Pseudo-Riemannian metrics on closed surfaces whose geodesic flows admit nontrivial integrals quadratic in momenta, and proof of the projective Obata conjecture for two-dimensional pseudo-Riemannian metrics

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Abstract. We describe all pseudo-Riemannian metrics on closed surfaces whose geodesic flows admit nontrivial integrals quadratic in momenta. As an application, we solve the Beltrami problem on closed surfaces, prove the nonexistence of quadratically-superintegrable metrics of nonconstant curvature on closed surfaces, and prove the two-dimensional pseudo-Riemannian version of the projective Obata conjecture.

1.   Introduction.

1.1.   Definitions and the statement of the problem.

Consider a pseudo-Riemannian metric $g = (g_{ij})$ on a surface M^2. A function $F : T^*M \to \mathbf{R}$ is called *an integral* of the geodesic flow of g, if $\{H, F\} = 0$, where $H := 1/2 \sum_{i,j} g^{ij} p_i p_j : T^*M \to \mathbf{R}$ is the kinetic energy corresponding to the metric. Geometrically, the condition $\{H, F\} = 0$ means that the function F is constant on the trajectories of the Hamiltonian system with the Hamiltonian H. We say that the integral F is *quadratic in momenta*, if in every local coordinate system (x, y) on M^2 it has the form

$$a(x,y)p_x^2 + b(x,y)p_x p_y + c(x,y)p_y^2 \tag{1}$$

in the canonical coordinates (x, y, p_x, p_y) on T^*M^2. Geometrically, the formula (1) means that the restriction of the integral to every cotangent space $T^*_{(x,y)}M^2 \equiv \mathbf{R}^2$

is a homogeneous quadratic function. As *trivial* examples of quadratic in momenta integrals we consider those proportional to the Hamiltonian H.

Similarly, we say that the integral is *linear* in momenta, if for every local coordinate system (x, y) on M^2 it has the form $\alpha(x, y)p_x + \beta(x, y)p_y$ in the canonical coordinates (x, y, p_x, p_y) on T^*M^2; an integral linear in momenta is *trivial*, if it is identically zero.

The importance of integrals quadratic in momenta other than the Hamiltonian for studying the metric was recognized long ago. Indeed, it was Jacobi's realization that the geodesic flow of the ellipsoid admitted such an 'extra' quadratic integral that allowed him to integrate the geodesics on the ellipsoid.

In the present paper we solve (see Model Examples 1, 2, 3 and Theorems 2, 3, 4 below) the following problem:

PROBLEM.   *Find all metrics of signature $(+, -)$ on closed 2-dimensional manifolds whose geodesic flows admit nontrivial integrals quadratic in momenta.*

Riemannian metrics whose geodesic flows admit integrals quadratic in momenta are quite well studied. Indeed, local description of such a metric in a neighborhood of almost every point is known since Liouville. Moreover, the Riemannian version (and, therefore, if the signature of g is $(-, -)$) of the problem above was solved. There exist two different approaches that lead to a solution: one, which is based on the ideas of Kolokoltsov [25], was realized in [25], [2], [29], see also [6], [7]. Alternative approach to the description of metrics whose geodesic flows admit nontrivial integrals quadratic in momenta is due to Kiyohara [22], see also [17], [23]. Our solution uses main ideas from both approaches.

Metrics whose geodesic flows admit integrals quadratic in momenta were studied in the framework of differential geometry (at least since Darboux [14]) and mathematical physics (at least since Birkhoff [5] and Whittaker [53]). We give two applications of our results in differential geometry and one application in mathematical physics. In differential geometry, we use the connection between integrals quadratic in momenta and geodesically equivalent metrics (we give the necessary definition in Section 2.1) to solve the natural generalization of the Beltrami problem for closed manifolds, and to prove the two-dimensional pseudo-Riemannian version of the projective Obata conjecture. In mathematical physics, we prove that all quadratically-superintegrable metrics on closed surfaces (the necessary definition is in Section 2.2) have constant curvature. This generalizes the result of [22], [29] to the pseudo-Riemannian metrics.

1.2. Metrics on the torus whose geodesic flows admit nontrivial integrals quadratic in momenta.

Locally, pseudo-Riemannian metrics admitting integrals quadratic in momenta were described[1] in [**9**, Theorem 1] and [**10**, Theorem 1]:

THEOREM 1 ([**9**], [**10**]). *Suppose a Riemannian or pseudo-Riemannian metric g on a connected surface M^2 admits an integral F quadratic in momenta such that $F \neq \text{const} \cdot H$ for all $\text{const} \in \mathbf{R}$. Then, in a neighbourhood of almost every point there exist coordinates x, y such that the metric and the integral are as in the following table:*

	Liouville case	*Complex-Liouville case*	*Jordan-block case*
g	$(X(x) - Y(y))(dx^2 + \varepsilon dy^2)$	$\Im(h)dxdy$	$\left(\widehat{Y}(y) + \dfrac{x}{2}Y'(y)\right)dxdy$
F	$\dfrac{X(x)p_y^2 + \varepsilon Y(y)p_x^2}{X(x) - Y(y)}$	$p_x^2 - p_y^2 + 2\dfrac{\Re(h)}{\Im(h)}p_xp_y$	$\varepsilon\left(p_x^2 - \dfrac{Y(y)}{\widehat{Y}(y) + \frac{x}{2}Y'(y)}p_xp_y\right)$

where $\varepsilon = \pm 1$, and $\Re(h)$ and $\Im(h)$ are the real and imaginary parts of a holomorphic function h of the variable $z := x + i \cdot y$.

REMARK 1. Within our paper, we understand "*almost every*" in the topological sense: a condition is fulfilled at almost every point, if the set of the points where it is fulfilled is everywhere dense.

We see that the metric g in the Jordan-block and Complex-Liouville cases always has indefinite signature $(+, -)$, and the metric g in the Liouville case has signature $(+, -)$ if and only if $\varepsilon = -1$. The Liouville case with $\varepsilon = 1$ was known to classics.

In Section 3, we repeat the proof of Theorem 1, because we will need most technical details from it in the proof of our main result, which is Theorem 2 below.

Let us now discuss the case when M^2 is closed. First of all, because of Euler characteristic, a closed surface admitting a pseudo-Riemannian metric of indefinite signature is homeomorphic to the torus or to the Klein bottle. Since a double cover of the Klein bottle is the torus, and the geodesic flow of the lift of a metric whose geodesic flow admits an integral quadratic in momenta also admits an integral quadratic in momenta, the most important case is when the surface is the torus. In Model Example 1 below we describe a class of pseudo-Riemannian metrics on the torus such that their geodesic flows admit nontrivial integrals quadratic in momenta. Theorem 2 claims that every metric such that its geodesic flow admits

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[1]As it mentioned in [**9**], [**10**], the essential part of the result appeared already in Darboux [**14**, Sections 592–594, 600–608].

a nontrivial integral quadratic in momenta is isometric to one from Model Example 1.

MODEL EXAMPLE 1.   We consider $\boldsymbol{R}^2$ with the standard coordinates (x, y), two linearly independent vectors $\xi = (\xi_1, \xi_2)$ and $\nu = (\nu_1, \nu_2)$, and two nonconstant functions X and Y of one variable (it is convenient to think that the variable of X is x and the variable of Y is y) such that

(a) for all $(x, y) \in \boldsymbol{R}^2$ we have $X(x) \neq Y(y)$, and
(b) for every $(x, y) \in \boldsymbol{R}^2$, $X(x + \xi_1) = X(x + \nu_1) = X(x)$ and $Y(y + \xi_2) = Y(y + \nu_2) = Y(y)$.

Next, consider the metrics $(X(x) - Y(y))(dx^2 + \varepsilon dy^2)$ on $\boldsymbol{R}^2$, where $\varepsilon = \pm 1$, and the action of the lattice $G := \{k \cdot \xi + m \cdot \nu \mid k, m \in \boldsymbol{Z}\}$ on $\boldsymbol{R}^2$. The action is free, discrete and preserves the metric and the quadratic integral $(X(x)p_y^2 + \varepsilon Y(y)p_x^2)/(X(x) - Y(y))$. Then, the geodesic flow of the induced metric on the quotient space $\boldsymbol{R}^2/G$ (homeomorphic to the torus) admits an integral quadratic in momenta. We will call such metrics *globally-Liouville*.
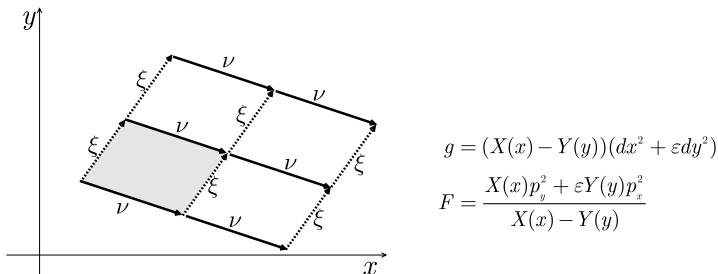

$$g = (X(x) - Y(y))(dx^2 + \varepsilon dy^2)$$

$$F = \frac{X(x)p_y^2 + \varepsilon Y(y)p_x^2}{X(x) - Y(y)}$$

Figure 1.   Vectors ξ and ν and a fundamental region (gray parallelogram) of the action of G from Model Example 1. The torus $\boldsymbol{R}^2/G$ can be identified with this parallelogram with glued opposite sides. Since the action of G preserves $X(x)$ and $Y(y)$, the metric g induces a metric on $\boldsymbol{R}^2/G$, and the integral F induces an integral quadratic in momenta.

THEOREM 2.   *Suppose a metric g on the two-torus T^2 admits an integral F quadratic in momenta. Assume the integral is not a linear combination of the square of an integral linear in momenta and the Hamiltonian. Then, (T^2, g) is globally Liouville, i.e., there exist X, Y, ξ, ν satisfying the conditions in the Model Example 1 above and a diffeomorphism $\phi : T^2 \to \boldsymbol{R}^2/G$ that takes g to the globally-Liouville metric $(X(x) - Y(y))(dx^2 + \varepsilon dy^2)$ on $\boldsymbol{R}^2/G$ and the integral F to the integral $\pm((X(x)p_y^2 + \varepsilon Y(y)p_x^2)/(X(x) - Y(y)))$.*

In the Riemannian case, Theorem 2 follows from [2], [22], see also [6], [7]. We see that the answer in the pseudo-Riemannian case is essentially the same (=

no new phenomena appear) as the answer in the Riemannian case. This similarity with the Riemannian case was unexpected: indeed, by Theorem 1, in the pseudo-Riemannian case (different from the Riemannian case) there are three different types of metrics admitting quadratic integrals. Moreover, the examples from papers [**13**], [**15**], [**47**], [**50**] show that, locally, the pair (metric, integral) can change the type, i.e., the pair (metric, integral) can be, for example, as in Liouville case from one side of a line, and as in Complex-Liouville case from another side of the line. But it appears that only one type, namely the Liouville, can exist on closed manifolds.

Moreover, as we show in Example 2, if the integral is the square of an integral linear in momenta, then the Jordan-block case is possible (even if the surface is closed). Moreover, the pair (metric, integral) can change the type: be of Jordan-block type in a neighborhood of one point, and of Liouville type in a neighborhood of another point. Moreover, one can modify Example 2(c) such that the set of the points such that the pair (metric, integral) changes the type is the direct product of the Cantor set and a circle.

1.3.　Metrics on the Klein bottle whose geodesic flows admit integrals quadratic in momenta.

The scheme of the description is the same as for the torus: in Model Example 2 we describe a big family of metrics on the Klein bottle whose geodesic flows admit integrals quadratic in momenta. Theorem 3 claims that every metric such that its geodesic flow admits an integral quadratic in momenta and such that the geodesic flow of the lift of the metric to the oriented cover admits no integral linear in momenta is as in Model Example 2.

MODEL EXAMPLE 2.　We consider $\boldsymbol{R}^2$ with the standard coordinates (x, y), constants $c \neq 0$, $d \neq 0$, two vectors $\xi = (c, 0)$ and $\nu = (0, d)$, and two nonconstant functions X and Y of one variable (it is convenient to think that the variable of X is x and the variable of Y is y) such that

(a)　for all $(x, y) \in \boldsymbol{R}^2$ we have $X(x) \neq Y(y)$, and
(b)　for every $(x, y) \in \boldsymbol{R}^2$, $X(x + c) = X(x)$ and $Y(y + d) = Y(-y) = Y(y)$.

Next, consider the metrics $(X(x) - Y(y))(dx^2 + \varepsilon dy^2)$ on $\boldsymbol{R}^2$ and the action of the group G generated by the transformations $(x, y) \mapsto (x + c, -y)$ and $(x, y) \mapsto (x, y + d)$. The action is free, discrete and preserves the metric and the quadratic integral $(X(x)p_y^2 + \varepsilon Y(y)p_x^2)/(X(x) - Y(y))$. Then, the geodesic flow of the induced metric on the quotient space $\boldsymbol{R}^2/G$ (homeomorphic to the Klein bottle) admits an integral quadratic in momenta. We will call such metrics *globally-(Klein)-Liouville*.
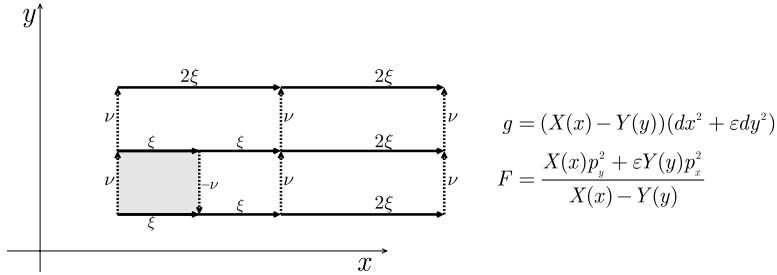
Figure 2.   Vectors ξ and ν and a fundamental region (gray rectangle) of the action of G from Model Example 2. The Klein bottle $\boldsymbol{R}^2/G$ can be identified with this rectangle with glued opposite sides: the horizontal sides are glued with preserving the orientation, and the vertical sides are glued with inverting the orientation. The action of G preserves the metric g and the integral F; hence, the geodesic flow of the induced metric on $\boldsymbol{R}^2/G$ admits an integral quadratic in momenta.

THEOREM 3.   *Suppose a metric g on the Klein bottle K^2 admits an integral F quadratic in momenta. Assume the lift of the integral to the oriented cover is not a linear combination of the lift of the Hamiltonian and the square of a function linear in momenta. Then, (K^2, g, F) is globally-(Klein)-Liouville, i.e., there exist X, Y, c, d satisfying the conditions in the Model Example 2 above and a diffeomorphism $\phi : K^2 \to \boldsymbol{R}^2/G$ that takes g to the globally-(Klein)-Liouville metric $(X(x) - Y(y))(dx^2 + \varepsilon dy^2)$ on $\boldsymbol{R}^2/G$ and F to the integral $\pm((X(x)p_y^2 + \varepsilon Y(y)p_x^2)/(X(x) - Y(y)))$.*

In the Riemannian case, Theorem 3 was proved in [**29**, Theorem 3]. We see that the answer in the pseudo-Riemannian case is essentially the same as the answer in the Riemannian case (similar to the torus).

The following example explains why we require that the *lift* of the integral (to the oriented cover) is not a linear combination of the lift of the Hamiltonian and the square of a function linear in momenta:

EXAMPLE 1.   As in the Model Example 2, we consider $\boldsymbol{R}^2$ with the standard coordinates (x, y), constants $c \neq 0$, $d \neq 0$, two vectors $\xi = (c, 0)$ and $\nu = (0, d)$, the function X of the variable x such that $X(x + c) = X(x)$. Different from the Model Example 2, by Y we denote a *constant* such that $X(x) \neq Y$ for all $x \in \boldsymbol{R}$.

Under this assumptions, the metric $(X(x) - Y)(dx^2 + \varepsilon dy^2)$ and the integral $(X(x)p_y^2 + \varepsilon Y p_x^2)/(X(x) - Y)$ induce a metric on the $K^2 := \boldsymbol{R}^2/G$, where G is the group generated by the mappings $(x, y) \mapsto (x + c, -y)$ and $(x, y) \mapsto (x, y + d)$, and an integral quadratic in momenta for the geodesic flow of this metric.

The lift of the integral to the oriented cover $T^2 := \boldsymbol{R}^2/G'$, where $G' := \{2k \cdot \xi + m \cdot \nu \mid k, m \in \boldsymbol{Z}\}$, is a linear combination of the Hamiltonian $(1/2)((p_x^2 +$

$\varepsilon p_y^2)/(X(x) - Y))$ and the square of the (linear in momenta) function p_y. Indeed, $F = p_y^2 + 2\varepsilon Y \cdot H$.

But, on K^2, the integral in *not* a linear combination of the Hamiltonian and of the square of a function linear in momenta. The formal proof of this observation in the Riemannian case can be found in [**29**, Sections 3, 4], the Riemannian proof can be easily generalized (using Theorem 2 of our paper) to the pseudo-Riemannian metrics. The main idea of the proof is that the function p_y does not generate a function on the Klein bottle, since the mapping $(x, y) \mapsto (x + c, -y)$ changes the sign of this function.

1.4. Metrics on the torus whose geodesic flows admit integrals linear in momenta.

In order to complete the description of the metrics of signature $(+, -)$ whose geodesic flows admit nontrivial integrals quadratic in momenta, we need to describe the metrics of signature $(+, -)$ on the torus such that their geodesic flows admit nontrivial integrals linear in momenta.

In the Riemannian case, metrics with geodesic flows admitting integrals linear in momenta can be considered as a partial case of the metrics whose geodesic flows admit integrals quadratic in momenta. Indeed, up to an isometry, any such metric is essentially as in Model Examples 1, 2 (see [**6**], [**7**]), the only difference is that the function X is constant. In particular, it implies that one can always slightly perturb a metric whose geodesic flow admits an integral linear in momenta such that the geodesic flow of the result admits an integral quadratic in momenta, but admits no integral linear in momenta.

It appears that in the pseudo-Riemannian case the situation is different.

Below, we construct a family of metrics on the torus whose geodesic flows admit integrals linear in momenta. In Examples 2, 3, we use the construction to show that in the pseudo-Riemannian case the following new (compared with the Riemannian case) phenomena appear:

- Example 2(a) shows that metric and the integral can be as in the Jordan-block case.
- Example 2(c) shows that the metric and the integral can be as in the Jordan-block case in one neighborhood and as in the Liouville case in another neighborhood.
- Example 3 shows the existence of a metric whose geodesic flow admits an integral linear in momenta, such that no small perturbation of this metric admits an integral quadratic in momenta which is not a linear combination of the square of an integral linear in momenta and the Hamiltonian.

CONSTRUCTION. We consider $\boldsymbol{R}^2$ with the standard coordinates x, y and the standard orientation, the vector fields $\xi := (1, 0)$, $\eta := (0, 1)$, and a smooth

foliation on $\boldsymbol{R}^2$ invariant with respect to the flow of the vector field ξ and with respect to the mapping $(x, y) \mapsto (x, y + 1)$. With the help of these data, we construct a metric of signature $(+, -)$ on $\boldsymbol{R}^2$ such that ξ is a Killing vector field for this metric.

At every point p, we consider two vectors $U_1(p)$ and $U_2(p)$ satisfying the following conditions:

- U_1 at every point is tangent to the leaf of the foliation containing this point,
- $(U_1(p), U_2(p))$ is an orthonormal positive basis for the flat metric $\hat{g} = dx^2 + dy^2$, that is
  - $|U_1|_{\hat{g}} = |U_2|_{\hat{g}} = 1$, $\hat{g}(U_1, U_2) = 0$,
  - the orientation given by the basis coincides with the standard orientation, see Figure 3.

Clearly, at every point there exist precisely two possibilities for such vector fields U_1, U_2 (the second possibility is $(-U_1, -U_2)$).
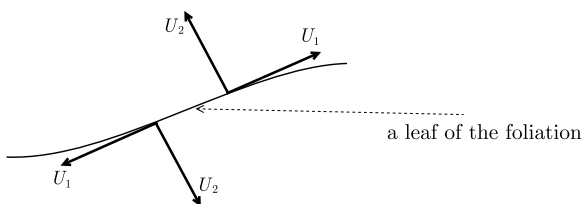

Figure 3.   A leaf of the foliation and two possibilities for the vectors U_1, U_2.

Now, consider the metric g such that in the basis (U_1, U_2) it has the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The metric clearly does not depend on the choice of vectors U_1, U_2 at every point, and is smooth. Since all objects we used to construct the metric are invariant with respect to the flow of ξ, the vector field ξ is Killing for the metric. Then, the geodesic flow of the metric admits an integral p_x linear in momenta. Since all objects are invariant with respect to the lattice $G = \{k \cdot \xi + m \cdot \eta \mid k, m \in \boldsymbol{Z}\}$, the metric induces a metric on the torus $\boldsymbol{R}^2/G$ whose geodesic flow admits an integral linear in momenta.

REMARK 2.   By construction, the leaves of the foliation are light-line geodesics.

EXAMPLE 2.   If the foliation is as on Figure 4(a), the square of the integral is as in the Jordan-block case. If the foliation is as on Figure 4(b), the square of the integral is as in the Liouville case. If the foliation is as on Figure 4(c), the square of the integral is as the Jordan-block case in an annulus $\{(x, y) \in \boldsymbol{R}^2 \mid y - [y] > 1/2\}$
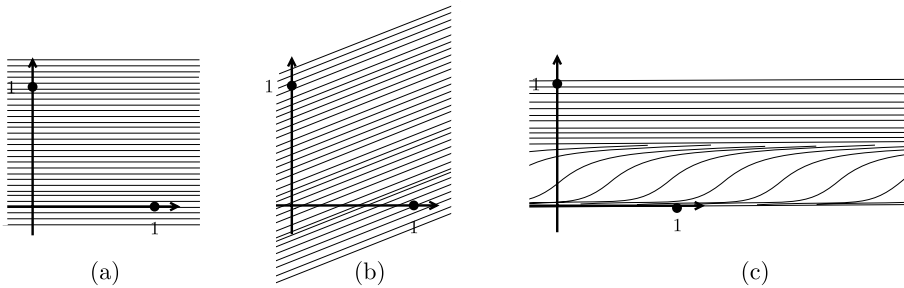
Figure 4.   The foliations from Example 2.

and as in Liouville case in the annulus $\{(x, y) \in \boldsymbol{R}^2 \mid y - [y] < 1/2\}$, where $[y]$ denotes the integer part of y.

EXAMPLE 3.   Let the foliation is as on Figure 5 (the restriction of the foliation to the annulus $\{(x, y) \mid x - [x] < 1/2\}$ is the so-called Reeb component). Then, the geodesic flow of no small perturbation of this metric admits an integral quadratic in momenta that is not a linear combination of the Hamiltonian and the square of an integral linear in momenta. Indeed, the Reeb component is stable with respect to small perturbations, and the light line geodesics of the metrics from Model Example 1 are winding on the torus and form no Reeb component.
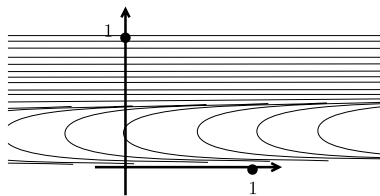

Figure 5.   The foliation from Example 3.

Let us now describe all metrics on closed manifolds whose geodesic flows admit nontrivial integrals linear in momenta.

MODEL EXAMPLE 3.   We consider $\boldsymbol{R}^2$ with the standard coordinates (x, y), the vectors $\xi := (1, 0)$ and $\nu := (0, 1)$, and three functions $K(y), L(y), M(y)$ of the variable y periodic with period 1 such that at every point $\det \left(\begin{smallmatrix} K & L \\ L & M \end{smallmatrix} \right) = KM - L^2 < 0$. Next, consider the metric $g = K(y)dx^2 + 2L(y)dxdy + M(y)dy^2$ on $\boldsymbol{R}^2$, and the action of the lattice $G := \{k \cdot \xi + m \cdot \nu \mid k, m \in \boldsymbol{Z}\}$ on $\boldsymbol{R}^2$. The action is free, discrete and preserves the metric and the integral p_x linear in momenta. Then, the geodesic flow of the induced metric on the quotient space $\boldsymbol{R}^2/G$ (homeomorphic to the torus) admits an integral linear in momenta.

THEOREM 4. *Let g be a metric of signature $(+, -)$ on the torus T^2 such that it is not flat. If the geodesic flow of g admits an integral linear in momenta, then the metric is as in Model Example 3, i.e., there exist functions $K(y), M(y), L(y)$ periodic with period 1 and a diffeomorphism $\phi : T^2 \to \mathbf{R}^2/G$ that takes the metric g to the metric $K(y)dx^2 + 2L(y)dxdy + M(y)dy^2$, and the integral to $\mathrm{const} \cdot p_x$.*

In Theorem 4, we assume that the metric g is not flat. For flat metrics, Theorem 4 is wrong, since the integral curves of the Killing vector field corresponding to the linear integral are non necessary closed curves for the flat metrics, but are closed curves in Model Example 3. We need therefore to describe separately flat metrics of signature $(+, -)$ on the torus.

By the *standard flat torus* we will consider $(\mathbf{R}^2/G, dxdy)$, where (x, y) are the standard coordinates on $\mathbf{R}^2$, and G is a lattice generated by two linearly independent vectors.

In Section 5.1 we will recall why every torus (T^2, g) such that the metric g is flat and has signature $(+, -)$ is isometric to a standard one.

2. Applications.

2.1. Application I: Beltrami problem on closed pseudo-Riemannian manifolds.

Two metrics g and $\bar{g}$ on one manifold are *geodesically equivalent*, if every (unparametrized) geodesic of the first metric is a geodesic of the second metrics. Investigation of geodesically equivalent metrics is a classical topic in differential geometry, see the surveys [1], [48] or/and the introductions to [36], [37], [27], [43], [42].

In particular, in 1865 Beltrami [3] asked[2] *to describe all pairs of geodesically equivalent Riemannian metrics on surfaces.* From the context it is clear that he considered this problem locally, in a neighbourhood of almost every point, but the problem has sense, and is even more interesting globally.

Geodesically equivalent metrics and quadratic integrals are closely related:

THEOREM 5. *Two metrics g and $\bar{g}$ on M^2 are geodesically equivalent, if and only if the following (quadratic in momenta) function*

$$F : TM^2 \to \mathbf{R}, \quad F(x_1, x_2, p_1, p_2) := \left(\frac{\det(g)}{\det(\bar{g})} \right)^{2/3} \cdot \sum_{i,j} \bar{g}^{ij} p_i p_j, \qquad (2)$$

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[2]Italian original from [3]: La seconda ... generalizzazione ... del nostro problema, vale a dire: riportare i punti di una superficie sopra un'altra superficie in modo che alle linee geodetiche della prima corrispondano linee geodetiche della seconda.

where we raised the indexes of $\bar{g}$ with the help of g, i.e., $\bar{g}^{ij} = g^{ki}\bar{g}_{km}g^{mj}$, is an integral of the geodesic flow of g. Moreover, $F = \text{const} \cdot H$ for a certain $\text{const} \in \mathbf{R}$ if and only if g and $\bar{g}$ are proportional with a constant coefficient of proportionality.

Theorem 5 above was essentially known to Darboux [14, Sections 600–608]; for recent proofs see [9, Corollary 1]. See also the discussion in [11, Section 2.4] and [8], [31], [32], [33], [34], [35].

Combining Theorems 2, 3, 4 with Theorem 5, we obtain a complete description of geodesically equivalent pseudo-Riemannian metrics on closed surfaces (up to a double cover).

2.2. Application II: every quadratically-superintegrable metric on a closed surface has constant curvature.

Recall that a metric on M^2 is called quadratically-superintegrable, if the geodesic flow of the metric admits three linearly independent integrals quadratic in momenta. Quadratically-superintegrable metrics were first considered by Koenigs [20]. Nowdays, investigation of quadratically-superintegrable metrics is a hot topic in mathematical physics due to various applications and deep mathematical structures behind it, see e.g. [18].

For example, the standard flat metric $dxdy$ on the 2-torus $\mathbf{R}^2/G$, where G is a lattice generated by two linearly independent vectors, is quadratically-superintegrable. Indeed, the Hamiltonian $H = 2p_xp_y$ and the quadratic in momenta functions $F_1 := p_x^2$, $F_2 := p_y^2$ are linearly independent integrals, and are invariant with respect to any lattice.

COROLLARY 1. *Let a metric g on a closed surface be quadratically-superintegrable. Then, it has constant curvature. If in addition the metric has signature $(+, -)$, then it is flat.*

In the proof of Corollary 1 we will need the following

LEMMA 1. *Let the metric g of signature $(+, -)$ on the two-torus T^2 admit an integral quadratic in momenta that is not a linear combination of the Hamiltonian and of the square of an integral linear in momenta. Then, there exists a Riemannian metric $\bar{g}$ geodesically equivalent to g.*

PROOF. By Theorem 2, without loss of generality we can assume that the metric g and the integral F are as in Model Example 1. Without loss of generality we can think that $X(x) > Y(y)$ for all $(x, y) \in \mathbf{R}^2$.

Let us cook with the help of H, F a Riemannian metric $\bar{g}$ geodesically equivalent to g. We put $X_{min} = \min_{x \in \mathbf{R}} X(x)$ and $Y_{max} = \max_{y \in \mathbf{R}} Y(y)$. Clearly, $X_{min} > Y_{max}$. We consider

$$\bar{F} := H + \frac{1}{X_{min} + Y_{max}} F_1 = \frac{\frac{1}{2} - \frac{Y}{X_{min} + Y_{max}}}{X - Y} p_x^2 + \frac{\frac{X}{X_{min} + Y_{max}} - \frac{1}{2}}{X - Y} p_y^2.$$

Since $X(x) > (X_{min} + Y_{max})/2 > Y(y)$ for every $(x, y) \in \boldsymbol{R}^2$, the integral $\bar{F}$ is positively defined (considered as a quadratic form on $T^* M^2$). Consider the metric $\bar{g}$ constructed by $\bar{F}$ with the help of Theorem 5. The metric is positively defined (i.e., is Riemannian), and is geodesically equivalent to g. Lemma 1 is proved. $\square$

PROOF OF COROLLARY 1.   The Riemannian version of Corollary 1 is known (see [**22**, Theorem 5.1] and [**29**, Lemma 3], see also [**40**, Theorem 6]). Then, without loss of generality we can assume that the metric has signature $(+, -)$.

Let H, F_1, F_2 be the linearly independent integrals quadratic in momenta. If both F_1 and F_2 are linear combinations of the square of integrals linear in momenta and the Hamiltonian, the metric admits two Killing vector fields implying that it has constant curvature.

Assume now that there exists an integral quadratic in momenta that is not a linear combination of the Hamiltonian and of the square of an integral linear in momenta. By Lemma 1, there exists a Riemannian metric $\bar{g}$ geodesically equivalent to g. The metric $\bar{g}$ is also quadratically-superintegrable. Indeed, as it was proved in [**32**, Lemma 1] (see also [**11**, Section 2.8] and [**26**, Lemma 3]), every metric geodesically equivalent to a quadratically-superintegrable metric is also quadratically-superintegrable. Then, by the Riemannian version of Corollary 1 (which is known, as we recalled above), the metric $\bar{g}$ has constant curvature. Then, by the Beltrami Theorem (see [**3**], [**41**]), the metric g also has constant curvature. The first part of Corollary 1 is proved.

If the metric has signature $(+, -)$, then the surface if the torus or the Klein bottle. By the Gauss-Bonnet Theorem, a metric of constant curvature on the torus or on the Klein bottle is flat. Corollary 1 is proved. $\square$

2.3.   Application III: Proof of projective Obata conjecture for two-dimensional pseudo-Riemannian metrics.

Let (M^n, g) be a pseudo-Riemannian manifold of dimension $n \geq 2$. Recall that a *projective transformation* of M^n is a diffeomorphism of the manifold that takes unparameterized geodesics to geodesics.

The goal of this paper is to prove the two-dimensional pseudo-Riemannian version of the following

PROJECTIVE OBATA CONJECTURE.   *Let a connected Lie group G act on a closed connected (M^n, g) of dimension $n \geq 2$ by projective transformations. Then, it acts by isometries, or for some $c \in \boldsymbol{R} \setminus \{0\}$ the metric $c \cdot g$ is the Riemannian*

metric of constant positive sectional curvature $+1$.

REMARK 3.   The attribution of conjecture to Obata is in folklore (in the sense we did not find a paper of Obata where he states this conjecture). Certain papers, for example [**16**], [**49**], [**54**], refer to this statement as to a classical conjecture. If we replace "closedness" by "completeness", the obtained conjecture is attributed in folklore to Lichnerowicz, see also the discussion in [**43**]. Local version of the projective Obata conjecture (recently solved in [**11**], [**44**]) is even more classical: it was explicitly asked by Sophus Lie [**28**].

For Riemannian metrics, projective Obata conjecture was proved in [**38**], [**39**], [**40**], [**43**]. Then, in dimension two we may assume that the signature of the metric is $(+, -)$, and that the manifold is covered by the torus T^2. Thus, the two-dimensional version of the projective Obata conjecture follows from

THEOREM 6.   *Let (T^2, g) be the two-dimensional torus T^2 equipped with a metric g of signature $(+, -)$. Assume a connected Lie group G acts on (T^2, g) by projective transformations. Then, G acts by isometries.*

Note that in the theory of geodesically equivalent metrics and projective transformations, dimension 2 is a special dimension: many methods that work in dimensions $n \geq 3$ do not work in dimension 2. In particular, the proof of the projective Obata conjecture in the Riemannian case was separately done for dimension 2 in [**38**], [**40**] and for dimensions greater than 2 in [**43**]. Moreover, recently an essential progress was achieved in the proof of the projective Obata conjecture in the pseudo-Riemannian case in dimensions $n \geq 3$, see [**19**], [**45**], [**46**]. This progress allows us to hope that it is possible to mimic (see [**19**, Section 1.2]) the Riemannian proof in the pseudo-Riemannian situation (assuming the dimension is $n \geq 3$). Thus, Theorem 6 closes an important partial case in the proof of projective Obata conjecture.

PROOF OF THEOREM 6.   Let g be a pseudo-Riemannian metric of signature $(+, -)$ of nonconstant curvature on T^2. We denote by $\mathrm{Proj}_0(T^2, g)$ the connected component of the group of projective transformations of (T^2, g), and by $\mathrm{Iso}_0(T^2, g)$ the connected component of the group of isometries. Clearly, $\mathrm{Proj}_0(T^2, g) \supseteq \mathrm{Iso}_0(T^2, g)$; our goal is to prove $\mathrm{Proj}_0(T^2, g) = \mathrm{Iso}_0(T^2, g)$.

We assume that $\mathrm{Proj}_0(T^2, g) \neq \mathrm{Iso}_0(T^2, g)$. Then, there exists a vector field v such that it is a projective vector field, but is not Killing vector field. (Recall that a vector field v is *projective*, if its local flow takes geodesics considered as unparameterized curved to geodesics). Then, by [**38**, Korollar 1], [**40**, Corollary 1], or [**52**], the quadratic in velocities function

$$I : TM \to \mathbf{R}, \quad I(\xi) := (\mathscr{L}_v g)(\xi, \xi) - \frac{2}{3}\operatorname{trace}\big(g^{-1}\mathscr{L}_v g\big)\, g(\xi, \xi),$$

where $\operatorname{trace}(g^{-1}\mathscr{L}_v g) := g^{ij}(\mathscr{L}_v g)_{ij}$, is a nontrivial (i.e., $\neq 0$) integral for the geodesic flow of g.

Suppose first I is not a linear combination of the energy integral $g(\xi, \xi)$ and of the square of an integral linear in velocities. Since closed manifolds do not allow vector fields v such that $\mathscr{L}_v g = \operatorname{const} \cdot g$ for $\operatorname{const} \neq 0$, I is not proportional to the energy integral $g(\xi, \xi)$. Then, by Lemma 1, there exists a *Riemannian* metric $\bar{g}$ geodesically equivalent to g.

Every projective vector field for g is also a projective vector field for $\bar{g}$ and vice versa, so that $\operatorname{Proj}_0(M, g) = \operatorname{Proj}_0(M, \bar{g})$. By the (already proved) Riemannian version of projective Obata conjecture we obtain that $\operatorname{Iso}_0(M, \bar{g}) = \operatorname{Proj}_0(M, \bar{g})$. Thus, $\operatorname{Proj}_0(M, g) = \operatorname{Iso}_0(M, \bar{g})$.

By [**41**, Corollary 1], see also [**24**], the dimensions of the Lie group of isometries of geodesically equivalent metrics coincide. Indeed, for every Killing vector field $\bar{K}$ for $\bar{g}$ the vector field $K^i := (\det g / \det \bar{g})^{1/(n+1)} \bar{g}^{ik} g_{kj} \bar{K}^j$ is a Killing vector field for g. Then, $\dim(\operatorname{Iso}_0(M, g)) = \dim(\operatorname{Iso}_0(M, \bar{g}))$ implying that $\operatorname{Iso}_0(M, g) = \operatorname{Proj}_0(M, g)$. Hence, the assumption that I is not a linear combination of the energy integral $g(\xi, \xi)$ and of the square of an integral linear in velocities leads to a contradiction. Thus, there exists a nontrivial integral linear in velocities. Finally, there exists a nontrivial Killing vector field that we denote by K.

Then, the group Proj_0 is at least two-dimensional (because it algebra contains K and v). The structures of possible Lie groups of projective transformations was understood already by S. Lie [**28**]. He proved that the for a 2-dimensional metric of nonconstant curvature the Lie algebra of Proj_0 is the noncommutative two dimensional algebra, or is $\mathfrak{sl}(3, \mathbf{R})$. In both cases there exists a projective vector field u such that the linear span $\operatorname{span}(u, K)$ is a two-dimensional noncommutative Lie algebra. Then, without loss of generality we can assume that $[K, u] = u$ or $[K, u] = K$.

Now, by Theorem 4, there exists a global coordinate system ($x \in (\mathbf{R}, \operatorname{mod} 1)$, $y \in (\mathbf{R}, \operatorname{mod} 1)$) such that in this coordinate system $K = \alpha \cdot (\partial/\partial x)$, where $\alpha \neq 0$. Assume $u(x, y) = u_1(x, y)(\partial/\partial x) + u_2(x, y)(\partial/\partial y)$. Without loss of generality we assume that $(u_1(0, 0), u_2(0, 0)) \neq (0, 0)$.

Let ϕ_t be the flow of K. Since $K = \alpha \cdot (\partial/\partial x)$, $\phi_t(x, y) = (x + \alpha t, y)$. Let us calculate the vector $d\phi_t(u(0, 0))$ for $t = 1/\alpha$ by two methods (and obtain two different results which gives us a contradiction).

First of all, since $\phi_{1/\alpha}$ is the identity diffeomorphism, $d\phi_t(u(0, 0)) = u(0, 0)$ for $t = 1/\alpha$.

The other method of calculating $d\phi_t(u(0, 0))$ is based on the commutative

relation $[K, u] = u$ or $[K, u] = K$.

Let us first assume that K, u satisfy $[K, u] = u$. In the coordinates, this condition reads $\alpha(\partial/\partial x)u_1 = u_1$ and $\alpha(\partial/\partial x)u_2 = u_2$ implying $u_1(x, 0) = u_1(0, 0) \cdot e^{x/\alpha}$ and $u_2(x, 0) = u_2(0, 0) \cdot e^{x/\alpha}$. Then,

$$d\phi_{1/\alpha}(u(0, 0)) = u_1(0, 0) \cdot e^{1/\alpha^2} \frac{\partial}{\partial x} + u_2(0, 0) \cdot e^{1/\alpha^2} \frac{\partial}{\partial y} = u(0, 0) \cdot e^{1/\alpha^2}.$$

Since $(u_1(0, 0), u_2(0, 0)) \neq (0, 0)$ we obtain that $d\phi_{1/\alpha}(u(0, 0)) \neq u(0, 0)$ which gives a contradiction. Thus, the commutative relation $[K, u] = u$ is not possible.

Let us now consider the second possible commutative relation $[K, u] = K$. In coordinates this relation reads $\alpha(\partial/\partial x)u_1 = \alpha$ and $\alpha(\partial/\partial x)u_2 = 0$ implying $u_1(x, 0) = u_1(0, 0) + x$. We again obtain that $d\phi_{1/\alpha}(u(0, 0)) \neq u(0, 0)$, which gives a contradiction. Thus, the commutative relation $[K, u] = K$ is also not possible. Finally, in all cases the existence of a nontrivial projective vector field on the torus T^2 equipped with a metric of nonconstant curvature leads to a contradiction.

Let us now consider the remaining case: we assume that g has constant curvature. By Gauss-Bonnet Theorem, a metrics of constant curvature on T^2 is flat. Then, as we show in Section 5.1, (T^2, g) is isometric to the *standard flat torus* $(\boldsymbol{R}^2/L, dxdy)$, where (x, y) are the standard coordinates on $\boldsymbol{R}^2$, and L is a lattice generated by two linearly independent vectors. In particular, all geodesics of the lift of the metric to $\boldsymbol{R}^2$ are the standard straight lines. Clearly, any projective transformation of $(\boldsymbol{R}^2/L, dxdy)$ generates a bijection $\phi : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ that commute with the lattice L and maps straight lines to straight lines. It is easy to see that the connected component of the group of such bijections consists of parallel translations, i.e., acts by isometries. Finally, $\mathrm{Proj}_0(\boldsymbol{R}^2/L, dxdy) = \mathrm{Iso}_0(\boldsymbol{R}^2/L, dxdy)$. Theorem 6 is proved. $\qquad\square$

2.4. One more possible application in mathematical physics.

An interesting possible application in mathematical physics is related to the Schrödinger equations on closed (pseudo-Riemannian) surfaces: as it was proved in [**9**, Theorem 5], the existence of an integral quadratic in momenta implies the existence of a differential operator of the second order that commute with the natural Schrödiger operator (i.e., essentially with the Beltrami-Laplace operator). This observation was applied with success in the Riemannian case (see, for example [**21**], [**30**]), and brought deep insight in the behavior of the quantum states of 2-dimensional Riemannian metrics. Global description of Riemannian metrics whose geodesic flows admit integrals quadratic in momenta played an important role in this result. In view of our results, one can try now to do the same in the signature $(+, -)$.

3. Local theory and the proof of Theorem 1.

3.1. Admissible coordinate systems and Birkhoff-Kolokoltsov forms.

Let g be a pseudo-Riemannian metric of signature $(+, -)$ on connected oriented M^2. Consider (and fix) two vector fields V_1, V_2 on M^2 such that

(A) $g(V_1, V_1) = g(V_2, V_2) = 0$ and
(B) $g(V_1, V_2) > 0$,
(C) the basis (V_1, V_2) is positive (i.e., induces the positive orientation).

Such vector fields always exist locally. Since locally there is precisely two possibilities in choosing the directions of such vector fields, the vector fields exist on a finite (at most, double-) cover of M^2.

We will say that a local coordinate system (x, y) is *admissible*, if the vector fields $(\partial/\partial x)$ and $(\partial/\partial y)$ are proportional to V_1, V_2 with positive coefficient of proportionality:

$$\frac{\partial}{\partial x} = \lambda_1(x, y)V_1(x, y), \quad \frac{\partial}{\partial y} = \lambda_2(x, y)V_2(x, y), \quad \text{where } \lambda_i > 0.$$

Obviously,

- admissible coordinates exist in a sufficiently small neighbourhood of every point,
- the metric g in admissible coordinates has the form

$$g = f(x, y)dxdy, \quad \text{where } f > 0, \tag{3}$$

- two admissible coordinate systems in one neighbourhood are connected by

$$\begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix} = \begin{pmatrix} x_{new}(x_{old}) \\ y_{new}(y_{old}) \end{pmatrix}, \quad \text{where } \frac{dx_{new}}{dx_{old}} > 0, \frac{dy_{new}}{dy_{old}} > 0. \tag{4}$$

REMARK 4.   For further use let us note that smooth local functions x, y form an admissible coordinate system, if and only if $V_1(x) > 0$, $V_2(y) > 0$, and $V_1(y) = V_2(x) = 0$ (where $V(h)$ denotes the derivative of the function h in the direction of the vector V).

LEMMA 2 ([**9**]).   *Let (x, y) be an admissible coordinate system for g. Let F given by (1) be an integral for g. Then,*

$$B_1 := \frac{1}{\sqrt{|a(x,y)|}}dx, \quad \left(\text{respectively, } B_2 := \frac{1}{\sqrt{|c(x,y)|}}dy\right)$$

*is a 1-form, which is defined at points such that $a \neq 0$ (respectively, $c \neq 0$).
Moreover, the coefficient a (respectively, c) depends only on x (respectively, y),
which in particular implies that the forms B_1, B_2 are closed.*

REMARK 5. The forms B_1, B_2 are not the direct analog of the "Birkhoff"
2-form introduced by Kolokoltsov in [**25**]. In a certain sense, they are real analogs
of the two branches of the square root of the Birkhoff form.

PROOF OF LEMMA 2. The first part of the statement, namely that

$$\frac{1}{\sqrt{|a(x,y)|}}dx, \quad \left(\text{respectively, } \frac{1}{\sqrt{|c(x,y)|}}dy\right)$$

transforms as a 1-form under admissible coordinate changes is evident: indeed,
after the coordinate change (4), the momenta transform as follows: $p_{x_{old}} = p_{x_{new}}(dx_{new}/dx_{old})$, $p_{x_{old}} = p_{x_{new}}(dx_{new}/dx_{old})$. Then, the integral F in the
new coordinates has the form

$$\underbrace{\left(\frac{dx_{new}}{dx_{old}}\right)^2 a}_{a_{new}} p_{x_{new}}^2 + \underbrace{\frac{dx_{new}}{dx_{old}}\frac{dy_{new}}{dy_{old}}b}_{b_{new}} p_{x_{new}}p_{y_{new}} + \underbrace{\left(\frac{dy_{new}}{dy_{old}}\right)^2 c}_{c_{new}} p_{y_{new}}^2.$$

Then, the formal expression $(1/\sqrt{|a|})dx_{old}$ (respectively, $(1/\sqrt{|c|})dy_{old}$) transforms
into

$$\frac{1}{\sqrt{|a|}}\frac{dx_{old}}{dx_{new}}dx_{new} \quad \left(\text{respectively, } \frac{1}{\sqrt{|c|}}\frac{dy_{old}}{dy_{new}}dy_{new}\right),$$

which is precisely the transformation law of 1-forms.

Let us prove that the coefficient a (respectively, c) depends only on x (respectively, y), which in particular implies that the forms B_1, B_2 are closed. If g is
given by (3), its Hamiltonian is

$$H = \frac{2p_x p_y}{f},$$

and the condition $\{H, F\} = 0$ reads

$$0 = \left\{ \frac{2p_x p_y}{f}, ap_x^2 + bp_x p_y + cp_y^2 \right\}$$

$$= \frac{2}{f^2} \left(p_x^3 (fa_y) + p_x^2 p_y (fa_x + fb_y + 2f_x a + f_y b) \right.$$
$$\left. + p_y p_x^2 (fb_x + fc_y + f_x b + 2f_y c) + p_y^3 (c_x f) \right),$$

i.e., is equivalent to the following system of PDE:

$$\begin{cases} a_y = 0, \\ fa_x + fb_y + 2f_x a + f_y b = 0, \\ fb_x + fc_y + f_x b + 2f_y c = 0, \\ c_x = 0. \end{cases} \tag{5}$$

Thus, $a = a(x)$, $c = c(y)$ implying that $B_1 := (1/\sqrt{|a|})dx$ and $B_2 := (1/\sqrt{|c|})dy$ are closed forms (assuming $a \neq 0$ and $c \neq 0$). Lemma 2 is proved. $\square$

REMARK 6.   For further use let us formulate one more consequence of equations (5): if $a \equiv c \equiv 0$ in a neighbourhood of a point, then $bf = $ const, implying $F - (\text{const}/2) \cdot H = 0$ in the neighborhood. If we consider (5) as a system of PDE on the unknown functions a, b, c, we see that the system is linear and of finite type. Then, vanishing of the solution corresponding to the integral $\hat{F} := (F - (\text{const}/2) \cdot H)$ in the neighborhood implies vanishing of the solution on the whole connected manifold. Thus, if $a \equiv c \equiv 0$ in a neighborhood of a point, then for a certain const $\in \mathbf{R}$ we have $F \equiv \text{const} \cdot H$ on the whole manifold.

REMARK 7.   For further use let us note that the set of the points where the form B_1 (B_2, resp.) is not defined coincides with the set of the points such that $a = 0$ ($c = 0$, resp.) and is invariant with respect to the (local) flow of the vector field V_2 (V_1, resp.)

A local coordinate system (x, y) will be called *perfect*, if it is admissible, and if in this coordinates system the coefficients a, c take values in the set $\{-1, 0, 1\}$ only.

LEMMA 3.   *Let F given by (1) be an integral for the geodesic flow of $g = f(x, y)dxdy$ such that $F \neq \text{const} \cdot H$ for all const $\in \mathbf{R}$. Then, almost every point p has a neighborhood U such that precisely one of the following conditions is fulfilled:*

(i) *$ac > 0$ at all points of U,*
(ii) *$ac < 0$ at all points of U,*

(iii) (a) $a = 0$ *and* $c \neq 0$ *at all points of* U*, or*

   (b) $a \neq 0$ *and* $c = 0$ *at all points of* U*.*

Moreover, there exists a perfect coordinate system $\tilde{x}, \tilde{y}$ *in a (possibly, smaller) neighborhood* $U'(p) \subseteq U(p)$ *of* p*. In the perfect coordinate system, the metric and the integral are given by*

$$g = \tilde{f}(\tilde{x}, \tilde{y})dxdy \quad and \quad F = \text{sign}(a(x,y))p_{\tilde{x}}^2 + \tilde{b}(\tilde{x}, \tilde{y})p_{\tilde{x}}p_{\tilde{y}} + \text{sign}(c(x,y))p_{\tilde{y}}^2,$$

$$where \quad \text{sign}(\tau) = \begin{cases} 1 & if \ \tau > 0, \\ -1 & if \ \tau < 0, \\ 0 & if \ \tau = 0. \end{cases}$$

PROOF OF LEMMA 3.   It is sufficient to prove the lemma assuming that M^2 is a small neighborhood W. We consider and fix admissible coordinates in this neighborhood. In this coordinates the coefficients a, b, c of the integral (1) are smooth functions.

We consider the following subsets of W :

- $W_{ac \neq 0} := \{q \in W \mid a(q)c(q) \neq 0\}$,
- $W_{a \neq 0, c = 0} := \{q \in W \mid a(q) \neq 0, c(q) = 0\}$,
- $W_{a = 0, c \neq 0} := \{q \in W \mid a(q) = 0, c(q) \neq 0\}$,
- $W_{a = 0, c = 0} := \{q \in W \mid a(q) = 0, c(q) = 0\}$.

The sets are clearly disjunkt, there union coincides with the whole W. We consider the set $W_{\text{perfect}} := W_{ac \neq 0} \cup int(W_{a = 0, c \neq 0}) \cup int(W_{a \neq 0, c = 0})$, where "$int$" denotes the set of inner points. The set W_{perfect} is open, and is everywhere dense in W. Indeed, it is open, since $W_{ac \neq 0}$, $int(W_{a = 0, c \neq 0})$, and $int(W_{a \neq 0, c = 0})$ are open. It is everywhere dense, since it is everywhere dense in the set $W_{ac \neq 0} \cup W_{a = 0, c \neq 0} \cup W_{a \neq 0, c = 0}$, and the remaining set $W_{a = 0, c = 0}$ does not contain any open nonempty set by Remark 6.

Now, by definition, every point of W_{perfect} has a neighborhood such that in this neighborhood one of the conditions (i)–(iii) is fulfilled. The first statement of the proposition is proved.

Let us now prove the second statement. Let $p_0 \in int(W_{a \neq 0, c = 0})$. In a simply-connected neighborhood $U(p_0) \subset W_{a \neq 0, c = 0}$, we consider the function

$$x_{new}(p) := \int_{p_0}^{p} B_1. \tag{6}$$

Since the form B_1 is closed, and $U(p_0)$ is simply-connected, the function x_{new} does not depend on the choice of the curve connecting the points p_0, p, and is therefore

well defined. The differential of the function x_{new} is precisely the 1-form B_1, and does not vanish at p_0. We have $V_1(x_{new}) = B_1(V_1) > 0$, $V_2(x_{new}) = B_1(V_2) = 0$. Since the coordinates (x, y) are admissible, $V_2(y) > 0$ and $V_1(y) = 0$. Then, by Remark 4, (x_{new}, y) is a local admissible coordinate system in a possibly smaller neighborhood $U' \subseteq U$ containing p_0.

REMARK 8. Let us note that, in the admissible coordinates the formula (6) looks

$$x_{new}(x) = \int_{x_0}^{x} \frac{1}{\sqrt{|a(t)|}} dt \tag{7}$$

implying that x_{new} is independent of y, i.e., $x_{new} = x_{new}(x)$.

In this coordinate system, the integral F is equal to

$$\left(\frac{dx_{new}}{dx}\right)^2 a p_{x_{new}}^2 + \frac{dx_{new}}{dx_{old}} b p_{x_{new}} p_y = \frac{a}{(\sqrt{|a|})^2} p_{x_{new}}^2 + \frac{b}{\sqrt{|a|}} p_{x_{new}} p_y$$

$$= \text{sign}(a) p_{x_{new}}^2 + b_{new} p_{x_{new}} p_y.$$

The cases $p_0 \in int(W_{a=0,c\neq 0})$, $p_0 \in W_{a\neq 0,c\neq 0}$ are similar: in the case $p_0 \in int(W_{a=0,c\neq 0})$, in the coordinate system (x, y_{new}) in a possibly smaller neighborhood of p_0, where

$$y_{new} := \int_{p_0}^{p} B_2, \tag{8}$$

the integral F is given by $b_{new} p_x p_{y_{new}} + \text{sign}(c) p_{y_{new}}^2$. In the case $p_0 \in W_{a\neq 0,c\neq 0}$, in the coordinate system (x_{new}, y_{new}), where x_{new} is given by (6) and y_{new} is given by (8), the integral F is given by $\text{sign}(a) p_{x_{new}}^2 + b_{new} p_{x_{new}} p_{y_{new}} + \text{sign}(c) p_{y_{new}}^2$. Lemma 3 is proved.   □

REMARK 9. If $a = 0$ ($c = 0$, resp.), the coordinate transformation of the form $(x_{new}(x), y)$ $((x, y_{new}(y))$, resp.) does not change the property of coordinates to be perfect. If $ac \neq 0$, the perfect coordinates are unique up to transformation $(x, y) \mapsto (x + \text{const}_1, y + \text{const}_2)$. In particular, if $ac \neq 0$, the vector fields $(\partial/\partial x)$ and $(\partial/\partial y)$, where x, y are local perfect coordinates, do not depend on the choice of local perfect coordinates, and therefore are well-defined globally, at all points such that $ac \neq 0$ (provided that V_1, V_2 satisfying (A,B,C) are globally defined).

3.2. Proof of Theorem 1.

By Lemma 3, almost every point of M^2 has a neighborhood such that in perfect coordinates the metrics and the integral are as in one of the following cases:

Case 1: $ac > 0$: The metric is $f(x, y)dxdy$, the integral is $\pm(p_x^2 + b(x, y)p_x p_y + p_y^2)$.

Case 2: $ac < 0$: The metric is $f(x, y)dxdy$, the integral is $\pm(p_x^2 + b(x, y)p_x p_y - p_y^2)$.

Case 3a: $c \equiv 0$: The metric is $f(x, y)dxdy$, the integral is $\pm(p_x^2 + b(x, y)p_x p_y)$.

Case 3b: $a \equiv 0$: The metric is $f(x, y)dxdy$, the integral is $\pm(b(x, y)p_x p_y + p_y^2)$.

We will carefully consider all four cases.

3.2.1. Case 1.

PROPOSITION 1. *Let the geodesic flow of a metric $g = f(x, y)dxdy$ admits an integral (1). Assume $ac > 0$ at the point p. Then, in the coordinates $(u, v) = ((x + y)/2, (x - y)/2)$, where (x, y) are perfect coordinates in a neighborhood of p,*

$$g = (U(u) - V(v))(du^2 - dv^2) \quad and \quad F = \pm\left(\frac{p_v^2 U(u) - p_u^2 V(v)}{U(u) - V(v)}\right), \qquad (9)$$

where U, V are certain functions of one variable.

PROOF. Without loss of generality a and c are positive in a neighborhood of p. Then, by Lemma 3, in perfect coordinates in a neighborhood of p the metric and the integral are $g = f(x, y)dxdy$, $F = p_x^2 + b(x, y)p_x p_y + p_y^2$. Then, the system (5) has the following simple form:

$$\begin{cases} (fb)_y + 2f_x = 0, \\ (fb)_x + 2f_y = 0, \end{cases} \quad \text{which is equivalent to} \quad \begin{cases} (fb + 2f)_x + (fb + 2f)_y = 0, \\ (fb - 2f)_x - (fb - 2f)_y = 0. \end{cases}$$

After the (non-admissible) change of coordinates $u = (x + y)/2$, $v = (x - y)/2$, the system has the form

$$\begin{cases} (fb + 2f)_u = 0, \\ (fb - 2f)_v = 0, \end{cases} \quad \text{which is equivalent to} \quad \begin{cases} fb + 2f = -4V(v), \\ fb - 2f = -4U(u) \end{cases}$$

for certain functions $U(u)$ and $V(v)$. Thus,

$$f = U(u) - V(v), \quad b = -2\frac{U(u) + V(v)}{U(u) - V(v)}.$$

Let us now calculate the metric in the integral in the new coordinates: substituting $dx = du + dv$, $dy = du - dv$ in the formula $g = f(x,y)dxdy = (U(u) - V(v))dxdy$, we obtain that in the new coordinates the metric is $(U(u) - V(v))(du^2 - dv^2)$. Substituting $p_x = ((\partial u/\partial x)p_u + (\partial v/\partial x)p_v) = (1/2)(p_u + p_v)$ and $p_y = ((\partial u/\partial y)p_u + (\partial v/\partial y)p_v) = (1/2)(p_u - p_v)$ in the formula $F = p_x^2 + bp_xp_y + p_y^2 = p_x^2 - 2(U(u) + V(v))/(U(u) - V(v))p_xp_y + p_y^2$, we obtain that in the new coordinates (u, v)

$$F = \frac{1}{2}\left(p_u^2 + p_v^2 - \frac{U(u) + V(v)}{U(u) - V(v)}(p_u^2 - p_v^2)\right) = \frac{U(u)p_v^2 - V(v)p_u^2}{U(u) - V(v)}.$$

We see that, in the new coordinates, the metric and the integral are as in (9). Proposition 1 is proved. $\qquad\square$

3.2.2.   Case 2.

PROPOSITION 2.   *Let the geodesic flow of a metric $g = f(x,y)dxdy$ admits an integral* (1). *Assume $ac < 0$ at the point p. Then, in perfect coordinates in a neighborhood of p,*

$$g = \Im(h)dxdy \quad and \quad F = \pm\left(p_x^2 - p_y^2 + 2\frac{\Re(h)}{\Im(h)}p_xp_y\right), \qquad (10)$$

where $\Re(h)$ and $\Im(h)$ are the real and the imaginary parts of a holomorphic function h of the variable $z = x + i \cdot y$.

PROOF.   Without loss of generality $a(p) > 0$, $c(p) < 0$. By Lemma 3, in perfect coordinates the metric and the integral are $g = f(x,y)dxdy$, $F = p_x^2 + b(x,y)p_xp_y - p_y^2$. Then, the system (5) has the following simple form:

$$\begin{cases} (fb)_y + 2f_x = 0, \\ (fb)_x - 2f_y = 0. \end{cases} \qquad (11)$$

We see that these equations are the Cauchy-Riemann conditions for the complex-valued function $fb + 2if$. Thus, for an appropriate holomorphic function $h = h(x + iy)$ we have $fb = (1/2)\Re(h)$, $f = \Im(h)$. Finally, the metric and the integral have the form (10). Proposition 2 is proved. $\qquad\square$

3.2.3.   Case 3.

In this case we prove two propositions: the first one is more general, and is the final step in the proof of Theorem 1. The second one requires additional assumptions, and will be used in the proof of Theorem 2.

PROPOSITION 3. *Let the geodesic flow of a metric $g = f(x, y)dxdy$ admits an integral (1). Then, the following two statements are true:*

(a) *If $a(p) \neq 0$, and $c(q) = 0$ at every point q of a small neighborhood of p, in perfect coordinates in a (possibly, smaller) neighborhood of p,*

$$g = \left(\widehat{Y}(y) + \frac{x}{2}Y'(y)\right)dxdy \quad and \quad F = \pm \left(p_x^2 - \frac{Y(y)}{\widehat{Y}(y) + \frac{x}{2}Y'(y)}p_x p_y\right), \quad (12)$$

where Y and $\widehat{Y}$ are functions of one variable.

(b) *If $c(p) \neq 0$, and $a(q) = 0$ at every point q of a small neighborhood of p, in perfect coordinates in a (possibly, smaller) neighborhood of p,*

$$g = \left(\widehat{X}(x) + \frac{y}{2}X'(x)\right)dxdy \quad and \quad F = \pm \left(p_y^2 - \frac{X(x)}{\widehat{X}(x) + \frac{y}{2}X'(x)}p_x p_y\right), \quad (13)$$

where X and $\widehat{X}$ are functions of one variable.

PROOF. The cases (a) and (b) are clearly analogous; without loss of generality we can assume $a(p) > 0$, $c \equiv 0$. By Lemma 3, in perfect coordinates the metric and the integral are $g = f(x, y)dxdy$, $F = p_x^2 + b(x, y)p_x p_y$. Then, the equation (5) has the following simple form:

$$\begin{cases} (fb)_y + 2f_x = 0, \\ (fb)_x = 0. \end{cases} \quad (14)$$

This system can be solved. Indeed, the second equation implies $fb = -Y(y)$. Substituting this in the first equation we obtain $Y'(y) = 2f_x$ implying

$$f = \frac{x}{2}Y'(y) + \widehat{Y}(y) \quad and \quad b = -\frac{Y(y)}{\frac{x}{2}Y'(y) + \widehat{Y}(y)}.$$

Finally, the metric and the integral are as in (12). Proposition 3(a) is proved. The proof of Proposition 3(b) is essentially the same. $\square$

PROOF OF THEOREM 1. Theorem 1 follows directly from Lemma 3 and Propositions 1, 2, 3. Indeed, by Lemma 3, almost every point has a neighborhood such that in this neighborhood the assumptions of one of Propositions 1, 2,

3 are fulfilled. Then, by Propositions 1, 2, 3 the metric and the integral are as in the table in Theorem 1. $\qquad\square$

We will also need a slightly less general version of normal form of metrics satisfying the assumption of Case 3.

Let us observe that the function Y from (12), or the function X from (13), can be given in invariant terms (i.e., they does not depend on the choice of a perfect coordinate system, and can be smoothly prolonged to the whole manifold). Indeed, consider the symmetric $(2,0)$-tensor $\tilde{F}^{ij}$ such that $F = \sum_{i,j} \tilde{F}^{ij} p_i p_j$ (if F is given by (1), the matrix of $\tilde{F}$ is $\left(\begin{smallmatrix} a & b/2 \\ b/2 & c \end{smallmatrix}\right)$). Transvecting $\tilde{F}^{ij}$ with g_{ij} we obtain the globally defined smooth function $L := \mathrm{trace}(\tilde{F}^i_j) := \sum_{i,j} \tilde{F}^{ij} g_{ij}$. Under assumptions of Case 3a, in the perfect coordinates, the function L is given by

$$
\begin{aligned}
L = \sum_{i,j} \tilde{F}^{ij} g_{ij} &= \mathrm{trace}\left(\begin{pmatrix} a & \dfrac{b}{2} \\ \dfrac{b}{2} & c \end{pmatrix} \begin{pmatrix} 0 & \dfrac{f}{2} \\ \dfrac{f}{2} & 0 \end{pmatrix} \right) \\
&= \mathrm{trace}\left(\begin{pmatrix} -\dfrac{Y}{4} & * \\ 0 & -\dfrac{Y}{4} \end{pmatrix} \right) = -\frac{Y}{2}.
\end{aligned} \tag{15}
$$

PROPOSITION 4. *Let the geodesic flow of a metric $g = f(x,y)dxdy$ admits an integral (1). Then, the following two statements are true:*

(a) *Suppose $a(p) \ne 0$, and $c(q) = 0$ at every point q of a small neighborhood of p. Assume $dL_{|p} \ne 0$, where L is given by (15). Then, in a (possibly, smaller) neighborhood of p, in perfect coordinates (x,y) such that $y(q) = -2L(q)$ for all q, the metric and the integral are given by*

$$
g = \left(Y(y) + \frac{x}{2} \right) dxdy \quad and \quad F = \pm \left(p_x^2 - \frac{y}{Y(y) + \dfrac{x}{2}} p_x p_y \right), \tag{16}
$$

where Y is a function of one variable.

(b) *Suppose $c(p) \ne 0$, and $a(q) = 0$ at every point q of a small neighborhood of p. Assume $dL_{|p} \ne 0$, where L is given by (15). Then, in a (possibly, smaller) neighborhood of p, in perfect coordinates (x,y) such that $x(q) = -2L(q)$ for all q, the metric and the integral are given by*

$$g = \left(X(x) + \frac{y}{2} \right) dx dy \quad and \quad F = \pm \left(p_y^2 - \frac{x}{X(x) + \frac{y}{2}} p_x p_y \right), \qquad (17)$$

where X is a function of one variable.

PROOF. The cases (a) and (b) are clearly analogous; without loss of generality we can assume $a(p) > 0$, $c \equiv 0$. In the perfect coordinates such that $y = -2L$, we have $g = f(x,y)dxdy$ and $F = p_x^2 - (y/f)p_x p_y$. Then, the system (5) is equivalent to the equation $2f_x = 1$. Thus, $f = Y(y) + x/2$. Proposition 4(a) is proved. The proof of Proposition 4(b) is similar. $\qquad\square$

4. Global theory and the main step in the proof of Theorem 2.

4.1. Notation, conventions, and the plan of the proof.
Within the whole section we assume that

- the surface is the torus T^2,
- g is a pseudo-Riemannian metric of signature $(+,-)$ on T^2.
- The vector fields V_1, V_2 satisfying conditions (A,B,C) from Section 3.1 are globally defined (the case when it is not possible will be considered in Section 5.4).
- F is a nontrivial integral of the geodesic flow of g. We will reserve notation x, y for admissible coordinates, or for perfect coordinates, and will denote the coefficients of the integral as in (1). As in Section 3.1, we will denote by B_1, B_2 the 1-forms $(1/\sqrt{|a|})dx$ and $(1/\sqrt{|c|})dy$.

As in Section 3.2.3., we denote by $\tilde{F}^{ij}$ the symmetric $(2,0)$-tensor corresponding to the integral F, and by $\tilde{F}^i_j$ the $(1,1)$-tensor $\tilde{F}^i_j := \sum_k \tilde{F}^{ik} g_{kj}$.

We will proceed according to the following plan:

1. In Section 4.2 we show that there exists no point such that $ac < 0$. This will imply that $\tilde{F}^i_j$ has real eigenvalues at every point of T^2.
2. By Remark 10, $\tilde{F}^i_j$ has only one eigenvalue (of algebraic multiplicity 2) at the points such that B_1 or B_2 is not defined. In Section 4.3, we show that this eigenvalue is constant on each connected component of the set such that B_1 or B_2 is not defined.
3. In Section 4.4 we show that the existence a point such that B_1 or B_2 is not defined implies that one of the eigenvalues of $\tilde{F}^i_j$ is constant on the whole manifold.
4. In Section 4.5, we show that if one of the eigenvalues of $\tilde{F}^i_j$ is constant, the quadratic integral F, or the lift of the quadratic integral to the appropriate double cover is a linear combination the square of a function linear in momenta

and the Hamiltonian. Later, in Corollary 6, we show that if the lift of the quadratic integral to a double cover is a linear combination of the lift of the Hamiltonian and the square of a integral linear in momenta, then the integral is a linear combination of the Hamiltonian and the square of an integral linear in momenta.

5. In Section 4.6 we show that if at every point B_1 and B_2 are defined, then the torus, the metric g, and the integral F are as in the Model Example 1.

These will prove Theorem 2 under the additional assumption that the vector fields V_1, V_2 exist on T^2. The case when this vector fields do not exist on T^2 will be considered later, in Section 5.4: we will prove that this case can not happen (if there exists an integral quadratic in momenta that is not a linear combination of the Hamiltonian and the square of an integral linear in momenta).

4.2.   At every point, the eigenvalues of $\tilde{F}^i_j$ are real.

LEMMA 4.   *There is no point $p \in T^2$ such that at this point $ac < 0$.*

PROOF.   Suppose at $p \in T^2$ we have $ac < 0$. Let W_0 be the connected component of the set

$$W := \{q \in T^2 \mid B_1 \text{ and } B_2 \text{ are defined}\}$$

containing the point p. At every $q \in W_0$ we have $ac < 0$. We consider the function $K : W_0 \to \mathbf{R}$, $K = 1/(g^*(B_1, B_2))$, where g^* is the scalar product on T^*T^2 induced by g.

In any perfect coordinates (x, y) we have $B_1 = dx$, $B_2 = dy$, and $g = \Im(h)dxdy$ by Proposition 2. Then, $K = \Im(h)$ for a holomorphic function h implying it is harmonic function. When we approach the boundary $\overline{W}_0 \setminus W_0$, the function K converges to 0. Indeed, in the admissible coordinates near a boundary point the function K is $f\sqrt{|ac|}$, and $ac \xrightarrow{\text{converges}} 0$ (because at least one of coefficients a, c is zero at the points of boundary).

Finally, by the maximum principle (for harmonic functions), the function h is identically zero, which clearly contradicts the assumptions. Lemma 4 is proved.

$\square$

COROLLARY 2.   *At every point of T^2, the eigenvalues of $\tilde{F}^i_j$ are real.*

PROOF.   The eigenvalues are the roots of the characteristic polynomial

$$\chi(t) = \det\left(\tilde{F}_j^i - t \cdot \delta_j^i\right) = \det\left(\begin{pmatrix} \dfrac{fb}{4} & \dfrac{af}{2} \\ \dfrac{cf}{2} & \dfrac{fb}{4} \end{pmatrix} - t \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$= t^2 - \frac{fb}{2}t + \frac{(fb)^2}{16} - \frac{acf^2}{4}. \tag{18}$$

The discriminant of $\chi(t)$ is $\mathscr{D} = (1/4)(fb/2)^2 - ((fb)^2/16 - acf^2/4) = acf^2/4$. We see that if $ac \geq 0$ (which is fulfilled by Proposition 2) the discriminant is nonnegative implying the eigenvalues of $\tilde{F}_j^i$ are real. Corollary 2 is proved. $\qquad\square$

REMARK 10.   For further use let us note that at the points such that $ac = 0$ the discriminant $\mathscr{D}$ of $\chi(t)$ given by (18) vanishes implying the tensor $\tilde{F}_j^i$ has only one eigenvalue (of algebraic multiplicity two), namely $fb/4$. At the points such that $ac > 0$ the discriminant $\mathscr{D} > 0$ implying the tensor $\tilde{F}_j^i$ has two different real eigenvalues.

4.3. The function $L := \sum_i \tilde{F}_i^i(:= \mathrm{trace}(\tilde{F}_j^i))$ is constant on each connected component of the set of the points such that B_1 or B_2 is not defined.

LEMMA 5.   *The function $L = \sum_i \tilde{F}_i^i$ ($:= \mathrm{trace}(\tilde{F}_j^i)$) is constant on each connected component of the set of the points such that B_1 is not defined.*

PROOF.   Let at the point p the form B_1 is not defined. We consider a small neighborhood $U(p)$ of p. Lemma 5 is a direct corollary of the following

STATEMENT.   *L is constant on each connected component of the set $\{q \in U(p) \mid B_1$ is not defined$\}$.*

Now, the above statement follows from the following two propositions:

PROPOSITION 5.   *Assume B_1 is not defined at every point of a neighborhood of p. Then, L is constant in this neighborhood.*

PROPOSITION 6.   *Assume every neighborhood of p has a point such that B_1 is defined. Then, for a certain neighborhood $U(p)$ the function L is constant on the connected component of the set $\{q \in U(p) \mid B_1$ is not defined$\}$ containing the point p.*

We will proceed as follows: we will first prove Proposition 5. Then, we prove a technical Proposition 7. Finally, we will use Propositions 5, 7 in the proof of Proposition 6.

PROOF OF PROPOSITION 5.   Our goal is to prove that $dL = 0$ at p. Without loss of generality, by Remark 6, we can assume $c \neq 0$ at the point p. We assume that $dL \neq 0$ at p, and find a contradiction.

We denote by W_0 the connected component of the set $W := \{q \in T^2 \mid B_1$ is not defined$\}$ containing p. We denote by $\alpha : (-\infty, +\infty) \to T^2$ the integral curve of V_2 such that $\alpha(0) = p$. Since W is invariant with respect to the flow of V_2, the curve α is a curve on W_0. Let us show that the curve α is periodic.

In a small neighborhood of every point of the curve, we have $a \equiv 0$ implying

$$
L = \text{trace}\left(\tilde{F}_j^i\right) = \text{trace}\left(\begin{pmatrix} 0 & \dfrac{b}{2} \\ \dfrac{b}{2} & c \end{pmatrix}\begin{pmatrix} 0 & \dfrac{f}{2} \\ \dfrac{f}{2} & 0 \end{pmatrix}\right) = \frac{fb}{2}.
$$

Then, the second equation of (5) implies that, on W, the function L is invariant with respect to the flow of V_2. Hence, at every point of the curve α we have $dL \neq 0$.

Then, the connected component of the set $\{q \in T^2 \mid L(q) = L(p)\}$ containing p coincides with the image of α. Since $\{q \in T^2 \mid L(q) = L(p)\}$ is compact, the image of α is compact implying the image of the curve is a closed circle.

The following cases are possible:

Case (a): For every $t \in \mathbf{R}$, the form B_2 is defined at the point $\alpha(t)$,
Case (b): There exists $t \in \mathbf{R}$ such that at the point $\alpha(t)$ the form B_2 is not defined.

Under assumptions of Case (a), let us construct a perfect coordinate system in a neighborhood $U(\alpha(t))$ of every point $\alpha(t)$. We assume that every neighborhood $U(\alpha(t))$ is sufficiently small and is homeomorphic to the disk.

As the first coordinate x we take the function $-2L$ (where $L = \sum_{i,j} \tilde{F}^{ij} g_{ij}$ as above). Since L is preserved by flow of V_2, its differential is not zero in a small neighborhood of every point $\alpha(t)$. Since $dL(V_2) = 0$, the coordinate x can be taken as the first admissible coordinate.

In order to construct the second coordinate y, we consider the curve $\gamma : [0, t+1] \to W$ connecting the points $p = \gamma(0)$ and $q \in U(c(t))$ the such that $\gamma_{|[0,t]} = \alpha_{|[0,t]}$, $\gamma(t+1) = q$, and such that $\gamma_{|[t,t+1]}$ lies in $U(\alpha(t))$. We put $y(q) := \int_\gamma B_2$.

The function y is well-defined, its differential is B_2 and is not zero at $\alpha(t)$. The local coordinates x, y are as in Proposition 4(b). Then, in this coordinates, the metric g is equal to $(X(x) + (1/2)y)dxdy$. Since $V_2(X) = 0$ locally, and since the functions $X(\alpha(t))$ coincides on the intersection of the neighborhoods $U(\alpha(t_0))$ and $U(\alpha(t_0 + \varepsilon))$ (for small ε), for every point of the curve α we have $X(\alpha(t)) = X(\alpha(0)) = X(p)$.

When t ranges from $-\infty$ to $+\infty$, the coordinate y also ranges from $-\infty$ to $+\infty$. Indeed, $\int_{\alpha_{|[0,t]}} B_2(\alpha'(t)) = \int_0^t B_{2|\alpha(s)}(V_{2|\alpha(s)})ds$, and $B_2(V_2)$ is positive and is therefore separated from zero on the compact set image(α).

Then, there exists t such that the value of y corresponding to $\alpha(t)$ is $-2X(p)$. At the point $\alpha(t)$, the metric $g = (X(x) + (1/2)y)dxdy$ is degenerate which contradicts the assumptions. Proposition 5 is proved under the additional assumptions of Case (a).

Let us now prove Proposition 5 under assumptions of Case (b): we assume that there exists t such that B_2 is not defined at the point $\alpha(t)$.

Let (t_{min}, t_{max}), where $t_{min} < 0 < t_{max} \in \boldsymbol{R}$, be the (open) interval such that

- B_2 is defined at $\alpha(t)$ for every $t \in (t_{min}, t_{max})$,
- B_2 is not defined at $\alpha(t_{min})$, and at $\alpha(t_{max})$.

As in the proof for Case (a), we construct a perfect local coordinate system x, y in a neighborhood $U(\alpha(t))$ of every point $\alpha(t)$, where $t \in (t_{min}, t_{max})$. We put $x(q) := -2L(q)$ and $y(q) := \int_\gamma B_2$, where $\gamma : [0, t+1] \to W$, $\gamma_{|[0,t]} = \alpha_{|[0,t]}$, $\gamma(t+1) = q$, and such that $\gamma_{|[t,t+1]}$ lies in $U(\alpha(t))$. We assume that the neighborhood $U(\alpha(t))$ is sufficiently small implying B_2 is defined at every point of $U(\alpha(t))$, and is homeomorphic to the disk.

By Proposition 4, in this coordinates, the metric is $(X(x) + (1/2)y)dxdy$. Let us show that the coordinate y converges to $-2X(p)$ when t converges to t_{max}.

In order to do this, we consider the scalar product on T^*T^2 induced by g (we will denote this scalar product by g^*). We consider the function $h := g^*(-2dL, B_2)$. This is indeed a function (i.e., h does not depend on the choice of an admissible coordinate system) which is defined at the points such that B_2 is defined. In admissible coordinates $(\tilde{x}, \tilde{y})$ in the neighborhood of the point $\alpha(t_{max})$, the function is given by $h = -2(1/\tilde{f}) \cdot (\partial L / \partial \tilde{x}) \cdot (1/\sqrt{|\tilde{c}|})$. Since $\tilde{c}(\alpha(t_{max})) = 0$, we have $h(\alpha(t)) \xrightarrow{t \to t_{max}} \pm\infty$. In the constructed above coordinates (x, y), we have $h(\alpha(t)) = 1/(X(p) + y(\alpha(t))/2) \cdot 1 \cdot 1$. Then, $X(p) + y(\alpha(t))/2 \xrightarrow{t \to t_{max}} 0$. Thus, $y(\alpha(t)) \xrightarrow{t \to t_{max}} -2X(p)$.

Similarly one can show that the same is true for t_{min}, namely $y(\alpha(t)) \xrightarrow{t \to t_{min}} -2X(p)$.

Since $y(\alpha(t)) = \int_0^t B_{2|\alpha(s)}(V_{2|\alpha(s)})ds$, and $B_{2|\alpha(s)}(V_{2|\alpha(s)})$ is positive for all $s \in (t_{min}, t_{max})$, the values of $y(\alpha(t))$ can not converge to the same number for $t \to t_{max}$ and for $t \to t_{min}$. The obtained contradiction proves Proposition 5. $\square$

PROPOSITION 7.   *The set $\{q \in T^2 \mid B_1$ or B_2 is defined in $q\}$ is connected.*

PROOF.   It is sufficient to prove that every point p has a neighborhood $U(p)$

such that the set $S(p) := \{q \in U(p) \mid B_1 \text{ or } B_2 \text{ is defined in } q\}$ is connected. We take a sufficiently small $U(p)$, and consider admissible coordinates x, y in $U(p)$. We assume that the neighborhood is small enough so we can connect every two points of this neighborhood by a geodesic.

If the set $S(p)$ is not connected, at every point $q \in U(p)$ we have $a(q) = 0$, or $c(q) = 0$. Without loss of generality we can assume that at every point of $U(p)$ we have $a = 0$. Then, the point p satisfies the assumptions of Proposition 5 above implying $L = fb/2 = \text{const}$ on $U(p)$.

Let us now consider the points $U(p) \setminus S(p)$. At every such point, $a = c = 0$ implying

$$\tilde{F}^{ij} = \begin{pmatrix} 0 & \dfrac{b}{2} \\ \dfrac{b}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{L}{f} \\ \dfrac{L}{f} & 0 \end{pmatrix} = \frac{L}{4} \cdot g^{ij}.$$

Thus, at such points, $F = (L/2)H = \text{const} \cdot H$. Without loss of generality we can assume that $\text{const} = 0$, otherwise we can replace F by $(F - \text{const} \cdot H)$.

We take 5 points $p_1, \ldots, p_5 \in U(p) \setminus S(p)$ such that $F_{|T_{p_i}^* T^2} = 0$ at these points. Since F is an integral, it vanishes on every geodesic passing through any of the points $p_1, \ldots, p_5$. Take a point $q \in U(p)$ in a small neighborhood of S, and connect this point with the points $p_1, \ldots, p_5$ by geodesics, see Figure 6. Let $\xi_1 \in T_{p_1}^* T^2, \ldots, \xi_5 \in T_{p_5}^* T^2$ be the vector-momenta of these geodesics at q. At almost every q, the tangent vectors of the geodesics are mutually nonproportional implying the vector-momenta ξ_i and ξ_j are not proportional for $i \neq j$.

Since F is an integral and $F_{|T_{p_i}^* T^2} \equiv 0$, we have $F(\xi_i) = 0$. Thus, the quadratic function $F_{|T_q T^2}$ vanishes in 5 mutually nonproportional points ξ_i. Hence,
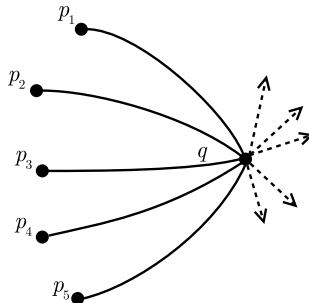

Figure 6. The geodesic connecting the points p_i with the point q, and their tangent vectors at the point q. For almost every q, the tangent vectors are mutually nonproportional.

$F_{|T^*qT^2} \equiv 0$. Thus, the restriction of F to a small neighborhood of p vanishes, which clearly contradicts the assumptions. The contradiction proves Proposition 7. $\qquad\square$

Combining Proposition 7, Remark 4, and Lemma 4, we obtain

COROLLARY 3. *Let $a > 0$ at a point. Then, at every point of T^2 we have $a \geq 0$, $c \geq 0$.*

PROOF OF PROPOSITION 6. We consider admissible coordinates x, y in a small neighborhood $U(p)$. We think that the point p has the coordinates $(x(p), y(p)) = (0, 0)$. In this coordinates, by Remark 4, the connected component of the set $\{q \in U(p) \mid B_1$ is not defined at $q\}$ containing p is one of the following sets (for a certain $\varepsilon > 0$):

$$W_{+\varepsilon} := \{q \in U(p) \mid 0 \leq x(q) \leq \varepsilon\}, \quad W_{-\varepsilon} := \{q \in U(p) \mid 0 \geq x(q) \geq -\varepsilon\},$$

$$\text{or} \ \ W_0 := \{q \in U(p) \mid x(q) = 0\}.$$

If the connected component of the set $\{q \in U(p) \mid B_1$ is not defined at $q\}$ containing p is $W_{+\varepsilon}$ or $W_{-\varepsilon}$, we are done by Proposition 5. We assume that the connected component of the set $\{q \in U(p) \mid B_1$ is not defined at $q\}$ containing p is W_0. Our goal is to prove that $\partial L/\partial y = 0$ for the points of this set.

Let us first observe that $da_{|q} = 0$ for every $q \in W_0$. Indeed, by Corollary 3, the function a accepts an extremum (minimum or maximum) at q.

Then, the second equation of (5) tells us that $\partial L/\partial y = 0$, i.e., L is constant on the set $\{q \in U(p) \mid x(q) = x(p)\}$. Proposition 6 and Lemma 5 are proved. $\quad\square$

REMARK 11. Since there is no essential difference between B_1 and B_2, the function L is constant on every connected component of the set $\{q \in T^2 \mid B_1$ or B_2 is not defined at $q\}$, as we claimed in the title of this section.

4.4. At a neighborhood of every point the metrics are Liouville, or one eigenvalue of $\tilde{F}^i_j$ is constant on the manifold.

Recall that integrals linear in momenta and Killing vector fields are closely related: the function $I = \alpha(x, y)p_x + \beta(x, y)p_y$ is an integral of the geodesic flow of g, if and only if the vector field $v = (\alpha, \beta)$ is a Killing vector field. Moreover, the mapping $I = \alpha(x, y)p_x + \beta(x, y)p_y \mapsto v = (\alpha, \beta)$ is coordinate-independent.

By Lemma 4, at every point of T^2 we have $ac \geq 0$.

LEMMA 6. *If there exists a point q such that at this point at least one of the forms B_1, B_2 is not defined, then one of the eigenvalues of $\tilde{F}^i_j$ is constant on the*

manifold.

PROOF.   We consider two sets:

$$W := \{p \in T^2 \mid B_1 \text{ and } B_2 \text{ are defined at } p\} \text{ and } T^2 \setminus W.$$

Assume $T^2 \setminus W \neq \varnothing$. At every point $s \in T^2$, we denote by $E_1(s) \leq E_2(s)$ the roots of the characteristic polynomial

$$\chi(t) := \det\left(\tilde{F}^i_j - t \cdot \delta^i_j\right)$$

at the point s counted with multiplicities. (By Corollary 2, the roots of the polynomials $\chi(t)$ are real). The functions E_1 and E_2 are at least continuous.

At the points of $T^2 \setminus W$, by Remark 10, we have $E_1 = E_2 = L/2$, where $L = \text{trace}(\tilde{F}^i_j)$. Then, by Remark 11, both functions E_1, E_2 are constant on each connected component of $T^2 \setminus W$.

Since W is open, and since $W \cup (T^2 \setminus W) = T^2$, in order to prove Lemma 6, it is sufficient to show that at least one of the functions E_1, E_2 is constant on every connected component of W.

We consider a point p such that at this point $ac > 0$, and denote by W_0 the connected component of W containing p.

At every point p_0 of W_0, we consider the vector fields $\partial/\partial x$, $\partial/\partial y$, where x, y are perfect coordinates in a neighborhood of p_0. Though the perfect coordinates are local coordinates, these vector fields are well defined at all points of W_0, see Remark 9. Moreover, at every point p_0 the vectors $\partial/\partial x$, $\partial/\partial y$ form a dual basis to the basis (B_1, B_2) in $T^*_{p_0} T^2$.

Let us show that the vector fields $\partial/\partial x$, $\partial/\partial y$ are complete on W_0. Since the basis (B_1, B_2) is dual to the basis $(\partial/\partial x, \partial/\partial y)$, it is sufficient to show that for every point q of the boundary $\partial W_0 := \overline{W}_0 \setminus W_0$ the integral $\int_p^q B_1 = \pm\infty$, or $\int_p^q B_2 = \pm\infty$. We consider admissible coordinates $\tilde{x}, \tilde{y}$ in a neighborhood of q. Without loss of generality, $\tilde{x}(q) = 0$ and $\tilde{a}(0) = 0$. As we explained in the proof of Lemma 5, the differential $d\tilde{a}_{|q} = 0$ implying $\tilde{a}(\tilde{x}) = \tilde{x}^2\alpha(x)$, where $\alpha(x)$ is a smooth function in a neigborhood of 0. Then, $\int_p^q B_1 = \text{const} + \int_{\tilde{x}_0}^0 (1/\sqrt{|\tilde{a}(s)|})ds = \text{const} \pm \int_{\tilde{x}_0}^0 (1/(|s|\sqrt{|\alpha(s)|}))ds = \pm\infty$.

Thus, the vector fields $\partial/\partial x$, $\partial/\partial y$ are complete on W_0.

We consider the local coordinates $u = (1/2)(x + y)$ and $v = (1/2)(x - y)$, and the corresponding vector fields $\partial/\partial u = (1/2)(\partial/\partial x + \partial/\partial y)$ and $\partial/\partial v = (1/2)(\partial/\partial x - \partial/\partial y)$. Since $\partial/\partial x$, $\partial/\partial y$ are complete, the vector fields $\partial/\partial u$, $\partial/\partial v$ are also complete.

The coordinates u, v are as in Proposition 1. Then, by Proposition 1, in the

coordinates (u, v), the metric and the integral have the form $(U(u) - V(v))(du^2 - dv^2)$ and $(U(u)p_v^2 - V(v)p_u^2)/(U(u) - V(v))$. Since $f = U(u) - V(v) > 0$, we have $U(u) > V(v)$.

Let us note that at every point of W_0, the local functions U and V have a clear geometric sense, and, therefore, are globally given at all points of W_0, and can be continuously prolonged up to the boundary. Indeed, in the coordinates (u, v) the matrix of $\tilde{F}_j^i$ is

$$\begin{pmatrix} -V(v) & 0 \\ 0 & -U(u) \end{pmatrix}.$$

Thus, $U = -E_1$ and $V = -E_2$.

Consider the action of the group $(\mathbf{R}^2, +)$ on W_0 generated by the vector fields $\partial/\partial u$ and $\partial/\partial v$. The action is well defined, since the vector fields commute and are complete. The action is transitive and locally-free. Then, W_0 is diffeomorphic to the torus, to the cylinder, or to $\mathbf{R}^2$. Since $T^2 \setminus W_0 \neq \varnothing$, W_0 can not be the torus.

Now suppose W_0 is a cylinder. Then, its boundary has at most two connected components. Each integral curve of at least one of the vector fields $\partial/\partial u$ and $\partial/\partial v$ is not closed. Without loss of generality, we assume that for every $p \in W_0$ the integral curve of the vector field $\partial/\partial v$ is not closed (i.e., it is the generator of the cylinder, or a standard winding on the cylinder. In the case the boundary of W_0 has two boundary components, the integral curve of $\partial/\partial v$ attracts to one component of the boundary for $t \to +\infty$, and to another component of the boundary for $t \to -\infty$).

For every boundary component, there exists a sequence of the points of any integral curve of $\partial/\partial v$ converging to a point of the boundary component. Indeed, the closure of W_0 is compact, so every sequence of points has a converging subsequence. We consider a converging subsequence of the sequence $\phi(0, p) = p$, $\phi(1, p), \phi(2, p), \phi(3, p), \ldots$ where $\phi : \mathbf{R} \times W_0 \to W_0$ denotes the flow of the vector field $\partial/\partial v$. Clearly, this sequence can not converge to a point of W_0. Then, it converges to a point of a boundary component. Since the function $E_1 = -U$ is constant along the integral curve, the value of E_1 on the boundary coincides with the value of E_1 at the point p. Similarly, the sequence points $\phi(0, p) = p$, $\phi(-1, p)$, $\phi(-2, p), \phi(-3, p), \ldots$ has a subsequence converging to another component of the boundary. Then, the value of E_1 on both components of the boundary coincides and is equal to the value of E_1 at every point of W_0. Then, the function E_1 is constant on W_0.

Let us use the same idea to show that W_0 can not be diffeomorphic to $\mathbf{R}^2$. Indeed, in this case ∂W_0 has one connected component, and the orbits of both

vector fields $\partial/\partial u$, $\partial/\partial v$ are not closed implying $U(u) = V(v)$ at every point, which clearly contradicts the assumptions.

Finally, one of the eigenvalues of $\tilde{F}_j^i$ is constant on W_0. Lemma 6 is proved.

□

4.5. If one eigenvalue of $\tilde{F}_j^i$ is constant, then there exists an integral linear in momenta.

By Lemma 6, we have the following two possibilities (not disjunkt):

(1) one of the eigenvalues of $\tilde{F}_j^i$ is constant,
(2) at every point $ac > 0$.

The goal of this section is to show that in the first case there exists an integral linear in momenta (at least on an appropriate double cover of the torus; later (in Section 5.2) we show that the integral exists already on the torus, see Corollary 6).

LEMMA 7. *Let one of the eigenvalues of $\tilde{F}_j^i$ is constant. Then, for a certain (at most, double) cover of the torus, the lift of the integral is a linear combination of the square of an integral linear in momenta and the lift of the Hamiltonian. Moreover, there exists no point q such that $F_{|T_q^*T^2} \equiv \text{const} \cdot H_{|T_q^*T^2}$.*

PROOF. Without loss of generality we can assume that one of the eigenvalues of $\tilde{F}_j^i$ is identically 0, otherwise we replace F by $F - \text{const} \cdot H$ for the appropriate const $\in \mathbf{R}$. Then, $\tilde{F}_j^i$ has rank at most 1.

Let $\tilde{F}_j^i \ne 0$ at a point q. We consider local coordinate (u, v) in $U(q)$ such that $\partial/\partial u$ lies in the kernel of $\tilde{F}_j^i$. In this coordinates, the (symmetric) matrix of $\tilde{F}^{ij}$ satisfies the equation

$$\begin{pmatrix} \tilde{F}^{11} & \tilde{F}^{12} \\ \tilde{F}^{21} & \tilde{F}^{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

implying $\tilde{F}^{11} = \tilde{F}^{12} = \tilde{F}^{21} = 0$. Then, in this coordinates $F = \tilde{F}^{22}p_v^2$ implying that the integral is locally the square of the function $\sqrt{F^{22}}p_v$, if $\tilde{F}^{22} > 0$, or $\sqrt{-F^{22}}p_v$, if $\tilde{F}^{22} < 0$. Then, the (linear in momenta) function $\sqrt{F^{22}}p_v$ (if $\tilde{F}^{22} > 0$) or $\sqrt{-F^{22}}p_v$ (if $\tilde{F}^{22} < 0$) in a local integral linear in momenta, and $\sqrt{F^{22}}(\partial/\partial v)$ (if $\tilde{F}^{22} > 0$) or $\sqrt{-F^{22}}(\partial/\partial v)$ (if $\tilde{F}^{22} < 0$) is a Killing vector field.

Let us show that the points such that $\tilde{F}_j^i = 0$ are isolated. Indeed, otherwise there exist two such points, say p_1 and p_2, in a sufficiently small neigborhood U. For every point q of this neighborhood we consider the geodesics connecting p_i with q. For almost every q, the geodesics intersect transversally at the point q, see Figure 7.
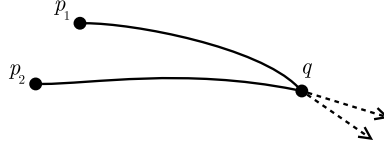
Figure 7. The geodesic connecting the points p_i with the point q, and their tangent vectors at the point q. At almost every q, the tangent vectors of the geodesics at the point q are linearly independent.

We denote by ξ_1, ξ_2 vector-momenta of these geodesics at the point q. Since $F_{|T^*_{p_i} T^2} \equiv 0$, we have $F(\xi_1) = F(\xi_2) = 0$ implying $F_{|T^*_{p_i} T^2} \equiv 0$. Since this is fulfilled for almost every point q of a small neighborhood, the integral F vanishes identically on two linearly independent vector-momenta, which impossible for the integral $F = \tilde{F}^{22} p_v^2$ (for $\tilde{F}^{22} \neq 0$).

Thus, the points q such that $F_{|T^*_q T^2} \equiv 0$ are isolated. Then, the set $N := \{q \in T^2 \mid F_{|T^*_q T^2} \equiv 0\}$ is discrete. Hence, the set $T^2 \setminus N = \{q \in T^2 \mid F_{|T^*_q T^2} \not\equiv 0\}$ is connected implying that $\tilde{F}^{ij}$ is zero or positive semi-definite everywhere, or zero or negative semi-definite everywhere. Without loss of generality we can think that $\tilde{F}^{ij}$ is zero or positive semi-definite everywhere, otherwise we replace F by $-F$.

Let us show that in *a small neighborhood* $U(p)$ *of every point* p *there exists precisely two integrals linear in momenta such that*

(a) *they are smooth at every points* $q \notin N$, *and*
(b) *the square of each of these integrals is equal to* F.

If $p \notin N$, the statement is evident: in the constructed above local coordinates u, v the integrals are $\pm\sqrt{F} = \pm\sqrt{\tilde{F}^{22} p_v^2} = \pm p_v \sqrt{\tilde{F}^{22}}$. Since every neighborhood has a point from $T^2 \setminus N$, in a neighborhood of every point there exist at most two such integrals. Thus, in order to prove the statement above we need to prove that in a neighborhood of every point from N there exists at least one such integral (the second one will be minus the first).

Let $p \in N$. We take a small neighborhood $U(p)$ homeomorphic to the disk, and consider $U(p) \setminus \gamma$, where γ is a geodesics starting at the point p, see Figure 8. Since $U(p) \setminus \gamma$ is simply-connected and contains no point from N, on $U(p) \setminus \gamma$ there exists an integral $I = \alpha(x, y)p_x + \beta(x, y)p_y$ linear in momenta such that $I^2 = F$. We consider the Killing vector field $v := (\alpha, \beta)$ corresponding to this integral. Since the value of this integral on each geodesic passing through p is zero, the Killing vector field (α, β) is orthogonal to geodesics containing p. Then, the qualitative behaviour of the vector field at the points of a small circle around p is as on Figure 8. Indeed, they are tangent to the level curves of the geodesic distance function to the point p, which are hyperbolas (one of them is on Figure
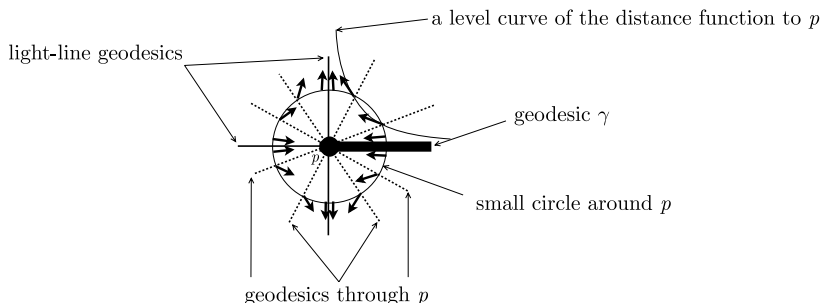
8) and light-line geodesics though p.

Figure 8.   Qualitative behaviour of the vector field v
at the points of a small circle around p.

We see that the vector field v is oriented in the same direction on the different sides of γ, implying that one can prolong the vector field to $U(p) \setminus \{p\}$. Then, there exists the integral I linear in momenta such that $I^2 = F$ in $U(p) \setminus p$ as we claimed.

Since in a small neighborhood $U(p)$ of every point p there exists precisely two integrals linear in momenta satisfying the conditions (a), (b) above, an integral linear in momenta satisfying the conditions (a), (b) above exists on T^2, or on the double cover of T^2. The first statement of Lemma 7 is proved.

Let us prove the second statement of Lemma 7: let us show that the set N is actually empty. Indeed, the index of the vector field v is negative at the points of N, see Figure 8, and is zero at all other points. But the sum of the indexes of any vector field on the torus must be zero.

Thus, there exists an integral linear in momenta satisfying the condition (b) above on the torus, or on the double cover of the torus. Lemma 7 is proved. $\square$

REMARK 12.   We would like to point out that, in order to define the index of a vector field v at a point q such that $v(q) = \vec{0}$, we actually do not need that the vector field is smooth at the point q. In fact, we did not find the local proof that the vector field v we used in the proof of Lemma 7 is smooth at the points of N (of cause, it is smooth at all other points; since later we proof that $v \neq \vec{0}$ at all points we obtain finally that it is smooth). Indeed, we can define the index of a zero of a vector field as the winding number of the vector field on a small circle around the point. This number is clearly independent of the choice of the small circle, and is used in the proof of the fact that the Euler characteristic of a surface is minus the sum of the indexes of all zeros of a vector field (which is used to show that N is empty set).

COROLLARY 4. *Let v be a nontrivial Killing vector field of a pseudo-Riemannian metric g on the torus T^2. Then, there is no point $p \in T^2$ such that $v = 0$ at p.*

PROOF. In the Riemannian case (and, therefore, if g has signature $(-,-)$), Corollary 4 is evident. Indeed, the Killing vector field preserves the complex structure corresponding to the metric, and is therefore holomorphic (with respect to the complex structure). By the Abel Lemma, it has no zeros.

Let now the signature of the metric be $(+,-)$. We consider the integral linear in momenta corresponding to the Killing vector field. It vanishes at the points where the Killing vector field vanishes. The square of this integral is an integral quadratic in momenta. If the linear integral is $\alpha(x,y)p_x + \beta(x,y)p_y$, its square is $F = \alpha^2 p_x^2 + 2\alpha\beta p_x p_y + \beta^2 p_y^2$, and the matrix $\tilde{F}^{ij}$ (such that $F = \sum_{i,j} \tilde{F}^{ij} p_i p_j$) is

$$\begin{pmatrix} \alpha^2 & \alpha\beta \\ \alpha\beta & \beta^2 \end{pmatrix}.$$

We see that its rang is ≤ 1 implying that 0 is a (constant) eigenvalue of $\tilde{F}^i_j$. Then, by Lemma 7, there exists no point such that $\alpha = \beta = 0$ implying there exists no point such that $v = 0$. Corollary 4 is proved. $\square$

REMARK 13. Actually, our final goal is to prove that a nontrivial integral linear in momenta exists already on the torus (and not on the double cover of the torus). We will do it later, in Section 5. By Corollary 6 (whose proof does not use Theorem 2, so no logical loop appears), the integral linear in momenta satisfying the condition (b) above exists already on the torus.

4.6. Proof of Theorem 2 under the assumption that the vector fields V_1, V_2 exist on the whole torus.

Let the geodesic flow of g of signature $(+,-)$ on the torus admits an integral quadratic in momenta; assume the integral is not a linear combination of the square of an integral linear in momenta and the Hamiltonian. As everywhere in Section 4, we assume that the vector fields V_1, V_2 satisfying conditions (A,B,C) from Section 3.1 exist on the whole torus. By Lemmas 6, 7, at every point of the manifold $ac > 0$.

We consider the vector fields $\partial/\partial u$, $\partial/\partial v$ from the proof of Lemma 6. These vector fields commute and never vanish. Then, they generate a locally free action of $(\mathbf{R}^2, +)$ on T^2. The stabilizer G of this actions is a subgroup of $(\mathbf{R}^2, +)$ with the following properties: it is

- discrete, and
- the quotient space is compact.

Then, it is a lattice, i.e., $G = \{k \cdot \xi + m \cdot \eta \mid (k, m) \in \mathbf{R}\}$ for certain linearly independent vectors ξ, η. Then, there exists a natural diffeomorphism $\phi : \mathbf{R}^2/G \to T^2$. We identify $\mathbf{R}^2/G$ and T^2 by this diffeomorphism and consider the lift of the metric and the integral to $\mathbf{R}^2$. By Proposition 1, in the coordinate system (u, v) on $\mathbf{R}^2$, the metric and the integral are $(U(u) - V(v))(du^2 - dv^2)$ and $\pm(U(u)p_v^2 - V(v)p_u^2)/(U(u) - V(v))$, i.e., are as in Model Example 1. Since the metric and the integral are preserved by the lattice, the functions U and V are preserved by the lattice as well. Thus, the metric on $\mathbf{R}^2/G$ are as in Model Example 1. Theorem 2 is proved (under the additional assumption that the vector fields V_1, V_2 exist on the whole torus).

5.   Proof of Theorem 4, final step of the proof of Theorem 2, and proof of Theorem 3.

5.1.   Flat metrics of signature $(+, -)$ on T^2, and their Killing vector fields.

By the Gauss-Bonnet Theorem, a metric of constant curvature on the torus is flat (= has zero curvature). Recall that by the *standard flat torus* we consider $(\mathbf{R}^2/G, dxdy)$, where (x, y) are the standard coordinates on $\mathbf{R}^2$, and G is a lattice generated by two linearly independent vectors.

It is well-known that every torus (T^2, g) such that the metric g is flat and has signature $(+, -)$ is isometric to a standard one. Indeed, by [**12**], the flat torus is geodesically complete implying its universal cover is isometric to $(\mathbf{R}^2, dxdy)$. The fundamental group of the torus, $(\mathbf{Z}^2, +)$, acts on $(\mathbf{R}^2, dxdy)$. The action is isometric, free, and discrete. It is easy to see that every orientation-preserving isometry of $(\mathbf{R}^2, dxdy)$ without fixed points is a translation. Then, $\mathbf{Z}^2$ acts as a lattice generated by two linearly independent vectors, and (T^2, g) is isometric to a certain $(\mathbf{R}^2/G, dxdy)$.

The space of Killing vector fields of $(\mathbf{R}^2, dxdy)$ is a 3-dimensional linear vector space generated by two translations $(1, 0) = \partial/\partial x$ and $(0, 1) = \partial/\partial y$, and the pseudo-rotation $(-x, y) = -x(\partial/\partial x) + y(\partial/\partial y)$. Then, the space of Killing vector fields on the flat torus $(\mathbf{R}^2/G, dxdy)$ is two-dimensional and is generated by the Killing vector fields $(1, 0) = \partial/\partial x$ and $(0, 1) = \partial/\partial y$. Note that, depending on the values of the constants $(\mathrm{const}_1, \mathrm{const}_2) \neq (0, 0)$, every integral curve of the Killing vector field $\mathrm{const}_1 \cdot \partial/\partial x + \mathrm{const}_2 \cdot \partial/\partial y$ is either a closed curve, or an everywhere dense winding on the torus.

5.2.   Killing vector fields on the torus of nonconstant curvature.

PROPOSITION 8.   *Let the metric g of nonconstant curvature on the torus T^2 admits a nonzero Killing vector field v. Then, there exists a free action of the group $(\mathbf{R}/\mathbf{Z}, +)$ on the torus such that the infinitesimal generator of this action is*

proportional to the Killing vector v with a constant coefficient of proportionality.

PROOF.   We denote by R the scalar curvature of g. By Corollary 4, the vector field v has no zeros on T^2. Then, the Killing vector field generates a locally-free action of the group $(\boldsymbol{R}, +)$. Let us prove that the Killing vector field (after an appropriate scaling) actually generates the action of the group $SO_1 = \boldsymbol{R}/\boldsymbol{Z}$ without fixed points.

Indeed, take a point p such that $dR \neq 0$, and consider the orbit of the Killing vector field containing the point. Since the flow of a Killing vector field preserves the curvature, at every point q of the orbit we have $R(q) = R(p)$ and $dR \neq 0$. Then, the orbit coincides with the connected component of the set $\{q \in T^2 \mid R(q) = R(p)\}$ containing the point p implying it is a circle.

We consider the action $\rho : \boldsymbol{R} \times T^2 \to T^2$ of the group $(\boldsymbol{R}, +)$ generated by the flow of the vector field. Since the orbit through p is a circle, for certain $t_0 > 0$ we have $\rho(t_0, p) = p$ and for no $t \in (0, t_0)$ $\rho(t, p) = p$. Without loss of generality we can think that $t_0 = 1$, otherwise we replace v by $t_0 \cdot v$.

Since the action ρ is isometric and orientation-preserving, it commutes with the exponential mapping $\exp : TT^2 \to T^2$. Then, the mapping $\rho(1, \cdot) : T^2 \to T^2$ is identity in a small neighborhood of the point p. Since the point p was arbitrary, for every point $q \in T^2$ we have $\rho(1, q) = q$ and $\rho(t, q) \neq q$ for $t \in (0, 1)$. Thus, the action of the group $(\boldsymbol{R}/\boldsymbol{Z}, +)$ is well-defined, and has no fixed points. Proposition 8 is proved. $\square$

COROLLARY 5.   *Let v be a nonzero Killing vector field on the torus (T^2, g), where g has signature $(+, -)$. Then, there exists no involution $\sigma : T^2 \to T^2$ without fixed point that preserves the orientation and the metric, and sends the vector field v to $-v$.*

PROOF.   If the metric g has constant curvature, as we have recalled in Section 5.1, the torus is isometric to $(\boldsymbol{R}^2/G, dxdy)$ for a lattice G generated by two linearly independent vectors ξ and η, and the Killing vector field is $\mathrm{const}_1 \cdot \xi + \mathrm{const}_2 \cdot \eta$ for $(\mathrm{const}_1, \mathrm{const}_2) \neq (0, 0)$. The involution σ without fixed points that preserves the orientation and the metric induces an isometry of $(\boldsymbol{R}^2, dxdy)$ without fixed points that preserves the orientation and the metric. Such isometry is a translation and can not send the Killing vector field $\mathrm{const}_1 \cdot \xi + \mathrm{const}_2 \cdot \eta$ to $-(\mathrm{const}_1 \cdot \xi + \mathrm{const}_2 \cdot \eta)$. Corollary 5 is proved under the assumption that g has constant curvature.

Assume now that the curvature of g is not constant. Then, by Proposition 8, the Killing vector field (after the appropriate scaling) generates a free action of $(\boldsymbol{R}/\boldsymbol{Z}, +)$ on T^2. We consider the quotient space $T^2/_{(\boldsymbol{R}/\boldsymbol{Z})}$. Since the action of $\boldsymbol{R}/\boldsymbol{Z}$ on T^2 is free, the quotient space is a 1-dimensional closed manifold, i.e., is

diffeomorphic to S^1. The orientation of the torus induces the orientation on S^1.

The involution σ of the torus preserves the action, the orientation, and sends v to $-v$. Then, it inverses the orientation of $S^1 = T^2/_{(\boldsymbol{R}/\boldsymbol{Z})}$. Then, it has a fix point. We consider the orbit of $\boldsymbol{R}/\boldsymbol{Z}$ corresponding to this point. The involution σ preserves this orbits and changes the direction of the vector field v on this orbit. Then, it has a fixed point which contradicts the assumptions. The contradiction proves Corollary 5. $\qquad\square$

COROLLARY 6. *Let F be a nontrivial integral quadratic in momenta for the geodesic flow of the metric g on the torus T^2 and $\pi : \widetilde{T}^2 \to T^2$ be a double cover of T^2. Assume the lift of the integral to $\widetilde{T}^2$ is a linear combination of the square of a function linear in momenta and the lift of the Hamiltonian. Then, the integral F is a linear combination of the square of an integral linear in momenta and the Hamiltonian.*

PROOF. We consider the involution $\sigma : \widetilde{T}^2 \to \widetilde{T}^2$ corresponding to the cover: $\sigma(\tilde{p}) = \tilde{q}$ if $\pi(\tilde{p}) = \pi(\tilde{q})$ and $\tilde{p} \ne \tilde{q}$. The involution preserves the lift of the Hamiltonian and of the integral.

We consider the function $I : T^*\widetilde{T}^2 \to \boldsymbol{R}$ linear in momenta such that $F = \mathrm{const}_1 \cdot H + \mathrm{const}_2 \cdot I^2$, where H and F denote the lift of the Hamiltonian and the integral. Since the integral F is nontrivial, $\mathrm{const}_2 \ne 0$ implying I is a nontrivial integral (linear in momenta). We consider the Killing vector field v corresponding to the integral. Since the involution σ preserves H and F, it preserves $I^2 = (1/\mathrm{const}_2)(F - \mathrm{const}_1 \cdot H)$. Since by Proposition 8 the vector field v vanishes at no point, either $d\sigma(v) = v$ for all points, or $d\sigma(v) = -v$ for all points. The second possibility is forbidden by Corollary 5. Then, $d\sigma(v) = v$ implying the integral I on $\widetilde{T}^2$ induces an integral I (linear in momenta) on $T^2 = \widetilde{T}^2/\sigma$ such that, on T^2, $F = \mathrm{const}_1 \cdot H + \mathrm{const}_2 \cdot I^2$. Corollary 6 is proved. $\qquad\square$

5.3. Proof of Theorem 4.

Let F be an integral linear in momenta of the geodesic flow of a metric g on the torus T^2. We denote by v the corresponding Killing vector field. We consider the action ρ of $(\boldsymbol{R}/\boldsymbol{Z}, +)$ on T^2 from Proposition 8, the quotient space $T^2/_{(\boldsymbol{R}/\boldsymbol{Z})}$ diffeomorphic to the circle, and the tautological projection $\pi : T^2 \to T^2/_{(\boldsymbol{R}/\boldsymbol{Z})} = S^1$. Let us construct a coordinate system ($x \in \boldsymbol{R}$ mod 1, $y \in \boldsymbol{R}$ mod 1) on T^2. We parametrize S^1 by ($Y \in \boldsymbol{R}$ mod 1), and put $y(q) := Y(\pi(q)) \in \boldsymbol{R}/\boldsymbol{Z}$). In order to construct the coordinate x, we consider a smooth section $c : S^1 \to T^2$ of the bundle. By definition of the section, for every $q \in T^2$ there exists a unique $t \in (\boldsymbol{R}$ mod 1) such that $\rho(t, q) \in \mathrm{image}(c)$. We put $x(q) = -t$.

By construction, in this coordinates, the vector field v is $\partial/\partial x$, and the corresponding integral linear in momenta is p_x. Let in this coordinates the metric

g is given by $g = K(x,y)dx^2 + 2L(x,y)dxdy + M(x,y)dy^2$. Since the metric has signature $(+,-)$, we have $KM - L^2 = \det\left(\begin{smallmatrix} K & L \\ L & M \end{smallmatrix}\right) < 0$. Thus, in order to prove Theorem 4, it is sufficient to show that the functions K, L, M are functions of the variable y only, i.e., $\partial K/\partial x = \partial L/\partial x = \partial M/\partial x = 0$.

We denote by $k(x,y), l(x,y), m(x,y)$ the components of the inverse matrix to g:

$$\begin{pmatrix} k & l \\ l & m \end{pmatrix} = \begin{pmatrix} K & L \\ L & M \end{pmatrix}^{-1}.$$

Evidently, $2H = k(x,y)p_x^2 + 2l(x,y)p_xp_y + m(x,y)p_y^2$, and the condition $\{F, 2H\} = 0$ reads

$$0 = \left\{ p_x, k(x,y)p_x^2 + 2l(x,y)p_xp_y + m(x,y)p_y^2 \right\}$$
$$= \frac{\partial k}{\partial x}p_x^2 + 2\frac{\partial l}{\partial x}p_xp_y + \frac{\partial m}{\partial x}p_y^2,$$

i.e., is equivalent to the condition $\partial k/\partial x = \partial l/\partial x = \partial m/\partial x = 0$. Then, the coefficients k, l, m depend on the variable y only, implying that the coefficients K, L, M also depend on the variable y only. Theorem 4 is proved. $\square$

5.4. Proof of Theorem 2 under the assumption that the vector fields V_1, V_2 do not exist on the torus.

We assume that the geodesic flow of the metric g on T^2 admits a nontrivial integral F quadratic in momenta that is not a linear combination of the Hamiltonian and an integral linear in momenta. Assume the vector fields V_1, V_2 satisfying assumptions (A,B,C) from Section 3.1 do not exist. We consider the double cover $\pi : \widetilde{T}^2 \to T^2$ such that V_1, V_2 satisfying (A,B,C) exist on $\widetilde{T}^2$. Then, by the proved part of Theorem 2, the lift of the metric to $\widetilde{T}^2$ is as in Model Example 1 (we identify $\widetilde{T}^2$ with $\mathbf{R}^2/G$ and the lift $\tilde{g}$ of the metric with the metric from Model Example 1). On the torus $\widetilde{T}^2$, the only possibility for the vector fields V_1, V_2 are (we consider the standard orientation on $\mathbf{R}^2$):

$$V_2 = \lambda\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right), \quad V_1 = \mu\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right),$$

where λ and μ are smooth functions on $\widetilde{T}^2$ such that for every $\tilde{p} \in \widetilde{T}^2$ we have $\lambda(\tilde{p})\mu(\tilde{p}) > 0$, and x, y are the standard coordinates on $\mathbf{R}^2$.

We consider the involution σ corresponding to the cover π, that it $\sigma(\tilde{p}) = \tilde{q}$ if and only if $\pi(\tilde{p}) = \pi(\tilde{q})$ and $\tilde{p} \neq \tilde{q}$. Since by assumptions the vector fields V_1, V_2

do not exist on T^2, and the involution preserves the orientation, the metric $\tilde{g}$, and the lift of the integral, we have

$$d\sigma\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) = -\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) \ \text{ and } \ d\sigma\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) = -\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)$$

implying

$$d\sigma\left(\frac{\partial}{\partial x}\right) = -\frac{\partial}{\partial x} \ \text{ and } \ d\sigma\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial y}. \tag{19}$$

But on the torus $\boldsymbol{R}^2/G$ there is no involution with no fixed point with the property (19). The contradiction shows that the situation assumed in this section, namely that the vector fields V_1, V_2 do not exist on T^2, is impossible. Theorem 2 is proved.

$\square$

5.5.   Proof of Theorem 3.

We assume that g is a metric of signature $(+, -)$ on the Klein bottle K^2 whose geodesic flow admits an integral quadratic in momenta. We also assume that the lift of the integral to the oriented cover is not a linear combination of the lift of the Hamiltonian and the square of a function linear in momenta. Our goal is to prove that (K^2, g) is as in Model Example 2.

We consider the oriented cover $\pi : T^2 \to K^2$, and the lift of the metric and the integral to T^2. They satisfy the assumptions in Theorem 2. Hence we can think that T^2, the lift of the metric, and the lift of the integral are as Model Example 1:

$$T^2 = \boldsymbol{R}^2/G, \quad g = (X(x) - Y(y))(dx^2 - dy^2), \ \text{ and } \ F = \frac{X(x)p_y^2 - Y(y)p_x^2}{X(x) - Y(y)},$$

where $G = \{k \cdot \xi + m \cdot \eta \mid k, m \in \boldsymbol{Z}\}$.

Next, consider the universal cover $\tilde{\pi} := \pi \circ P : \boldsymbol{R}^2 \to K^2$, where P is the canonical projection from $\boldsymbol{R}^2$ to $\boldsymbol{R}^2/G$. We consider the action of the fundamental group of the Klein bottle on $\boldsymbol{R}^2$ corresponding to $\tilde{\pi}$. Recall that the fundamental group of K^2 is generated by two elements, say A and B, satisfying the relation $ABA^{-1}B = \boldsymbol{1}$:

$$\pi_1(K^2) = \langle A, B | ABA^{-1}B = \boldsymbol{1} \rangle. \tag{20}$$

This action has the following properties:

(a) It preserves the metric and the integral,

(b) It is free and discrete.

Let us show that the condition (a) implies the condition

(a′) For every element $\alpha \in \pi_1(K^2)$ we have

$$d\alpha\left(\frac{\partial}{\partial y}\right) = \pm\frac{\partial}{\partial y}, \quad d\alpha\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial x}. \tag{21}$$

Indeed, since at every point $(x, y) \in \boldsymbol{R}^2$ the factor $X(x) - Y(y) \neq 0$, and since every nonempty level $\{X = \mathrm{const}_1\}$ intersects with every nonempty level $\{Y = \mathrm{const}_2\}$, without loss of generality we can think that $X(x) > Y(y)$ for all $(x, y) \in \boldsymbol{R}^2$.

Now, in the coordinates x, y, the matrix of F_j^i is $\begin{pmatrix} -Y(y) & \\ & -X(x) \end{pmatrix}$, so $\tilde{F}_j^i$ has eigenvalues $-X(x), -Y(y)$.

Since the action preserves the metric and the integral, it preserves the eigenvalues X, Y and the eigenspaces $\mathrm{span}(\partial/\partial y)$ and $\mathrm{span}(\partial/\partial x)$ of this eigenspaces. Since $g(\partial/\partial x, \partial/\partial x) = X(x) - Y(y)$, and α preserves X and Y, we have that $g(d\alpha(\partial/\partial x), d\alpha(\partial/\partial x)) = X(x) - Y(y)$ implying $d\alpha(\partial/\partial x) = \pm\partial/\partial x$. The proof that $d\alpha(\partial/\partial y) = \pm\partial/\partial y$ is similar.

Thus, the action preserves the standard flat metric $dx^2 + dy^2$ on $\boldsymbol{R}^2$. Then, the fundamental group of K^2 acts as a crystallographic group. From the classification of crystallographic groups [**4**, Section 1.7], it follows that every action of the group (20) on $\boldsymbol{R}^2$ satisfying (a′, b) is generated by A, $A(x, y) = (x + c, -y)$ and B, $B(x, y) = (x, y + d)$ for certain $c \neq 0 \neq d$, i.e., is as the Model Example 2. Theorem 3 is proved. □

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