J. Math. Soc. Japan
Vol. 64, No. 1 (2012) pp. 91–106
doi: 10.2969/jmsj/06410091

The gap hypothesis for finite groups which have an abelian quotient group not of order a power of 2

Dedicate to Professor Krzysztof Pawałowski on his 60th birthday

By Toshio Sumi

(Received June 23, 2010)

Abstract. For a finite group G, an $\mathscr{L}(G)$-free gap G-module V is a finite dimensional real G-representation space satisfying the two conditions: (1) $V^L = 0$ for any normal subgroup L of G with prime power index. (2) $\dim V^P > 2\dim V^H$ for any $P < H \le G$ such that P is of prime power order. A finite group G not of prime power order is called a gap group if there is an $\mathscr{L}(G)$-free gap G-module. We give a necessary and sufficient condition for that G is a gap group for a finite group G satisfying that $G/[G,G]$ is not a 2-group, where $[G,G]$ is the commutator subgroup of G.

1. Introduction.

Let G be a finite group and p a prime. In this paper we regard the trivial group as a p-group. We denote by $\mathscr{P}_p(G)$ the set of p-subgroups of G, let $O^p(G)$, called the Dress subgroup of type p, be the smallest normal subgroup of G whose index is a power of p, possibly 1, and denote by $\mathscr{L}_p(G)$ the set of subgroups L of G which contain $O^p(G)$. We denote by $\pi(G)$ the set of prime divisors of the order of G. Set

$$\mathscr{P}(G) = \bigcup_{p\in\pi(G)} \mathscr{P}_p(G) \quad \text{and} \quad \mathscr{L}(G) = \bigcup_{p\in\pi(G)} \mathscr{L}_p(G).$$

Let V be a G-module, which means a finite dimensional real G-representation space. For a set $\mathscr{F}$ of subgroups of G, we say that V is *$\mathscr{F}$-free* if $V^H = \{0\}$ for all $H \in \mathscr{F}$. An $\mathscr{L}(G)$-free G-module V is called a *gap G-module* if

$$\dim V^P > 2\dim V^H$$

for all pairs (P, H) of subgroups of G with $P \in \mathscr{P}(G)$ and $P < H$. The inequality arose from equivariant surgery theory [**7**], [**8**], [**2**], [**5**]. A finite group G not of

2000 *Mathematics Subject Classification.* Primary 57S17; Secondary 20C15.
Key Words and Phrases. gap group, gap module, representation.

prime power order is called a *gap group* if there is a gap G-module. The purpose of this paper is to study which finite groups are gap groups.

Let us recall that the following groups are gap groups:

- Any nontrivial finite perfect group [**4**].
- Any finite group with $\mathscr{P}(G) \cap \mathscr{L}(G) = \emptyset$ and $O^2(G) = G$ [**3**].
- The symmetric group S_n of degree $n \geq 6$ [**1**].
- Any finite group with $\mathscr{P}(G) \cap \mathscr{L}(G) = \emptyset$ such that $O^p(G) \neq G$ for at least two odd primes p [**6**].
- Any extension of a gap group by a group of odd order [**6**].
- Any finite group which has a gap quotient group [**10**].

In this paper we give a characterization of gap groups with $O^{p_0}(G) \neq G$ for a unique odd prime p_0. The main theorems are as follows.

Theorem 1.1. *Suppose that $\mathscr{P}(G) \cap \mathscr{L}(G) = \emptyset$, $O^2(G) \neq G$ and $O^{p_0}(G) \neq G$ for a unique odd prime p_0. Then G is a gap group if and only if every subgroup K with $O^2(G) \lhd K \leq G$ and $[K : O^2(G)] = 2$ is a gap group.*

Theorem 1.2. *Suppose that $\mathscr{P}(G) \cap \mathscr{L}(G) = \emptyset$, $[G : O^2(G)] = 2$ and $O^{p_0}(G) \neq G$ for a unique odd prime p_0. Then G is a gap group if and only if there is an element of G outside $O^2(G)$ of order 2^a, $a \geq 2$ or*

$$\sum \frac{2}{|C_G(g)/O^2(C_G(g))|} < 1,$$

where the sum is taken over all representatives of conjugacy classes (g) of elements g of G outside $O^2(G)$ of order 2 such that $O^2(C_G(g))$ is a p_0-group.

This paper is organized as follows. In Section 2, we consider $\mathscr{L}(G)$-free G-modules V satisfying that

$$\dim V^P \geq 2 \dim V^H$$

for all pairs (P, H) of subgroups of G with $P \in \mathscr{P}(G)$ and $P < H$. For a gap G-module W, the complexification of W satisfies the gap hypothesis:

$$\dim_{\boldsymbol{C}} V^P > 2 \dim_{\boldsymbol{C}} V^H$$

for all pairs (P, H) of subgroups of G with $P \in \mathscr{P}(G)$ and $P < H$. To show that G is a gap group it suffices to show that there is an $\mathscr{L}(G)$-free complex G-module W satisfying the gap hypothesis. In Section 3, we discuss gap complex modules by decomposition of submodules in the complex representation ring and give the proof of Theorem 1.1. In Section 4, we consider modules V induced from modules over cyclic subgroups C and estimate the integer

$$\dim_{\boldsymbol{C}} V^P - 2\dim_{\boldsymbol{C}} V^H,$$

which corresponds with the number of the fixed point set $(P\backslash G/C)^{H/P}$ if $[H : P] = 2$. Finally, in Section 5, we prove Theorem 1.2 and show its corollaries.

2. Nonnegative modules.

We denote by $\mathscr{D}(G)$ the set of all pairs (P, H) of subgroups of G such that $P < H \leq G$ and $P \in \mathscr{P}(G)$. For a G-module V, we define a function $d_V : \mathscr{D}(G) \to \boldsymbol{Z}$ by

$$d_V(P, H) = \dim V^P - 2\dim V^H.$$

We say that V is *positive* (resp. *nonnegative*) at (P, H) if $d_V(P, H)$ is positive (resp. nonnegative), and that V is *positive* (resp. *nonnegative*) on $\mathscr{E}$ if V is positive (resp. nonnegative) at any element of $\mathscr{E}$ for a subset $\mathscr{E}$ of $\mathscr{D}(G)$. Further we briefly say that V is *positive* (resp. *nonnegative*) if V is positive (resp. nonnegative) on $\mathscr{D}(G)$. Then an $\mathscr{L}(G)$-free G-module V is a gap module if and only if V is positive.

Let $\boldsymbol{R}[G]$ be the real regular representation space. For a finite group G, we define the G-module

$$V(G) = (\boldsymbol{R}[G] - \boldsymbol{R}) - \bigoplus_{p\in\pi(G)} \big(\boldsymbol{R}[G/O^p(G)] - \boldsymbol{R}\big).$$

If G is a group of prime power order, then $V(G) = \{0\}$ holds. Laitinen and Morimoto [**3**] show that $V(G)$ is an $\mathscr{L}(G)$-free nonnegative G-module.

PROPOSITION 2.1 ([**3**, Theorem 2.3]). *Let $(P, H) \in \mathscr{D}(G)$. $d_{V(G)}(P, H) = 0$ implies that $P \in \mathscr{L}(G)$ or $O^q(G)P = G$ for any odd prime q and $[H : P] = [O^2(G)H : O^2(G)P] = 2$.*

We set

$$\mathscr{D}^2(G) = \big\{(P, H) \in \mathscr{D}(G) \mid [H : P] = [O^2(G)H : O^2(G)P] = 2 \text{ and } O^q(G)P = G \text{ for all odd primes } q\big\}.$$

The induced G-module $\mathrm{Ind}_K^G V$ of a nonnegative K-module V is nonnegative on $\mathscr{D}^2(G)$. We construct a gap module by assembling nonnegative modules.

For an element x of G, let $\psi(x)$ be the set of odd primes q such that there exists a subgroup N of G satisfying $x \in N$ and $O^q(N) \neq N$. For a finite group G, we define the subset $E_2(G)$ of $G \smallsetminus O^2(G)$ as the set of elements x of order 2 such that $|\psi(x)| > 1$ or $O^2(C_G(x)) \notin \mathscr{P}(G)$, and define $E_4(G)$ as the set of elements x of $G \smallsetminus O^2(G)$ of order a power of 2 greater than 2 with $|\psi(x)| > 0$ [**11**]. Recall that the trivial group is a p-group by our convention. The sets $E_2(G)$ and $E_4(G)$ are invariant subsets of G with respect to the conjugation by elements of G. Set $E(G) = E_2(G) \cup E_4(G)$.

Proposition 2.2 ([**11**, Propositions 4.1, 4.2 and 4.5]). *Suppose that $\mathscr{P}(G) \cap \mathscr{L}(G) = \emptyset$. For each $h \in E(G)$, there is an $\mathscr{L}(G)$-free nonnegative G-module W_h such that W_h is positive at $(P, H) \in \mathscr{D}^2(G)$ if $H \smallsetminus P$ meets the conjugacy class (h) of h in G.*

Note that $G \smallsetminus (O^2(G) \cup E(G))$ is an invariant subset of G with respect to the conjugation by elements of G. Let S be the set of conjugacy classes of elements of 2-power order which do not lie in $O^2(G) \cup E(G)$. We denote by $\mathscr{E}^2(G)$ the subset of $\mathscr{D}^2(G)$ consisting of (P, H) such that $E(G) \cap H \smallsetminus P$ is empty and $O^2(C_G(h))$ is a subgroup of P if (x) contains an element h of $H \smallsetminus P$ for any $(x) \in S$ and in addition P is a q-group if $\psi(h) = \{q\}$.

Theorem 2.3. *If $\mathscr{P}(G) \cap \mathscr{L}(G) = \emptyset$ then there exists an $\mathscr{L}(G)$-free nonnegative G-module which is positive on $\mathscr{D}(G) \smallsetminus \mathscr{E}^2(G)$.*

Proof. Let T_1 be a complete set of all representatives of conjugacy classes of elements of $E(G)$. By Proposition 2.2, for each $h \in T_1$, we take an $\mathscr{L}(G)$-free nonnegative G-module W_h such that W_h is positive at $(P, H) \in \mathscr{D}^2(G)$ if $H \smallsetminus P$ meets the conjugacy class (h) of h in G. Let T_2 be a complete set of all representatives of conjugacy classes of elements g of $G \smallsetminus (O^2(G) \cup E(G))$ of order 2 such that $C_G(g)$ is not a 2-group, and put $H_g = O^2(C_G(g))\langle g \rangle$ for $g \in T_2$. Note that $O^2(C_G(g)) \in \mathscr{P}(G)$ and H_g is not a 2-group for $g \in T_2$ since $g \notin E_2(G)$. Put

$$V = V(G) \oplus \bigoplus_{h \in T_1} W_h \oplus \bigoplus_{g \in T_2} \mathrm{Ind}_{H_g}^G V(H_g)$$

which is an $\mathscr{L}(G)$-free nonnegative G-module. We show that V is positive on $\mathscr{D}(G) \smallsetminus \mathscr{E}^2(G)$. Let $(P, H) \in \mathscr{D}(G) \smallsetminus \mathscr{E}^2(G)$. If $(P, H) \notin \mathscr{D}^2(G)$ then $V(G)$ is positive at (P, H) and so is V. Suppose that $(P, H) \in \mathscr{D}^2(G)$. Then $E(G) \cap H \smallsetminus P$ is not empty or $O^2(C_G(h))$ is not a subgroup of P for some element h of $H \smallsetminus P$

of 2-power order. If $E(G) \cap H \smallsetminus P$ is not empty then W_h is positive at (P, H) for $h \in T_1$ with $(h) \cap H \smallsetminus P \neq \emptyset$ by Proposition 2.2 and thus so is V. Suppose that $E(G) \cap H \smallsetminus P$ is empty. Then there is an element h of $H \smallsetminus P$ of 2-power order such that $O^2(C_G(h))$ is not a subgroup of P. In particular $O^2(C_G(h))$ is not the trivial group and then $C_G(h)$ is not a 2-group. Furthermore, since $h \notin E(G)$, the element h has order 2 and thus $h \in T_2$. For $g \in T_2$ with $(g) \cap H \smallsetminus P \neq \emptyset$, the equation $d_{\mathrm{Ind}_{H_g}^G V(H_g)}(P, H) = 0$ implies that $P \geq O^2(H_g) = O^2(C_G(g))$ by [**11**, Lemma 4.3]. Thus we take $g \in T_2$ such that $(g) = (h)$ and then $\mathrm{Ind}_{H_g}^G V(H_g)$ is positive at (P, H). Therefore V is also positive at (P, H). We complete the proof. $\square$

3. Gap complex modules.

A gap module means a real G-representation space which is positive. By seeing the complexification and realification, there is an $\mathscr{L}(G)$-free gap G-module if and only if there is an $\mathscr{L}(G)$-free complex G-module W such that

$$\dim_{\boldsymbol{C}} W^P > 2 \dim_{\boldsymbol{C}} W^H$$

for $(P, H) \in \mathscr{D}(G)$. So, we also use the same words, a gap module, a nonnegative module etc. for complex modules. For an arbitrary element X of the complex representation ring $R(G)$ of G, we define a function $d_X \colon \mathscr{D}(G) \to \boldsymbol{Z}$ by

$$\bigl(\dim_{\boldsymbol{C}} U^P - \dim_{\boldsymbol{C}} V^P\bigr) - 2\bigl(\dim_{\boldsymbol{C}} U^H - \dim_{\boldsymbol{C}} V^H\bigr),$$

where U and V are complex G-modules such that $U - V$ represents X. Note that the set of d_X for $X \in R(G)$ is a complex vector space.

For a complex G-module U, we put

$$U_{\mathscr{L}(G)} = (U - U^G) - \bigoplus_{p \in \pi(G)} (U - U^G)^{O^p(G)}.$$

This G-module $U_{\mathscr{L}(G)}$ is the maximal $\mathscr{L}(G)$-free G-submodule of U.

Let $\mathrm{Irr}_{\mathscr{L}(G)}(G)$ be the subset of $R(G)$ consisting of isomorphism classes of $\mathscr{L}(G)$-free irreducible complex G-modules and let $\mathrm{CycInd}_{\mathscr{L}(G)}(G)$ be the subset of $R(G)$ consisting of isomorphism classes of complex modules $(\mathrm{Ind}_C^G \xi)_{\mathscr{L}(G)}$ for cyclic subgroups C of G and C-modules ξ.

We have the following proposition.

Proposition 3.1. *Let G be a finite group. The following are equivalent.*

(1) *G is a gap group.*
(2) *There exist nonnegative integers k_V for $V \in \mathrm{Irr}_{\mathscr{L}(G)}(G)$ such that*

$$\sum_{V \in \mathrm{Irr}_{\mathscr{L}(G)}(G)} k_V d_V(P,H) > 0$$

for any $(P,H) \in \mathscr{D}(G)$.
(3) *There exist rational numbers q_W for $W \in \mathrm{CycInd}_{\mathscr{L}(G)}(G)$ such that*

$$\sum_{W \in \mathrm{CycInd}_{\mathscr{L}(G)}(G)} q_W d_W(P,H) > 0$$

for any $(P,H) \in \mathscr{D}(G)$.
(4) *There exist integers n_W for $W \in \mathrm{CycInd}_{\mathscr{L}(G)}(G)$ such that*

$$\sum_{W \in \mathrm{CycInd}_{\mathscr{L}(G)}(G)} n_W d_W(P,H) > 0$$

for any $(P,H) \in \mathscr{D}(G)$.

PROOF. An $\mathscr{L}(G)$-free G-module W can be written as

$$\sum_{V \in \mathrm{Irr}_{\mathscr{L}(G)}(G)} k_V V$$

in $R(G)$ for some nonnegative integers k_V and then

$$d_W(P,H) = \sum_{V \in \mathrm{Irr}_{\mathscr{L}(G)}(G)} k_V d_V(P,H).$$

Thus (1) and (2) are equivalent. By the same way it is easy to see that (3) implies (2). Furthermore clearly (3) and (4) are equivalent. For each $V \in \mathrm{Irr}_{\mathscr{L}(G)}(G)$, there exist rational numbers $q_{V,W}$ for $W \in \mathrm{CycInd}_{\mathscr{L}(G)}(G)$ such that

$$V = \sum_{W \in \mathrm{CycInd}_{\mathscr{L}(G)}(G)} q_{V,W} W$$

in $R(G)$ by Artin's theorem [**9**, Section 9.2 Corollary]. Thus (2) implies (3). $\square$

We give a proof of Theorem 1.1.

PROOF OF THEOREM 1.1. If G is a gap group, then all subgroups K with $O^2(G) \triangleleft K \leq G$ and $K \neq O^2(G)$ are gap groups by [**6**, Proposition 3.1]. We show the converse. Suppose that all subgroups K with $O^2(G) \triangleleft K \leq G$, possessing nontrivial cyclic quotients $K/O^2(G)$ and $[K : O^2(G)] = 2$ are gap groups. Take an $\mathscr{L}(K)$-free gap K-module W_K for each such subgroup K. Put

$$V = V(G) \oplus \bigoplus_K \mathrm{Ind}_K^G W_K.$$

We show that V is an $\mathscr{L}(G)$-free gap G-module. Recall that $\mathrm{Ind}_K^G W_K$ is an $\mathscr{L}(G)$-free nonnegative G-module since W_K is an $\mathscr{L}(K)$-free nonnegative K-module [**11**, Lemma 2.4]. Thus V is $\mathscr{L}(G)$-free and nonnegative. Let $(P, H) \in \mathscr{D}(G)$. Note that V is a sum of nonnegative G-modules. If $(P, H) \notin \mathscr{D}^2(G)$ then

$$d_V(P, H) \geq d_{V(G)}(P, H) > 0.$$

Suppose that $(P, H) \in \mathscr{D}^2(G)$. Since $O^{p_0}(G)P = G$, the group P is a nontrivial p_0-group. Thus $[O^2(G)H : O^2(G)] = 2$ and then

$$d_V(P, H) = \sum_K d_{\mathrm{Ind}_K^G W_K}(P, H) \geq d_{\mathrm{Ind}_{O^2(G)H}^G W_{O^2(G)H}}(P, H).$$

It holds that $PeO^2(G)H \in (P\backslash G/O^2(G)H)^{H/P}$ and in particular the set $(P\backslash G/O^2(G)H)^{H/P}$ is not empty. Since W_K is a gap K-module,

$$d_V(P, H) \geq d_{W_{O^2(G)H}}(P, H) > 0$$

by Proposition 4.1.

Therefore we have V is positive at (P, H) and thus V is an $\mathscr{L}(G)$-free gap G-module. □

4. Induced modules and double cosets.

In this section we estimate values of the function d_V for $V \in \mathrm{CycInd}_{\mathscr{L}(G)}(G)$.

Let K, P and H be subgroups of G with $[H : P] = 2$. Then H/P can act on the set $P\backslash G/K$ via $hP \cdot PgK = PhgK$. We frequently compute the number $d_{\mathrm{Ind}_K^G W}(P, H)$ for $(P, H) \in \mathscr{D}^2(G)$ by the following formula.

Proposition 4.1 ([**6**, Lemma 0.6]). *Let K, P and H be subgroups of G with $[H:P]=2$. For a (complex) K-module W, it holds that*

$$d_{\mathrm{Ind}_K^G W}(P,H)=\sum_{PgK\in(P\backslash G/K)^{H/P}} d_W(K\cap g^{-1}Pg, K\cap g^{-1}Hg).$$

Furthermore $PgK\in(P\backslash G/K)^{H/P}$ if and only if $[K\cap g^{-1}Hg : K\cap g^{-1}Pg]=2$.

The following two lemmas are obtained by direct calculation.

Lemma 4.2. *Let G be a finite group such that $G/O^2(G)$ is abelian. For $(P,H)\in\mathscr{D}^2(G)$, if K is a subgroup of $O^2(G)P$, then $d_{(\mathrm{Ind}_K^G W)_{\mathscr{L}(G)}}(P,H)=0$ for any K-module W.*

Proof. Let $(P,H)\in\mathscr{D}^2(G)$. Recall that $O^p(G)P=G$ for any odd prime p. Since

$$\dim(U_{\mathscr{L}(G)})^K=(\dim U^K-\dim U^G)-\sum_{p\in\pi(G)}\left(\dim U^{O^p(G)K}-\dim U^G\right)$$

for a subgroup K of G and a G-module U, then

$$d_{(\mathrm{Ind}_K^G W)_{\mathscr{L}(G)}}(P,H)=d_{\mathrm{Ind}_K^G W}(P,H)-d_{\mathrm{Ind}_K^G W}(O^2(G)P,O^2(G)H).$$

By Proposition 4.1 it suffices to show that both $(P\backslash G/K)^{H/P}$ and $(O^2(G)P\backslash G/K)^{O^2(G)H/O^2(G)P}$ are empty. First we show that $(O^2(G)P\backslash G/K)^{O^2(G)H/O^2(G)P}$ is empty. Suppose that $O^2(G)PgK\in(O^2(G)P\backslash G/K)^{O^2(G)H/O^2(G)P}$. Then $O^2(G)HgK=O^2(G)PgK$. There is an element a of $H\smallsetminus P$ such that $a\in O^2(G)PgKg^{-1}$. Since $G/O^2(G)$ is abelian and $K\le O^2(G)P$, it holds that $a\in O^2(G)P$. It is a contradiction against $[O^2(G)H:O^2(G)P]=2$. Therefore the set $(O^2(G)P\backslash G/K)^{O^2(G)H/O^2(G)P}$ is empty and then so is $(P\backslash G/K)^{H/P}$, since the identity map over G induces a map from $(P\backslash G/K)^{H/P}$ to $(O^2(G)P\backslash G/K)^{O^2(G)H/O^2(G)P}$. □

Lemma 4.3. *Let G be a finite group with $[G:O^2(G)]=2$ and C a cyclic subgroup of G of even order such that $C\cap O^2(G)$ has odd order. Let ξ_j be an irreducible complex C-module whose character sends x^k to $\exp((2jk\pi\sqrt{-1})/|C|)$ $(0\le j<|C|)$. Then*

$$d_{(\mathrm{Ind}_C^G \xi_j)_{\mathscr{L}(G)}}(P,H) = \begin{cases} -|(P\backslash G/C)^{H/P}|+1, & j=0 \\ |(P\backslash G/C)^{H/P}|-1, & j=|C|/2 \\ 0, & j \neq 0, |C|/2 \end{cases}$$

for any $(P,H) \in \mathscr{D}^2(G)$ *with* $d_{\mathrm{Ind}_C^G V(C)}(P,H)=0$.

PROOF. Let x be an element of G such that $\langle x\rangle = C$. It holds that

$$\begin{aligned} & d_{(\mathrm{Ind}_C^G \xi_j)_{\mathscr{L}(G)}}(P,H) \\ &= d_{\mathrm{Ind}_C^G \xi_j}(P,H) - d_{\mathrm{Ind}_C^G \xi_j}(O^2(G)P, O^2(G)H) \\ &= \sum_{PgC \in (P\backslash G/C)^{H/P}} d_{\xi_j}\left(g^{-1}Pg \cap C, g^{-1}Hg\cap C\right) - d_{\mathrm{Ind}_C^G \xi_j}(O^2(G), G) \\ &= \sum_{PgC \in (P\backslash G/C)^{H/P}} d_{\xi_j}(O^2(G)\cap C, C) - d_{\xi_j}(O^2(G)\cap C, C) \\ &= -\frac{|(P\backslash G/C)^{H/P}|-1}{|C|/2} \sum_{k=1}^{|C|/2} \chi_{\xi_j}(x^{2k-1}) \\ &= -\frac{2(|(P\backslash G/C)^{H/P}|-1)}{|C|} \sum_{k=1}^{|C|/2} \exp\left(\frac{2j(2k-1)\pi\sqrt{-1}}{|C|}\right) \\ &= -\frac{2(|(P\backslash G/C)^{H/P}|-1)}{|C|} \exp\left(\frac{2j\pi\sqrt{-1}}{|C|}\right) \sum_{k=1}^{|C|/2} \exp\left(\frac{4j(k-1)\pi\sqrt{-1}}{|C|}\right) \\ &= \begin{cases} -|(P\backslash G/C)^{H/P}|+1, & j=0 \\ |(P\backslash G/C)^{H/P}|-1, & j=|C|/2 \\ 0, & j\neq 0, |C|/2. \end{cases} \end{aligned}$$

□

LEMMA 4.4. *Let G be a finite group, P a subgroup of G of odd order, H a subgroup of G with $[H:P]=2$, and C a cyclic subgroup of G of even order. Let a be an element of C of order 2. Then $(P\backslash G/C)^{H/P}=\emptyset$ if and only if the conjugacy class (a) in G does not meet with H. Furthermore, if $bab^{-1}\in H$ for $b \in G$, then*

$$(P\backslash G/C)^{H/P} = P\backslash PbC_G(a)/C.$$

In particular, if $b^{-1}Pb \geq O^2(C_G(a))$ and $|C|/2$ is odd, then

$$|(P\backslash G/C)^{H/P}| = \frac{|C_G(a)|}{2|O^2(C_G(a))|}.$$

Proof. Let $PgC \in (P\backslash G/C)^{H/P}$. Take an element h of H of order 2. Then $h \notin P$. Then $PgC = PhgC$ which implies that $xh \in gCg^{-1}$ for some $x \in P$. An element of $\langle xh \rangle$ of order 2 forms $x'h = gag^{-1}$ for some $x' \in P$. Since any elements of H of order 2 are conjugate in H, there is $y \in H$ such that $y^{-1}hy = gag^{-1}$. Thus $(P\backslash G/C)^{H/P}$ is not empty if and only if a and h are conjugate in G. We may assume that $y \in P$ since if necessary we may replace y by yh. If $bab^{-1} = h$ for some $b \in G$, then $g \in y^{-1}bC_G(a) \subset PbC_G(a)$ since $y^{-1}bab^{-1}y = gag^{-1}$ and thus $(P\backslash G/C)^{H/P} = P\backslash PbC_G(a)/C$.

Suppose that $b^{-1}Pb \geq O^2(C_G(a))$ and $|C|/2$ is odd. Set $P' = b^{-1}Pb$ and $H' = b^{-1}Hb$. The group C is a subgroup of $C_G(a)$, $a \in H'$ and $P' \geq O^2(C_G(a))$. Since a map

$$(P' \cap C_G(a))\backslash C_G(a) \to P'\backslash P'C_G(a)$$

sending $(P' \cap C_G(a))q$ to $P'q$ is a $C_G(a)$-bijection as right $C_G(a)$-sets, we have

$$\begin{aligned} P\backslash PbC_G(a)/C &\cong P'\backslash P'C_G(a)/C \cong (P' \cap C_G(a))\backslash C_G(a)/C \\ &= O^2(C_G(a))\backslash C_G(a)/C \cong C_G(a)/O^2(C_G(a))C \end{aligned}$$

and thus

$$|(P\backslash G/C)^{H/P}| = \frac{|C_G(a)|}{2|O^2(C_G(a))|}. \qquad \square$$

Let $\mathrm{CycInd}^{\mathrm{out}}_{\mathscr{L}(G)}(G)$ be the subset of $\mathrm{CycInd}_{\mathscr{L}(G)}(G)$ consisting of isomorphism classes of complex modules over cyclic subgroups C of G with $C \not\leq O^2(G)$. We extend Proposition 3.1 slightly.

Proposition 4.5. *Let G be a finite group with $\mathscr{P}(G) \cap \mathscr{L}(G) = \emptyset$. The following are equivalent.*

(1) *G is a gap group.*
(2) *There exist integers k_V for $V \in \mathrm{Irr}_{\mathscr{L}(G)}(G)$ such that*

$$\sum_{V \in \mathrm{Irr}_{\mathscr{L}(G)}(G)} k_V d_V(P,H) > 0$$

for any $(P,H) \in \mathscr{E}^2(G)$.

(3) *There exist rational numbers q_W for $W \in \mathrm{CycInd}^{\mathrm{out}}_{\mathscr{L}(G)}(G)$ such that*

$$\sum_{W \in \mathrm{CycInd}^{\mathrm{out}}_{\mathscr{L}(G)}(G)} q_W d_W(P,H) > 0$$

for any $(P,H) \in \mathscr{E}^2(G)$.

(4) *There exist integers n_W for $W \in \mathrm{CycInd}^{\mathrm{out}}_{\mathscr{L}(G)}(G)$ such that*

$$\sum_{W \in \mathrm{CycInd}^{\mathrm{out}}_{\mathscr{L}(G)}(G)} n_W d_W(P,H) > 0$$

for any $(P,H) \in \mathscr{E}^2(G)$.

Proof. We denote by $\boldsymbol{C}[G]$ the regular representation space. For an $\mathscr{L}(G)$-free complex G-module V, $-V + m\boldsymbol{C}[G]_{\mathscr{L}(G)}$ becomes an $\mathscr{L}(G)$-free G-module for a sufficiently large integer m, since $\boldsymbol{C}[G]_{\mathscr{L}(G)}$ contains every $\mathscr{L}(G)$-free irreducible complex G-module. Let U be an $\mathscr{L}(G)$-free nonnegative complex G-module which is positive on $\mathscr{D}(G) \smallsetminus \mathscr{E}^2(G)$ by Theorem 2.3. We may assume that U has $\boldsymbol{C}[G]_{\mathscr{L}(G)}$ as a submodule since if necessary we replace U by $U \oplus \boldsymbol{C}[G]_{\mathscr{L}(G)}$. For $(P,H) \in \mathscr{D}(G)$, if $(P,H) \notin \mathscr{E}^2(G)$ then U is positive at this pair (P,H) and thus there is an integer m such that $X + mU$ is an $\mathscr{L}(G)$-free G-module which is positive on $\mathscr{D}(G) \smallsetminus \mathscr{E}^2(G)$ for $X \in R(G)$. In particular, if $d_X(P,H) > 0$ for an arbitrary $(P,H) \in \mathscr{E}^2(G)$, then $X + mU$ is an $\mathscr{L}(G)$-free gap G-module for some m. Therefore (1) and (2) are equivalent.

Let C be a cyclic subgroup of $O^2(G)$ and ξ a complex C-module. For $(P,H) \in \mathscr{D}^2(G)$, the set $(P\backslash G/C)^H$ is empty, since $[H : P] = [O^2(G)H : O^2(G)P] = 2$. Thus $d_{(\mathrm{Ind}_C^G \xi)_{\mathscr{L}(G)}}(P,H) = 0$. Therefore, it holds that $d_W(P,H) = 0$ for any $(P,H) \in \mathscr{E}^2(G)$ and any $W \in \mathrm{CycInd}_{\mathscr{L}(G)}(G) \smallsetminus \mathrm{CycInd}^{\mathrm{out}}_{\mathscr{L}(G)}(G)$. The remaining part of the proof is similar as the proof of Proposition 3.1 and so we omit it. □

5. Proof of Theorem 1.2.

Throughout this section we let G be a finite group such that $\mathscr{P}(G) \cap \mathscr{L}(G) = \emptyset$, $[G : O^2(G)] = 2$, and $O^{p_0}(G) \neq G$ for a unique odd prime p_0. Then $E_2(G)$ is a subset of elements g of G outside $O^2(G)$ of order 2 such that $O^2(C_G(g))$ is not a p_0-group and $E_4(G)$ is the set of elements of order a power of 2 greater than 2.

For a subset $\mathscr{E}$ of $\mathscr{D}(G)$ suppose that there is an $\mathscr{L}(G)$-free nonnegative G-module W such that W is positive on $\mathscr{D}(G) \smallsetminus \mathscr{E}$. If an $\mathscr{L}(G)$-free G-module V satisfies that V is positive on $\mathscr{E}$, then $V \oplus (\dim V + 1)W$ is a gap G-module. Thus we give a condition for that there is an $\mathscr{L}(G)$-free G-module V which is positive on $\mathscr{E}^2(G)$.

We write $\boldsymbol{x} \geq \boldsymbol{y}$ (resp. $\boldsymbol{x} > \boldsymbol{y}$), if $x_i \geq y_i$ (resp. $x_i > y_i$) for any i, where $\boldsymbol{x} = {}^t(x_1, \dots, x_k)$ and $\boldsymbol{y} = {}^t(y_1, \dots, y_k)$. Let $\boldsymbol{Q}$ be the set of rational numbers.

Let S_1 be a complete set of representatives of conjugacy classes of elements of 2-power order which does not lie in $O^2(G)$ and S_3 the set consisting of all elements of S outside $E(G)$. Fix a Sylow 2-subgroup G_2 of G. We may assume that $C_{G_2}(x)$ is a Sylow 2-subgroup of $C_G(x)$ for each $x \in S_1$ without loss of generality. Since it is not easy to determine $\psi(x)$ for $g \in G$, we consider a weaker condition than one of $E(G)$. Note that p_0 contains in $\psi(x)$ for any element x of G, since $O^{p_0}(G) \neq G$. Let S_2 be the set consisting of all elements g of S_1 such that $|g| \geq 4$, or $|g| = 2$ and $O^2(C_G(g))$ is a p_0-group. Clearly, $S_3 \subset S_2 \subset S_1$. Also note that P is a nontrivial p_0-group for $(P, H) \in \mathscr{D}^2(G)$, since $O^{p_0}(G)P = G$. Let $\mathscr{E}_0^2(G)$ be the set consisting of all $(P, H) \in \mathscr{D}^2(G)$ such that $O^2(C_G(g))$ is a subgroup of P for an arbitrary element g of $H \smallsetminus P$ of order 2. $H \smallsetminus P$ has no element of order divisible by 4 since $|H| = 2{p_0}^i$ for some $i > 0$ and all elements of $H \smallsetminus P$ of order 2 are conjugate by Sylow's theorem. It holds that

$$\mathscr{E}^2(G) \subset \mathscr{E}_0^2(G) \subset \mathscr{D}^2(G) \subset \mathscr{D}(G).$$

Set $n = |\mathrm{CycInd}^{\mathrm{out}}_{\mathscr{L}(G)}(G)|$ and $m = |\mathscr{E}_0^2(G)|$. Further, let $D = (d_V(P, H))$ be an $m \times n$ matrix whose columns correspond to $V \in \mathrm{CycInd}^{\mathrm{out}}_{\mathscr{L}(G)}(G)$ and rows correspond to $(P, H) \in \mathscr{E}_0^2(G)$.

Since $[G : O^2(G)] = 2$, we have

$$\sum_{a \in S_2} \frac{|G_2|}{|C_{G_2}(a)|} \leq \sum_{a \in S_1} \frac{|G_2|}{|C_{G_2}(a)|} \leq |G_2| \left(1 - \frac{|G_2|}{|O^2(G) \cap G_2|}\right) = \frac{|G_2|}{2}$$

and thus

$$\sum_{a \in S_2} \frac{2}{|C_{G_2}(a)|} \leq \sum_{a \in S_1} \frac{2}{|C_{G_2}(a)|} \leq 1.$$

In particular, if $E(G)$ is not empty, then

$$\sum_{a \in S_2} \frac{2}{|C_{G_2}(a)|} < 1.$$

Note that any element of S_2 has order 2. Let $S_2 = \{x_1, \dots, x_r\}$ and put $s_j = |C_{G_2}(x_j)/\langle x_j \rangle|$ for $1 \leq j \leq r$. Then

$$\sum_{j=1}^{r} s_j^{-1} \le 1.$$

Put

$$A = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r) = \begin{pmatrix} s_1 - 1 & -1 & -1 & \cdots & -1 \\ -1 & s_2 - 1 & -1 & \cdots & -1 \\ -1 & -1 & s_3 - 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 \\ -1 & -1 & \cdots & -1 & s_r - 1 \end{pmatrix}.$$

THEOREM 5.1. *The following are equivalent.*

(1) *G is a gap group.*
(2) $\sum_{j=1}^{r} s_j^{-1} \neq 1$.
(3) $Z(G) := \{\boldsymbol{y} \in \boldsymbol{Z}^r \mid A\boldsymbol{y} = \boldsymbol{0},\ \boldsymbol{y} \ge \boldsymbol{0},\ \boldsymbol{y} \neq \boldsymbol{0}\}$ *is empty.*

PROOF. Let $(P, H) \in \mathscr{E}_0^2(G)$. Since $|P|$ is odd and $[H : P] = 2$, there is a unique element x_j of S_2 such that $(x_j) \cap H \smallsetminus P \neq \emptyset$. Let C be a cyclic subgroup C of G with $C \not\le O^2(G)$. Since $[G : O^2(G)] = 2$, if $gCg^{-1} \cap H \smallsetminus P$ has an element of order 2 for an element $g \in G$, the order $|C|$ is not divisible by 4. Thus if $|C|$ is divisible by 4, then $(P\backslash G/C)^H$ is empty. In addition, by Lemma 4.3, for $V \in \mathrm{CycInd}^{\mathrm{out}}_{\mathscr{L}(G)}(G)$, it holds that

$$d_V(P, H) = -1,\ 0,\ \pm\bigl(|(P\backslash G/C)^{H/P}| - 1\bigr),$$

where V is an $\mathscr{L}(G)$-free G-module induced from a cyclic subgroup C with $C \not\le O^2(G)$.

If $(x_j) \cap C = \emptyset$ if and only if $d_V(P, H) = -1$, since $|(O^2(G)P\backslash G/C| = 1$. Otherwise, suppose that $a := g^{-1}x_j g \in C$ for some $g \in G$. Then $|C|/2$ is odd since $[G : O^2(G)] = 2$. Let $yx_jy^{-1} \in H \smallsetminus P$ for some $y \in G$. Since $(yg)a(yg)^{-1} \in H \smallsetminus P$, P contains $O^2(C_G((yg)a(yg)^{-1}))$. By [**11**, Lemma 4.3], it holds that $d_{\mathrm{Ind}_C^G V(C)}(P, H) = 0$. Thus,

$$\bigl|(P\backslash G/C)^{H/P}\bigr| = \frac{|C_G(x_j)|}{2|O^2(C_G(x_j))|} = \frac{s_j}{2}$$

by Lemma 4.4. Let D' be the matrix with r rows by removing all duplicate rows from the $m \times n$ matrix $D = (d_V(P, H))$. The inequality $D\boldsymbol{x} > \boldsymbol{0}$ is equivalent to the inequality $D'\boldsymbol{x} > \mathbf{0}$. Vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r$ are appeared in column vectors of the matrix D', and each column vector of D' is $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r$, $\mathbf{0}$, or ${}^t(-1, \ldots, -1)$. If necessary, permuting the column vectors of D', we may assume that $D' = (A, A')$ for some $r \times (n-r)$ matrix A'.

Suppose that $\sum_{j=1}^r s_j^{-1} \neq 1$, that is, $\sum_{j=1}^r s_j^{-1} < 1$. Setting $t = 1 - \sum_{j=1}^r s_j^{-1} > 0$, we have

$$D' \begin{pmatrix} (ts_1)^{-1} \\ \vdots \\ (ts_r)^{-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = A \begin{pmatrix} (ts_1)^{-1} \\ \vdots \\ (ts_r)^{-1} \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} > \mathbf{0}.$$

Thus there is a vector $\boldsymbol{x} \geq \mathbf{0}$ in $\boldsymbol{Q}^n$ such that $D\boldsymbol{x} > \mathbf{0}$ which implies that G is a gap group by Proposition 4.5. Furthermore, there is a vector $\boldsymbol{x} > 0$ such that $A\boldsymbol{x} > \mathbf{0}$. Thus for an arbitrary nonzero vector $\boldsymbol{y} \geq 0$, it holds that ${}^t\boldsymbol{x}A\boldsymbol{y} > 0$ and so $Z(G)$ is an empty set. Therefore (2) implies both (1) and (3).

Conversely suppose that $\sum_{j=1}^r s_j^{-1} = 1$. Then A is a singular symmetric matrix. In fact, $A\boldsymbol{y} = \mathbf{0}$ for $\boldsymbol{y} = {}^t(s_1^{-1}, \ldots, s_r^{-1}) \in \boldsymbol{Q}^r$. Thus $\boldsymbol{y} \in Z(G)$ which means that (3) implies (2). If there is a vector $\boldsymbol{x} > 0$ such that $A\boldsymbol{x} > \mathbf{0}$, then ${}^t\boldsymbol{y}(A\boldsymbol{x})$ must be positive, since $\boldsymbol{y} > \mathbf{0}$, but $({}^t\boldsymbol{y}A)\boldsymbol{x} = 0$, a contraction. Thus (1) implies (2). $\square$

PROOF OF THEOREM 1.2. Note that $\sum_{a \in S_1}(2/|C_{G_2}(a)|) \leq 1$. In particular if $E_4(G) \neq \emptyset$ then

$$\sum_{j=1}^r s_j^{-1} < \sum_{a \in S_1} \frac{2}{|C_{G_2}(a)|} \leq 1$$

and thus G is a gap group by Theorem 5.1. Suppose that $E_4(G) = \emptyset$. The set S_2 is a complete set of conjugacy classes (g) of elements g of G outside $O^2(G)$ of order 2 such that $O^2(C_G(g))$ is a p_0-group. Thus the assertion immediately follows from Theorem 5.1, since

$$\sum_{j=1}^r s_j^{-1} = \sum_{a \in S_2} \frac{2}{|C_G(a)/O^2(C_G(a))|}. \qquad \square$$

Let A be a finite abelian group and h an element of order 2. Let D be a finite group generated by A and h with relation $hah = a^{-1}$ for any $a \in h$. We call the group D a generalized dihedral group.

Theorem 5.1 implies the following corollaries.

COROLLARY 5.2. *If one of the following properties holds, then G is a gap group.*

(1) *$E(G)$ is not empty.*
(2) *There are two elements of $G_2 \setminus O^2(G)$ of order 2 which are conjugate in G but not conjugate in G_2.*
(3) *G_2 is not a generalized dihedral group.*

Here G_2 is a Sylow 2-subgroup of G.

PROOF. If (1) or (2) holds, then it is easy to see that $\sum_{j=1}^{r} s_j^{-1} < 1$ and thus G is a gap group. Now we show the case (3). Suppose that G is not a gap group. Then $E_4(G) = \emptyset$ by (1). Thus any element of $G_2 \setminus O^2(G)$ has order 2. Therefore G_2 is a generalized dihedral group. □

COROLLARY 5.3. *Let K be a finite group such that $O^p(K) \neq K$ for $p = 2$ and a unique odd prime p and $\mathscr{L}(K) \cap \mathscr{P}(K) = \emptyset$ and K_2 a Sylow 2-subgroup of K. If $O^2(K) \cap K_2$ is not abelian, then K is a gap group.*

PROOF. Suppose that K is not a gap group. By Theorem 1.1, there is a subgroup L of G with $[L : O^2(K)] = 2$ which is not a gap group. Then a Sylow 2-subgroup L_2 of L is a generalized dihedral group by Corollary 5.2 (3). Since $O^2(L) = O^2(K)$, the group $O^2(K) \cap K_2 = O^2(L) \cap L_2$ is abelian. □

By the similar proof of Theorem 5.1 replacing S_2 and $\mathscr{E}_0^2(G)$ by S_3 and $\mathscr{E}^2(G)$ respectively, we have the following theorem and omit the proof.

THEOREM 5.4. *The group G is a gap group if and only if*

$$\sum_{g \in S_3} \frac{2}{|C_{G_2}(g)|} \neq 1.$$

COROLLARY 5.5. *Suppose that $\sum_{j=1}^{r} s_j^{-1} = 1$. Then $S_2 = S_3$. Letting g be an element of G outside $O^2(G)$ of order 2 and N a subgroup of G, if $g \in N$ and $O^p(N) \neq N$ for an odd prime p, then $p = p_0$.*

PROOF. By the assumption, G is not a gap group. Then

$$\sum_{g\in S_2} \frac{2}{|C_{G_2}(g)|} = \sum_{g\in S_3} \frac{2}{|C_{G_2}(g)|} = 1$$

and thus $S_2 = S_3$. Let g be an element of G outside $O^2(G)$ of order 2 and N a subgroup of G such that $g \in N$. Since an element conjugate to g lies in S_3 then $\psi(g) = \{p_0\}$. Thus if $O^p(N) \neq N$ for an odd prime p, then $p = p_0$. □

References

[1] K. H. Dovermann and M. Herzog, Gap conditions for representations of symmetric groups, J. Pure Appl. Algebra, **119** (1997), 113–137.

[2] W. Lück and I. Madsen, Equivariant L-theory. I, Math. Z., **203** (1990), 503–526.

[3] E. Laitinen and M. Morimoto, Finite groups with smooth one fixed point actions on spheres, Forum Math., **10** (1998), 479–520.

[4] E. Laitinen, M. Morimoto and K. Pawałowski, Deleting-Inserting Theorem for smooth actions of finite nonsolvable groups on spheres, Comment. Math. Helv., **70** (1995), 10–38.

[5] M. Morimoto, Equivariant surgery theory: deleting-inserting theorems of fixed point manifolds on spheres and disks, K-Theory, **15** (1998), 13–32.

[6] M. Morimoto, T. Sumi and M. Yanagihara, Finite groups possessing gap modules, In: Geometry and Topology: Aarhus, 1998, **258**, Contemp. Math., Amer. Math. Soc., Providence, RI, 2000, pp. 329–342.

[7] T. Petrie, Pseudoequivalences of G-manifolds, In: Algebraic and Geometric Topology, Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976, Part 1, Proc. Sympos. Pure Math., **XXXII**, Amer. Math. Soc., Providence, RI, 1978, pp. 169–210.

[8] T. Petrie and J. D. Randall, Transformation groups on manifolds, Monogr. Textbooks Pure Appl. Math., **82**, Marcel Dekker Inc., New York, 1984.

[9] J.-P. Serre, Linear Representations of Finite Groups, Springer-Verlag, New York, 1977, translated from the second French edition by Leonard L. Scott, Graduate Texts in Math., **42**.

[10] T. Sumi, Gap modules for direct product groups, J. Math. Soc. Japan, **53** (2001), 975–990.

[11] T. Sumi, Gap modules for semidirect product groups, Kyushu J. Math., **58** (2004), 33–58.

Toshio Sumi

Faculty of Design
Kyushu University
4-9-1, Shiobaru, Minami-ku
Fukuoka 815-8540, Japan
E-mail: sumi@design.kyushu-u.ac.jp