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Extension spaces for superharmonic functions and Jensen measures

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Abstract. Let Ω be a Brelot space satisfying the domination principle and has a positive potential. We define the superharmonic extension property (SEP) for Ω and introduce a sufficient condition under which Ω has the SEP property. We give a characterization of the extreme Jensen measures in a Brelot space with SEP property.

1. Introduction.

Let Ω be a connected, locally compact, second countable Hausdorff space. Suppose that Ω has a system of harmonic functions satisfying Brelot axioms [**3**]. That is, the system of harmonic functions have the sheaf property and there exist a base of connected open sets (called regular) where Dirichlet problem is uniquely solvable, moreover the limit of an increasing sequence of harmonic functions defined on a domain is either infinity or harmonic. A lower semicontinuous, extended real valued function u on an open set $V \subset \Omega$ is called superharmonic if u never takes the value $-\infty$, is not identically ∞ on any connected component of V and for any regular open set $\omega \subset \overline{\omega} \subset V$, u satisfies the inequality

$$u(x) \geq \int u \, d\rho_x^{\omega}, \quad \forall x \in \omega,$$

where ρ_x^{ω} is the harmonic measure of ω at x. The set of all superharmonic (respectively nonnegative superharmonic) function on V will be denoted by $SH(V)$ (respectively $SH^+(V)$). A superharmonic function $p \geq 0$, is called a potential if any harmonic function $h \leq p$ also satisfies $h \leq 0$. Suppose that Ω has a potential $p > 0$, and also assume that the constant functions are harmonic. Moreover, assume that the space Ω satisfies the domination principle. That is, given a locally bounded potential which is harmonic off some closed set $A \subset \Omega$, and given a

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superharmonic function $u \in SH^+(\Omega)$ such that $u \geq p$ on A, then $u \geq p$ on Ω. To give an example of such spaces, let Ω be a connected open subset of $\boldsymbol{R}^n$ for $n \geq 2$ such that Ω has a Green function defined on it in case $n = 2$. Ω with the classical harmonic functions is a Brelot space satisfying all the above assumptions.

Definition 1.1. Let Ω be a Brelot space and let $x \in \Omega$. A Jensen measure on Ω at $x \in \Omega$ is a Radon probability measure supported by a compact set such that

$$u(x) \geq \int u \, d\mu, \quad \forall u \in SH(\Omega).$$

The set of all Jensen measures on Ω at the point x will be denoted by $J_x(\Omega)$. And the set of all Jensen measures on Ω at the point x with support in the set K is denoted by $J_x(\Omega, K)$.

For a Euclidean open subset U, in ([**5**]) B. Cole and T. J. Ransford proved that for $x \in U$ the harmonic measures ρ_x^D at x, for $D \subset\subset U$, are extreme element of $J_x(U)$. The characterization of the extreme elements of $J_x(U)$ was completed by S. Roy [**8**]. He proved that if μ is an extreme element of $J_x(U)$, then μ has the form $\mu = \delta_x^{\mathscr{C}V}$ where V is a finely open set such that $V \subset\subset U$. Roy's proof depends on the fact that the Euclidean space has some extension property for the superharmonic functions. In this article we focus on this extension property (call it SEP property) in Brelot spaces Ω and introduce a condition which is sufficient for Ω to have SEP property. And then we extend Roy's result to any Brelot spaces with SEP property.

Definition 1.2. The Brelot space Ω is said to have the SEP property (Superharmonic Extension Property) if for every compact set $K \subset \Omega$, and for every superharmonic function $u \in SH(\Omega)$, there exists a superharmonic function $v \in SH(\Omega)$ such that v is bounded below and $v = u$ on K.

In Theorem 3.1 we show that if Ω has suitable cover of open sets, then Ω has SEP property. More precisely, we show that Ω has the SEP if there exists a sequence of open sets ω_n such that for each n, ω_n is inner-outer regular (look at Definition 2.1) such that $\overline{\omega}_n \subset \omega_{n+1} \subset \overline{\omega}_{n+1} \subset \Omega$ and $\Omega = \bigcup_n \omega_n$.

2. Preliminary.

Let E be a subset of Ω and $u \in SH(\Omega)$ with $u \geq 0$ on E. The reduced function of u relative to E is defined as the following:

$$\mathscr{R}_u^E(x) = \inf\{v(x) : v \in SH^+(\Omega),\ v \geq u \text{ on } E\}$$

The lower semicontinuous regularization of $\mathscr{R}_u^E$, denoted by $\hat{\mathscr{R}}_u^E$, is called the balayage function of u relative to E.

If A is a Borel subset of Ω and $\mu \in M_+(\Omega)$ (the space of all positive Radon measures on Ω, such that $\mu(p) < \infty$ for some strictly positive continuous potential p), then there exists a unique Radon measure μ^A (called balayaged measure) such that

$$\int u\, d\mu^A = \int \hat{\mathscr{R}}_u^A\, d\mu, \quad \forall u \in SH^+(\Omega).$$

For details, see [**4**]. An important special case is the case when $\mu = \delta_x$ for some $x \in \Omega$. In this case δ_x^A is defined by the relation:

$$\int u\, d\delta_x^A = \hat{\mathscr{R}}_u^A(x), \quad \forall u \in SH^+(\Omega). \tag{1}$$

The fine topology on Ω is the coarsest topology on Ω which is finer than the given topology and makes every superharmonic function continuous.

In Equation (1), when $A = \mathscr{C}\omega\ (= \Omega - \omega)$, where ω is finely open set containing x, the measure $\delta_x^{\mathscr{C}\omega}$ is called the fine harmonic measure of ω at x.

A finely lower semicontinuous, extended real valued function f on a finely open set $U \subset \Omega$, is called finely superharmonic in U if $f > -\infty$, finite on a dense subset of U and $\forall$ finely open set $V \subset\subset U$ we have the inequality

$$f(x) \geq \int f\, d\delta_x^{\mathscr{C}V}, \quad \forall x \in U.$$

Let ω be an open subset of Ω. Let f be an extended real valued function on $\partial\omega$. The upper solution $\overline{\mathscr{H}}_f^\omega$ on ω is defined to be the lower envelop of all superharmonic functions u in ω such that $u > -\infty$ and

$$\liminf_{x \in \omega, x \to y} u(x) \geq f(y), \quad \forall y \in \partial\omega.$$

The lower solution $\underline{\mathscr{H}}_f^\omega$ is define to be $-\overline{\mathscr{H}}_{-f}^\omega$. If the function f is continuous then $\overline{\mathscr{H}}_f^\omega, \underline{\mathscr{H}}_f^\omega$ are equal and harmonic in ω. The common function is denoted by $\mathscr{H}_f^\omega$ and called the general solution of Dirichlet problem on ω with boundary data f. For $x \in \omega$ the functional $f \mapsto \mathscr{H}_f^\omega(x)$ is just a Radon measure μ_x^ω, called the harmonic measure of ω at x, identical to ρ_x^ω in the case ω is a regular set. Moreover $\mu_x^\omega = \delta_x^{\mathscr{C}\omega}$ and $H_f^\omega(x) = \int f\, d\mu_x^\omega = \int f\, d\delta_x^{\mathscr{C}\omega}$. The point $x \in \partial\omega$ is called regular point of ω if for each $f \in C(\partial\omega)$ (the continuous functions on the boundary of ω)

we have

$$\lim_{y\in\omega, y\to x} \mathscr{H}_f^{\omega}(y) = f(x).$$

DEFINITION 2.1. A relatively compact open set ω will be called inner-outer regular if its boundary points are regular with respect to both ω and $\Omega - \overline{\omega}$.

DEFINITION 2.2. For $x \in \Omega$ and a convex cone $\mathscr{A}$ of lower semi continuous functions on Ω we define the set $J_x(\mathscr{A})$ by the following relation:

$$J_x(\mathscr{A}) = \left\{\mu \in M_+(\Omega) : \int u\, d\mu \le u(x),\ \forall u \in \mathscr{A}\right\}.$$

An important special case in the above definition is the case when $\mathscr{A} = SH^+(\Omega)$. In this case the set $J_x(SH^+(\Omega))$ is a compact (in the w*-topology) convex subset of $M_+(\Omega)$. For details, see [**4**].

3. Superharmonic extension.

THEOREM 3.1. *If Ω has a countable cover of inner-outer regular open sets ω_n such that $\overline{\omega}_n \subset \omega_{n+1} \subset \overline{\omega}_{n+1} \subset \Omega$ and $\Omega = \bigcup_n \omega_n$. Then Ω has SEP property.*

PROOF. Let K be a compact subset of Ω and $u \in SH(\Omega)$. Let ω be an inner-outer regular domain such that $K \subseteq \omega \subset \overline{\omega} \subset \Omega$. Let ω_1, ω_2 be regular open sets such that $\overline{\omega} \subset \omega_1 \subset \overline{\omega}_1 \subset \omega_2 \subset \overline{\omega}_2 \subset \Omega$. Since u is lower semicontinuous, it is bounded below on K. So, without loss of generality we can assume that $u \ge 0$ on K. Define the function $\overline{u}$ on ω_2 by the relation $\overline{u} = (\hat{R}_u^K)_{\omega_2}$. Clearly, $\overline{u}$ has the following properties:

(i) $\overline{u}$ is a positive superharmonic function on ω_2.
(ii) $\overline{u}$ is harmonic on $\omega_2 \backslash K$, in particular continuous on $\partial\omega$.
(iii) $\overline{u} = u$ on K.
(iv) $\overline{u}$ tends to zero on the boundary of ω_2.

Let h_1 be the general Dirichlet solution on $\Omega - \overline{\omega}$ with boundary data $\overline{u}$ on $\partial\omega$ and 1 at Alexandroff point. Let h_2 be the general Dirichlet solution on $\Omega - \overline{\omega}$ with boundary data 0 on $\partial\omega$ and 1 at Alexandroff point. Since ω is outer regular then h_1 tends to $\overline{u}$ on $\partial\omega$ and h_2 tends to zero on $\partial\omega$, furthermore $0 < h_2 \le 1$. Now define the harmonic function H as the following:

$$H = h_1 - Ah_2, \text{ where } A > \sup\left\{\frac{h_1(x) - \overline{u}(x)}{h_2(x)} : x \in \partial\omega_1\right\}.$$

We claim that

$$\overline{u} \geq H \quad \text{on the open set } \omega_1 \backslash \overline{\omega}. \tag{2}$$

Because $\overline{u} - H$ is a lower bounded superharmonic function (actually harmonic) on $\omega_1 \backslash \overline{\omega}$, it is enough to show that

$$\liminf_{y \to x} (\overline{u} - H)(y) \geq 0, \quad \forall x \in \partial(\omega_1 \backslash \overline{\omega}) = \partial(\omega_1) \cup \partial(\omega).$$

If $x \in \partial(\omega_1)$, then

$$\begin{aligned}
\liminf_{y \to x} \{\overline{u}(y) - H(y)\} &= \liminf_{y \to x} \{\overline{u}(y) - h_1(y) + Ah_2(y)\} \\
&= \lim_{y \to x} (\overline{u}(y) - h_1(y)) + \lim_{y \to x} Ah_2(y) \\
&\qquad \text{(because } \overline{u}, h_1, h_2 \text{ are continuous on } \partial\omega_1) \\
&= \overline{u}(x) - h_1(x) + Ah_2(x) \\
&\geq 0 \qquad \text{(by the choice of } A)
\end{aligned}$$

If $x \in \partial\omega$, then

$$\begin{aligned}
\liminf_{y \to x} \{\overline{u}(y) - H(y)\} &= \liminf_{y \to x} \{\overline{u}(y) - h_1(y) + Ah_2(y)\} \\
&\geq \liminf_{y \to x} \overline{u}(y) + \liminf_{y \to x} (-h_1(y)) + \liminf_{y \to x} Ah_2(y) \\
&= \overline{u}(x) - \lim_{y \to x} h_1(y) + A \lim_{y \to x} h_2(y) \\
&\qquad \text{(since } \overline{u} \text{ is continuous at } x) \\
&= \overline{u}(x) - \overline{u}(x) + 0 \qquad \text{(by the construction of } h_1, h_2) \\
&= 0
\end{aligned}$$

The claim is proved.
Define the function v on Ω as the following:

$$v(x) = \begin{cases} H(x) & \text{if } x \in \Omega - \overline{\omega} \\ \overline{u}(x) & \text{if } x \in \overline{\omega}. \end{cases}$$

Now we show that the function v is the required function. It is clear that $v = \overline{u} = u$ on K. Moreover, v is bounded below. In fact $v = \overline{u} \geq 0$ on $\overline{\omega}$ and

$v = H = h_1 - Ah_2 \geq -A$ on $\Omega - \overline{\omega}$ (since $h_1 > 0$ and $h_2 \leq 1$). It remains to show that v is superharmonic on Ω. Obviously v is super harmonic in the open sets ω and $\Omega - \overline{\omega}$. So, we only need to check that:

(a) v is lower semicontinuous on $\partial\omega$.
(b) $\forall x \in \partial\omega$ there exists a local basis (at x) of regular open sets $\sigma \ni x$ such that

$$v(x) \geq \int v \, d\rho_x^\sigma. \tag{3}$$

If $x \in \partial\omega$, then

$$\begin{aligned}
\liminf_{y \to x} v(y) &= \min\left\{ \liminf_{y\in\omega\, y\to x} \overline{u}(y), \liminf_{y\in\Omega-\overline{\omega}\, y\to x} H(y)\right\} \\
&\geq \min\left\{\overline{u}(x), \liminf_{y\in\Omega-\overline{\omega}\, y\to x} H(y)\right\} && (\overline{u} \text{ is continuous at } x) \\
&= \min\{\overline{u}(x), \overline{u}(x)\} && (\text{by the construction of } H) \\
&= \overline{u}(x) = v(x).
\end{aligned}$$

So,

$$\liminf_{y\to x} v(y) \geq v(x) \; \forall \, x \in \partial\omega.$$

Therefore, v is lower semicontinuous and (a) is proved.

To prove (b), let $x \in \partial\omega$ and let δ be a regular open set such that $x \in \delta \subseteq \overline{\delta} \subseteq \omega_1$. By (2), we have that $H \leq \overline{u}$ on $\omega_1 \backslash \overline{\omega}$, So $v \leq \overline{u}$ on δ. As a result we have that for any regular domain σ with $x \in \sigma \subset \delta$,

$$\int v \, d\rho_x^\sigma \leq \int \overline{u} \, d\rho_x^\sigma \leq \overline{u}(x) = v(x).$$

The second inequality holds because $\overline{u}$ is superharmonic on ω_1. This proves (b). The proof is complete. □

4. Extreme Elements of $J_x(\Omega)$.

For a convex set A, denote the set of the extreme elements of A by *ext* A. We recall that Mokobodzki's theorem (see [**4**]) gives a characterization of the extreme elements of $J_x(SH^+(\Omega))$. Explicitly,

$$ext\, J_x(SH^+(\Omega)) = \left\{\delta_x^A : A\, \text{Borel} \subset \Omega\right\} \cup \{\delta_x\}. \tag{4}$$

The following result illustrates the relation between the classes $J_x(SH^+(\Omega))$, $J_x(\Omega)$.

Lemma 4.1. *Suppose that Ω has SEP property. For $x \in \Omega$, we have*

$$\left\{\mu \in J_x(SH^+(\Omega)) : supp\, \mu \subset\subset \Omega \text{ and } \mu(\Omega) = 1\right\} = J_x(\Omega).$$

Proof. First of all, it is clear that

$$J_x(\Omega) \subset \left\{\mu \in J_x(SH^+(\Omega)) : \operatorname{supp} \mu \subset\subset \Omega \text{ and } \mu(\Omega) = 1\right\}.$$

For the converse inclusion, let $\mu \in J_x(SH^+(\Omega))$ be such that $supp\, \mu \subset\subset \Omega$ and $\mu(\Omega) = 1$. To show that $\mu \in J_x(\Omega)$ we only need to show that every $u \in SH(\Omega)$ satisfies the inequality $u(x) \geq \int u\, d\mu$. Let $u \in SH(\Omega)$, and let K be a compact subset of Ω containing $supp\, \mu \cup \{x\}$. Since Ω has the SEP property, there exists a superharmonic function $v \in SH(\Omega)$ such that v is bounded below and $v = u$ on K. Let a be a positive real number such that $v + a \geq 0$. Now $u(x) + a = v(x) + a \geq \int (v + a)\, d\mu = \int_K v\, d\mu + a = \int_K u\, d\mu + a = \int u\, d\mu + a$. Thus we have $u(x) \geq \int u\, d\mu$. □

Lemma 4.2. *Suppose that Ω has the SEP property. For $x \in \Omega$, we have*

$$ext\, J_x(\Omega) = ext\, J_x(SH^+(\Omega)) \cap J_x(\Omega).$$

Proof. First, notice that $J_x(\Omega) \subset J_x(SH^+(\Omega))$ which implies that $ext\, J_x(SH^+(\Omega)) \cap J_x(\Omega) \subset ext\, J_x(\Omega)$. For the converse inclusion, let $\mu \in ext\, J_x(\Omega)$. To show that $\mu \in ext\, J_x(SH^+(\Omega))$, suppose that $\mu = (1/2)(\mu_1 + \mu_2)$ where μ_1, μ_2 are in $J_x(SH^+(\Omega))$. We must show that $\mu = \mu_1 = \mu_2$. First, we show that the measures μ_1, μ_2 are in fact in $J_x(\Omega)$. Since $\mu_i(\Omega) \leq 1$ for $i = 1, 2$ and $1 = \mu(\Omega) = (1/2)(\mu_1(\Omega) + \mu_2(\Omega))$, we have that $\mu_1(\Omega) = \mu_2(\Omega) = 1$. Furthermore, the relation $\mu = (1/2)(\mu_1 + \mu_2)$ implies that for $i = 1, 2$, $supp\, \mu_i \subset supp\, \mu \subset\subset \Omega$. Thus, by Lemma 4.1 we have that $\mu_1, \mu_2 \in J_x(\Omega)$. Considering the relation $\mu = (1/2)(\mu_1 + \mu_2)$ and the fact that $\mu \in ext\, J_x(\Omega)$ and $\mu_1, \mu_2 \in J_x(\Omega)$, we get that $\mu = \mu_1 = \mu_2$. □

Recall that the class of finely harmonic measures $FH_x(\Omega)$ is defined as:

$$FH_x(\Omega) = \left\{\delta_x^{\mathscr{C}\omega} : x \in \omega,\ \omega \text{ finely domain} \subset\subset \Omega\right\} \cup \{\delta_x\}.$$

The following theorem gives a characterization of the extreme elements of $J_x(\Omega)$.

THEOREM 4.1. *Suppose that Ω has the SEP property. For $x \in \Omega$, we have*

$$ext\, J_x(\Omega) = FH_x(\Omega). \tag{5}$$

PROOF. Let $\mu \in ext\, J_x(\Omega)$. The above lemma implies that $\mu \in ext\, J_x(SH^+(\Omega)) \cap J_x(\Omega)$. Now, by Mokobodzki's theorem we have that $\mu = \delta_x$ or $\mu = \delta_x^A$, $x \notin A$ for some Borel set $A \subset \Omega$. Obviously if $\mu = \delta_x$ then μ belongs to the right hand side of (5). Suppose that $\mu = \delta_x^A, x \notin A$. Notice that $\delta_x^A = \delta_x^B$ where $B = b(A) = \{y : \delta_y^A = \delta_y\}$, is the base of A ([**6**, Theorem 4.7]). Moreover, $\mu = \delta_x^B = \delta_x^{\mathscr{C}\omega}$ where ω is the finely connected component of $\Omega - B$ which contains x ([**6**, Theorem 12.7]). Since $\mu \in J_x(\Omega)$, $supp\,\mu = \partial\omega \subset\subset \Omega$, therefore $\omega \subset\subset \Omega$. Hence μ is in the right hand side of (5) and we have

$$ext\, J_x(\Omega) \subseteq \left\{\delta_x^{\mathscr{C}\omega} : x \in \omega,\ \omega \text{ finely domain} \subset\subset \Omega\right\} \cup \{\delta_x\}.$$

The converse inequality: Let μ be in the right hand side of (5). The result is obvious if $\mu = \delta_x$. Suppose that $\mu = \delta_x^{\mathscr{C}\omega}$ for some finely domain ω containing x. It is clear that the measure $\mu = \delta_x^{\mathscr{C}\omega}$ is a finely harmonic probability Borel measure supported by the set $\partial_f\omega \subset \overline{\omega} \subset\subset \Omega$. If $u \in SH(\Omega)$, then u is a finely superharmonic function on Ω. Therefore, by definition of finely superharmonic function, we have the inequality $u(x) \geq \int u\, d\delta_x^{\mathscr{C}\omega}$. This implies that $\mu \in J_x(\Omega)$.

To show that $\mu = \delta_x^{\mathscr{C}\omega}$ is an extreme element of $J_x(\Omega)$, suppose that $\mu = \delta_x^{\mathscr{C}\omega} = (1/2)(\mu_1 + \mu_2)$, where $\mu_1, \mu_1 \in J_x(\Omega)$. This relation implies that for $i = 1, 2$ the measure μ_i does not charge polar sets and $supp\,\mu_i \subset supp\,\delta_x^{\mathscr{C}\omega} \subset \mathscr{C}\omega$.

Let u be a nonnegative superharmonic function. For $i = 1, 2$ we have

$$\begin{aligned}
\int u\, d\mu_i &= \int_{\mathscr{C}\omega} u\, d\mu_i \\
&= \int_{\mathscr{C}\omega} \mathscr{R}_u^{\mathscr{C}\omega}\, d\mu_i \\
&= \int_{\mathscr{C}\omega} \hat{\mathscr{R}}_u^{\mathscr{C}\omega}\, d\mu_i && \text{(because } \mu_i \text{ does not charge polar sets)} \\
&= \int \hat{\mathscr{R}}_u^{\mathscr{C}\omega}\, d\mu_i \\
&\leq \hat{\mathscr{R}}_u^{\mathscr{C}\omega}(x) && \text{(because } \mu_i \in J_x(\Omega)\text{)} \\
&= \int u\, d\delta_x^{\mathscr{C}\omega}. && \text{(by definition of } \delta_x^{\mathscr{C}\omega}\text{)}
\end{aligned}$$

Thus, for $i = 1, 2$ we have $\mu_i(u) \leq \delta_x^{\mathscr{C}\omega}(u) = (1/2)(\mu_1(u) + \mu_2(u))$, which implies that $\mu_1(u) = \mu_2(u)$ for all positive continuous superharmonic functions u. So, $\mu_1(u_1 - u_2) = \mu_2(u_1 - u_2)$ for all positive continuous superharmonic functions u_1, u_2. Because $\{SH^+(\Omega) - SH^+(\Omega)\} \cap C(\Omega)$ is dense in $C(\Omega)$, the space of continuous functions on Ω, we get $\mu_1 = \mu_1 = \delta_x^{\mathscr{C}\omega}$. Hence $\mu = \delta_x^{\mathscr{C}\omega} \in ext(J_x(\Omega))$. $\square$

Characterizing $ext\, J_x(\Omega)$ can be used to give integral representations for the elements of $J_x(\Omega)$ in the sense of Choquet. First, let's recall Choquet theorem: if X is a compact convex metrizable subset of a locally convex space E, and $x \in X$. Then there exists a probability Radon measure σ on X carried by $ext\, X$ such that

$$f(x) = \int_X f\, d\sigma, \quad \forall f \in E^*,$$

where E^* is the set of all continuous linear functionals on E.

Let $x \in \Omega$. Let $\mu \in J_x(\Omega)$, so $\mu \in J_x(\Omega, K)$ for some compact set K. The set $J_x(\Omega, K)$ is a convex compact metrizable (in w*-topology) subset of the locally convex space $C^*(\Omega)$. So, by Choquet theorem, there exists a Radon probability measure σ on $J_x(\Omega, K)$, supported by $ext\, J_x(\Omega, K)$, such that

$$L(\mu) = \int_{ext\, J_x(\Omega,K)} L(\nu)\, d\sigma(\nu), \quad \forall L \in C^{**}(\Omega). \tag{6}$$

Obviously, σ can be extended to $J_x(\Omega)$. The set $ext\, J_x(\Omega, K)$ is a subset of the Borel set $ext\, J_x(\Omega)$, so σ is carried by $ext\, J_x(\Omega)$. Moreover, the continuous linear functionals on $C^*(\Omega)$ are the functions $f \in C(\Omega)$, so (6) becomes

$$\int f\, d\mu = \int_{FH_x(\Omega)} \left(\int_\Omega f\, d\nu \right) d\sigma(\nu), \quad \forall f \in C(\Omega).$$

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