

Gromov hyperbolicity of Denjoy domains with hyperbolic and quasihyperbolic metrics

By Peter HÄSTÖ, Henri LINDÉN, Ana PORTILLA,
José M. RODRÍGUEZ, and Eva TOURÍS

(Received Feb. 13, 2009)
(Revised Aug. 3, 2010)

Abstract. We derive explicit and simple conditions which in many cases allow one to decide, whether or not a Denjoy domain endowed with the Poincaré or quasihyperbolic metric is Gromov hyperbolic. The criteria are based on the Euclidean size of the complement. As a corollary, the main theorem allows us to deduce the non-hyperbolicity of any periodic Denjoy domain.

1. Introduction.

In the 1980s Mikhail Gromov introduced a notion of abstract hyperbolic spaces, which have thereafter been studied and developed by many authors. Initially, the research was mainly centered on hyperbolic group theory [8], but lately researchers have shown an increasing interest in more direct studies of spaces endowed with metrics used in geometric function theory [4], [6], [7], [21], [22].

One of the primary questions is naturally whether a metric space (X, d) is hyperbolic in the sense of Gromov or not. The most classical examples, mentioned in every textbook on this topic, are metric trees, the classical Poincaré hyperbolic metric developed in the unit disk and, more generally, simply connected complete Riemannian manifolds with sectional curvature $K \leq -k^2 < 0$.

However, it is not easy to determine whether a given space is Gromov hyperbolic or not. In recent years several investigators have been interested in showing that metrics used in geometric function theory are Gromov hyperbolic. For

2000 *Mathematics Subject Classification.* Primary 30F45; Secondary 53C23, 30C99.

Key Words and Phrases. Poincaré metric, hyperbolic metric, quasihyperbolic metric, Gromov hyperbolic, Denjoy domain.

The first author was supported in part by the Academy of Finland.

The third and fifth authors were supported in part by a grant from Ministerio de Ciencia e Innovación (MTM 2009-12740-C03-01), Spain.

The third, fourth and fifth authors were supported in part by two grants from Ministerio de Ciencia e Innovación (MTM 2009-07800 and MTM 2008-02829-E), Spain.

The fifth author was supported in part by a grant from U.C.III M./C.A.M. (CCG08-UC3M/ESP-4516), Spain.

instance, the Klein-Hilbert and Kobayashi metrics are Gromov hyperbolic (under particular conditions on the domain of definition, see [5], [11] and [3]); the Gehring-Osgood j -metric is Gromov hyperbolic; and the Vuorinen j -metric is not Gromov hyperbolic except in the punctured space (see [9]). Also, in [12] the hyperbolicity of the conformal modulus metric μ and the related so-called Ferrand metric λ^* , is studied.

Since the Poincaré metric is also the metric giving rise to what is commonly known as the hyperbolic metric when speaking about open domains in the complex plane or in Riemann surfaces, it could be expected that there is a connection between the notions of hyperbolicity. For simply connected subdomains Ω of the complex plane, it follows directly from the Riemann mapping theorem that the metric space (Ω, h_Ω) is in fact Gromov hyperbolic. However, as soon as simply connectedness is omitted, there is no immediate answer to whether the space h_Ω is hyperbolic or not. The question has lately been studied in [1] and [13]–[20].

The related quasihyperbolic metric has also recently been a topic of interest regarding the question of Gromov hyperbolicity. In [6], Bonk, Heinonen and Koskela found necessary and sufficient conditions for when a planar domain D endowed with the quasihyperbolic metric is Gromov hyperbolic. This was extended by Balogh and Buckley, [4]: they found two different necessary and sufficient conditions which work in Euclidean spaces of all dimensions and also in metric spaces under some conditions.

In this article we are interested in Denjoy domains. In this case the result of [6] says that the domain is Gromov hyperbolic with respect to the quasihyperbolic metric if and only if the domain is the conformal image of an inner uniform domain (see Section 3). Although this is a very nice characterization, it is somewhat difficult to check that a domain is inner uniform, since we need to construct uniform paths connecting every pair of points.

In this paper we show that it is necessary to look at paths joining only a very small (countable) number of points when we want to determine the Gromov hyperbolicity. This allows us to derive simple and very concrete conditions on when the domain is Gromov hyperbolic. However, the main purpose of the results on the quasihyperbolic metric is that they suggest methods for proving the corresponding results for the hyperbolic metric, which is the main contribution of the paper. To the best of our knowledge, this is the first time that Gromov hyperbolicity of any class of infinitely connected domains has been obtained from conditions on the Euclidean size of the complement of the domain. It means that we are relating Euclidean conditions to properties of non-Euclidean metrics.

The main results in this article are the following:

THEOREM 1.1. *Let Ω be a Denjoy domain with $\Omega \cap \mathbf{R} = (-\infty, 0) \cup$*

$\bigcup_{n=1}^{\infty} (a_n, b_n)$, $b_n \leq a_{n+1}$ for every n , and $\lim_{n \rightarrow \infty} a_n = \infty$.

(1) The metrics k_{Ω} and h_{Ω} are Gromov hyperbolic if

$$\liminf_{n \rightarrow \infty} \frac{b_n - a_n}{a_n} > 0.$$

(2) The metrics k_{Ω} and h_{Ω} are not Gromov hyperbolic if

$$\lim_{n \rightarrow \infty} \frac{b_n - a_n}{a_n} = 0.$$

In the case

$$0 = \liminf_{n \rightarrow \infty} \frac{b_n - a_n}{a_n} < \limsup_{n \rightarrow \infty} \frac{b_n - a_n}{a_n},$$

which is not covered by the previous theorem, one can construct examples to show that the metrics k_{Ω} and h_{Ω} may or may not be Gromov hyperbolic. In this sense our result is optimal.

In this theorem the most relevant and difficult part is the second one, whereas the first one is a kind of converse. Both of them joined even provide a characterization when the limit exists. Consider the following example: $\Omega := \mathbb{C} \setminus \bigcup_{n=1}^{\infty} \{(\log n)^{\alpha} n^{\beta} A^n\}$ with $\lim_{n \rightarrow \infty} (\log n)^{\alpha} n^{\beta} A^n = \infty$; Theorem 1.1 gives directly that Ω is hyperbolic if and only if $A > 1$.

The main difficulty in the proof is that it is impossible to determine the precise location of the geodesics with these metrics (we do not even have an explicit expression for the Poincaré density).

It is interesting to note that in the case of Denjoy domains many of the results seem to hold for both the hyperbolic and the quasihyperbolic metrics. In fact, we know of no planar domain which is Gromov hyperbolic with respect to one of these metrics, but not the other.¹

In the previous theorem, the boundary components have a single accumulation point, at ∞ , and the accumulation happens only from one side. It turns out that if this kind of domain is not Gromov hyperbolic, then we cannot mend the situation by adding some boundary to the other side of the accumulation point, as the following theorem shows.

¹After the completion of this paper, we have been able to prove that no such domain exists, see [10, Theorem 4.6].

THEOREM 1.2. *Let Ω be a Denjoy domain with $(-\infty, 0) \subset \Omega$ and let $F \subseteq (-\infty, 0]$ be closed. If k_Ω is not Gromov hyperbolic, then neither is $k_{\Omega \setminus F}$; if h_Ω is not Gromov hyperbolic, then neither is $h_{\Omega \setminus F}$.*

One might think that the assumption $F \subseteq (-\infty, 0]$ is superfluous; however the following example shows that the conclusion is false in general when we consider a closed set F not contained in the negative half axis: let Ω be as in Theorem 1.1(2) and $F := [0, \infty)$. Then Ω is not hyperbolic, but $\Omega \setminus F = \mathbf{C} \setminus F$ is hyperbolic, since it is simply connected.

Theorem 1.2, in particular, allows to deduce the same conclusions as Theorem 1.1(2), removing the technical hypothesis $(-\infty, 0) \subset \Omega$.

If E_0 is any closed set contained in the open set $\{z = x + iy \in \mathbf{C} : x, y \in (0, 1)\}$ and $E_{m,n} := E_0 + m + in$, then it is clear that $\mathbf{C} \setminus \cup_{m,n \in \mathbf{Z}} E_{m,n}$ is not Gromov hyperbolic, since its isometry group contains a subgroup isomorphic to \mathbf{Z}^2 (a non-hyperbolic group).

It might be reasonable to think that any periodic Denjoy domain is hyperbolic, since its isometry group is (in the generic case) isomorphic to \mathbf{Z} , which is a hyperbolic group. However, in the following example we prove the non-hyperbolicity of any periodic Denjoy domain (as a direct consequence of Theorem 4.2):

EXAMPLE 1.3. Let $E_0 \subset [0, t)$ be closed, $t > 0$, set $E_n := E_0 + tn$ for $n \in \mathbf{N}$ or $n \in \mathbf{Z}$, and $\Omega := \mathbf{C} \setminus \cup_n E_n$. Then h_Ω and k_Ω are not Gromov hyperbolic.

2. Definitions and notation.

By \mathbf{H} we denote the upper half plane, $\{z \in \mathbf{C} : \text{Im } z > 0\}$, and by \mathbf{D} the unit disk $\{z \in \mathbf{C} : |z| < 1\}$. For $D \subset \mathbf{C}$ we denote by ∂D and \overline{D} its boundary and closure, respectively. For $z \in D \subsetneq \mathbf{C}$ we denote by $\delta_D(z)$ the distance to the boundary of D , $\min_{a \in \partial D} |z - a|$. Finally, we denote by c , C , c_j and C_j generic constants which can change their value from line to line and even in the same line.

Recall that a domain $\Omega \subset \mathbf{C}$ is said to be of *hyperbolic type* if it has at least two finite boundary points. The universal cover of such domain is the unit disk \mathbf{D} . In Ω we can define the Poincaré metric, i.e. the metric obtained by projecting the metric $ds = 2|dz|/(1 - |z|^2)$ of the unit disk by any universal covering map $\pi : \mathbf{D} \rightarrow \Omega$. Equivalently, we can project the metric $ds = |dz|/\text{Im } z$ of the upper half plane \mathbf{H} . Therefore, any simply connected subset of Ω is isometric to a subset of \mathbf{D} . With this metric, Ω is a geodesically complete Riemannian manifold with constant curvature -1 , in particular, Ω is a geodesic metric space. By λ_Ω we denote the density of the Poincaré metric in Ω , i.e. the positive function such that $\lambda_\Omega^2(z)(dx^2 + dy^2)$ is the Poincaré metric in Ω . The Poincaré metric is natural and useful in complex analysis; for instance, any holomorphic function between two

domains is Lipschitz with constant 1, when we consider the respective Poincaré metrics.

The quasihyperbolic metric is the distance induced by the density $1/\delta_\Omega(z)$. By k_Ω and h_Ω we denote the quasihyperbolic and Poincaré distance in Ω , respectively. Length (of a curve) will be denoted by the symbol $\ell_{d,\Omega}$, where d is the metric with respect to which length is measured. If it is clear which metric or domain is used, either one or both subscripts in $\ell_{d,\Omega}$ might be omitted. The subscript Eucl is used to denote the length with respect to the Euclidean metric. Also, as most of the proofs apply to both the quasihyperbolic and the Poincaré metrics, we will use the symbol κ as a “dummy metric” symbol, where it can be replaced by either k or h .

It is well known that for every domain Ω of hyperbolic type

$$\lambda_\Omega(z) \leq \frac{2}{\delta_\Omega(z)} \quad \forall z \in \Omega, \quad \ell_{h,\Omega}(\gamma) \leq 2\ell_{k,\Omega}(\gamma) \quad \forall \gamma \subset \Omega,$$

and that for all domains $\Omega_1 \subset \Omega_2$ we have $\lambda_{\Omega_1}(z) \geq \lambda_{\Omega_2}(z)$ for every $z \in \Omega_1$.

A geodesic metric space (X, d) is said to be *Gromov δ -hyperbolic*, if

$$d(w, [x, z] \cup [z, y]) \leq \delta$$

for all $x, y, z \in X$; corresponding geodesic segments $[x, y]$, $[y, z]$ and $[x, z]$; and $w \in [x, y]$. If this inequality holds, we also say that the geodesic triangle is *δ -thin*, so Gromov hyperbolicity can be reformulated by requiring that all geodesic triangles are thin. In order to simplify the notation, we say that d is Gromov-hyperbolic (instead of (X, d) is Gromov-hyperbolic).

A *Denjoy domain* $\Omega \subset \mathbf{C}$ is a domain whose boundary is contained in the real axis. Since $\Omega \cap \mathbf{R}$ is an open set contained in \mathbf{R} , it is the union of pairwise disjoint open intervals; as each interval contains a rational number, this union is countable. Hence, we can write $\Omega \cap \mathbf{R} = \cup_{n \in \Lambda} (a_n, b_n)$, where Λ is a countable index set, $\{(a_n, b_n)\}_{n \in \Lambda}$ are pairwise disjoint, and it is possible to have $a_{n_1} = -\infty$ for some $n_1 \in \Lambda$ and/or $b_{n_2} = \infty$ for some $n_2 \in \Lambda$.

In order to study Gromov hyperbolicity, we consider the case where Λ is countably infinite, since if Λ is finite then h_Ω and k_Ω are easily seen to be Gromov hyperbolic by Proposition 3.5, below.

3. Some classes of Denjoy domains which are Gromov hyperbolic.

The quasihyperbolic metric is traditionally defined in subdomains of Euclidean n -space \mathbf{R}^n , i.e. open and connected subsets $\Omega \subsetneq \mathbf{R}^n$. However, a more abstract setting is also possible, as Bonk, Heinonen and Koskela showed in [6]. They show that if (X, d) is any locally compact, rectifiably connected and non-

complete metric space, then the quasihyperbolic metric k_X can be defined as usual, using the weight $1/\text{dist}(x, \partial X)$.

Given a real number $A \geq 1$, a rectifiable curve $\gamma: [0, 1] \rightarrow \Omega$ is called *A-uniform for the metric d* if

$$\begin{aligned} \ell_d(\gamma) &\leq A d(\gamma(0), \gamma(1)) \quad \text{and} \\ \min\{\ell_d(\gamma|[0, t]), \ell_d(\gamma|[t, 1])\} &\leq A \text{dist}_d(\gamma(t), \partial\Omega), \quad \text{for all } t \in [0, 1]. \end{aligned}$$

Moreover, a locally compact, rectifiably connected noncomplete metric space is said to be *A-uniform* if every pair of points can be joined by an *A-uniform* curve. The abbreviations “*A-uniform*” and “*A-inner uniform*” (without mention of the metric) mean *A-uniform* for the Euclidean metric and Euclidean inner metric, respectively.

Uniform domains are intimately connected to domains which are Gromov hyperbolic with respect to the quasihyperbolic metric (see [6, Theorems 1.12, 11.3]). Specifically, for a Denjoy domain Ω these results imply that k_Ω is Gromov hyperbolic if and only if Ω is the conformal image of an inner uniform domain.

Here we will use the generalized setting in [6] to show that for Denjoy domains with the quasihyperbolic metric it actually suffices to consider the intersection of the closed upper (or lower) halfplane with the actual domain. The same result holds for the Poincaré metric:

LEMMA 3.1. *Let $E \subset \mathbf{R}$ be a closed set with at least two points, and denote by $\Omega = \mathbf{C} \setminus E$ and $\Omega_0 = \Omega \cap \{z \in \mathbf{C} \mid \text{Im } z \geq 0\} = \Omega \cap \overline{\mathbf{H}^2}$. Then the metric space Ω_0 , with the restriction of the Poincaré or the quasihyperbolic metric in Ω , is δ -Gromov hyperbolic, with some universal constant δ .*

PROOF. We deal first with the quasihyperbolic metric. As the upper halfplane is uniform in the classical case, the same curve of uniformity (which is an arc of a circle orthogonal to \mathbf{R}) can be shown to be an *A-uniform* curve in the sense of [6] for the set Ω_0 , for some absolute constant A . Hence Ω_0 is *A-uniform*. By [6, Theorem 3.6] it then follows that the space (Ω_0, k_{Ω_0}) is Gromov hyperbolic. (Note that k_{Ω_0} is the same as k_Ω restricted to Ω_0 .)

We also have that Ω_0 is hyperbolic with the restriction of the Poincaré metric h_Ω , since it is isometric to a geodesically convex subset of the unit disk (in fact, for every pair of points in Ω_0 , there is just one geodesic contained in Ω_0 joining them). Therefore, Ω_0 has $\log(1 + \sqrt{2})$ -thin triangles, as the unit disk does (see, e.g. [2, p. 130]). \square

DEFINITION 3.2. Let Ω be a Denjoy domain of hyperbolic type. Then $\Omega \cap \mathbf{R} = \cup_{n \geq 0} (a_n, b_n)$ for some pairwise disjoint intervals. We say that a curve in Ω is

a *fundamental geodesic* if it is a geodesic (with respect to the metric considered in Ω) joining $x_{0,n} \in (a_0, b_0)$ and $x_n \in (a_n, b_n)$, $n > 0$, which is contained in the closed halfplane $\overline{\mathbf{H}^2} = \{z \in \mathbf{C} : \text{Im } z \geq 0\}$. We denote by γ_n a fundamental geodesic corresponding to n . Some examples are shown in Figure 1.

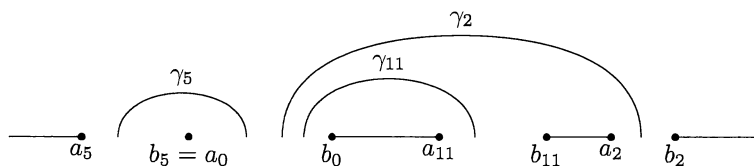


Figure 1. Fundamental geodesics for $n = 2, 5, 11$.

The next result was proven for the hyperbolic metric in [1, Theorem 5.1]. In view of Lemma 3.1 one can check that the same proof carries over to the quasihyperbolic metric.

By a *bigon* we mean a polygon with two edges.

We say that an inequality holds *quantitatively* if it holds with a constant depending only on the constants in the assumptions.

THEOREM 3.3. *Let Ω be a Denjoy domain of hyperbolic type and denote by κ_Ω the Poincaré or the quasihyperbolic metric. Then the following conditions are quantitatively equivalent:*

- (1) κ_Ω is δ -hyperbolic.
- (2) There exists a constant c_1 such that for every choice of fundamental geodesics $\{\gamma_n\}_{n=1}^\infty$ we have $\kappa_\Omega(z, \mathbf{R}) \leq c_1$ for every $z \in \cup_{n \geq 1} \gamma_n$.
- (3) There exists a constant c_2 such that for a fixed choice of fundamental geodesics $\{\gamma_n\}_{n=1}^\infty$ we have $\kappa_\Omega(z, \mathbf{R}) \leq c_2$ for every $z \in \cup_{n \geq 1} \gamma_n$.
- (4) There exists a constant c_3 such that every geodesic bigon in Ω with vertices in \mathbf{R} is c_3 -thin.

Note that the case $\Omega \cap \mathbf{R} = \cup_{n=0}^N (a_n, b_n)$ is also covered by the theorem.

COROLLARY 3.4. *Let Ω be a Denjoy domain of hyperbolic type and denote by κ_Ω either the Poincaré or the quasihyperbolic metric. If there exist a constant C and a sequence of fundamental geodesics $\{\gamma_n\}_{n \geq 1}$ with $\ell_{\kappa, \Omega}(\gamma_n) \leq C$, then κ_Ω is δ -Gromov hyperbolic, and δ depends only on C .*

If Ω has only finitely many boundary components, then it is always Gromov hyperbolic, in a quantitative way:

PROPOSITION 3.5. *Let Ω be a Denjoy domain of hyperbolic type with $\Omega \cap \mathbf{R} =$*

$\cup_{n=0}^N(a_n, b_n)$, and denote by κ_Ω either the Poincaré or the quasihyperbolic metric. Then κ_Ω is δ -Gromov hyperbolic, where δ is a constant which only depends on N and

$$c_0 = \sup_n \kappa_\Omega((a_n, b_n), (a_{n+1}, b_{n+1})).$$

Note that we do not require $b_n \leq a_{n+1}$ (although the intervals $\{(a_n, b_n)\}_n$ are, as always, pairwise disjoint).

PROOF. Let us consider the geodesics g_n^* with respect to κ_Ω joining (a_n, b_n) and (a_{n+1}, b_{n+1}) in $\Omega^+ := \Omega \cap \overline{\mathbf{H}^2}$. Then $\ell_\Omega(g_n^*) \leq c_0$ for $0 \leq n \leq N-1$.

By Theorem 3.3, we only need to prove that there exists a constant c , which only depends on c_0 and N , such that $\kappa_\Omega(z, \mathbf{R}) \leq c$ for every $z \in \cup_{n=1}^N \gamma_n$.

For each $0 \leq n \leq N-1$, let us consider the geodesic polygon P in Ω^+ , with the following edges: $\gamma_n, g_0^*, \dots, g_{n-1}^*$, and the geodesics joining their endpoints which are contained in $(a_0, b_0), \dots, (a_n, b_n)$. Since $(\Omega^+, \kappa_\Omega)$ is δ_0 -Gromov hyperbolic, where δ_0 is a constant which only depends on c_0 , by Lemma 3.1, and P is a geodesic polygon in Ω^+ with at most $2N+2$ sides, P is $2N\delta_0$ -thin. Therefore, given any $z \in \gamma_n$, there exists a point $w \in \cup_{k=0}^{N-1} g_k^* \cup \mathbf{R}$ with $\kappa_\Omega(z, w) \leq 2N\delta_0$. Since $\ell_\Omega(g_k^*) \leq c_0$ for $0 \leq k \leq N-1$, there exists $x \in \mathbf{R}$ with $\kappa_\Omega(x, w) \leq c_0/2$. Hence, $\kappa_\Omega(z, \mathbf{R}) \leq \kappa_\Omega(z, x) \leq 2N\delta_0 + c_0/2$, and we conclude that κ_Ω is δ -Gromov hyperbolic. \square

THEOREM 3.6. Let Ω be a Denjoy domain with $\Omega \cap \mathbf{R} = \cup_{n=0}^\infty(a_n, b_n)$, $(a_0, b_0) = (-\infty, 0)$ and $b_n \leq a_{n+1}$ for every n . Suppose that $b_n \geq Ka_n$ for a fixed $K > 1$ and every n . Then h_Ω and k_Ω are δ -Gromov hyperbolic, with δ depending only on K .

PROOF. Fix n and consider the domain

$$\Omega_n = \frac{1}{a_n} \Omega = \left\{ \frac{x}{a_n} \mid x \in \Omega \right\}.$$

If we define $D := \mathbf{C} \setminus ([0, 1] \cup [K, \infty))$, then $D \subset \Omega_n$, and $\ell_{k, \Omega_n}(\gamma) \leq \ell_{k, D}(\gamma)$ for every curve $\gamma \subset D$. The circle $\sigma := S^1(0, (1+K)/2)$ goes around the boundary component $[0, 1]$ in D and has finite quasihyperbolic length:

$$\ell_{k, D}(\sigma) \leq \int_\sigma \frac{|dz|}{\frac{K-1}{2}} = 2\pi \frac{K+1}{K-1}.$$

Consider the shortest fundamental geodesics joining (a_0, b_0) with (a_n, b_n) , with the Poincaré and the quasihyperbolic metrics, γ_n^h and γ_n^k , respectively. Then,

$$\ell_{k,\Omega}(\gamma_n^k) = \ell_{k,\Omega_n} \left(\frac{1}{a_n} \gamma_n^k \right) \leq \ell_{k,\Omega_n}(\sigma) \leq \ell_{k,D}(\sigma) \leq 2\pi \frac{K+1}{K-1},$$

$$\ell_{h,\Omega}(\gamma_n^h) \leq \ell_{h,\Omega}(\gamma_n^k) \leq 2 \ell_{k,\Omega}(\gamma_n^k) \leq 4\pi \frac{K+1}{K-1}.$$

Therefore, by Corollary 3.4, h_Ω and k_Ω are δ -Gromov hyperbolic (and δ depends only on K). □

PROOF OF THEOREM 1.1(1). If $\liminf_{n \rightarrow \infty} (b_n - a_n)/a_n > 0$, then we can choose $K > 1$ so that $(b_n - a_n)/a_n > K - 1$ for every n , whence $b_n > Ka_n$. Thus the previous theorem implies the claims. □

4. Some classes of Denjoy domains which are not Gromov hyperbolic.

To use the characterization of Bonk, Heinonen and Koskela [6], one would need to show that the domain is not the conformal image of an inner uniform domain. However, this seems to be very difficult. Let us prove that the domain is not A -inner uniform. We will then use the ideas to provide a direct proof for the claim.

So, suppose for a contradiction that the domain is A -inner uniform for some fixed $A > 0$. We define $s_n := \max_{1 \leq m \leq n} (b_m - a_m)$. It is clear that s_n is an increasing sequence and the assumption of the theorem implies that $\lim_{n \rightarrow \infty} s_n/a_n = 0$. If we define $g_n := \sqrt{s_n/a_n}$, then $b_m - a_m \leq a_n g_n^2$ for every $1 \leq m \leq n$ and $\lim_{n \rightarrow \infty} g_n = 0$.

Since $g_n > 0$, we can choose a subsequence $\{g_{n_k}\}$ with $g_{n_k} \geq g_m$ for every $m \geq n_k$; consider a fixed n from the sequence $\{n_k\}$. Set $c_n = (b_n + a_n)/2$, the mid-point of (a_n, b_n) . We define $x_n = c_n + ic_n g_n$ and $y_n = c_n - ic_n g_n$. Since $[x_n, y_n] \subset \Omega$, we have $\ell_{\text{Eucl},\Omega}([x_n, y_n]) = 2c_n g_n$. Let γ be an A -inner uniform curve joining x_n and y_n , and let $z \in \gamma \cap \mathbf{R}$. Since $|x_n - z|, |y_n - z| \geq c_n g_n$, we conclude by the uniformity of the curve that $\delta_\Omega(z) \geq c_n g_n/A$. On the other hand, the uniformity of γ also implies that $|z - c_n| \leq 2Ac_n g_n$.

We may assume that n is so large that $c_n > 2Ac_n g_n$. Then z lies in the positive real axis, which means that $z \in (a_m, b_m)$ for some $m \geq 1$. If $m \leq n$, then we have $b_m - a_m \leq s_n = a_n g_n^2 < c_n g_n^2$. For $m > n$ we have $b_m - a_m \leq g_m^2 a_m \leq g_n^2 a_m$. However, since $a_m < z \leq c_n + 2Ac_n g_n < 2c_n$, we obtain $b_m - a_m < 2c_n g_n^2$ also in this case.

Since $\delta_\Omega(z) \leq (b_m - a_m)/2$, we conclude that $c_n g_n/A \leq c_n g_n^2$. Since $g_n \rightarrow 0$ and A is a constant, this is a contradiction. Hence the assumption that an A -inner uniform curve exists was false, and we can conclude that the domain is not inner uniform.

For the proof of our theorem, we need the following well-known fact:

LEMMA 4.1. *Let γ be a rectifiable curve in a domain $D \subset \mathbf{R}^n$ with end-point $a \in D$ and Euclidean length s . Then $\ell_{k,D}(\gamma) \geq \log(1 + s/\delta_D(a))$.*

PROOF. For completeness we give the proof even though it is well-known. Let $z \in \partial D$ be a point with $\delta_D(a) = |a - z|$. Without loss of generality we assume that $z = 0$. By monotonicity $\ell_{k,D}(\gamma) \geq \ell_{k,\mathbf{R}^n \setminus \{0\}}(\gamma)$. Further, it is clear that $\ell_{k,\mathbf{R}^n \setminus \{0\}}(\gamma) \geq \ell_{k,\mathbf{R}^n \setminus \{0\}}(|a|, |a| + s)$, whence the estimate by integrating the density $1/|x|$. \square

PROOF OF THEOREM 1.1(2). By Theorem 4.6 of [10] the space (Ω, h_Ω) is Gromov hyperbolic if and only if (Ω, k_Ω) is Gromov hyperbolic, quantitatively.² Therefore it suffices to prove the theorem for the quasihyperbolic metric. We consider two cases: either $\{b_m - a_m\}_m$ is bounded or unbounded. We start with the latter case.

Define $s_n := \max_{1 \leq m \leq n} (b_m - a_m)$ and $g_n := \sqrt{s_n/a_n}$. Then $b_m - a_m \leq a_n g_n^2$ for every $1 \leq m \leq n$ and $\lim_{n \rightarrow \infty} g_n = 0$. Since $g_n > 0$, we can choose a subsequence $\{g_{n_k}\}$ with $g_{n_k} \geq g_m$ for every $m \geq n_k$. Since $\{b_m - a_m\}_m$ is not bounded we may, moreover, choose the sequence so that $g_n^2 = (b_n - a_n)/a_n$ for every $n \in \{n_k\}$. Also we may assume that $g_n \leq 1$ for all values of n considered. Fix now n from the sequence $\{n_k\}$. As before, we conclude that $b_m - a_m \leq a_n g_n^2$ for $m \leq n$ and $b_m - a_m \leq a_m g_m^2 \leq a_m g_n^2$ for $m > n$.

Consider $x \in (a_n, b_n)$ which lies on the shortest fundamental geodesic γ_n joining $(-\infty, 0)$ with (a_n, b_n) . Define an angle $\theta = \arctan g_n \in (0, \pi/2)$ and a set

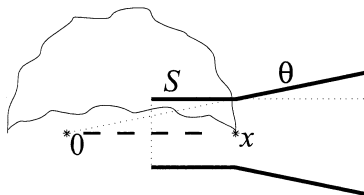


Figure 2. The set S .

²Originally, we gave separate proofs for the two metrics. However, in the revised version of this paper we have utilized the result from our newer investigation, [10].

$$S = \left[\frac{1}{2}x + ixg_n, x + ixg_n \right] \cup \{x + ixg_n + te^{i\theta} \mid t \geq 0\}.$$

The set S is shown in Figure 2. Notice that any point $\zeta \in S$ satisfies $g_n \operatorname{Re} \zeta \leq \operatorname{Im} \zeta \leq 2g_n \operatorname{Re} \zeta$. It is clear that γ_n hits the set $S \cup [(1/2)x + ixg_n, (1/2)x]$. We claim that it in fact hits S . Assume to the contrary that this is not the case. Then it hits $[(1/2)x + ixg_n, (1/2)x]$. Let γ' denote a part of γ_n connecting x and this segment which does not intersect S . Since Ω is a Denjoy domain, $b \mapsto \delta_\Omega(a + ib)$ is increasing in $b > 0$. Hence $\ell_{k,\Omega}(\gamma') \geq \ell_{k,\Omega}([(1/2)x + ixg_n, x + ixg_n])$. Since the gap size in $[(1/2)x, x]$ is at most $a_n g_n^2$, we have $\delta_\Omega(w) \leq \sqrt{x^2 g_n^2 + a_n^2 g_n^4} \leq \sqrt{2} x g_n$ for $w \in [(1/2)x + ixg_n, x + ixg_n]$. Hence

$$\ell_{k,\Omega}(\gamma_n) \geq \ell_{k,\Omega} \left(\left[\frac{1}{2}x + ixg_n, x + ixg_n \right] \right) \geq \frac{\frac{1}{2}x}{\sqrt{2} x g_n} = \frac{C}{g_n}.$$

We next construct another path σ and show that it is in the same homotopy class as the supposed geodesic, only shorter. Let z be the midpoint of gap n and let σ be the curve $[z, z + iz] \cup [z + iz, -z + iz] \cup [-z + iz, -z]$. Using $b_n - a_n = a_n g_n^2$ we easily calculate

$$\ell_{k,\Omega}(\sigma) \leq \log \left(\frac{2z}{a_n g_n^2} \right) + C \leq 2 \log \left(\frac{1}{g_n} \right) + C$$

with an absolute constant C . The curve σ joins $(-\infty, 0)$ and (a_n, b_n) ; therefore $\ell_{k,\Omega}(\gamma_n) \leq \ell_{k,\Omega}(\sigma)$. But this contradicts the previously derived bounds for the lengths as $g_n \rightarrow 0$.

Therefore the supposition that γ_n does not intersect S was wrong, so we conclude that $\gamma_n \cap S \neq \emptyset$. Let now $\zeta \in S \cap \gamma_n$. We claim that $k_\Omega(\zeta, \mathbf{R}) \rightarrow \infty$, which means the domain is not Gromov hyperbolic, by Theorem 3.3. Let $\xi \in \Omega \cap \mathbf{R}$; chose m so that $\xi \in (a_m, b_m)$. Let α be a curve joining ξ and ζ .

If $0 < m \leq n$, then the size of (a_m, b_m) is at most $a_m g_m^2$, so $\delta_\Omega(\xi) \leq a_m g_m^2$. Then α has Euclidean length at least $\operatorname{Im} \zeta \geq x g_n$, so by Lemma 4.1, $\ell_{k,\Omega}(\alpha) \geq c \log(C/g_n)$. As $g_n \rightarrow 0$, this bound tends to ∞ . If, on the other hand, $m > n$, then the Euclidean length of α is at least

$$d(\xi, \zeta) \geq d(\xi, S) \geq \xi \sin \theta \geq \frac{1}{2} \xi \tan \theta = \frac{1}{2} \xi g_n,$$

and the size of the gap is at most $a_m g_n^2$. By Lemma 4.1 this implies that $\ell_{k,\Omega}(\alpha) \geq c \log(C/g_n)$. As $g_n \rightarrow 0$, this bound again tends to ∞ .

It remains to consider $m = 0$, i.e., $\xi < 0$. We consider only the case $\zeta \in [(1/2)x + ixg_n, x + ixg_n]$, since the other case is similar. Now the Euclidean length of α is at least $(1/2)x$. As before, $\delta_\Omega(\zeta) \leq \sqrt{2}xg_n$, and so the length of the curve is at least

$$\log \frac{\frac{1}{2}x}{\sqrt{2}xg_n} = \log \frac{1}{2\sqrt{2}g_n} \rightarrow \infty.$$

Hence in every case we get a lower bound which tends to infinity as $g_n \rightarrow 0$; hence (Ω, k_Ω) is not Gromov hyperbolic, by Theorem 3.3.

This takes care of the case when $\{b_m - a_m\}_m$ is unbounded. Assume next that $\sup_m(b_m - a_m) = M < \infty$. In this case it is difficult to work with bigons, since we do not get a good control on what the geodesics look like; the problem with the previous argument is that we cannot choose $g_{n_k}^2 = (b_{n_k} - a_{n_k})/a_{n_k}$ in our sequence, and consequently we do not get a good bound on the length of the curve σ , as defined above.

To get around this we consider a geodesic triangle. Assume for a contradiction that k_Ω is δ -Gromov hyperbolic.

Fix $R \gg M^2$ and set $w_\pm = \pm iR$. Let γ_0 be the geodesic segment joining w_+ and w_- . Choose $t > 0$ so large that $k_\Omega(\gamma_0, H_t) > \delta$, where $H_t = \{z \in \mathbf{C} \mid \operatorname{Re} z > t\}$. Let $w \in \Omega \cap \mathbf{R}$ be a point in $H_{2 \max\{t, R\}}$ chosen so that the nearest boundary point of every point on the segment $[w_+, w]$ has smaller real part than w . Let $\gamma_+ \subset \overline{\mathbf{H}^2}$ be a geodesic joining w and w_+ .

Let us show that the geodesic γ_+ does not dip below the ray from w through w_+ . Suppose to the contrary that γ_+ intersects the ray at points w_1 and w_2 and lies below the ray in between. Let $\tilde{\gamma}_+$ be the part of the geodesic between w_1 and w_2 . Let L be a line perpendicular to $[w_1, w_2]$. Since L is almost vertical, and the gaps in the boundary are relatively small, we find that δ_Ω increases as we move on L from $L \cap \gamma_+$ to $L \cap [w_1, w_2]$. Moreover, the Euclidean length of $[w_1, w_2]$ is also smaller than that of $\tilde{\gamma}_+$. Therefore $\ell_{k, \Omega}(\tilde{\gamma}_+) > \ell_{k, \Omega}([w_1, w_2])$, a contradiction since $\tilde{\gamma}_+$ is a geodesic.

Similarly, we construct γ_- and conclude that it is a geodesic. Choose now $\zeta \in \gamma_+ \cap H_{\max\{t, R\}}$ with $\operatorname{Im} \zeta = \sqrt{R}$. Since $\gamma_0 \cup \gamma_+ \cup \gamma_-$ is a geodesic triangle, it should be possible to connect ζ with some point in $\gamma_0 \cup \gamma_-$ using a path of length δ . By the definition of t , $k_\Omega(\zeta, \gamma_0) > \delta$. If α is a path connecting ζ and γ_- , then it crosses the real axis at some point ξ . If ξ lies in (a_m, b_m) , $m > 0$, then $\ell_{k, \Omega}(\alpha) \geq C \log \sqrt{R}/M$, by Lemma 4.1. Otherwise, $\xi \in (-\infty, 0)$. This case is handled as in the first case of the proof. In each case we see that $k_\Omega(\zeta, \gamma_-) > \delta$ provided R is large enough. But this means that Ω is not Gromov hyperbolic, which finishes the proof. \square

In Theorem 1.1(2) the gaps (a_n, b_n) and (a_{n+1}, b_{n+1}) are separated by the boundary component $[b_n, a_{n+1}]$. We easily see from the proofs that it would have made no difference if this boundary component had some gaps, as long as they at most comparable to the lengths of the adjacent gaps, (a_n, b_n) and (a_{n+1}, b_{n+1}) . Thus we get the following stronger theorem by the same proofs. (In the proofs we can assume that $(-\infty, 0) \subset \Omega$, by using Theorem 1.2).

THEOREM 4.2. *Let Ω be a Denjoy domain with $\Omega \cap \mathbf{R} = \cup_{n=0}^\infty (a_n, b_n)$ and $\limsup_{n \rightarrow \infty} a_n = \infty$. Suppose $G: \mathbf{R}^+ \rightarrow \mathbf{R}^+$ is a function with $\lim_{x \rightarrow \infty} G(x) = 0$. If $b_n - a_n \leq a_n G(a_n)$ for every $a_n > 0$, then κ_Ω , the hyperbolic or quasihyperbolic metric, is not Gromov hyperbolic.*

The function G plays the role of g_n^2 in the proof of Theorem 1.1(2).

REMARK 4.3. The condition $\Omega \cap \mathbf{R} = \cup_{n=0}^\infty (a_n, b_n)$ (without the hypothesis $b_n \leq a_{n+1}$ for every n) allows any topological behavior; for instance, $\partial\Omega$ can contain a countable sequence of Cantor sets.

Let $E_0 \subset [0, t)$ be closed, $t > 0$, set $E_n := E_0 + tn$ for $n \in \mathbf{N}$, and $\Omega := \mathbf{C} \setminus \cup_n E_n$. Then Ω satisfies the hypotheses of Theorem 4.2 for $G(x) = t/x$. From this we deduce Example 1.3, the non-hyperbolicity of periodic Denjoy domain, in the case the index set is \mathbf{N} . The case with index set \mathbf{Z} follows from this and Theorem 1.2.

We then move to the proof of the final claim. Again, the result from [10] has allowed us to significantly simplify the proof from our original circulated preprint.

PROOF OF THEOREM 1.2. By Theorem 4.6 of [10] the space (Ω, h_Ω) is Gromov hyperbolic if and only if (Ω, k_Ω) is Gromov hyperbolic, quantitatively. Therefore we present a proof only for the case of (Ω, k_Ω) . Since k_Ω is not Gromov hyperbolic, by Proposition 3.5, we conclude that Ω has countably infinitely many boundary components: $\Omega \cap \mathbf{R} = \cup_{n=0}^\infty (a_n, b_n)$. Without loss of generality we can assume that $(-\infty, 0) \subseteq (a_1, b_1)$.

Let us consider fundamental geodesics γ_n of k_Ω joining the midpoint c_0 of (a_0, b_0) with the midpoint c_n of (a_n, b_n) for $n \geq 2$. Since γ_n is contained in $\{z \in \mathbf{C} : c_0 \leq \operatorname{Re} z \leq c_n\}$, and $k_{\Omega \setminus F} = k_\Omega$ in $\{z \in \mathbf{C} : \operatorname{Re} z \geq \inf_{n \geq 2} a_n\}$, we deduce that γ_n is also a fundamental geodesic with the metric $k_{\Omega \setminus F}$.

Since k_Ω is not Gromov hyperbolic, there exist points $z_k \in \gamma_{n_k}$ with $\lim_{k \rightarrow \infty} k_\Omega(z_k, \mathbf{R}) = \infty$ by Theorem 3.3. Since γ_{n_k} are also fundamental geodesics with the metric $k_{\Omega \setminus F}$, we deduce that $\lim_{k \rightarrow \infty} k_{\Omega \setminus F}(z_k, \mathbf{R}) \geq \lim_{k \rightarrow \infty} k_\Omega(z_k, \mathbf{R}) = \infty$. Consequently, $(\Omega \setminus F, k_{\Omega \setminus F})$ is not Gromov hyperbolic. \square

ACKNOWLEDGEMENTS. We would like to thank the referee for exceptionally useful comments and suggestions.

References

- [1] V. Alvarez, A. Portilla, J. M. Rodríguez and E. Tourís, Gromov hyperbolicity of Denjoy domains, *Geom. Dedicata*, **121** (2006), 221–245.
- [2] J. W. Anderson, *Hyperbolic Geometry*, Springer-Verlag, London, 1999.
- [3] Z. M. Balogh and M. Bonk, Gromov hyperbolicity and the Kobayashi metric on strictly pseudoconvex domains, *Comment. Math. Helv.*, **75** (2000), 504–533.
- [4] Z. M. Balogh and S. M. Buckley, Geometric characterizations of Gromov hyperbolicity, *Invent. Math.*, **153** (2003), 261–301.
- [5] Y. Benoist, Convexes hyperboliques et fonctions quasisymtriques, *Publ. Math. Inst. Hautes Études Sci.*, **97** (2003), 181–237.
- [6] M. Bonk, J. Heinonen and P. Koskela, Uniformizing Gromov hyperbolic spaces, *Astérisque*, **270**, 2001.
- [7] M. Bonk and O. Schramm, Embeddings of Gromov hyperbolic spaces, *Geom. Funct. Anal.*, **10** (2000), 266–306.
- [8] E. Ghys and P. de la Harpe, *Sur les Groupes Hyperboliques d’après Mikhael Gromov*, *Progr. Math.*, **83**, Birkhäuser, 1990.
- [9] P. A. Hästö, Gromov hyperbolicity of the j_G and \tilde{j}_G metrics, *Proc. Amer. Math. Soc.*, **134** (2006), 1137–1142.
- [10] P. Hästö, A. Portilla, J. M. Rodríguez and E. Tourís, Gromov hyperbolic equivalence of the hyperbolic and quasihyperbolic metrics in Denjoy domains, *Bull. London Math. Soc.*, **42** (2010), 282–294.
- [11] A. Karlsson and G. A. Noskov, The Hilbert metric and Gromov hyperbolicity, *Enseign. Math.*, **48** (2002), 73–89.
- [12] H. Lindén, Gromov hyperbolicity of certain conformal invariant metrics, *Ann. Acad. Sci. Fenn. Math.*, **32** (2007), 279–288.
- [13] A. Portilla, J. M. Rodríguez and E. Tourís, Gromov hyperbolicity through decomposition of metric spaces II, *J. Geom. Anal.*, **14** (2004), 123–149.
- [14] A. Portilla, J. M. Rodríguez and E. Tourís, The topology of balls and Gromov hyperbolicity of Riemann surfaces, *Differential Geom. Appl.*, **21** (2004), 317–335.
- [15] A. Portilla, J. M. Rodríguez and E. Tourís, The role of funnels and punctures in the Gromov hyperbolicity of Riemann surfaces, *Proc. Edinburgh Math. Soc. (2)*, **49** (2006), 399–425.
- [16] A. Portilla, J. M. Rodríguez and E. Tourís, A real variable characterization of Gromov hyperbolicity of flute surfaces, *Osaka J. Math.*, to appear.
- [17] A. Portilla and E. Tourís, A characterization of Gromov hyperbolicity of surfaces with variable negative curvature, *Publ. Mat.*, **53** (2009), 83–110.
- [18] J. M. Rodríguez and E. Tourís, Gromov hyperbolicity through decomposition of metric spaces, *Acta Math. Hungar.*, **103** (2004), 53–84.
- [19] J. M. Rodríguez and E. Tourís, A new characterization of Gromov hyperbolicity for Riemann surfaces, *Publ. Mat.*, **50** (2006), 249–278.
- [20] J. M. Rodríguez and E. Tourís, Gromov hyperbolicity of Riemann surfaces, *Acta Math. Sinica*, **23** (2007), 209–228.
- [21] J. Väisälä, Hyperbolic and uniform domains in Banach spaces, *Ann. Acad. Sci. Fenn. Math.*, **30** (2005), 261–302.
- [22] J. Väisälä, Gromov hyperbolic spaces, *Expo. Math.*, **23** (2005), 187–231.

Peter HÄSTÖ

Department of Mathematical Sciences
P.O. Box 3000
FI-90014 University of Oulu
Finland
E-mail: peter.hasto@helsinki.fi

Henri LINDÉN

Department of Mathematics and Statistics
P.O. Box 64
FI-00140 University of Helsinki
Finland
E-mail: hlinden@iki.fi

Ana PORTILLA

St. Louis University (Madrid Campus)
Avenida del Valle 34
28003 Madrid
Spain
E-mail: aportil2@slu.edu

José M. RODRÍGUEZ

Departamento de Matemáticas
Universidad Carlos III de Madrid
Avenida de la Universidad 30
28911 Leganés, Madrid
Spain
E-mail: jomaro@math.uc3m.es

Eva TOURÍS

Departamento de Matemáticas
Universidad Carlos III de Madrid
Avenida de la Universidad 30
28911 Leganés, Madrid
Spain
E-mail: etouris@math.uc3m.es