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The necessary and sufficient condition for the group of leaf preserving diffeomorphisms to be simple

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Abstract. Let $\mathscr{F}$ be a C^∞-foliation on a compact C^∞-manifold M. We consider the group of all leaf preserving C^∞-diffeomorphisms of $(M, \mathscr{F})$ which are isotopic to the identity through leaf preserving C^∞-diffeomorphisms. Then we show that the group is simple if and only if all leaves of $\mathscr{F}$ are dense.

1. Introduction and statement of result.

Let M be a connected C^∞-manifold without boundary and let $D_c^\infty(M)$ denote the group of all C^∞-diffeomorphisms of M which are isotopic to the identity through C^∞-diffeomorphisms with compact support. M. Herman [**3**] and W. Thurston [**5**] showed that $D_c^\infty(M)$ is perfect, that is, every element of $D_c^\infty(M)$ is represented by a product of commutators. By combining this with the result of D. Epstein [**1**], we know that $D_c^\infty(M)$ is a simple group. Here a group G is said to be simple if G contains no non-trivial proper normal subgroups.

Let $\mathscr{F}$ be a C^∞-foliation on M. A diffeomorphism $f : M \to M$ is called a leaf preserving diffeomorphism if f maps each leaf of $\mathscr{F}$ to itself. We denote by $D_{L,c}^\infty(M, \mathscr{F})$ the group of all leaf preserving C^∞-diffeomorphisms of $(M, \mathscr{F})$ which are isotopic to the identity through leaf preserving C^∞-diffeomorphisms with compact support. T. Rybicki [**4**] and T. Tsuboi [**6**] showed that the group is perfect. We don't have the criterion whether the group is simple since it does not satisfy the conditions requested in Epstein [**1**]. In this paper we consider the simplicity of the groups of leaf preserving diffeomorphisms for foliations.

We have the following theorem.

Theorem. *Let $\mathscr{F}$ be a C^∞-foliation on a compact m-dimensional C^∞-manifold M. Then $D_L^\infty(M, \mathscr{F})$ is a simple group if and only if all leaves of $\mathscr{F}$ are dense.*

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2. Proof of Theorem.

For a given group G and $g \in G$, let C_g denote the union of conjugate classes of g and g^{-1} and $(C_g)^k$ be the set of elements represented as a product of k conjugates of g or g^{-1}. Then note that G is simple if $G = \bigcup_{k=1}^{\infty}(C_g)^k$ for any element $g \in G - \{e\}$.

First we review the following lemma due to T. Tsuboi [**8**] which plays a key role to prove Theorem.

Lemma (Lemma 3.1 of [**8**]). *Let M be a C^∞-manifold and U an open ball in M. Suppose that $g \in D_c^\infty(M)$ satisfies $g(U) \cap U = \emptyset$. Then any commutator $[a,b]$ for $a, b \in D_c^\infty(U)$ can be written as a product of 4 conjugates of g or g^{-1}, that is, $[a,b] \in (C_g)^4$.*

Proof. Put $c = g^{-1}ag$. Then since $cb = bc$, we have

$$\begin{aligned} aba^{-1}b^{-1} &= gcg^{-1}bgc^{-1}g^{-1}b^{-1} \\ &= gcg^{-1}c^{-1}cbgc^{-1}b^{-1}bg^{-1}b^{-1} \\ &= g(cg^{-1}c^{-1})(bcgc^{-1}b^{-1})(bg^{-1}b^{-1}). \end{aligned}$$

This completes the proof. □

Remark. The above lemma holds for leaf preserving diffeomorphism groups.

Proof of "if" part. Take any non-trivial element g of $D_L^\infty(M, \mathscr{F})$. Then there is an open foliated ball $U \subset \overline{U} \neq M$ satisfying $g(U) \cap U = \emptyset$. Here a foliated ball U means a neighborhood homeomorphic to $\operatorname{int} D^q \times \operatorname{int} D^{m-q}$ such that $\operatorname{int} D^q \times \{pt\}$ is a plaque of the q-dimensional foliation $\mathscr{F}$ in U. Then, since M is compact, we can take a finite cover of open foliated balls $\mathscr{U} = \{U_j\}$ $(j = 1, 2, \dots, k)$ of M containing U such that each U_j is diffeomorphic to U via a leaf preserving diffeomorphism in $D_L^\infty(M, \mathscr{F})$. In fact, given any two points on the same leaf, there is a leaf preserving diffeomorphism which maps one to the other. Therefore the union of all the images of U by leaf preserving diffeomorphisms is saturated. Being open, the union coincides with M since the foliation $\mathscr{F}$ is minimal. Take any $f \in D_L^\infty(M, \mathscr{F})$. Since f is isotopic to the identity in $D_L^\infty(M, \mathscr{F})$, we have the following decomposition:

(1) $f = f_1 \circ f_2 \circ \cdots \circ f_\ell$, $(f_i \in D_L^\infty(M, \mathscr{F}))$ and
(2) each f_i $(i = 1, 2, \dots, \ell)$ is C^1-close to the identity.

Then by using the partition of unity subordinate to the cover $\mathscr{U}$, we have the following decomposition for each f_i:

(1) $f_i = f_{i,1} \circ f_{i,2} \circ \cdots \circ f_{i,k}$ and
(2) $f_{i,j} \in D^\infty_{L,c}(U_j, \mathscr{F}|_{U_j})$.

On the one hand, $f_{i,j}$ is written as a product of at most 2 commutators of elements in $D^\infty_{L,c}(U_j, \mathscr{F}|_{U_j})$ (see Proposition 3.3 of [**2**] and also Theorem 2.1 of [**7**]). That is, $f_{i,j} = [a_{i,j}, b_{i,j}][c_{i,j}, d_{i,j}]$, where $a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j} \in D^\infty_{L,c}(U_j, \mathscr{F}|_{U_j})$. As U_j is diffeomorphic to U by a leaf preserving diffeomorphism, say $h_j \in D^\infty_L(M, \mathscr{F})$, $h_j[a_{i,j}, b_{i,j}]h_j^{-1}$ and $h_j[c_{i,j}, d_{i,j}]h_j^{-1}$ are supported in U. From Lemma and Remark, $h_j[a_{i,j}, b_{i,j}]h_j^{-1}$ and $h_j[c_{i,j}, d_{i,j}]h_j^{-1}$ are written as a product of 4 conjugates of g or g^{-1}, hence $[a_{i,j}, b_{i,j}]$ and $[c_{i,j}, d_{i,j}]$ are so. Thus $f_{i,j}$ is written as a product of 8 conjugates of g or g^{-1}. Finally f is written as a product of $8k\ell$ conjugates of g or g^{-1}, hence $f \in (C_g)^{8k\ell}$. This completes the proof. □

Proof of "only if" part. If there is a non-dense leaf L in $\mathscr{F}$, the closure $\overline{L}$ of L in M is a proper saturated subset of M. Then the subgroup G of $D^\infty_L(M, \mathscr{F})$ consisting of leaf preserving diffeomorphisms fixing $\overline{L}$ pointwise is a non-trivial proper normal subgroup of $D^\infty_L(M, \mathscr{F})$. For, there are an open subset $U (\subset M - \overline{L})$ and a non-trivial element $f \in G$ satisfying $\mathrm{supp}(f) \subset U$. Therefore G is a non-trivial normal subgroup. Thus $D^\infty_L(M, \mathscr{F})$ is not simple. This completes the proof. □

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