem B ([**10**]). *Let* $1<p_j<\infty$*,* $j=1,\dots,m$*, and* $1/p=1/p_1+\cdots+1/p_m$*. Then the inequality*

$$\|\mathscr{M}(\vec{f})\|_{L^p(\nu_{\vec{\omega}})} \le C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i)} \tag{1.15}$$

holds if and only if $\nu_{\vec{\omega}}$ satisfies the $A_{\vec{p}}$ condition.

The above theorem can be used to obtain the boundedness for multilinear Calderón-Zygmund operators on the weighted spaces.

THEOREM C ([**10**]). *Let T be an m-linear Calderón-Zygmund operator, satisfying* (1.6), (1.7), *$1/p = (1/p_1) + \cdots + (1/p_m)$, and $\vec{\omega}$ satisfy the $A_{\vec{p}}$ condition, $1 < p_i < \infty$. Then*

$$\big\|T(\vec{f})(x)\big\|_{L^p(\nu_{\vec{\omega}})} \le C \prod_{i=1}^m \big\|f_i\big\|_{L^{p_i}(\omega_i)}. \tag{1.16}$$

We are now in a position to state the theorems in [**11**] and [**12**], which our work mainly builds on and improves significantly. We note that in [**12**], the maximal operator is defined by

$$T_*(\vec{f})(x) = \sup_{\delta>0} \left| \int_{\boldsymbol{R}^n} K_\delta(x, y_1, \ldots, y_m) f_1(y_1) \cdots f_m(y_m) d\vec{y} \right|,$$

where $K_\delta(x, y_1, \ldots, y_m) = \phi\big(\sqrt{|x-y_1|^2 + \cdots + |x-y_m|^2}/2\delta\big) K(x, y_1, \ldots, y_m)$ and $\phi(x)$ is a smooth function on $\boldsymbol{R}^n$, which vanishes if $|x| \le 1/4$ and is equal to 1 if $|x| > 1/2$. By proving the following multiple Coltlar's inequality

$$\tilde{T}_*(\vec{f})(x) \le C_\eta \bigg(\big(M(|T(\vec{f})|^\eta)(x)\big)^{(1/\eta)} + (A+W) \prod_{i=1}^m M f_i(x) \bigg),$$

where M denote the Hardy-Littlewood maximal function, the boundedness on weighed Lebesgue spaces for the maximal operator as below was obtained in [**12**]. It can be seen that the results obtained in [**12**] actually hold for T_* defined in (1.12). Hence we have

THEOREM D ([**12**]). *Let T be an m-linear Calderón-Zygmund operator, satisfying* (1.9), (1.10), *$1/p = (1/p_1) + \cdots + (1/p_m)$, $\vec{\omega}$ satisfy the $A_{\vec{p}}$ condition, $1 < p_i < \infty$. Then*

$$\big\|T_*(\vec{f})(x)\big\|_{L^p(\nu_{\vec{\omega}})} \le C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i)}. \tag{1.17}$$

There are many equivalent approaches to define weighted Hardy spaces (see [**4**], [**13**]).

DEFINITION 1.3 ([**13**]). Let f be a distribution, $\omega \in A_\infty$, and $0 < p \leq \infty$. We say f belongs to $H^p_\omega(\boldsymbol{R}^n)$ if there exists a $\Phi \in \mathscr{S}$ with $\int \Phi \neq 0$ so that $M_\Phi f \in L^p_\omega(\boldsymbol{R}^n)$, where $M_\Phi f(x)$ is defined by

$$M_\Phi f(x) = \sup_{t>0} |(f * \Phi_t)(x)|.$$

We define the norm of f in $H^p_\omega(\boldsymbol{R}^n)$ by $\|f\|_{H^p_\omega(\boldsymbol{R}^n)} = \|M_\Phi f\|_{L^p_\omega(\boldsymbol{R}^n)}$.

We will appeal to the atomic decomposition characterization of weighted Hardy spaces to establish the boundedness of the operators. As if $\omega \in A_p$ for $1 < p < \infty$, then $\omega \in A_r$ for all $r > p$ and $\omega \in A_q$ for some $1 < q < p$, we will use $q_\omega := \inf\{q > 1 : \omega \in A_q\}$ to denote the critical index of ω.

DEFINITION 1.4. Assume that $\omega \in A_\infty$ with critical index q_ω. Let $[\cdot]$ be the greatest integer function. If $p \in (0,1]$, $q \in [q_\omega, \infty)$, $s \in \boldsymbol{Z}$ satisfying $s \geq [n(q_\omega/p - 1)]$, a real-valued function $a(x)$ is called (p,q,s)-atom centered at x_0 with respect to ω (or $\omega - (p,q,s)$-atom centered at x_0) if

(a) $a \in L^q_\omega(\boldsymbol{R}^n)$ and is supported in a cube Q centered at x_0;
(b) $\|a\|_{L^q_\omega} \leq \omega(Q)^{(1/q)-(1/p)}$;
(c) $\int_{\boldsymbol{R}^n} a(x) x^\alpha dx = 0$ for every multi-index α with $|\alpha| \leq s$.

When $q = \infty$, L^∞_ω will be taken to mean L^∞ and $\|f\|_{L^\infty_\omega} = \|f\|_{L^\infty}$.

From [**4**], we know that if $w \in A_q$, $0 < p \leq 1 \leq q \leq \infty$, $p \neq q$, and $f \in H^p_\omega(\boldsymbol{R}^n)$, there exist a sequence a_i of $\omega - (p, q, [n(q_w/p) - 1])$-atoms, and a sequence λ_i of real numbers with $\sum |\lambda_i|^p \leq C\|f\|^p_{H^p_\omega}$ such that $f = \sum \lambda_i a_i$ both in the sense of distributions and in the H^p_ω norm.

THEOREM E ([**11**], [**12**]). *Let* $1 < q_1, \ldots, q_m$, $q < \infty$, $0 < p_1, \ldots, p_m$, $p \leq 1$, *satisfying*

$$\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{1}{q} \tag{1.18}$$

and

$$\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}. \tag{1.19}$$

Let T be an m-linear Calderón-Zygmund operator satisfying (1.9), (1.10) *and* (1.11) *with*

$$N = \max_{1\le i\le m}[n((q_i)_\omega/p_i - 1)],\ [(q_i/p_i - 1)mn],$$

and T_* be its corresponding maximal operator, then

(i) *If $\omega \in A_{q_1} \cap \cdots \cap A_{q_m}$, then*

$$\left\|T(\vec{f})(x)\right\|_{L^p_\omega} \le C\prod_{i=1}^m \left\|f_i\right\|_{H^{p_i}_\omega} \tag{1.20}$$

and

$$\left\|T_*(\vec{f})(x)\right\|_{L^p_\omega} \le C\prod_{i=1}^m \left\|f_i\right\|_{H^{p_i}_\omega}. \tag{1.21}$$

(ii) *If for each i, $\omega_i \in A_1$, let*

$$\nu_{\vec{\omega}} = \prod_{j=1}^m \omega_j^{p/p_j}, \tag{1.22}$$

then

$$\left\|T(\vec{f})(x)\right\|_{L^p_{\nu_{\vec{\omega}}}} \le C\prod_{i=1}^m \left\|f_i\right\|_{H^{p_i}_{\omega_i}} \tag{1.23}$$

and

$$\left\|T_*(\vec{f})(x)\right\|_{L^p_{\nu_{\vec{\omega}}}} \le C\prod_{i=1}^m \left\|f_i\right\|_{H^{p_i}_{\omega_i}}. \tag{1.24}$$

We will apply Theorem 1.1 to the multilinear Calderón-Zygmund operator and its maximal operator on weighted Hardy spaces, and get the following theorem.

Theorem 1.2. *Let $1 < q_1, \dots, q_m$, $q < \infty$, $0 < p_1, \dots, p_m$, $p \le 1$, satisfying*

$$\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{1}{q}$$

and

$$\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}.$$

Let T be an m-linear Calderón-Zygmund operator related to K and assume that T admits an extension that maps $L^{q_1} \times \cdots \times L^{q_m}$ into L^q, satisfying (1.9), (1.10) *and* (1.11) *with $N \geq \max_{1\leq i\leq m}\{[(1/p_i - 1)mn]\}$ and T_* be its corresponding maximal operator. We have the following results*:

If for each i, $\omega_i \in A_\infty$, let

$$\nu_{\vec{\omega}} = \prod_{j=1}^{m} \omega_j^{p/p_j},$$

then

$$\left\|T(\vec{f})(x)\right\|_{L^p_{\nu_{\vec{\omega}}}} \leq C \prod_{i=1}^{m} \left\|f_i\right\|_{H^{p_i}_{\omega_i}} \tag{1.25}$$

and

$$\left\|T_*(\vec{f})(x)\right\|_{L^p_{\nu_{\vec{\omega}}}} \leq C \prod_{i=1}^{m} \left\|f_i\right\|_{H^{p_i}_{\omega_i}}. \tag{1.26}$$

Remark 1.3. Note that the condition $\omega_i \in A_1$ in Theorem E is replaced by the condition $\omega_i \in A_\infty$. Thus Theorem 1.2 improves the results in Theorem E essentially and significantly. We also obtain a bigger random of N in (1.11), which allows more Calderón-Zygmund operators having such boundness properties on Hardy spaces.

2. Proof of the Theorems.

Proof of Theorem 1.1. We first prove that when $\omega_i \in A_q$, $1 < q < \infty$,

$$\int_Q \nu_{\vec{\omega}} \geq \prod_{i=1}^{m} \left(\frac{\int_Q \omega_i}{[\omega_i]_{A_q}} \right)^{\theta_i}. \tag{2.1}$$

Once this is proved, as $[\omega]_{A_q} \downarrow [\omega]_{A_\infty}$ for any $\omega \in A_q$, $1 \leq q < \infty$, and the fact that $A_\infty = \bigcup_{q>1} A_q$ (see [**2**]), take the limit $q \to \infty$ in (2.1), and the theorem can be obtained.

We now prove (2.1). Set

$$\nu_{\vec{\omega}} = \prod_{i=1}^{m} \omega_i^{\theta_i}. \tag{2.2}$$

By Hölder's inequality,

$$\begin{aligned} |Q| &= \int_Q \left(\nu_{\vec{\omega}} \cdot \frac{1}{\nu_{\vec{\omega}}}\right)^{1/q} \\ &\le \left(\int_Q \nu_{\vec{\omega}}\right)^{1/q} \left(\int_Q \nu_{\vec{\omega}}^{-q'/q}\right)^{1/q'} \\ &= \left(\int_Q \nu_{\vec{\omega}}\right)^{1/q} \left(\int_Q \prod_{i=1}^{m} \omega_i^{-\theta_i(q'/q)}\right)^{1/q'} \\ &\le \left(\int_Q \nu_{\vec{\omega}}\right)^{1/q} \prod_{i=1}^{m} \left(\int_Q \omega_i^{-q'/q}\right)^{\theta_i/q'}. \end{aligned} \tag{2.3}$$

If $\omega_i \in A_q$, by definition,

$$\sup_Q \frac{\int_Q \omega_i \big(\int_Q \omega_i^{-q'/q}\big)^{q/q'}}{|Q|^q} \le [\omega_i]_{A_q}. \tag{2.4}$$

Then we have

$$\int_Q \nu_{\vec{\omega}} \ge \frac{|Q|^q}{\prod_{i=1}^m \big(\int_Q \omega_i^{-q'/q}\big)^{q/q'} \theta_i} \ge \frac{|Q|^q}{\prod_{i=1}^m \Big(\frac{[\omega_i]_{A_q}|Q|^q}{\int_Q \omega_i}\Big)^{\theta_i}} = \prod_{i=1}^{m} \left(\frac{\int_Q \omega_i}{[\omega_i]_{A_q}}\right)^{\theta_i}. \tag{2.5}$$

This is just (2.1) and Theorem 1.1 is thus proved. □

PROOF OF COROLLARY 1.1. We give two methods to prove Corollary 1.1.

Method 1:

For $\sum_{i=1}^m \theta_i < 1$, we denote $\varepsilon = 1 - \sum_{i=1}^m \theta_i > 0$. Repeat the step in (2.3) and using Hölder's inequality (since $\theta_1 + \cdots + \theta_m + \varepsilon = 1$), we get

$$
\begin{aligned}
|Q| &= \int_Q \left(\nu_{\vec{\omega}} \cdot \frac{1}{\nu_{\vec{\omega}}}\right)^{1/q} \\
&\le \left(\int_Q \nu_{\vec{\omega}}\right)^{1/q} \left(\int_Q \nu_{\vec{\omega}}^{-q'/q}\right)^{1/q'} \\
&\le \left(\int_Q \nu_{\vec{\omega}}\right)^{1/q} \prod_{i=1}^{m} \left(\int_Q \omega_i^{-q'/q}\right)^{\theta_i/q'} |Q|^{\varepsilon/q'}. \qquad (2.6)
\end{aligned}
$$

Thus by (2.4) and (2.6), we get

$$
\begin{aligned}
\int_Q \nu_{\vec{\omega}} &\ge \frac{|Q|^q}{|Q|^{(1-\sum_{i=1}^m \theta_i)q/q'} \prod_{i=1}^m \left(\int_Q \omega_i^{-q'/q}\right)^{q/q'\theta_i}} \\
&\ge \frac{|Q|^q}{|Q|^{(1-\sum_{i=1}^m \theta_i)q/q'} \prod_{i=1}^m \left(\frac{[\omega_i]_{A_q}|Q|^q}{\int_Q \omega_i}\right)^{\theta_i}} \ge |Q|^{1-\sum_i \theta_i} \prod_{i=1}^m \left(\frac{\int_Q \omega_i}{[\omega_i]_{A_q}}\right)^{\theta_i}. \qquad (2.7)
\end{aligned}
$$

Taking the limit $q \to \infty$ in (2.7) yields the desired results.

Method 2:

By using Theorem 1.1 directly, we get

$$
\begin{aligned}
\int_Q \prod_{i=1}^m {\omega_i}^{\theta_i} &= \int_Q \prod_{i=1}^m {\omega_i}^{\theta_i} 1^{1-\sum_i \theta_i} \ge \prod_{i=1}^m \left(\frac{\int_Q \omega_i}{[\omega_i]_{A_\infty}}\right)^{\theta_i} \left(\frac{\int_Q 1}{[1]_{A_\infty}}\right)^{1-\sum_i \theta_i} \\
&= |Q|^{1-\sum_i \theta_i} \prod_{i=1}^m \left(\frac{\int_Q \omega_i}{[\omega_i]_{A_\infty}}\right)^{\theta_i}.
\end{aligned}
$$

Proof of Theorem 1.2. We follow the main idea in [**11**] and [**12**] to prove this theorem.

We use the atomic decomposition of H^p_ω spaces [**4**], and we will firstly consider finite sums of atoms. Assume $f_i = \sum_{k_i} \lambda_{i,k_i} a_{i,k_i}$, where a_{i,k_i} are w_i-(p_i, ∞, s)-atoms. This means that they are supported in cubes Q_{i,k_i}, $|a_{i,k_i}| \le \omega(Q_{i,k_i})^{-1/p_i}$ and

$$
\int_{Q_{i,k_i}} a_{i,k_i}(x) x^\alpha dx = 0, \quad |\alpha| \le s, \ s \ge [n((q_i)_\omega/p_i - 1)]. \qquad (2.8)
$$

We first consider the m-linear Calderón-Zygmund operator and prove (1.25).

Denote by c_{i,k_i} and $l(Q_{i,k_i})$ the center and the side length of Q_{i,k_i}, and let $Q^*_{i,k_i} = 8\sqrt{n}Q_{i,k_i}$. We write

$$\begin{aligned} &T(f_1,\dots,f_m)(x)\\ &\quad= \sum_{k_1}\cdots\sum_{k_m}\lambda_{1,k_1}\cdots\lambda_{m,k_m}T(a_{1,k_1},\dots,a_{m,k_m})(x)\\ &\quad\le \sum_{k_1}\cdots\sum_{k_m}|\lambda_{1,k_1}|\cdots|\lambda_{m,k_m}|\,|T(a_{1,k_1},\dots,a_{m,k_m})(x)|\chi_{Q^*_{1,k_1}\cap\cdots\cap Q^*_{m,k_m}}\\ &\qquad+\sum_{k_1}\cdots\sum_{k_m}|\lambda_{1,k_1}|\cdots|\lambda_{m,k_m}|\,|T(a_{1,k_1},\dots,a_{m,k_m})(x)|\chi_{{Q^*_{1,k_1}}^c\cup\cdots\cup{Q^*_{m,k_m}}^c}\\ &\quad= I_1(x)+I_2(x). \end{aligned} \tag{2.9}$$

We first estimate I_2. This part has been estimated in [**11**], but we note that it can actually be obtained by following [**7**] in the unweighted situation and the easy fact that if $|a_{i,k_i}| \le \omega(Q_{i,k_i})^{-1/p_i}$, then $\int_{Q_{i,k_i}}|a_{i,k_i}| \le |Q_{i,k_i}|\omega_i(Q_{i,k_i})^{-1/p_i}$. We then can get

$$I_2(x) \le C\prod_{i=1}^m\left(\sum_{k_i}\frac{\omega_i(Q_{i,k_i})^{-1/p_i}|Q_{i,k_i}|^{1+(N+1)/nm}}{(|x-c_{i,k_i}|+l(Q_{i,k_i}))^{n+(N+1)/m}}\right). \tag{2.10}$$

We set the m-linear Calderón-Zygmund operator satisfies (1.11) with $N \ge \max_{1\le i\le m}\{[(1/p_i-1)mn]\}$ (accordingly, we assume a_{i,k_i} are (p_i,∞,s)-atoms with $s\ge N$). Then we have

$$\|I_2\|_{L^p_{\nu_{\vec{\omega}}}} \le C\prod_{i=1}^m\left(\sum_{k_i}|\lambda_{i,k_i}|^{p_i}\right)^{1/p_i}. \tag{2.11}$$

So it remains to estimate $I_1(x)$. For fixed $k_1,\dots,k_m$, assume that $Q^*_{1,k_1}\cap\cdots\cap Q^*_{m,k_m}\neq\emptyset$, otherwise there is nothing to prove. Assume that the side length of the cube Q_{α,k_α} is the smallest among the side lengths of the cubes $Q_{1,k_1},\dots,Q_{m,k_m}$. We take a cube $G_{k_1,\dots,k_m}$ such that

$$Q^*_{1,k_1}\cap\cdots\cap Q^*_{m,k_m}\subset G_{k_1,\dots,k_m}\subset G^*_{k_1,\dots,k_m}\subset Q^{**}_{1,k_1}\cap\cdots\cap Q^{**}_{m,k_m}$$

and

$$\omega_i\big(G_{k_1,\dots,k_m}\big) \geq C\omega_i\big(Q^*_{\alpha,k_\alpha}\big),$$

$1 < i < m$.

Let k be a positive integer, $q = kp$, $q_i = kp_i$, then $1/q = \sum_i (1/q_i)$,

$$\nu_{\vec{\omega}} = \prod_{i=1}^{m} \omega_i^{p/p_i} = \prod_{i=1}^{m} \omega_i^{q/q_i}. \tag{2.12}$$

We assume k to be big enough such that $q > 1$, and $\omega_i \in A_{q_i}$. Set $\vec{\omega} = (\omega_1, \dots, \omega_m)$. By [**10**], we know $\vec{\omega}$ satisfies the $A_{\vec{q}}$ condition. According to (1.6) and Theorem C, we have

$$\begin{aligned}
&\frac{1}{\nu_{\vec{\omega}}(G_{k_1,\dots,k_m})} \int_{G_{k_1,\dots,k_m}} \big|T(a_{1,k_1}, \dots, a_{m,k_m})(x)\big| \nu_{\vec{\omega}}(x)dx \\
&\leq C\nu_{\vec{\omega}}(G_{k_1,\dots,k_m})^{-1/q} \prod_{i=1}^{m} \|a_{i,k_i}\|_{L^{q_i}_{\omega_i}} \\
&\leq C\bigg(\int_{G_{k_1,\dots,k_m}} \prod_{i=1}^{m} \omega_i^{q/q_i}\bigg)^{-1/q} \prod_{i=1}^{m} \omega_i(Q^*_{\alpha,k_\alpha})^{(1/q_i)-(1/p_i)} \\
&\leq C\prod_{i=1}^{m} \bigg(\frac{\int_{G_{k_1,\dots,k_m}} \omega_i}{[\omega_i]_{A_\infty}}\bigg)^{-q/q_i \cdot 1/q} \prod_{i=1}^{m} \omega_i(Q^*_{\alpha,k_\alpha})^{(1/q_i)-(1/p_i)} \\
&\leq C\prod_{i=1}^{m} \big(\omega_i(Q^*_{\alpha,k_\alpha})\big)^{-q/q_i \cdot 1/q} \prod_{i=1}^{m} \omega_i(Q^*_{\alpha,k_\alpha})^{(1/q_i)-(1/p_i)} \\
&\leq C\prod_{i=1}^{m} \omega_i(Q^*_{\alpha,k_\alpha})^{-1/p_i}.
\end{aligned} \tag{2.13}$$

To estimate I_1, we need the following lemma in [**11**].

Lemma 2.1. *Let $0 < p \leq 1$. Then there is a constant $C = C(p)$ such that for all finite collections of cubes $\{Q_k\}_{k=1}^m$ in $\boldsymbol{R}^n$ and all nonnegative functions $g_k \in L_\omega$ with* $\operatorname{supp} g_k \subset Q_k$ *we have*

$$\bigg\|\sum_{k=1}^{m} g_k\bigg\|_{L^p(\omega)} \leq C\bigg\|\sum_{k=1}^{m} \frac{1}{\omega(Q_k)} \int_{Q_k} g_k(x)\omega(x)dx\chi_{\tilde{Q}_k}\bigg\|_{L^p(\omega)}. \tag{2.14}$$

We note that when $\omega = 1$, this lemma was proved in [**7**].

By Lemma 2.1, we obtain

$$\begin{aligned}\|I_1\|_{L^p_{\nu_{\vec\omega}}} &\le C\Big\| \sum_{k_1}\cdots\sum_{k_m} |\lambda_{1,k_1}|\cdots|\lambda_{m,k_m}| \prod_{i=1}^m \omega_i(Q^*_{\alpha,k_\alpha})^{-1/p_i}\chi_{G^*_{k_1,\dots,k_m}}\Big\|_{L^p_{\nu_{\vec\omega}}} \\ &\le C\prod_{i=1}^m \Big\|\Big(\sum_{k_i}|\lambda_{i,k_i}|\omega_i(Q^*_{\alpha,k_\alpha})^{-1/p_i}\omega_i^{1/p_i}\chi_{G^*_{k_1,\dots,k_m}}\Big)\Big\|_{L^{p_i}} \\ &\le C\prod_{i=1}^m\Big(\sum_{k_i}|\lambda_{i,k_i}|^{p_i}\Big)^{1/p_i}. \end{aligned} \tag{2.15}$$

Summing (2.11) and (2.15), we get

$$\|T(\vec f)\|_{L^p_{\nu_{\vec\omega}}} \le C\prod_{i=1}^m\Big(\sum_{k_i}|\lambda_{i,k_i}|^{p_i}\Big)^{1/p_i}, \tag{2.16}$$

for $f_i = \sum_{k_i}\lambda_{i,k_i}a_{i,k_i}$, where a_{i,k_i} are (p_i,∞,s)-atoms.

Thus so far we have proved (2.16) when $\omega_i \in A_{kp_i}$. If $\omega_i \in A_\infty$, then there exists r_i, $1 < r_i < \infty$, such that $\omega_i \in A_{r_i}$. Let k be big enough that $kq_i > r_i$, $1 < r_i < \infty$. Then the above argument shows the same boundedness of the m-linear Calderón-Zygmund operator for such weights ω_i.

Note that T (also T_*) is an operator bounded from $\prod_{i=1}^m L^{2m}(w_i)$ to $L^2(\nu_{\vec\omega})$ (if we replace T by T_*, the following argument still works from (2.19) to (2.21)). For general $f_i \in L^{2m}(w_i)\cap H^{p_i}(w_i)$, by the n-dimensional version of Theorem II.3.6 (see [**4**]), there exist sequences of (p_i,∞)-atoms a_{i,k_i} and sequences of numbers λ_{i,k_i} satisfying

$$\sum_{k_i=1}^\infty |\lambda_{i,k_i}|^{p_i} \le C\|f_i\|^{p_i}_{H^{p_i}(w_i)}, \tag{2.17}$$

and

$$f_i(x) = \sum_{k_i=1}^\infty \lambda_{i,k_i}a_{i,k_i}(x) \ \text{ a.e. and in } \ L^{2m}(w_i). \tag{2.18}$$

Set $g_{i,N} = \sum_{k_i=1}^N \lambda_{i,k_i}a_{i,k_i}(x)$. Then, since $g_{i,N}$ tends to f_i in $L^{2m}(w_i)$, and T is bounded from $\prod_{i=1}^m L^{2m}(w_i)$ to $L^2(\nu_{\vec\omega})$, we see that $T(\vec{g_N})$ tends to $T(\vec f)$ in

$L^2(\nu_{\vec{\omega}})$, where $\vec{g_N} = (g_{1,N}, \ldots, g_{m,N})$ and $\vec{f} = (f_1, \ldots, f_m)$. Taking a subsequence if necessary, we may assume

$$\lim_{n\to\infty} T(\vec{g_N}) = T(\vec{f}) \text{ a.e.} \tag{2.19}$$

Now, from (2.16) and (2.17) we have

$$\left(\int |T(\vec{g_N})(x)|^p \nu_{\vec{\omega}}(x)dx\right)^{1/p} \leq C\prod_{i=1}^m \left(\sum_{k_i=1}^N |\lambda_{i,k_i}|^{p_i}\right)^{1/p_i} \leq C\prod_{i=1}^m \|f_i\|_{H^{p_i}(w_i)}. \tag{2.20}$$

Thus, by (2.19) and Fatou's lemma we get

$$\left(\int |T(\vec{f})(x)|^p \nu_{\vec{\omega}}(x)dx\right)^{1/p} \leq C\prod_{i=1}^m \|f_i\|_{H^{p_i}(w_i)}, \quad f_i \in L^{2m}(w_i) \cap H^{p_i}(w_i). \tag{2.21}$$

Since $L^{2m}(w_i)\cap H^{p_i}(w_i)$ is dense in $H^{p_i}(w_i)$, we can extend T from $L^{2m}(w_i)\cap H^{p_i}(w_i)$ to $H^{p_i}(w_i)$, by using (2.21), and still get

$$\left\|T(\vec{f})(x)\right\|_{L^p_{\nu_{\vec{\omega}}}} \leq C\prod_{i=1}^m \|f_i\|_{H^{p_i}_{\omega_i}}.$$

This establishes (1.25).

We now prove (1.26) for the maximal operator. The step will be almost identical as above, as the multilinear Calderón-Zygmund operators and the maximal operators share many similar properties. We also use the atomic decomposition of H^p_ω spaces and we only need to consider finite sums of atoms for H^p_ω. We write

$$\begin{aligned}
&T_*(f_1, \ldots, f_m)(x)\\
&\quad= \sum_{k_1}\cdots\sum_{k_m} \lambda_{1,k_1}\cdots\lambda_{m,k_m} T_*(a_{1,k_1}, \ldots, a_{m,k_m})(x)\\
&\quad\leq \sum_{k_1}\cdots\sum_{k_m} |\lambda_{1,k_1}|\cdots|\lambda_{m,k_m}|\,|T_*(a_{1,k_1}, \ldots, a_{m,k_m})(x)|\chi_{Q^*_{1,k_1}\cap\cdots\cap Q^*_{m,k_m}}\\
&\qquad+ \sum_{k_1}\cdots\sum_{k_m} |\lambda_{1,k_1}|\cdots|\lambda_{m,k_m}|\,|T_*(a_{1,k_1}, \ldots, a_{m,k_m})(x)|\chi_{{Q^*_{1,k_1}}^c\cup\cdots\cup {Q^*_{m,k_m}}^c}\\
&\quad= I_1(x) + I_2(x).
\end{aligned} \tag{2.22}$$

The estimate of I_2 will be the same as in [**12**]. We omit its proof and only have to estimate $I_1(x)$.

For fixed $k_1, \dots, k_m$, we still assume that $Q^*_{1,k_1} \cap \cdots \cap Q^*_{m,k_m} \neq \emptyset$. We introduce the same $G_{k_1,\dots,k_m}$ and Q^*_{α,k_α} as in the above proof. Also set $q = kp$, $q_i = kp_i$. Let k be big enough such that $q > 1$, and assume again $\omega_i \in A_{q_i}$. Set $\vec{\omega} = (\omega_1, \dots, \omega_m)$. By [**10**], we know $\vec{\omega}$ satisfies the $A_{\vec{q}}$ condition. According to (1.6) and Theorem D, we have

$$\begin{aligned} &\frac{1}{\nu_{\vec{\omega}}(G_{k_1,\dots,k_m})} \int_{G_{k_1,\dots,k_m}} |T_*(a_{1,k_1}, \dots, a_{m,k_m})(x)| \nu_{\vec{\omega}}(x) dx \\ &\leq C \bigg(\prod_{i=1}^m \omega_i(G_{k_1,\dots,k_m})^{q/q_i} \bigg)^{-1/q} \prod_{i=1}^m \omega_i(Q^*_{\alpha,k_\alpha})^{(1/q_i)-(1/p_i)} \\ &\leq C \prod_{i=1}^m \omega_i(Q^*_{\alpha,k_\alpha})^{-1/p_i}. \end{aligned} \tag{2.23}$$

By Lemma 2.1 again,

$$\begin{aligned} \|I_1\|_{L^p_{\nu_{\vec{\omega}}}} &\leq C \bigg\| \sum_{k_1} \cdots \sum_{k_m} |\lambda_{1,k_1}| \cdots |\lambda_{m,k_m}| \prod_{i=1}^m \omega_i(Q^*_{\alpha,k_\alpha})^{-1/p_i} \chi_{G^*_{k_1,\dots,k_m}} \bigg\|_{L^p_{\nu_{\vec{\omega}}}} \\ &\leq C \prod_{i=1}^m \bigg\| \bigg(\sum_{k_i} |\lambda_{i,k_i}| \omega_i(Q^*_{\alpha,k_\alpha})^{-1/p_i} \omega_i^{1/p_i} \chi_{G^*_{k_1,\dots,k_m}} \bigg) \bigg\|_{L^{p_i}} \\ &\leq C \prod_{i=1}^m \bigg(\sum_{k_i} |\lambda_{i,k_i}|^{p_i} \bigg)^{1/p_i}. \end{aligned} \tag{2.24}$$

We can follow the same steps as in the case of the m-linear Calderón-Zygmund operator T, and the proof will be omitted. We then get (1.26) and the proof of Theorem 1.2 is completed. □

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