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Necessary and sufficient conditions for the existence of an n-subtle cardinal

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Abstract. We extend the work of Abe in [**1**], to show that the strong partition relation $C \to (n+2)^{n+1}_{<\text{-reg}}$, for every $C \in \text{WNS}^*_{\kappa,\lambda}$, is a consequence of the existence of an n-subtle cardinal. We then build on Kanamori's result in [**10**], that the existence of an n-subtle cardinal is equivalent to the existence of a set of ordinals containing a homogeneous subset of size $n+2$ for each regressive coloring of $n+1$-tuples from the set. We use this result to show that a seemingly weaker relation, in the context of $P_\kappa\lambda$ is also equivalent. This relation is a new type of regressive partition relation, which we then attempt to characterize.

1. Introduction.

Subtle cardinals were introduced in an unpublished manuscript of 1969 by R. Jensen and K. Kunen, as part of a group of combinatorial large cardinal hypotheses between weakly compact and measurable in consistency strength. The other significant member of this group is the *ineffable* cardinals.

In 1975, J. Baumgartner published his monumental study of the n-subtle and n-ineffable cardinals, where he proves a key lemma [**3**, Theorem 4.1] relating n-subtle cardinals to indescribability, using it to reduce n-ineffable cardinals to a sort of composition of n-subtle cardinals and indescribability. This line of study is continued in [**9**] and [**2**].

In that study, Baumgartner also proved that both subtle and ineffable cardinals have equivalent formulations in terms of a regressive partition relation and a $\diamondsuit$-like statement.

Around the same time as Jensen and Kunen were introducing subtle cardinals, the study of strong combinatorial hypotheses on the structure $P_\kappa\lambda = \{x \subseteq \lambda : |x| < \kappa\}$ began with the introduction of supercompact cardinals by Solovay and Reinhardt, which are equivalent to the existence of a normal ultrafilter on $P_\kappa\lambda$ for every $\lambda \geq \kappa$. Soon after it was realized that the rich variety of strong combinatorial

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hypotheses on $P_\kappa\lambda$ was worth studying for its own sake.

The initial studies in this direction were done by Jech [**8**], Menas [**12**], and Magidor [**11**] who found properly intertwined hierarchies with supercompactness being the common limit. Although Jech, Menas, and Magidor had considered regressive functions on $P_\kappa\lambda$, they had used *choice* functions ($f(x_1, x_2, \ldots, x_n) \in x_1$) for this. By the late 1980's, the work of Carr and Pelletier [**6**], [**4**], [**5**] had convinced most researchers that using $f(x_1, \ldots, x_n) < x_1$ where $<=_{def} \{\langle x, y\rangle : x \subset y \text{ and } |x| < |y \cap \kappa|\}$, was the "right" way to generalize regressive functions to $P_\kappa\lambda$.

While Carr and others found satisfying analogues of ineffable and weakly compact cardinals in the $P_\kappa\lambda$ context, it was not so with subtle cadinals. In 2005, Y. Abe gave a $P_\kappa\lambda$ version of subtlety which satisfies a key lemma analogous to Baumgartner's [**1**]. Unlike Carr's analogues, however, Abe's notion, called "strongly subtle", was in fact no stronger than subtle.

In Section 3, we prove that the generalization of Abe's lemma to strongly n-subtle yields a strong partition-theoretic consequence of n-subtlety, which may be equivalent.

On the other hand, in [**10**], Kanamori had showed that the existence of a n-subtle cardinal is actually equivalent to a weaker regressive partition relation than that given by Baumgartner. In Section 4, we give an (ostensibly) weaker condition for the existence of a n-subtle cardinal, involving a new type of partition relation ("ordertype-regressive") on $P_\kappa\lambda$.

In the final section, we attempt to characterize this partition relation.

2. Preliminaries.

Our set theoretic terminology is standard: $\alpha, \beta, \ldots$ stand for ordinals, while $\kappa, \lambda, \ldots$ usually denote cardinals. Also, $[X]^n$ denotes the set of unordered n-tuples from X.

We first introduce the partition relation and regressive partition relation for cardinals and subsets of cardinals, which were first studied by Ramsey and extended into the transfinite by Erdős and Rado [**7**], who introduced the "arrow" notation:

- $A \to (m)^n_k$ means:

For all $f : [A]^n \to k$, there exists $H \in [A]^m$ homogeneous for f, i.e. there exists a γ such that for every $\alpha_1, \alpha_2, \ldots, \alpha_n \in [H]^n$, $f(\alpha_1, \alpha_2, \ldots, \alpha_n) = \gamma$.

This γ will be called "the color of H".

- For a set A of ordinals, $A \to (m)^n_{\text{reg}}$ means:

For all $f : [A]^n \to A$ such that $f(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \alpha_1$ ("f is regressive"), there exists $H \in [A]^m$ homogeneous for f, i.e. there exists γ such that for all $\alpha_1, \alpha_2, \ldots, \alpha_n \in [H]^n$, $f(\alpha_1, \alpha_2, \ldots, \alpha_n) = \gamma$.

Again, this γ will be called "the color of H".

We will also need the following notations and facts concerning ideals over κ:

DEFINITION 2.1. An *ideal* I on κ is a subset of $P(\kappa)$ satisfying:

(i) $\forall X(X \in I \wedge Y \subseteq X \to Y \in I)$, and
(ii) $\forall X \forall Y(X \in I \wedge Y \in I \to X \cup Y \in I)$.

DEFINITION 2.2. An ideal I is *κ-complete* if it is closed under unions of size $< \kappa$: $\forall \langle X_\alpha : \alpha < \gamma \rangle \in [I]^{<\kappa}$, $\bigcup X_\alpha \in I$.

DEFINITION 2.3. An ideal is *normal* if it is closed under diagonal unions: $\forall \langle X_\alpha : \alpha < \kappa \rangle \in [I]^\kappa$, $\nabla X_\alpha \doteq \{\beta : \exists \alpha < \beta (\beta \in X_\alpha)\} \in I$.

I_κ denotes the ideal of bounded subsets of κ.

NS_κ denotes the ideal of nonstationary subsets of κ (the dual of the club filter). It is also the least normal ideal.

For any ideal $I \subseteq P(\kappa)$, I^* denotes the filter dual to I, and I^+ the sets not in the ideal. Using these notations, we can extend our partition relation notation in two ways:

- $I^+ \to (m)^n_k$ means:

For all $X \in I^+, X \to (m)^n_k$.

And also:

- $A \to (J^+)^n_k$ means:

For all $f : [A]^n \to k$, there exists $H \in J^+$ homogeneous for f, i.e. $\exists \gamma (\forall \alpha_1, \alpha_2, \ldots, \alpha_n \in [H]^n, f(\alpha_1, \alpha_2, \ldots, \alpha_n) = \gamma)$.

These notations can also be used with the regressive partition relation. For example, Fodor's theorem is expressible as: "I is normal iff $I^+ \to (I^+)^1_{\mathrm{reg}}$".

For every one of the combinatorial properties P we will define for κ in this paper, we say that $X \subseteq \kappa$ has property P if the same sentence holds with κ replaced by X. Also, if κ has one of these properties P, then the corresponding ideal I of sets *not* having the property P will generally be proper, κ-complete, and also normal.

In his seminal paper [**3**], Baumgartner studied the n-subtle cardinals, generalizations of the subtle cardinals introduced by Kunen and Jensen. There Baumgartner showed that n-subtle cardinals have equivalent characterizations in terms of regressive partition relations and a $\diamondsuit$-like statement:

DEFINITION 2.4. κ is *n-subtle* iff:

(i) $\mathrm{NS}^*_\kappa \to (n+2)^{n+1}_{\mathrm{reg}}$ (i.e. $\forall C \in \mathrm{NS}^*_\kappa, C \to (n+2)^{n+1}_{\mathrm{reg}}$).
(ii) $\mathrm{NS}^*_\kappa \to (\gamma)^{n+1}_{\mathrm{reg}}$, for any $\gamma < \kappa$.
(iii) $\forall C \in \mathrm{NS}^*_\kappa, \forall \langle S_{\alpha_1,\ldots,\alpha_n} : \alpha_1 < \cdots < \alpha_n < \kappa \rangle$ where $S_{\alpha_1,\ldots,\alpha_n} \subseteq \alpha_1$, $\exists \beta_1 < \cdots < \beta_{n+1}$ all in C satisfying $S_{\beta_2,\ldots,\beta_{n+1}} \cap \beta_1 = S_{\beta_1,\ldots,\beta_n}$.

3. Subtlety in the $P_\kappa\lambda$ context.

To get large cardinals beyond measurable cardinals, set theorists entertained elementary embeddings $j : V \to M$ in which M is closed under ever longer sequences. This led to the notions of λ-compactness and λ-supercompactness, the latter of which turned out to have a characterization analogous to measurability, in terms of normal ultrafilters on the set $P_\kappa\lambda = \{x \subseteq \lambda : |x| < \kappa\}$.

This prompted the consideration of various $P_\kappa\lambda$-generalizations of the partition relations and $\diamondsuit$-like statements which had been considered for cardinals. However, each cardinal notion can be generalized in multiple ways, and the equivalences which were found between partition relations and $\diamondsuit$-like statements on cardinals often do not carry through to $P_\kappa\lambda$. The result is that we have a much more muddled picture for these two-cardinal notions.

Some partition-theoretic large cardinal axioms have obvious $P_\kappa\lambda$ generalizations which reflect most of the properties of the cardinal version. In the case of weakly compact and n-ineffable, these analogues are called $Part_n(\kappa,\lambda)$ and λ-n-ineffable, and they form a properly intertwined hierarchy with λ-supercompactness (see Menas [**12**]). In contrast, as we will see, the notion of a subtle cardinal is not so easily generalized to the $P_\kappa\lambda$ context.

We first define the basic notions of the κ-complete and normal and the well-known bounded and nonstationary ideals, which behave very much like the corresponding notions for subsets of cardinals.

DEFINITION 3.1. An *ideal* I on $P_\kappa\lambda$ is a subset of $P(P_\kappa\lambda)$ satisfying the same conditions (i) and (ii) for ideals on cardinals.

The definition of κ-completeness for these ideals is also identical.

DEFINITION 3.2. An ideal I is *fine* if $\forall \alpha \in \lambda(\{y : \alpha \notin y\} \in I)$.

The first two fine ideals we define are analogous to the ideals I_κ and NS_κ on κ, defined earlier:

DEFINITION 3.3. $\mathrm{I}_{\kappa,\lambda} = \{X \subseteq P_\kappa\lambda : \exists y \in P_\kappa\lambda \forall x \in X(y \not\subseteq x)\}$ is the set of bounded subsets of $P_\kappa\lambda$, and

$$\mathrm{NS}_{\kappa,\lambda} = \{X \subseteq P_\kappa\lambda : \exists C(C \text{ is club } \wedge C \cap X = \emptyset)\}$$

where a set $X \subseteq P_\kappa\lambda$ is called *club* if it is:

(i) unbounded: $\forall x \in P_\kappa\lambda, \exists y \in X(x \subseteq y)$, and
(ii) closed: for every chain in X $x_0 \subseteq x_1 \subseteq \cdots \subseteq x_\alpha \subseteq \cdots$ of length $< \kappa$, $\bigcup x_\alpha \in X$.

Thus $\mathrm{NS}_{\kappa,\lambda}$ denotes the set of *non-stationary* subsets of $P_\kappa\lambda$. It is also the least normal ideal on $P_\kappa\lambda$ ([**4**]):

Definition 3.4. An ideal I is *normal* if it is fine and closed under diagonal unions: $\forall\langle X_\alpha : \alpha < \lambda^{<\kappa}\rangle \in {}^\lambda I$, $\nabla X_\alpha \doteq \{x : \exists\alpha \in x(x \in X_\alpha)\} \in I$.

Normality is again equivalent to the regressive partition relation $I^+ \to (I^+)^1_{\mathrm{reg}}$ (this is the $P_\kappa\lambda$ version of Fodor's theorem). Here, the subscript "reg" means we are quantifying over all functions such that $f(x) \in x, \forall x \in P_\kappa\lambda$.

In the early 1970's, Menas proved that the most obvious generalization of subtlety to the $P_\kappa\lambda$ context is a consequence of subtlety in the cardinal context (for the case $n = 1$):

Definition 3.5 ([**12**]). $X \subseteq P_\kappa\lambda$ is *λ-subtle* if $\forall C \in \mathrm{NS}^*_{\kappa,\lambda}, \forall\langle S_x : x \in P_\kappa\lambda\rangle$ where $S_x \subseteq x$, $\exists x \subsetneq y$ both in $C \cap X$ such that $S_y \cap x = S_x$.

Theorem 3.6 ([**12**]). *$\forall\lambda \geq \kappa$, if κ is subtle, then κ is λ-subtle.*

Here, we use the convention "κ is λ-subtle" to mean that $P_\kappa\lambda$ is λ-subtle (and thus the corresponding ideal is proper). A similar convention is used for any property of subsets of $P_\kappa\lambda$. The converse of this theorem is not true, as noted by Usuba [**13**], but the exact consistency strength of its negation is not known.

A relatively straightforward generalization of the proof given in [**3**] gives a partition relation sufficient for λ-subtlety analogous to that for subtlety:

Theorem 3.7. *$X \subseteq P_\kappa\lambda$ is λ-subtle if $\forall C \in \mathrm{NS}^*_{\kappa,\lambda}, C \cap X \to (3)^2_{\mathrm{reg}}$.*

Prior to the 1980's, researchers in set theory had generalized "regressive" functions to the $P_\kappa\lambda$ setting as *choice functions*, so

$$f(\alpha_1, \ldots, \alpha_n) < \alpha_1 \text{ and } S_\alpha \subseteq \alpha \text{ were generalized to:}$$
$$f(x_1, \ldots, x_n) \in x_1 \text{ and } S_x \subseteq x.$$

As various conjectures about these generalizations languished unsolved, a new generation of set theorists, beginning with Donna Carr [**4**], [**5**], began using the

ordering $<=_{def} \{\langle x,y\rangle : x \subset y \text{ and } |x| < |y \cap \kappa|\}$ on $P_\kappa\lambda$, investigating the "$<$-regressive" notions

$$f(x_1,\dots,x_n) < x_1 \text{ and } S_x \subseteq P_{\kappa_x}x,$$

where $\kappa_x \doteq |x \cap \kappa|$, and so $P_{\kappa_x}x \doteq \{z : z < x\}$.

Thus, "$P_\kappa\lambda \to (n+1)^n_{<\text{-reg}}$" means:

For all $f : [P_\kappa\lambda]^n_< \to P_\kappa\lambda$ such that $f(x_1,x_2,\dots,x_n) < x_1$, there is a chain $H = \{x_1,x_2,\dots,x_{n+1}\}$ such that $\forall i \le n, x_i < x_{i+1}$, and $\exists\gamma(\forall\{x_1,x_2,\dots,x_n\} \in [H]^n_<,\ f(x_1,x_2,\dots,x_n) = \gamma)$ (i.e, H is homogeneous in color γ), where $[X]^n_< \doteq \{\{x_1,x_2,\dots,x_n\} \in [X]^n : x_1 < x_2 < \cdots < x_n\}$.

Note that whenever we seek a "homogeneous" set of finite size in the $P_\kappa\lambda$ context, we necessarily mean a homogeneous *chain*, since any antichain trivially satisfies the homogeneity condition.

These ideas also led naturally to a new version of "normality":

DEFINITION 3.8. An ideal $I \subseteq P(P_\kappa\lambda)$ is called *strongly normal* if:

$$\forall\langle X_x : x \in P_\kappa\lambda\rangle \in {}^{P_\kappa\lambda}I,\ \nabla_< X_\alpha \doteq \{x : \exists y < x(x \in X_y)\} \in I.$$

There is also the following generalization of Fodor's lemma, and a minimal strongly normal ideal:

THEOREM 3.9 ([**6**]). *I is strongly normal iff $I^+ \to (I^+)^1_{<\text{-reg}}$.*

If $f : P_\kappa\lambda \to P_\kappa\lambda$, let $C_f = \{x \in P_\kappa\lambda : f\text{``}P_{\kappa_x}x \subseteq P_{\kappa_x}x\}$. Then:

$$\text{WNS}_{\kappa,\lambda} = \{X \subseteq P_\kappa\lambda : X \cap C_f = \emptyset \text{ for some } f : P_\kappa\lambda \to P_\kappa\lambda\}$$

is the least strongly normal ideal on $P_\kappa\lambda$. It is proper iff κ is Mahlo or $\kappa = \nu^+$, with $\nu^{<\nu} = \nu$ [**6**].

Using these ideas, Abe in 2005 [**1**] defined a stronger version of subtlety for subsets of $P_\kappa\lambda$, called "strongly subtle". Abe then showed this property is equivalent to subtlety for cardinals. We now generalize this definition and theorem to arbitrary n-tuples. Note that "strongly 1-subtle" is just Abe's "strongly subtle".

DEFINITION 3.10. κ is *λ-strongly n-subtle* (or just *strongly n-subtle*, if the value of λ is clear) if $\forall C \in \text{WNS}^*_{\kappa,\lambda}, \forall\langle S_{x_1,\dots,x_n} : x_1,\dots,x_n \in [P_\kappa\lambda]^n_<\rangle$ where $S_{x_1,\dots,x_n} \subseteq P_{\kappa_{x_1}}x_1$, $\exists H \in [C]^{n+1}_<$ such that $S_{x_1,\dots,x_n} = S_{y_1,\dots,y_n} \cap P_{\kappa_{x_1}}x_1$, whenever $x_1,\dots,x_n$ and $y_1,\dots,y_n$ are both in $[H]^n_<$, and $x_1 < y_1$ or $x_1 = y_1$.

THEOREM 3.11. *For all $\lambda \geq \kappa$ and $n < \omega$, κ is λ-strongly n-subtle if κ is n-subtle.*

The proof is a straightforward generalization of Abe's proof for the case $n = 1$ (Theorem 2.6, Proposition 2.7 in [**1**]).

In the same paper, Abe also proves a $P_\kappa\lambda$ version of the key lemma of Baumgartner relating subtlety and indescribability. To describe this lemma, we first need a $P_\kappa\lambda$ -definition of indescribability. For this, we need an appropriate model to do the "describing", which is one whose universe of discourse is essentially "V_λ up to κ", or the sets in V_λ of size (hereditarily) less than κ, defined as follows:

(i) $V_0(\kappa, \lambda) = \lambda$
(ii) $V_{\alpha+1}(\kappa, \lambda) = P_\kappa(V_\alpha(\kappa, \lambda)) \cup V_\alpha(\kappa, \lambda)$
(iii) $V_\alpha(\kappa, \lambda) = \cup_{\beta<\alpha} V_\beta(\kappa, \lambda)$, for any limit ordinal $\alpha \leq \kappa$

DEFINITION 3.12 ([**3**]). $X \subseteq P_\kappa\lambda$ is Π^m_n-*indescribable* iff for every sentence ϕ (of type Π^m_n), and every $R \subseteq V_\kappa(\kappa, \lambda)$, if $\langle V_\kappa(\kappa, \lambda), \in, R\rangle \models \phi$, then there exists an $x \in X$ such that $|x \cap \kappa| = x \cap \kappa$ and $\langle V_{\kappa_x}(\kappa_x, x), \in, R \cap V_{\kappa_x}(\kappa_x, x)\rangle \models \phi$.

The ideal of non-Π^m_n-indescribable subsets of $P_\kappa\lambda$ is denoted Π^m_n, as it is for the cardinal version of indescribable. The difference between these notions/ideals and the indescribability mentioned earlier (for cardinals and subsets of cardinals) will always be clear from the context.

LEMMA 3.13 ([**1**]). *If $X \subseteq P_\kappa\lambda$ is strongly subtle and $S_x \subseteq P_{\kappa_x}x$ for all $x \in P_\kappa\lambda$, then $\{x \in X : \{y \in X \cap P_{\kappa_x}x : S_y = S_x \cap P_{\kappa_y}y\}$ is not Π^n_m-indescribable for some n, m (as a subset of $P_{\kappa_x}x$)$\}$ is not strongly subtle.*

Again, Abe proves only the case for strongly 1-subtle, but his proof may possibly be generalized to any $n < \omega$ by following Baumgartner's original proof for n-subtle subsets of κ:

CONJECTURE 3.14. *If $X \subseteq P_\kappa\lambda$ is strongly n-subtle and $\overline{S} = \langle S_{x_1,\ldots,x_n} \subseteq P_{\kappa_{x_1}}x_1 : x_1, \ldots, x_n \in [P_\kappa\lambda]^n_<\rangle$. Let $Y = \{x \in X : \forall H \subseteq P_{\kappa_x}x$ such that H is homogeneous for $\overline{S}$, H is not Π^n_m-indescribable, for some $n, m\}$. Then $X \setminus Y$ is not strongly n-subtle.*

where "H is homogeneous for $\overline{S}$" means $S_{x_1,\ldots,x_n} = S_{y_1,\ldots,y_n} \cap P_{\kappa_{x_1}}x_1$, whenever $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are both in $[H]^n_<$, and $x_1 \leq y_1$.

Using this lemma, we now prove that a strong regressive partition relation on $P_\kappa\lambda$ is also implied by subtlety. The reader can verify that, if the above conjecture holds, then we may change subtle to n-subtle and 2 to $n + 1$ in the following theorem:

MAIN THEOREM 1. *If $X \subseteq P_\kappa\lambda$ is strongly subtle then $\forall C \in \mathrm{WNS}^*_{\kappa,\lambda}, C \cap X \to (3)^2_{<\text{-reg}}$.*

PROOF. Assume X is strongly subtle. Let $f : [P_\kappa\lambda]^2_< \to P_\kappa\lambda$ be a $<$-regressive function, i.e. $\forall\langle x, y\rangle \in [P_\kappa\lambda]^2_<, f(x, y) < x$. Define $f_y : P_{\kappa_y}y \to P_{\kappa_y}y$ by $f_y(x) = f(x, y)$.

Let $p : [\lambda]^2 \to \lambda$ be any injective function, and set

$$C \doteq \{x \in P_\kappa\lambda : p\text{“}[x]^2 \subseteq x\}.$$

If $y_0 \in P_\kappa\lambda$, then recursively define, for each $i \in \omega$, $y_{i+1} = y_i \cup p\text{“}[y_i]^2$. So for any $\alpha < \beta$ both in $y_\omega = \bigcup_{i<\omega} y_i$, α and β are both in y_i for some i, and so $p(\alpha, \beta) \in y_{i+1}$. Since κ is subtle, and hence regular and uncountable, $y_\omega \in P_\kappa\lambda$. Therefore $y_\omega \in C$, proving that C is unbounded. C is obviously closed, so $C \in \mathrm{NS}^*_{\kappa,\lambda}$. Also note that by the same reasoning, $C \cap P_{\kappa_y}y \in \mathrm{NS}^*_{\kappa_y,y}$, whenever $y \in C$ and $\mathrm{cf}(\kappa_y) > \omega$.

Define $\mathrm{ot}(x)$ to be the ordertype of x as a set of ordinals, well-ordered by $\in$. In what follows, if $x < y$ are both in $P_\kappa\lambda$, then for each $i < \mathrm{ot}(y)$, define y_i to be the i^{th} element in y, and x_i to be the i^{th} element in x, if x has an i^{th} element, and to be the first (least) element of x, otherwise.

Now define $\phi : [P_\kappa\lambda]^2_< \to P_\kappa\lambda$ by:

$$\phi(x, y) = \big\{p(x_{s(i)}, y_{s(i)}) : i < \mathrm{ot}(y)\big\},$$

where $s(i) = i + 1$ if $i < \omega$ and $s(i) = i$ otherwise.

We claim that ϕ is injective: Assume $\phi(x', y') = \phi(x, y)$. Since $|x| < |y|$, there will be exactly one ordinal which equals x_i for multiple i (for $i = 0$ and $\mathrm{ot}(x) < i < \mathrm{ot}(y)$), and similarly with x'. So, using the injectiveness of p, $x_0 = x'_0$. Also using the injectiveness of p, it is easy to see that the other elements of x and x' are the same, as well as the elements of y and y'. So by extensionality, $x' = x$ and $y' = y$.

Note for every $y \in C$, $\phi\text{“}[P_{\kappa_y}y]^2 \subseteq P_{\kappa_y}y$. We therefore define $S_y = \phi\text{“}f_y$ for all $y \in C$ (and $S_y = \emptyset$, otherwise).

By Abe's Lemma, $Y = \{y \in X : \{x \in P_{\kappa_y}y : S_x = S_y \cap P_{\kappa_x}x\}$ is not totally indescribable in $P_{\kappa_y}y\}$ is not strongly subtle. So, $X \setminus Y = \{y \in X : \{x \in P_{\kappa_y}y : S_x = S_y \cap P_{\kappa_x}x\}$ is totally indescribable in $P_{\kappa_y}y\}$ is strongly subtle (totally indescribable means Π^n_m-indescribable for every $m, n \in \omega$). Let $y \in (X \setminus Y) \cap C \cap \{y : \kappa_y$ is a strongly inaccessible cardinal$\}$. (This last set is in $\mathrm{WNS}^*_{\kappa,\lambda}$ [**6**], so the intersection is nonempty since the non-strongly subtle sets form a strongly normal ideal). Let $Z_y = \{x \in P_{\kappa_y}y : S_x = S_y \cap P_{\kappa_x}x\} \cap C$, so Z_y is totally

indescribable in $P_{\kappa_y} y$, and so in particular is in $\mathrm{WNS}^+_{\kappa_y,y}$ since the non-totally indescribable sets form a strongly normal ideal.

Now let $g_y : Z_y \to P_{\kappa_y} y$ be given by $g_y(x) = f(x,z)$ for some (any) $z \in Z_y$ such that $x < z$. g_y is evidently $<$-regressive, and is well-defined: Such a z exists since κ_y is a strongly inaccessible cardinal. If z and z' are two such members of Z_y, then since $S_z = S_y \cap P_{\kappa_z} z$, $S_{z'} = S_y \cap P_{\kappa_{z'}} z'$, and x is in both $P_{\kappa_z} z$ and $P_{\kappa_{z'}} z'$, we get that $\phi(x, f(x,z)) \in S_z$ and $\phi(x, f(x,z')) \in S_{z'}$ are both members of S_y. Thus, $\phi(x, f(x,z)) = \phi(x, f(x,y)) = \phi(x, f(x,z'))$. But ϕ is injective, so $z' = z$ and $f(x,z) = f(x,y) = f(x,z')$.

Finally, we apply the fact that $Z_y \in \mathrm{WNS}^+_{\kappa_y,y}$ and the strong version of Fodor's lemma to get a subset $H_y \in \mathrm{WNS}^+_{\kappa_y,y}$ homogeneous for g_y. So, $\forall x \in H_y \forall z \in H_y$, if $x < z$, then $g_y(x) = g_y(z)$, and by the inaccessibility of y, $\exists z' > z$ such that $z' \in H_y$ also. So $f(x,z) = f(x,z') = f(z,z')$, meaning $\{x, z, z'\}$ is a homogeneous chain for f. □

Remark 3.15. A stronger conclusion has actually been derived: We always find not just a homogeneous chain of length $n+1$ for every $<$-regressive coloring of $[P_\kappa \lambda]^n_<$, but a homogeneous set which is in $\mathrm{WNS}^+_{\kappa_y,y}$, for *many* $y \in P_\kappa \lambda$ (at least $\mathrm{WNS}^+_{\kappa,\lambda}$ many). In particular, we will always get a homogeneous chain of any length $\gamma < \kappa$.

The converse of Main Theorem 1 is also true, as Usuba notes:

Theorem 3.16 ([**13**]). *If* $\forall C \in \mathrm{WNS}^*_{\kappa,\lambda}, C \cap X \to (3)^2_{<\text{-reg}}$, *then* $X \subseteq P_\kappa \lambda$ *is strongly subtle.*

Proof. Let $\langle S_x \subseteq P_{\kappa_x} x : x \in P_\kappa \lambda \rangle$, and $C \in \mathrm{WNS}^*_{\kappa,\lambda}$. We may assume that either $\forall x \in X (\emptyset \in S_x)$ or $\forall x \in X (\emptyset \notin S_x)$. Define $f : [C \cap X]^2_< \to P_\kappa \lambda$ by:

$$f(x,y) = \begin{cases} \text{an arbitrary element of } S_x \triangle (S_y \cap P_{\kappa_x} x) & \text{if } S_x \neq S_y \cap P_{\kappa_x} x \\ 0 & \text{else} \end{cases}.$$

Note that by our assumption, $f(x,y) \neq \emptyset$ whenever the first clause applies. By the premise, we obtain $x < y < z$ forming a homogeneous set for f. If $f(x,y) = f(x,z) = f(y,z) = \emptyset$, then $S_x = S_y \cap P_{\kappa_x} x$ and we are done. If $f(x,y) = f(x,z) = f(y,z) = a \neq \emptyset$, then either we have $a \in S_x \setminus S_y$, in which case $a \in S_z \setminus S_y$ and so $a \in S_x \cap S_z$, or we have $a \in (S_y \cap P_{\kappa_x} x) \setminus S_x$, in which case $a \in S_y \setminus S_z$ and so $a \notin S_x \cup S_z$. Either case is a contradiction. □

In the case $X = P_\kappa \lambda$, collecting these results together with Abe's (Theorem 3.11 above) yields:

THEOREM 3.17. *The following are equivalent*:

(i) κ *is subtle.*
(ii) $\mathrm{WNS}^*_{\kappa,\lambda} \to (3)^2_{<\text{-reg}}$ *for every* $\lambda \geq \kappa$.
(iii) κ *is* λ*-strongly subtle for every* $\lambda \geq \kappa$.

4. A minimal condition for the existence of a subtle cardinal.

In [**10**], Kanamori proved that the existence of an n-subtle cardinal is a consequence of the existence of a cardinal κ containing a homogeneous subset of size $n+1$ for each regressive coloring of $[\kappa]^n$, instead of a homogeneous subset of each club subset of κ. In other words, one could remove the quantification over all club sets in the statement of "there exists an n-subtle cardinal", and still have a proposition which is just as strong. The only other change necessary is to exclude 0 and 1, since these would trivially yield a homogeneous set.

The following is essentially Kanamori's theorem:

THEOREM 4.1. *For all* $n \geq 2$, *If* κ *is the least cardinal such that* $(\kappa \setminus 2) \to (n+1)^n_{\mathrm{reg}}$, *then* κ *is* $(n-1)$*-subtle.*

Given the current state of the literature, which mostly deals with choice-regressive (reg) and $<$-regressive ($<$ -reg) functions on $P_\kappa\lambda$, one would naturally try as a $P_\kappa\lambda$ analogue to Kanamori's theorem something like one or more of the following propositions:

CONJECTURE 1a. *If* κ *is the least ordinal such that, for some* $\lambda > \kappa$, $P_{[2,\kappa)}\lambda \to (n+1)^n_{\mathrm{reg}}$, *then* $\forall C \in \mathrm{NS}^*_{\kappa,\lambda}$, $C \to (n+1)^n_{\mathrm{reg}}$.

CONJECTURE 1b. *If* κ *is the least ordinal such that, for some* $\lambda > \kappa$, $P_{[2,\kappa)}\lambda \to (n+1)^n_{<\text{-reg}}$, *then* $\forall C \in \mathrm{WNS}^*_{\kappa,\lambda}$, $C \to (n+1)^n_{<\text{-reg}}$.

where $P_{[\gamma,\kappa)}\lambda \doteq \{x \subset \lambda : \gamma \leq \mathrm{ot}(x) < \kappa\}$.

It is necessary to exclude singleton sets and $\emptyset$ in Conjectures 1a and 1b, as it was in Kanamori's theorem, so that the premise is not merely a consequence of ZFC, since if we include singleton sets then we are forced, for any x containing as a subset some $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ of ordertype ω, into $f(\{\alpha_1\}, x) = 0$, and so (to avoid a homogeneous set of size 3) $f(\{\alpha_1, \alpha_2\}, x) = 1$ and $f(\{\alpha_1, \alpha_2, \alpha_3\}, x) = 2$, etc. But now $f(\{\alpha_1, \alpha_2, \alpha_3, \ldots\}, x) = n$, for some $n \in \omega$, so $\{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n+1}\}, \{\alpha_1, \alpha_2, \alpha_3, \ldots\}$, and x form a homogeneous chain of length 3. A similar chain of length $n+1$ can be constructed for any f with domain $[P_\kappa\lambda]^n_{\subseteq}$ or $[P_\kappa\lambda]^n_{<}$, where $n \geq 2$.

However, in trying to prove these conjectures, an unexpected $P_\kappa\lambda$-analogue

of Kanamori's threorem was found involving a heretofore unstudied notion of regressiveness, although one which has equal claim to be the natural generalization to $P_\kappa\lambda$:

DEFINITION 4.2. A function on $[P_\kappa\lambda]^n$ is *ordertype-regressive* (*otreg*) if $\forall x_1, x_2, \ldots, x_n \in [P_\kappa\lambda]^n$ such that $x_1 \subsetneq x_2 \subsetneq \cdots \subsetneq x_n$, $f(x_1, x_2, \ldots, x_n) \in \mathrm{ot}(x_1)$.

Although it is easily seen for (choice) regressive functions that only those partition relations with κ a cardinal need be considered, in the ordertype-regressive setting it is not immediately obvious that the same applies. So, provisionally, when we say $P_\kappa\lambda$ in the context of ordertype-regressive partition relations, we are technically speaking of the set $\{x \subseteq \lambda : \mathrm{ot}(x) < \kappa\}$ (the fact that, when κ is a cardinal, $\{x \subseteq \lambda : \mathrm{ot}(x) < \kappa\} = \{x \subseteq \lambda : |x| < \kappa\}$ justifies this overloading of the notation).

4.1. Construction of j_n^α.

In this subsection, we construct functions j_n^α, for every ordinal α, $n \in \omega$, witnessing $P_{[\alpha,\alpha+1)}\lambda \nrightarrow (n+1)^n_{\mathrm{otreg}}$. These functions are used in the proof of the main theorem in the following subsection.

Here $P_{[\alpha,\gamma)}\lambda \rightarrow (n+1)^n_{\mathrm{otreg}}$ is defined similarly to $P_{[\alpha,\gamma)}\lambda \rightarrow (n+1)^n_{\mathrm{reg}}$, only the quantification over all regressive (choice) functions $f : [P_{[\alpha,\gamma)}\lambda]^n_\subset \rightarrow \lambda$ is replaced by a quantification over all ordertype-regressive functions $f : [P_{[\alpha,\gamma)}\lambda]^n_\subset \rightarrow \gamma$.

To construct the functions j_n^α, we construct $j_2^\alpha : [P_{[\alpha,\alpha+1)}\lambda]^2 \rightarrow \alpha$ containing no homogeneous set of size 3, and set $j_n^\alpha(x_1, x_2, \ldots, x_n) = j_2^\alpha(x_1, x_2)$, since if $\{h_1, h_2, \ldots, h_{n+1}\}$ is a homogeneous set for j_n^α, then it is immediate that $\{h_1, h_2, h_3\}$ is homogeneous for j_2^α.

Whenever $x_i \subsetneq x_j$, define $\mu_{ij} = \min(x_j \setminus x_i)$ and $\epsilon_{ij} = \min(\beta \in x_i : \beta > \mu_{ij})$, and let $\phi : \alpha \times \alpha \hookrightarrow |\alpha|$ be any injection.

$$\text{Define } \; j_2^\alpha(x_i, x_j) = \phi(\mathrm{ot}(x_i \restriction \mu_{ij}), \mathrm{ot}(x_j \restriction \epsilon_{ij})).$$

THEOREM 4.3. *j_2^α witnesses $P_{[\alpha,\alpha+1)}\lambda \nrightarrow (3)^2_{\mathrm{otreg}}$.*

PROOF. Let $x_i \subsetneq x_j$ and $\mathrm{ot}(x_i) = \mathrm{ot}(x_j) = \alpha$. We first show $\mathrm{ot}(x_i \restriction \mu_{ij}) < \alpha$: If not, then μ_{ij} is greater than every ordinal $\alpha \in x_i$, so that x_j is an end-extension of x_i, which contradicts the assumption that they have the same ordertype. So $\mathrm{ot}(x_i \restriction \mu_{ij}) < \alpha$. This implies that ϵ_{ij} exists. Since $\epsilon_{ij} \in x_i$ by the definition, we have $\epsilon_{ij} \in x_j$, hence $x_j \restriction \epsilon_{ij}$ cannot have ordertype α since it does not contain ϵ_{ij} itself. Therefore, j_2^α is well-defined, and j_2^α is ordertype-regressive simply because it outputs a number less than $|\alpha| \leq \alpha = \mathrm{ot}(x_i)$.

Suppose that $H = \{x_1, x_2, x_3\}$ with $x_1 \subseteq x_2 \subseteq x_3$ is a homogeneous set for

j_2^α. Then by the injectivity of ϕ, $\mathrm{ot}(x_1 \restriction \mu_{12}) = \mathrm{ot}(x_1 \restriction \mu_{13}) = \mathrm{ot}(x_2 \restriction \mu_{23})$, and $\mathrm{ot}(x_2 \restriction \epsilon_{12}) = \mathrm{ot}(x_3 \restriction \epsilon_{13}) = \mathrm{ot}(x_3 \restriction \epsilon_{23})$.

Therefore, $x_1 \restriction \mu_{12} = x_1 \restriction \mu_{13}$. Also, $x_1 \restriction \mu_{12} = x_2 \restriction \mu_{12}$, $x_1 \restriction \mu_{13} = x_3 \restriction \mu_{13}$, and $x_2 \restriction \mu_{23} = x_3 \restriction \mu_{23}$ by the definition of μ_{ij}. So, since $\mathrm{ot}(x_1 \restriction \mu_{12}) = \mathrm{ot}(x_2 \restriction \mu_{23})$, we have:

$$x_3 \restriction \mu_{13} = x_1 \restriction \mu_{13} = x_1 \restriction \mu_{12} = x_2 \restriction \mu_{12} = x_2 \restriction \mu_{23} = x_3 \restriction \mu_{23}$$

The fact that $x_3 \restriction \mu_{13} = x_3 \restriction \mu_{23}$ shows that $\mu_{13} = \mu_{23}$, since both are members of x_3. So $\mu_{13} = \mu_{23} < \mu_{12}$ since $\mu_{23} \leq \mu_{12}$ by $x_1 \subseteq x_2 \subseteq x_3$, and $\mu_{23} \neq \mu_{12}$ since $\mu_{23} \notin x_2$ and $\mu_{12} \in x_2$.

Now $\epsilon_{23} = \mu_{12}$ since $\mu_{23} < \mu_{12}$ and $x_2 \restriction \mu_{23} = x_2 \restriction \mu_{12}$. So $\epsilon_{13} > \epsilon_{23}$, since $\epsilon_{13} \geq \epsilon_{23}$ by $x_1 \subsetneq x_2$ and $\mu_{13} = \mu_{23}$, and $\epsilon_{13} \neq \epsilon_{23} = \mu_{12}$ since $\epsilon_{13} \in x_1$ and $\mu_{12} \notin x_1$.

So $\mathrm{ot}(x_3 \restriction \epsilon_{13}) > \mathrm{ot}(x_3 \restriction \epsilon_{23})$ since $\epsilon_{23} \in x_2 \subseteq x_3$. So $j_2^\alpha(x_1, x_3) \neq j_2^\alpha(x_2, x_3)$, using the injectivity of ϕ. So H is not homogeneous. □

4.2. The main theorem.

MAIN THEOREM 2. *If $n > 1$ and κ is the least ordinal such that, for some $\lambda > \kappa$, $P_{[2,\kappa)}\lambda \to (n+1)^n_{\mathrm{otreg}}$, then $(\kappa \setminus 2) \to (n+1)^n_{\mathrm{reg}}$ (and so κ is $(n-1)$-subtle).*

PROOF. As in the proof of Kanamori's theorem, we begin by noting that $\kappa > \omega$, i.e. $P_{[2,\omega)}\lambda \nrightarrow (n+1)^n_{\mathrm{otreg}}$: Witness the ordertype-regressive function $f : [P_{[2,\omega)}\lambda]^n \to \omega$ given by $f(x_1, x_2, \ldots, x_n) = \mathrm{ot}(x_1) - 1$. Since every x_i is finite, $x_1 \subsetneq x_2$ implies $\mathrm{ot}(x_1) - 1 \lneq \mathrm{ot}(x_2) - 1$, so no chain of size $n+1$ is homogeneous.

Also as in Kanamori's proof, we can assume that, if we can always find a homogeneous set, then we can always find a homogeneous set consisting only of infinite sets:

LEMMA 4.4. *If $P_{[2,\kappa)}\lambda \to (n+1)^n_{\mathrm{otreg}}$, then $P_{[\omega,\kappa)}\lambda \to (n+1)^n_{\mathrm{otreg}}$.*

PROOF. If $f : [P_{[\omega,\kappa)}\lambda]^n \to \kappa$ witnesses $P_{[\omega,\kappa)}\lambda \nrightarrow (n+1)^n_{\mathrm{otreg}}$, then we can assume by renumbering the outputs $< \omega$ that $f(x_1, x_2, \ldots, x_n) > 1$ always.
Define $f' : [P_{[2,\kappa)}\lambda]^n \to \kappa$ by:

$$f'(x_1, x_2, \ldots, x_n) = \begin{cases} \mathrm{ot}(x_1) - 1 & \text{if } \mathrm{ot}(x_n) < \omega \\ 0 & \text{if } |\{i : \mathrm{ot}(x_i) \geq \omega\}| \text{ is odd and } \notin \{n, 0\} \\ 1 & \text{if } |\{i : \mathrm{ot}(x_i) \geq \omega\}| \text{ is even and } \notin \{n, 0\} \\ f(x_1, x_2, \ldots, x_n) & \text{if } \mathrm{ot}(x_1) \geq \omega \end{cases}$$

Suppose $x_1 \subsetneq x_2 \subsetneq \cdots \subsetneq x_{n+1}$ is a homogeneous set for f'. By the last clause, we must have $\mathrm{ot}(x_1) < \omega$, and by the first clause, we must have $\mathrm{ot}(x_{n+1}) \geq \omega$. So, there exists $j \leq n$ such that $\mathrm{ot}(x_j) < \omega$ but $\mathrm{ot}(x_{j+1}) \geq \omega$. But now we have $f'(x_1, x_2, \ldots, x_n) \neq f'(x_2, x_3, \ldots, x_{n+1})$:

The latter has one more argument x_i such that $\mathrm{ot}(x_i) \geq \omega$ than the former (namely x_{n+1}). So, either:

$$f'(x_2, x_3, \ldots, x_{n+1}) = f(x_2, x_3, \ldots, x_{n+1}) > 1 \quad (\text{if } \mathrm{ot}(x_2) \geq \omega), \text{ or}$$

$$f'(x_2, x_3, \ldots, x_{n+1}) = 0 < f'(x_1, x_2, \ldots, x_n) = \mathrm{ot}(x_1) - 1 \quad (\text{if } \mathrm{ot}(x_n) < \omega), \text{ or}$$

$$f'(x_2, x_3, \ldots, x_{n+1}) = f'(x_1, x_2, \ldots, x_n) + 1 (\mathrm{mod} 2)$$

(if both fall under the second and third clauses).

So f' witnesses $P_{[2,\kappa)}\lambda \nrightarrow (n+1)^n_{\mathrm{otreg}}$. □

We are now ready for the main argument.

Let $g : [\kappa \setminus \omega]^n \to \kappa$ be any regressive function, and for all $\xi < \kappa$, let $g_\xi : [P_{[\omega,\xi)}\lambda]^n \to \xi$ be an ordertype-regressive function which witnesses $P_{[\omega,\xi)}\lambda \nrightarrow (n+1)^n_{\mathrm{otreg}}$. We may assume (by doubling all finite outputs) that $\mathrm{ran}(g)$ and $\mathrm{ran}(g_\xi)$ contain no odd finite numbers.

For the case $n > 2$, define an ordertype-regressive function $f : [P_{[\omega,\kappa)}\lambda]^n \to \kappa$ by:

$$f(x_1, x_2, \ldots, x_n) = \begin{cases} g(\mathrm{ot}(x_1), \mathrm{ot}(x_2), \ldots, \mathrm{ot}(x_n)) & \text{if } \mathrm{ot}(x_1) < \mathrm{ot}(x_2) < \cdots < \mathrm{ot}(x_n) \\ g_{\mathrm{ot}(x_1)+1}(x_1, x_2, \ldots, x_n) & \text{if } \mathrm{ot}(x_1) = \mathrm{ot}(x_2) = \cdots = \mathrm{ot}(x_n) < \kappa \\ & \quad (\text{or } \kappa - 1 \text{ if it exists}) \\ j^{\kappa-1}_n(x_1, x_2, \ldots, x_n) & \text{if } \mathrm{ot}(x_1) = \mathrm{ot}(x_2) = \cdots = \mathrm{ot}(x_n) = \kappa - 1 \\ \mathrm{type}\{\mathrm{ot}(x_1), \mathrm{ot}(x_2), \ldots, \mathrm{ot}(x_n)\} & \text{otherwise} \end{cases}$$

where $\mathrm{type}\,\{\beta_1, \beta_2, \ldots, \beta_n\} = 3^{\psi_1} 5^{\psi_2} \cdots p_n^{\psi_{n-1}}$, where p_i is the i^{th} prime, and

$$\psi_i = \begin{cases} 0 & \text{if } \beta_i < \beta_{i+1} \\ 1 & \text{if } \beta_i = \beta_{i+1} \end{cases}.$$

Let $H = \{h_1, h_2, \ldots, h_{n+1}\}$ be our hypothetical homogeneous set of color $\eta < \kappa$.

If η is an odd finite number, then the fourth clause is used on every n-tuple

from H. So, there exists $i < n$ such that either:

$$\mathrm{ot}(h_{i-1}) < \mathrm{ot}(h_i) = \mathrm{ot}(h_{i+1}) \quad \text{or} \quad \mathrm{ot}(h_{i-1}) = \mathrm{ot}(h_i) < \mathrm{ot}(h_{i+1})$$

In either case, by the unique decomposition into primes,

$$\mathrm{type}\{\mathrm{ot}(h_1), \mathrm{ot}(h_2), \ldots, \mathrm{ot}(h_n)\} \neq \mathrm{type}\{\mathrm{ot}(h_2), \mathrm{ot}(h_3), \ldots, \mathrm{ot}(h_{n+1})\}.$$

So it must be the case that either

$$\mathrm{ot}(h_1) < \mathrm{ot}(h_2) < \cdots < \mathrm{ot}(h_{n+1}) \quad \text{or} \quad \mathrm{ot}(h_1) = \mathrm{ot}(h_2) = \cdots = \mathrm{ot}(h_{n+1}).$$

The latter case is impossible by the second and third clauses in the definition of f: If $\mathrm{ot}(h_i) < \kappa - 1$, it contradicts our assumption that $g_{\mathrm{ot}(h_i)+1} : [P_{[\omega,\mathrm{ot}(h_i)+1)}\lambda]^n \to \xi$ is an otreg function which witnesses $P_{[\omega,\mathrm{ot}(h_i)+1)}\lambda \nrightarrow (n+1)^n_{\mathrm{otreg}}$. If $\mathrm{ot}(h_i) = \kappa - 1$, it contradicts the fact that $j_n^{\kappa-1}$ witnesses $P_{[\kappa-1,\kappa)}\lambda \nrightarrow (n+1)^n_{\mathrm{otreg}}$.

Therefore, $1 < \mathrm{ot}(h_1) < \mathrm{ot}(h_2) < \cdots < \mathrm{ot}(h_{n+1})$, and, by the first clause, the $n+1$ ordinals $\{\mathrm{ot}(h_1), \mathrm{ot}(h_2), \ldots, \mathrm{ot}(h_{n+1})\}$ form a homogeneous set for g. Since g was an arbitrary regressive function, this shows that $(\kappa \setminus 2) \to (n+1)^n_{\mathrm{reg}}$.

For the case $n = 2$, we again let $g : [\kappa\setminus\omega]^2 \to \kappa$ be any regressive function, and $\forall \xi < \kappa$, let $g_\xi : [P_{[\omega,\xi)}\lambda]^2 \to \xi$ be an ordertype-regressive function which witnesses $P_{[\omega,\xi)}\lambda \nrightarrow (3)^2_{\mathrm{otreg}}$.

This time we consider the ordertype-regressive function $h : [P_{[\omega,\kappa)}\lambda]^2 \to \kappa$ defined by:

$$h(x_1,x_2) = \begin{cases} f^e_\kappa(g(\mathrm{ot}(x_1),\mathrm{ot}(x_2))) & \text{if } \mathrm{ot}(x_1) < \mathrm{ot}(x_2) \\ f^o_{\mathrm{ot}(x_1)}(g_{\mathrm{ot}(x_1)+1}(x_1,x_2)) & \text{if } \mathrm{ot}(x_1) = \mathrm{ot}(x_2) < \kappa \\ & \quad (\text{or } \kappa - 1 \text{ if it exists}) \\ f^o_{\mathrm{ot}(x_1)}(j_2^{\kappa-1}(x_1,x_2)) & \text{if } \mathrm{ot}(x_1) = \mathrm{ot}(x_2) = \kappa - 1 \end{cases}$$

where f^e_ν and f^o_ν are bijections from ν to the even and odd ordinals less than ν, respectively.

Let $H = \{h_1, h_2, h_3\}$ be our hypothetical homogeneous set, of color η. If η is an odd ordinal, then $\mathrm{ot}(h_1) = \mathrm{ot}(h_2) = \mathrm{ot}(h_3)$. This ordertype cannot be $\kappa - 1$ (if it exists), since that would contradict our theorem about $j_2^{\kappa-1}$, and cannot be $< \kappa - 1$ either, since that would contradict our assumption about $g_{\mathrm{ot}(x_1)+1}$. So it must be that $\mathrm{ot}(h_1) < \mathrm{ot}(h_2) < \mathrm{ot}(h_3)$.

Therefore, by clause 1, the three ordinals $\{\mathrm{ot}(h_1), \mathrm{ot}(h_2), \mathrm{ot}(h_3)\}$ form a

homogeneous set for g. Since g was an arbitrary regressive function, this shows that $(\kappa \setminus \omega) \to (3)^2_{\mathrm{reg}}$. □

Main Theorem 2 is analogous to Kanamori's theorem, and Conjectures 1a/1b. It seems natural to now ask about the quantified version of this partition relation, $\mathrm{NS}^*_{\kappa,\lambda} \to (n+1)^n_{\mathrm{otreg}}$, and its relationship to $n-1$-subtlety. A result in this direction has been discovered by Usuba:

THEOREM 4.5 ([**13**]). *Suppose κ is not subtle but $P_\kappa\gamma$ is γ-subtle for some $\gamma > \kappa$. If λ is the least such that $P_\kappa\lambda$ is λ-subtle, then $\mathrm{NS}^*_{\kappa,\lambda} \to (3)^2_{\mathrm{otreg}}$.*

4.3. Consequences.

We now collect the equivalences resulting from Main Theorem 2:

COROLLARY 4.6. *The following are equivalent*:

(i) $P_{[2,\kappa)}\lambda \to (n+2)^{n+1}_{\mathrm{otreg}}$ *for some* $\lambda \geq \kappa$.
(ii) $P_{[2,\kappa)}\lambda \to (n+2)^{n+1}_{\mathrm{otreg}}$ *for every* $\lambda \geq \kappa$.
(iii) $(\kappa \setminus 2) \to (n+2)^{n+1}_{\mathrm{reg}}$.
(iv) $\kappa \geq$ *the least n-subtle cardinal.*

PROOF. Main Theorem 2 says (i) → (iii), and Kanamori's theorem (iii) → (iv). To show that (iv) → (ii), note that if κ is greater than or equal to the least n-subtle ordinal, $\lambda \geq \kappa$, and $f : [P_{[2,\kappa)}\lambda]^{n+1} \to \kappa$ is ordertype-regressive, then let $g : [\kappa \setminus 2]^n \to \kappa$ be defined by the values of f on the "central chain" of initial segments of λ, i.e. let $g(\alpha_1, \ldots, \alpha_n) = f([0, \alpha_1), \ldots, [0, \alpha_n))$, whenever $2 < \alpha_1 < \cdots < \alpha_n < \kappa$. Then g is regressive (in the original sense). So, by hypothesis, we have some $H = \{\alpha_1, \ldots, \alpha_{n+1}\}$ which is homogeneous for g. So $H' = \{[0, \alpha_1), \ldots, [0, \alpha_{n+1})\}$ is homogeneous for f. (ii) → (i) is trivial. □

Partition relations such as (i), (ii) and (iii) above posit a homogeneous set of fixed type, i.e., one which is independent of κ, λ, or whatever set is the resource (left side) of the partition relation. Such partition relations are generally upward persistent in both κ and λ. Therefore we are interested only in the *least* κ, λ, for which they are satisfied. In the case of (i), (ii) and (iii), we showed these are $\kappa = \lambda =$ the least n-subtle cardinal.

5. Characterizing otreg relations.

There is a close relationship between regressive partition relations on $\kappa \setminus 2$, and ordertype-regressive partition relations on $P_{[2,\kappa)}\lambda$; more so than with choice regressive functions on $P_{[2,\kappa)}\lambda$, it appears. This is in part because on any chain $C \subseteq P_{[\eta,\kappa)}\lambda$, an ordertype regressive function looks like a regressive function on

the set $\{\mathrm{ot}(x) : x \in C\} \subseteq \kappa$.

We now attempt to characterize the ordertype-regressive partition relation on subsets of $P_\kappa\lambda$. We first establish a strict upper bound on the size of homogeneous sets we can expect:

THEOREM 5.1. $\forall n > 0, \forall\kappa, \forall\lambda > \kappa(P_\kappa\lambda \nrightarrow (\mathrm{I}^+_{\kappa,\lambda})^n_{\mathrm{otreg}})$.

PROOF. As negations of partition relations are upward persistent in n, we must only show the case $n = 1$. The ordertype-regressive function $f : P_\kappa\lambda \to \kappa$ given by:

$$f(x) = \begin{cases} \sup\{\gamma : \gamma \subset x\} & \text{if } x \text{ is not an ordinal} \\ 0 & \text{else} \end{cases}$$

has no unbounded homogeneous set, because if H is a homogeneous set whose color is any $\gamma \in [1, \kappa)$, then no $x \in \widehat{\gamma + 1} \doteq \{x : \gamma + 1 \subset x\}$ is in H, so H is not unbounded. Also, any set which is homogeneous of color 0 cannot contain any $x \supseteq \{\kappa\}$, since no ordinal member of $P_\kappa\lambda$ contains κ. □

As with all partition relations, if $X \to (I^+)^n_{\mathrm{otreg}}$ and $X \subseteq Y$, then $Y \to (I^+)^n_{\mathrm{otreg}}$, whether X and Y are subsets of κ or $P_\kappa\lambda$. In particular, there is no $X \subseteq P_\kappa\lambda$ such that $X \to (\mathrm{I}^+_{\kappa,\lambda})^1_{\mathrm{otreg}}$. This is another way of saying that no ideal is "normal" in the ordertype-regressive sense, since for every $X \subseteq P_\kappa\lambda$ there is an ordertype-regressive coloring of X with only homogeneous sets smaller than X (bounded if X is unbounded, and of lesser cardinality if X is bounded). This is in contrast to the choice-regressive and $<$-regressive partition relations, where this "pressing down" process has fixed points (normal ideals) of $\mathrm{NS}^+_{\kappa,\lambda}$ and $\mathrm{WNS}^+_{\kappa,\lambda}$, respectively.

QUESTION 5.2. If κ is n-subtle, what is the maximum "size" homogeneous set (greater than $n+2$, yet bounded in $P_\kappa\lambda$) we can be guaranteed to find for any ordertype-regressive function on $P_\kappa\lambda$?

To begin to answer this ill-defined question, we first realize that we can always find a homogeneous chain of length γ, for any $\gamma < \kappa$: For any ordertype-regressive coloring function $f : P_\kappa\lambda \to \kappa$, $f \restriction \kappa$ is an ordinary regressive function. So, by definition (ii) above of n-subtlety, there must be a homogeneous subchain of length γ in the branch $\kappa \subset P_\kappa\lambda$. Therefore we can add to our list of equivalences at the end of the previous section:

(v) $P_\kappa\lambda \to (\gamma)^{n+1}_{\mathrm{otreg}}$, for every $\gamma < \kappa_0$, where κ_0 is the least n-subtle cardinal.

However:

LEMMA 5.3. *If κ is the least n-subtle cardinal, then for any positive n, $P_\kappa\lambda \nrightarrow (\kappa)^{n+1}_{\mathrm{otreg}}$.*

PROOF. There must exist a regressive function $f : [\kappa]^n \to \kappa$ for which there is no unbounded homogeneous subset. Otherwise κ is an *almost n-ineffable* cardinal, and many n-subtle cardinals exist below κ. (see [**3**]).

Now define an ordertype-regressive function $g : [P_\kappa\lambda]^n \to \kappa$ by:

$$g(x_1, x_2, \ldots, x_n) = \begin{cases} 2 \cdot f(\mathrm{ot}(x_1), \mathrm{ot}(x_2), \ldots, \mathrm{ot}(x_n)) & \text{if } \mathrm{ot}(x_1) < \mathrm{ot}(x_2) < \cdots < \mathrm{ot}(x_n) \\ 0 & \text{if } \mathrm{ot}(x_1) = \mathrm{ot}(x_2) = \cdots = \mathrm{ot}(x_n) \\ \mathrm{type}\{\mathrm{ot}(x_1), \mathrm{ot}(x_2), \ldots, \mathrm{ot}(x_n)\} & \text{otherwise} \end{cases}$$

Now suppose, for a contradiction, that H is a chain of length κ, homogeneous for g. The color of H cannot be 0, since the ordertypes of elements in H are cofinal in κ. The type function as in the proof of Main Theorem 2 prevents the color of H from being an odd finite ordinal. Finally, if the color of H is $2 \cdot \gamma$, then $H' = \{\mathrm{ot}(x) : x \in H\}$ is unbounded in κ, since it contains κ distinct ordertypes. But then $f \restriction [H']^n = \{\gamma\}$, contradicting our assumption about f. □

Further refinements on the answer to this question can probably be found among the various generalizations of the basic partition relation. We mention one possibility in particular, similar to the remark at the end of Section 2, since its resolution may shed light on the relationship between the λ-subtle and λ-ineffable ideals:

CONJECTURE 5.4. *If $\lambda \geq \kappa \geq$ the least n-subtle cardinal, then for any ordertype-regressive function $f : [P_{[2,\kappa)}\lambda]^{n+1} \to \kappa$, $\gamma < \kappa$, and $\eta < \lambda$, there is a homogeneous set isomorphic to some $H \in \mathrm{NS}^+_{\gamma,\eta}$.*

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